Unawareness†

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Abstract

I construct a state space model with unawareness following Aumann (1976). Dekel, Lipman and Rustichini (1998) show that standard state space models are incapable of representing unawareness. My model circumvents this impossibility result by allowing the agent to have a subjective state space and a subjective information structure when he is unaware of some possibilities. This is achieved by modelling information as a pair, consisting of both factual information and *awareness information*, which captures the agent's frame of mind. The model exhibits nice properties parallel to those in the standard model. A characterization of common knowledge with unawareness is supplied.

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"There are things we know that we know. There are known unknowns - that is to say, there are things that we now know we don't know. But there are also unknown unknowns. There are things we do not know we don't know."

U.S. Secretary of Defense Donald Rumsfeld

1 Introduction

Unawareness refers to things that one does not know, that one does not know that one does not know, and so on *ad infinitum*. In real life, *formulating* a decision problem, including recognizing all relevant uncertainties and available options, is at least as important as finding the solution to the formulated problem. Being unaware of some aspects of the situation is a common problem encountered at this stage. A recent example is the 9/11 attack. Prior to it, we did not know that terrorists might use civilian aircraft as a weapon, and more importantly, we did not know that we did not know this. We were simply unaware of this possibility.

Unawareness, or unforeseen contingencies, has attracted much attention, especially in the literature of preferences for flexibility and incomplete contracts. The idea that one prefers to retain some flexibility when anticipating unforeseen contingencies has been explored in Kreps (1979, 1992) and many subsequent papers (Dekel, Lipman and Rustichini 2001, Ergin 2002, Nehring 1999, Kraus and Sagi 2002, Ozdenoren 2002). Unawareness is also a potential reason for contractual incompleteness. Intuitively, if parties are aware that they may be unaware of some possibilities, an incomplete contract combined with a reputational device may turn out to be the best thing to do (Kreps 1992). However, a rigorous analysis linking unawareness, preferences for flexibility and incomplete contracts has yet to be developed.

This paper deals with information structures with unawareness. Dekel, Lipman and Rustichini (1998a) (henceforth DLR) show that standard state space models, including possibility correspondence models, are incapable of handling unawareness. Logical systems of unawareness have been explored in Modica and Rustichini (1994, 1999) and Halpern (2001). However, a set-theoretic model that allows nontrivial unawareness following Aumann (1976) is still lacking. In this paper, I provide such a model.

As an illustration, consider the following episode: Sherlock Holmes and Watson are investigating a crime. A horse has been stolen and the keeper was killed. From the narration of the local police, Holmes notices the dog in the stable did not bark that night and hence concludes that there was no intruder in the stable. Watson, on the other hand, although he also knows the dog did not bark – he mentioned this fact to Holmes – somehow does not come up with the inference that there was no intruder.[1]

The feature I would like to capture in this story is the following. Watson is unaware of the possibility that there was no intruder. However, he does understand that

[1]The Watson story was introduced into economics by Bacharach (1985) and was subsequently discussed in almost every paper dealing with unawareness.

the dog would bark if and only if there were an intruder and would have deduced that there was an intruder had the dog actually barked. In fact, had someone asked Watson, "Could there have been an intruder in the stable that night?" He would have recognized his negligence and replied, "Of course not, the dog did not bark!"

Intuitively, Watson has a frame of mind that contains all his reasoning. When he is unaware of the uncertainty of an intruder, his frame of mind contains nothing regarding an intruder. Consequently, although he knows the dog did not bark, this information does not "ring a bell" in his mind. To model this, I let Watson have a *subjective state space* and a *subjective information structure* that characterize his frame of mind at each "full" state.[2] Let $a = (a', \Delta)$, $b = (b', \Delta)$, where a' stands for "there was an intruder," b' stands for "there was no intruder" and Δ stands for "*cogito ergo sum*," an implicit specification in every full state of the world.[3] The full state space is $\{a, b\}$. The dog barked in a and did not bark in b. Watson knows whether the dog barked, which generates the information partition $\{\{a\}, \{b\}\}$. In b, Watson is unaware of the uncertainty of an intruder, and hence a and b are beyond his frame of mind. His subjective state space is represented by the singleton set $\{\Delta\}$. Since a and b do not belong to this space, the information partition $\{\{a\}, \{b\}\}$ is not recognized. To Watson, the information he has is just the trivial partition of the subjective state space $\{\{\Delta\}\}$. Not only does Watson not know that the event "there was no intruder"$(=\{b\})$ occurred; but the event is also not even present in Watson's mind.

Suppose someone asks Watson, "Could there have been an intruder in the stable that night?" The question, although it does not reveal any *facts*, is informative in the sense that it reminds Watson of the *uncertainty* of an intruder. Now Watson adds the specification of whether there was an intruder to his subjective state space, updates it to the full state space $\{a, b\}$, and hence recognizes the information partition $\{\{a\}, \{b\}\}$, obtaining the knowledge "there was no intruder" as a result of simply being asked a question. This is a novel feature of information processing with unawareness. In standard models, information has always been restricted to the revelation of new facts. However, with nontrivial unawareness, since "unknown unknowns" affect the outcomes of the agent's decision, a message revealing these unknown unknowns is informative, even if it merely turns them into "known unknowns."[4]

This observation is the key to modelling unawareness. With nontrivial unawareness, information should include both the relevant uncertainties themselves and the resolution of uncertainties. I refer to information about relevant uncertainties as *awareness*

[2] For clarity, I refer to the "objective" state space and an "objective" state as the *full state space* and a *full state* respectively.

[3] Δ is a technical convenience to avoid letting Watson have an empty subjective state space. See Section 3.2 for details.

[4] Aragones, Gilboa, Postlewaite and Schmeidler (2003) give a complexity argument for such "fact-free learning." They show that, finding a set of regressors to obtain a given R^2 value in linear regression is an NP-complete problem. Loosely speaking, one can view their model as providing a possible explanation of why human beings have unawareness problem.

information and information about the resolution of uncertainties as *factual information*. The agent's information processing is restricted to those uncertainties *of which he is aware*. More specifically, the agent has a *subjective state space* and a *subjective information structure*. Each *subjective state* only specifies the resolution of those uncertainties contained in the agent's awareness information. Assuming the agent always knows what he is aware of,[5] his subjective information structure is captured by *subjective factual information*, the factual information restricted to the uncertainties contained in his awareness information.

To construct the model, I explore a particular product structure of the state space.[6] Intuitively, one can think of a relevant uncertainty as a question to which the agent would like to know the answer in order to make a better decision. A full state corresponds to a complete list of answers, one for each question. Likewise, a subjective state corresponds to an incomplete list that only contains answers to the questions of which the agent is aware. It is then natural to represent awareness information by the collection of sets of answers and to define the corresponding state space as the Cartesian product of these sets. The factual information is captured by a *full possibility correspondence* defined over the full state space as in the standard model. The subjective factual information is captured by the projection of the factual information on the corresponding subjective state space defined by the awareness information. For example, in the Watson story, the relevant question is "was there an intruder?" and the full state space $\{a, b\} = \{(a', \Delta), (b', \Delta)\}$ is the Cartesian product of $\{a', b'\}$ and $\{\Delta\}$. In b, Watson's awareness information is only $\{\{\Delta\}\}$, and hence his subjective state space is $\{\Delta\}$, the Cartesian product of the singleton set $\{\Delta\}$. Consequently, Watson does not recognize his factual information $\{b\}$, but only recognizes the projection of it on his subjective state space, which is again $\{\Delta\}$.

Each subjective state leaves some relevant questions unanswered, and hence corresponds to many full states that only differ in their answers to the questions the subjective state omits and coincide with the subjective state on the answers to the questions the latter does include. Thus a subjective state as a singleton set only contains as much factual information as the corresponding *set* of full states. For instance, the subjective state Δ conveys the same facts as the event $\{a, b\}$ in the full state space. One implication of this observation is that when the agent is unaware of something, his frame of mind only contains a "vague" picture of possible worlds. The subjective state space is incomplete with respect to the full state space in the sense of omitting "dimensions," not omitting "points." In other words, unawareness is more than the inability to imagine some possible scenarios; rather, it is the inability to imagine *any* scenario *precisely*.

The same facts have multiple representations in this model. This is because in the presence of nontrivial unawareness, information content in events consists of awareness

[5]For a precise formulation, see the "rational awareness" assumption in section 3.1.

[6]The product structure does not impose real limitations on the model. I give an intuitive account of why this is true in section 3. See footnotes 11 and 14. More rigorously, one can construct the unawareness model based on an arbitrary full state space, which, under mild conditions, is isomorphic to the product model. The details are available in the supplementary note.

information in addition to factual information. For instance, $\{\Delta\}$ and $\{a, b\}$ are different events even though they convey the same facts. In b, Watson knows the former, but not the latter. It is natural to characterize the agent's knowledge of an event to be "the set of full states where the agent *is aware of* the event, and is sure that all facts contained in the event are true." That is, with unawareness, the knowledge hierarchy should be characterized within the subjective state spaces. I then show the model captures the essence of unawareness and satisfies the analogues of the standard properties of knowledge.

I further consider the multi-agent case where agents reason about each other's awareness information as well as factual information: *everyone reasons about everyone's frames of mind within his own frame of mind*. Intuitively, an implicit assumption of common knowledge is that everyone is aware of the event involved, everyone knows that everyone is aware of the event involved, and so on. Extending the product model to the multi-agent setting, I clarify the restrictions this "common awareness" requirement imposed on the generalized information structure. The resulting characterization of common knowledge is a natural generalization of the classic characterization of Aumann's.

The rest of the paper is organized as follows: Section 2 reviews the possibility correspondence model and DLR's impossibility results. Section 3 presents the model of unawareness, which I dub "the product model" for the use of the product structure. Section 4 characterizes the knowledge hierarchy with nontrivial unawareness. Section 5 deals with multi-agent information processing with unawareness. I comment on the modelling approach of the product model in Section 6. Section 7 concludes. Proofs not found in the text are collected in Appendix A. Appendix B contains the details of the multi-agent model.

2 A Review of the Standard Model

The standard model, also known as the possibility correspondence model, consists of a state space Ω and a possibility correspondence $P : \Omega \to 2^{\Omega} \setminus \{\emptyset\}$. Each state $\omega \in \Omega$ completely specifies the resolution of all relevant uncertainties. An "event" in the ordinary usage of the term corresponds to a set of states in the model. For instance, the informal idea of the event that "there was an intruder" is formally taken to be the set of states where there was an intruder.

With this formulation, one can identify logical relations with set operations: set inclusion "$\subseteq$", set intersection "$\cap$", set union "$\cup$" and set complement (with respect to the state space) "$\setminus$" correspond to logical consequence "$\rightarrow$", conjunction "$\wedge$", disjunction "$\vee$" and negation "$\neg$" respectively.[7]

In general, the agent does not observe the true state, but is only informed of an event. Such information is factual information since it concerns only the resolution of uncertainties. A particularly nice information structure is an *information partition*,

[7]See Aumann (1976), Fagin, Halpern, Moses and Vardi (1995), Geanakoplos (1990, 1992) and Rubinstein (1998) for details about the set-theoretic approach to modelling knowledge.

which, roughly speaking, means that the state space is the disjoint union of some events, and the agent is informed of the corresponding event at every state.

Formally, a possibility correspondence P is used to characterize the agent's information structure. P associates each state ω with a nonempty event $P(\omega)$, which is interpreted as the agent's information at ω. The idea is, at ω, the agent considers $P(\omega)$ to be the set of possible states.

Definition 1 *P induces an* **information partition** *over the state space if (1) for any $\omega \in \Omega$, $\omega \in P(\omega)$; and (2) for any $\omega, \omega' \in \Omega$, $\omega' \in P(\omega)$ implies $P(\omega') = P(\omega)$.*

If P induces an information partition, then (Ω, P) is an *information partition model*. Otherwise, it is a *non-partitional* model.

Knowledge is characterized as "truth in all possible states." Intuitively, if there was no intruder in every state Holmes considers possible, then for Holmes, there is no uncertainty left regarding this event, i.e., he knows "there was no intruder." Formally, for any event $E \subseteq \Omega$, define the knowledge operator $K : 2^\Omega \to 2^\Omega$ by

$$K(E) = \{\omega : P(\omega) \subseteq E\}$$

$K(E)$ is the set of states in which the agent knows E, and hence is interpreted as the event "the agent knows E." To see that it makes sense to interpret K as knowledge, consider the following properties. For any $E, F \subseteq \Omega$,

K1 *Necessitation*: $K(\Omega) = \Omega$

K2 *Monotonicity*: $E \subseteq F \Rightarrow K(E) \subseteq K(F)$

K3 *Conjunction*: $K(E) \cap K(F) = K(E \cap F)$[8]

K4 *The axiom of knowledge*: $K(E) \subseteq E$

K5 *The axiom of transparency*: $K(E) \subseteq KK(E)$[9]

K6 *The axiom of wisdom*: $\neg K(E) \subseteq K\neg K(E)$

A statement like "A is A" is universally true and the agent should know this. Indeed, a tautology is represented by the event Ω, and hence necessitation corresponds to knowledge of tautologies. Monotonicity says the agent is able to perform logical deductions. If E implies F and the agent knows E, then he knows F. In other words, the agent knows the logical consequences of his knowledge. Conjunction says the agent knows the events E and F if and only if he knows the event "E and F." K1-3 are basic properties of knowledge that make the characterization sensible. Without them, it is

[8]Note that conjunction implies monotonicity.

[9]In places where there is no risk of confusion, I omit the parentheses when applying the operators.

not clear what it means "to know" something. In contrast, the next three axioms K4-6 reflect the agent's *rationality* in information processing. The axiom of knowledge says if the agent knows something, it must be true. The axiom of transparency says if the agent knows something, he knows that he knows it. The axiom of wisdom says if the agent does not know something, he knows that he does not know it.

Theorem 1 *(Geanakoplos 1990, Rubinstein 1998) In the possibility correspondence model (Ω, P), knowledge satisfies K1-3. It satisfies K4-6 if P induces an information partition.*

It is obvious that the axiom of wisdom prevents an information partition model from having nontrivial unawareness. DLR further show that *any* possibility correspondence model is incapable of handling unawareness and that it is the state space specification that causes the problem. DLR introduce an unawareness operator: $U : 2^{\Omega} \to 2^{\Omega}$, where $U(E)$ is the set of states where the agent is unaware of E, and hence is interpreted as the event "the agent is unaware of E." They consider three intuitive properties of unawareness: for any event $E \subseteq \Omega$,

U1 *Plausibility*: $U(E) \subseteq \neg K(E) \cap \neg K \neg K(E)$

U2 *AU introspection*: $U(E) \subseteq UU(E)$

U3 *KU introspection*: $KU(E) = \emptyset$

Plausibility says if one is unaware of something, then one does not know it, and does not know that one does not know it. AU introspection says if one is unaware of something, then one must be unaware of the possibility of being unaware of it. KU introspection says under no circumstances can one know exactly what one is unaware of.

DLR show that the combination of these three axioms implies one critical property of unawareness: whenever the agent is unaware of something, he must not know the state space. That is, U1-3 imply $U(E) \subseteq \neg K(\Omega)$ for any $E \subseteq \Omega$. But then adding necessitation or monotonicity eliminates nontrivial unawareness.

Remark 1

DLR further show that a triviality result holds for all standard state space models. Roughly speaking, a standard state space model is one that implicitly embeds certain logical properties through the use of set operations. Among these implicit logical assumptions, the most interesting one concerns *negation*. In the standard state space, negation of an event in the logical sense is identified by taking the set complement of the set representation of the event *with respect to the state space*. This amounts to assuming that there are only two truth values for any atomic sentence in the underlying logical system. However, it is precisely a third truth value that defines unawareness: from an outside observer's point of view, when the agent is unaware of something, it is neither true nor false for the agent. Therefore, to have nontrivial unawareness in a set-theoretic model, one needs to have states where atomic sentences can be assigned some truth value other than true or false. In particular, negation cannot be identified with the usual set-complement operation.

This key observation points out that the standard state space model has awareness information built into it. Therefore, a natural solution is to introduce multiple standard state spaces with different degrees of awareness.

3 The Product Model

3.1 The primitives

Let $\mathcal{D}^* = \{D_i\}_{i \in A}$, where each D_i is the set of possible resolutions of uncertainty i and A is an arbitrary index set for all relevant uncertainties. For concreteness, one can think of A as the index set of relevant "questions," and D_i the set of "answers" for question i. Without loss of generality, assume D_i is non-empty for all $i \in A$. I call $\mathcal{D}^*$ *the collection of full awareness information*.

The full state space Ω^* is defined as the Cartesian product of all sets in the collection of full awareness information. That is,

$$\Omega^* = \times_{i \in A} D_i \equiv \times \mathcal{D}^*$$

A *full state* $\omega^* \in \Omega^*$ is an A-tuple which specifies resolutions of all uncertainties in A.

Let $W^* : \Omega^* \to 2^{\mathcal{D}^*}$ be the *awareness function*. $W^*(\omega^*)$ is the collection of sets of possible resolutions for those uncertainties of which the agent is aware at ω^*.[10] Let P^* be the *full possibility correspondence* defined over the full state space Ω^* as in the standard model. $W^*(\omega^*)$ and $P^*(\omega^*)$ represent the agent's *awareness information* and *factual information* at ω^* respectively.[11] I refer to the pair (W^*, P^*) as the *generalized information structure*.

At ω^*, the agent's frame of mind is characterized by a *subjective state space* $\Omega(\omega^*)$ and a *subjective information structure*. The subjective state space only specifies the

[10]For instance, if the agent is aware of uncertainties 1 and 3 at ω^*, then $W^*(\omega^*) = \{D_1, D_3\}$.

[11]The difference between the formulation of the state space in this model and the standard formulation is the use of the product structure. The product structure may create impossible states, but this problem can be easily handled by letting the agent be fully aware and informed of the true state at the impossible full states. For instance, suppose there are two uncertainties: whether it is raining and whether there is a tornado. Suppose a tornado never happens without it raining. Normally one would model this by a state space consisting of three states: not raining, raining without a tornado, raining with a tornado. The product structure imposes a fourth state: not raining with a tornado. Call this state s_4 and let $W^*(s_4) = \mathcal{D}^*$ and $P^*(s_4) = \{s_4\}$. Then this impossible state would not interfere with the agent's information processing, and hence would not affect the interesting portion of the agent's knowledge hierarchy.

resolution of those uncertainties of which the agent is aware and is naturally defined as the Cartesian product of all sets in the awareness information:

$$\Omega(\omega^*) = \times W^*(\omega^*)$$

Every subjective state $\omega \in \Omega(\omega^*)$ leaves some questions (those corresponding to sets of answers not included in $W^*(\omega^*)$) unanswered, and hence is only a "blurry" picture of the environment compared to a full state in the full state space. Different subjective state spaces blur the full state space in different ways. Mathematically, each subjective state space is the projection of the full state space on a (weakly) lower-dimensional space. I say one state space is *finer* than another if it has weakly more dimensions. Let $\mathcal{S} = \{\Omega = \times\mathcal{D} : \mathcal{D} \subseteq \mathcal{D}^*\}$ be the collection of all possible subjective state spaces generated by the collection of full awareness information. For any $\Omega \in \mathcal{S}$, let $\mathbb{P}^{\Omega}$ be the projection operator that yields the projection on Ω of an element or a subset of Ω' that is finer than Ω.

When the agent has unawareness problem, he may not fully comprehend all the facts his factual information $P^*(\omega^*)$ reveals, since they may involve resolution of the uncertainties of which he is unaware. Intuitively, the agent can only recognize the answers to the questions of which he is aware. For example, although the information "the dog did not bark that night" provides an answer to the question "was there an intruder that night?" Watson missed it since he did not ask himself the question in the first place.

Formally, I introduce a family of *subjective possibility correspondences* $\{P_{\omega^*}\}_{\omega^* \in \Omega^*}$ to characterize the *subjective factual information* structure in the agent's frame of mind at each $\omega^* \in \Omega^*$. The above intuition suggests taking the projection of the factual information onto the corresponding subjective state space. For any $\omega^* \in \Omega^*$,[12]

$$P_{\omega^*}(\omega) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*)) \quad \text{for all} \quad \omega \in \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*)) \tag{3.1}$$

Let $s(\omega^*) = \mathbb{P}^{\Omega(\omega^*)}(\omega^*)$ denote the subjective state in the agent's frame of mind corresponding to the underlying true state ω^*. In other words, $s(\omega^*)$ is the "true state" to the extent the agent understands. At ω^*, the agent's subjective factual information is $P_{\omega^*}(s(\omega^*)) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*))$, which is equal to $P^*(\omega^*)$ only if the agent is aware of all relevant uncertainties.

I restrict attention to rational information structures:

Definition 2 *In the product model, the generalized information structure* (W^*, P^*) *is* **rational** *if it satisfies:*

1. *Partition:* P^* *induces an information partition over* Ω^**;*

2. *Rational awareness: For any* $\omega^*, \omega_1^* \in \Omega^*$, $\omega_1^* \in P^*(\omega^*)$ *implies*

[12] The image of the subjective possibility correspondence at subjective states the agent knows to be false is irrelevant for characterizing the agent's knowledge hierarchy. Hence I leave it unspecified.

$$W^*(\omega^*) \subseteq \bigcup_{\left\{\omega_2^*:\ \mathbb{P}^{\Omega(\omega^*)}(\omega_2^*)=\mathbb{P}^{\Omega(\omega^*)}(\omega_1^*)\right\}} W^*(\omega_2^*)$$

Rational awareness rules out the case where the agent has different awareness information in different subjective states he considers possible, according to his own frame of mind. Consider the following example:

$$\begin{aligned}
&\Omega^* = \times\mathcal{D}^* = \times\left\{\{a_1, a_2\}, \{b_1, b_2\}\right\} = \{a_1b_1, a_2b_1, a_1b_2, a_2b_2\} \\
&W^*(a_1b_1) = \mathcal{D}^*, W^*(a_1b_2) = \left\{\{b_1, b_2\}\right\}, W^*(a_2b_1) = \left\{\{a_1, a_2\}\right\} \\
&P^*(a_1b_1) = \{a_1b_1, a_2b_1\}, P^*(a_2b_1) = \{a_2b_1\}
\end{aligned}$$

Rational awareness is violated in a_1b_1, but not in a_2b_1. At a_1b_1, the agent is aware of both questions a and b. But since he knows that if the true state were a_2b_1, he would be unaware of question b, then the fact that he is aware of question b ought to enable him to exclude a_2b_1. On the other hand, having (a_1b_1, a_2b_1) as the possibility set at a_2b_1 does not violate rational awareness. At a_2b_1, the agent considers two subjective states possible, a_1 and a_2. He is indeed aware of question a at a "sub"-a_1 state. According to the agent's frame of mind at a_2b_1, a_1 and a_2 are indistinguishable as far as his awareness information is concerned.[13]

Rational generalized information in the product model is the analogue of rational information (i.e. information partition) in the standard model. The essence of rationality in information processing is that the agent can exclude any state in which he receives different information. In the standard model, this argues for an information partition, so that the agent excludes any state in which he receives different factual information. In the product model, since information includes both awareness information and factual information, the additional condition – *rational awareness* – is needed. Under rational generalized information structure, the agent excludes any *subjective* states in which he has different *subjective* factual information.[14]

The triple (Ω^*, W^*, P^*) consists of the primitives of the product model of unawareness.[15] The model extends the information structure from a possibility correspondence

[13]Technically, the rational awareness assumption assures that the agent's frame of mind is completely characterized by a subjective state space and a subjective *factual information* structure without having to refer to a subjective *generalized information* structure. For more about this assumption, see Appendix B and footnote 25.

[14]I assume that except for being constrained by his current awareness, the agent knows the awareness function. The argument is completely analogous to the argument that the agent knows the possibility correspondence in the standard model. The generalized information structure is just a coding system. If the agent is not sure about $W^*(\omega^*)$ (or the subjective version of it), one can always split ω^* into, say, ω_1^*, ω_2^*, so that the agent knows $W^*(\omega_1^*)$ and $W^*(\omega_2^*)$. The information structure is preserved as long as ω_1^* and ω_2^* always belong together in the possibility sets. Combined with the product structure of the state space, this may create impossible states. But as argued in footnote 11, this is not a problem.

[15]Alternatively, one may think of $(\mathcal{D}^*, W^*, P^*)$ as the primitives of the model, since the state space is in fact a derived concept.

defined over the full state space (P^*) to a pair (W^*, P^*), specifying a subjective state space at each full state in addition to the possibility correspondence. When (W^*, P^*) is rational, the "subjective model" at each full state is a "local" information partition model $(\Omega(\omega^*), P_{\omega^*})$, which, under nontrivial unawareness, is a coarser version of the "full model" (Ω^*, P^*) in the sense that each subjective state in $\Omega(\omega^*)$ corresponds to an event in Ω^*.

As an illustration, consider the following example of an agent with an episodic hearing problem that causes him to hear a lot of noise when he experiences the problem. The agent sits in a room and listens to determine whether it is raining outside. If he experiences the hearing problem, he cannot tell whether it rains; otherwise he can. The agent is never aware of the hearing problem. Let r, nr, p, np denote "it is raining," "it is not raining," "experiencing the hearing problem," "not experiencing the hearing problem" respectively. Then the model is the following:

The full awareness information: $\mathcal{D}^* = \{\{r, nr\}, \{p, np\}\}$
The full state space: $\Omega^* = \times\mathcal{D}^* = \{(r,p), (r,np), (nr,p), (nr,np)\}$
The awareness function: for any $\omega^* \in \Omega^*$,
$W^*(\omega^*) = \{\{r, nr\}\}$
The full possibility correspondence P^* induces the following information partition over the full state space:
$\{\{(r,p), (nr,p)\}, \{(r,np)\}, \{(nr,np)\}\}$

At (r,p), the agent's factual information is the partition element $\{(r,p), (nr,p)\}$. Since he is unaware of the hearing problem, the agent does not realize that the fact he cannot tell whether it rains indicates he has a hearing problem. To the agent, the factual information is simply $\{r, nr\}$. The subjective possibility correspondence at (r,p), denoted by $P_{(r,p)}$, maps the set of subjective states to the set of nonempty subsets of the subjective state space. That is:

$$P_{(r,p)}(r) = P_{(r,p)}(nr) = \mathbb{P}^{\{r,nr\}}(\{(r,p), (nr,p)\}) = \{r, nr\}$$

Hence the agent's subjective factual information at (r,p) is $P_{(r,p)}(s[(r,p)]) = P_{(r,p)}(r) = \{r, nr\}$.

3.2 The events

In the standard model, where information is limited to factual information, events only differ in the facts they convey. In the product model, under generalized information, events can differ in awareness, in facts, or in both. For instance, in the hearing problem example, the events "it rains, and there is a possibility that I have a hearing problem" and "it rains" are different events because they involve different levels of awareness. By construction, every subjective state has awareness built into it, and hence can be conveniently used to represent these "subjective" events: an event is *the set of subjective states that contain the same awareness information as the event, and in which the event is true.*

Let E be a nonempty subset of some subjective state space. By construction, one can identify its space and hence its awareness information. Let it be denoted $\mathcal{D}_E$, i.e. $E \subseteq \times\mathcal{D}_E$. But the empty set, being a subset of any space, creates both technical difficulties and conceptual inconsistency. The empty set represents an impossible event or a logical contradiction. Intuitively, there are many different contradictions that involve different sets of issues. For example, the statements "it rains and it does not rain" and "it rains, it does not rain, and I have a hearing problem" are both logical contradictions, yet the former contains less awareness than the latter. Presumably an agent who is unaware of the possibility of the hearing problem cannot perceive the latter.

In light of this, for any state space $\Omega \in \mathcal{S}$, let $\emptyset_\Omega$ denote the empty set associated with Ω. Intuitively, this object is the empty set tagged with the awareness information. It behaves in the same way as the usual empty set, except that it is confined to its state space. To incorporate this new object in set operations, I extend the usual set inclusion, intersection, union and complement notions (for notational ease, I use the conventional symbols for these operations): For any state space Ω and any sets $E, F \neq \emptyset_\Omega,\ E, F \subseteq \Omega$, the set inclusion, intersection, union and complement notions are defined in the usual way, except that for disjoint E and F, $E \cap F = \emptyset_\Omega$ instead of $\emptyset$. In addition, for any $E \subseteq \Omega,\ \emptyset_\Omega \cup E = E,\ \emptyset_\Omega \cap E = \emptyset_\Omega,\ E \setminus \emptyset_\Omega = E$, and for any E such that $E \neq \emptyset$ and $E \subseteq \Omega,\ \emptyset_\Omega \subseteq E$.[16]

In association with the issue of multiple empty sets, it would be convenient if the empty state space could be avoided. However, one can also imagine interesting cases where the agent is unaware of all relevant uncertainties, such as the Watson story. An easy way to get around this is to include a question 0 which is, in principle, relevant to all decision problems and to which the agent always knows the answer. Denote the answer $\{\Delta\}$. Then one can redefine $\Omega^* = \{\Delta\} \times \mathcal{D}^*$ and $\Omega(\omega^*) = \{\Delta\} \times W(\omega^*)$. For interpretation, think of question 0 and $\{\Delta\}$ as "do I exist?" and "*cogito ergo sum.*"[17] Thus, without loss of generality, I assume all state spaces are non-empty.

The collection of events in this model is:

$$\mathcal{E}^p = \{E \neq \emptyset : E \subseteq \Omega \text{ for some } \Omega \in \mathcal{S}\}$$

Note $\{\emptyset_\Omega\}_{\Omega \in \mathcal{S}}$ are elements of $\mathcal{E}^p$. Hence this collection consists of all perceivable events, including the impossible events, in some subjective state space. With this construction,

[16] Another way to think about the multiple empty sets is that an event E in the model is actually a pair $(\times\mathcal{D}_E, E)$ where $E \subseteq \times\mathcal{D}_E$. The first object in the pair, $\times\mathcal{D}_E$, specifies the awareness in E while the second object represents the involved facts. In particular, $\emptyset_\Omega$ should be interpreted as the pair $(\Omega, \emptyset)$. Then the usual set inclusion can be extended to this space by letting $(\Omega, E) \subseteq (\Omega', F)$ iff $\Omega = \Omega'$ and $E \subseteq F$, and similarly for other set operations. For $E \neq \emptyset$, $\mathcal{D}_E$ is uniquely identified from it and hence is redundant.

[17] For simplicity, I omit Δ when there is no risk of confusion.

$\mathcal{D}_E$ is well-defined for every $E \in \mathcal{E}^p$. In the rest of the paper, I use E, F, G to denote generic events in $\mathcal{E}^p$.

Events that convey the same facts but contain different awareness play a critical role in an environment with unawareness.

Definition 3 *F is said to be an* **elaboration** *of E and E is said to be a* **reduction** *of F, if*

$$\mathcal{D}_F \supseteq \mathcal{D}_E \text{ and } F = \left\{\omega \in \times\mathcal{D}_F : \mathbb{P}^{\times\mathcal{D}_E}(\omega) \in E\right\}$$

Denote it by $E \leq F$.

For example, $\{(r, p), (r, np)\}$ is an elaboration of $\{r\}$. Since the logical relations between events concern only facts, they are preserved by elaborations. Thus one can identify the logical relations between two arbitrary events by checking their elaborations that live in the same space. Formally, for any $E \in \mathcal{E}^p$ and Ω satisfying $\mathcal{D}_E \subseteq \mathcal{D}_\Omega$, let E_Ω denote the elaboration of E in Ω. That is, E_Ω is the unique event satisfying $E_\Omega \subseteq \Omega$ and $E \leq E_\Omega$. In the above example, $\{(r, p), (r, np)\} = \{r\}_{\Omega^*}$. I define the following *extended set relations and operations* on elements of $\mathcal{E}^p$:

Logical consequence is identified using *extended set inclusion*:

$$E \subseteq_* F \Leftrightarrow E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \subseteq F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)}$$

Conjunction is identified using *extended set intersection*:

$$E \cap_* F = E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \cap F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)}$$

Disjunction is identified using *extended set union*:

$$E \cup_* F = E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \cup F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)}$$

Negation of an event involves the same amount of awareness but the facts are reversed. It is naturally identified with the set complement operation with respect to the corresponding *subjective state space*:

$$\neg E = \times\mathcal{D}_E \setminus E$$

The extended set operations reduce to the usual ones for events from the same space. Apparently, each state space $\Omega \in \mathcal{S}$ is a standard state space satisfying the standard logical assumptions.

3.3 The knowledge and unawareness operators

The *unawareness operator* $U : \mathcal{E}^p \to 2^{\Omega^*}$ is:

$$U(E) = \{\omega^* \in \Omega^* : \mathcal{D}_E \nsubseteq W^*(\omega^*)\} \tag{3.2}$$

That is, the agent is unaware of an event if and only if the event contains awareness information that the agent does not have.

There are some subtle issues concerning knowledge in the presence of unawareness. The agent processes information *within his frame of mind*. Intuitively, for any event E, only if there is some version of E – either E itself or some elaboration of it – in the agent's frame of mind, can he reason about it. In addition, the agent's frame of mind also constrains his reasoning about his own knowledge. For instance, recall the hearing problem example. The agent knows it rains only if it rains and he does not experience the hearing problem. The event "**the agent** *knows it rains"* (perhaps from the modeler's perspective) is the singleton set $\{(r, np)\}$. However, at (r, np), since the agent is unaware of the hearing problem, *from his perspective* the event "**I** *know it rains"* is represented by the singleton set $\{r\}$ in the subjective state space. In other words, there are two types of knowledge in this model: *objective knowledge* and *subjective knowledge*. $\{(r, np)\}$ is the objective event "**the agent** knows it rains," while $\{r\}$ is the subjective event "**I** know it rains" in the agent's frame of mind at (r, np). Intuitively, in higher-order knowledge such as "the agent knows that *he knows it rains*," the event "he knows it rains" should refer to the subjective knowledge. To obtain an objective account for what the agent knows and does not know, one needs to examine the subjective knowledge hierarchy *in the agent's frame of mind full state by full state.*

The subjective model $(\Omega(\omega^*), P_{\omega^*})$ provides the natural model for characterizing the agent's *subjective knowledge* at ω^*. Let $\emptyset_E \equiv \emptyset_{\times \mathcal{D}_E}$ denote the empty set that lies in the same state space as E.

$$\tilde{K}_{\omega^*}(E) = \begin{cases} \left\{\omega \in \Omega(\omega^*) : P_{\omega^*}(\omega) \subseteq E_{\Omega(\omega^*)}\right\} & \text{if } \mathcal{D}_E \subseteq W^*(\omega^*) \\ \emptyset_E & \text{if } \mathcal{D}_E \nsubseteq W^*(\omega^*) \end{cases} \tag{3.3}$$

Note that $E_{\Omega(\omega^*)}$ is the version of E in the agent's frame of mind provided he is aware of E. (3.3) says the agent knows E if he is aware of E and E is true in all *subjective* states that he considers possible. Thus $\tilde{K}_{\omega^*}(E)$ is interpreted as "knowledge of E in the agent's frame of mind."

When the agent is unaware of E – that is, when $\mathcal{D}_E \nsubseteq W^*(\omega^*)$ – the event $E_{\Omega(\omega^*)}$ is not defined and $\tilde{K}_{\omega^*}(E)$ is empty. Since the events "the agent knows E" and "the agent does not know E" should contain at least as much awareness information as E itself, it is plausible to use the empty set that contains the same awareness as E.

By iterating subjective knowledge operator, one obtains *subjective higher-order knowledge*, i.e. the knowledge hierarchy *in the agent's frame of mind*:

$$\tilde{K}^n_{\omega^*}(E) = \tilde{K}_{\omega^*}\tilde{K}^{n-1}_{\omega^*}(E) \tag{3.4}$$

The *objective n-th order knowledge* is obtained by putting together the relevant pieces of the subjective knowledge hierarchies:

$$K_n(E) = \left\{\omega^* \in \Omega^* : s(\omega^*) \in \tilde{K}^n_{\omega^*}(E)\right\} \tag{3.5}$$

$$(\neg K)_n(E) = \left\{\omega^* \in \Omega^* : s(\omega^*) \in (\neg\tilde{K}_{\omega^*})^n(E)\right\} \tag{3.6}$$

$K_n(E)$ is the objective knowledge "the agent knows that he knows $\cdots$ he knows E." Similarly, $(\neg K)_n(E)$ represents the event "the agent does not know that he does not know $\cdots$ he does not know E." When $n = 1$, equation (3.5) is just

$$K(E) = \left\{\omega^* \in \Omega^* : \mathcal{D}_E \subseteq W^*(\omega^*),\ P_{\omega^*}(s(\omega^*)) \subseteq E_{\Omega(\omega^*)}\right\}$$

which is equivalent to:

$$K(E) \;\; = \;\; \{\omega^* \in \Omega^* : \mathcal{D}_E \subseteq W^*(\omega^*),\ P^*(\omega^*) \subseteq E_{\Omega^*}\} \tag{3.7}$$

Equation (3.7) has an intuitive interpretation: The agent knows E if and only if he is aware of E and E is true in all possible full states. Note that $\mathcal{D}_E \subseteq W^*(\omega^*)$ and $P^*(\omega^*) \subseteq E_{\Omega^*}$ imply $P_{\omega^*}(s(\omega^*)) \subseteq_* E$, which is logically equivalent to the characterization of knowledge in the standard model.

If the agent is fully aware in all full states, then he has the same subjective model – the full model – in all full states. The subjective knowledge operator $\tilde{K}$ then reduces to the usual K in the standard model and all subjective knowledge hierarchies become identical and coincide with the objective knowledge hierarchy. In that case the product model simply reduces to the standard model.

4 Information Processing with Unawareness

Recall that in the standard information partition model (Ω, P), the agent's knowledge hierarchy is completely characterized by the following: for any event $E \subseteq \Omega$,

1. $K(E) = \{\omega \in \Omega : P(\omega) \subseteq E\} = KK(E)$;
2. $\neg K(E) = K\neg K(E)$.

That is, for any event, the agent either knows it or does not know it, and he always knows whether he knows it. It turns out that in the product model, the following is true:

Theorem 2 *In the product model (Ω^*, W^*, P^*), let (W^*, P^*) be rational. Then the agent's knowledge hierarchy is completely characterized by the following: for any $E \in \mathcal{E}^p$,*

1. $U(E) = \{\omega^* \in \Omega^* : \mathcal{D}_E \nsubseteq W^*(\omega^*)\} = \neg K(E) \cap (\neg K)_2(E)$;
2. $K(E) = \{\omega^* \in \Omega^* : \mathcal{D}_E \subseteq W^*(\omega^*),\ P^*(\omega^*) \subseteq E_{\Omega^*}\} = K_2(E)$;
3. $\neg U(E) \cap \neg K(E) = \neg(\neg K)_2(E)$.

In words, theorem 2 says that for any event, the agent is either unaware of it, in which case he does not know it, and he does not know that he does not know it; or he is aware of it, in which case he either knows it or does not know it, while always knows whether he knows it. Note that the knowledge hierarchy can be directly derived from the pair (W^*, P^*) without referring to the subjective models.

The result follows from two lemmas. First, consider the following properties of unawareness and knowledge: for all $E,\ F \in \mathcal{E}^p$,

U0* *Symmetry*: $U(E) = U(\neg E)$

U1′ *Strong plausibility*: $U(E) \subseteq \bigcap_{n=1}^{\infty} (\neg K)_n(E)$

U2* *AU introspection*: $U(E) \subseteq UU(E)$

U3′ *Weak KU introspection*: $U(E) \cap KU(E) = \emptyset_{\Omega^*}$

K1* *Subjective necessitation*: $\omega^* \in K(\Omega(\omega^*))$ for all $\omega^* \in \Omega^*$

K2* *Generalized monotonicity*: $E \subseteq_* F,\ \mathcal{D}_E \supseteq \mathcal{D}_F \Rightarrow K(E) \subseteq K(F)$[18]

K3* *Conjunction*: $K(E) \cap K(F) = K(E \cap_* F)$[19]

Symmetry is proposed by Modica and Rustichini (1999). It says that one is unaware of an event if and only if one is unaware of the negation of it. The other three unawareness properties correspond to the three axioms proposed by DLR, with slight modifications of plausibility and KU introspection. Strong plausibility strengthens DLR's plausibility axiom. Plausibility requires that whenever one is unaware of something, one does not know it and does not know that one does not know it. I require such a lack of knowledge to be extended to an arbitrarily high order.[20] DLR's KU introspection states that the agent never knows exactly what he is unaware of. I consider a slightly weaker version that says the agent does not know exactly what he is unaware of when he is actually unaware of it. In other words, the difference between KU introspection and weak KU introspection is that weak KU introspection allows the agent to have false knowledge of his being unaware of a particular event.

$K1^* - 3^*$ are natural analogues of K1-3 in the context of nontrivial unawareness. Recall that necessitation says the agent always knows the state space: $K(\Omega^*) = \Omega^*$, which means the agent knows tautological statements. This seems to be a reasonable property of knowledge. After all, how can one not know things like, "If there was an intruder,

[18] This property implies the agent is no longer logically omniscient: he knows the logical consequences of his knowledge *only if he is aware of these events*.

[19] Parallel to the standard model, conjunction implies generalized monotonicity.

[20] DLR consider the weaker property plausibility, which is sufficient for the negative result in which they are interested. However, when it comes to providing *positive* results in a model that deals with unawareness, strong plausibility, or an even stronger property that equates unawareness with the lack of knowledge of all orders which I discuss shortly, seems to be more interesting.

then there was an intruder," even if one has no idea whether there was an intruder or not? However, it seems plausible to say Watson does not know this tautology when the dog did not bark, because the *idea* of an intruder simply never occurs to him. The suitable generalization of necessitation to an environment with unawareness is that the agent knows all tautological statements *of which he is aware*, which is precisely subjective necessitation.[21]

The essence of monotonicity is the intuitive notion that *knowledge is monotonic with respect to the information content of events*. In the standard model, one event is more informative than another if and only if it conveys more facts. In the product model, an event is more informative than another if and only if it contains both more facts and *more awareness*. Another way to look at it is, monotonicity means the agent knows the logical consequences of his knowledge, while generalized monotonicity, which explicitly takes into account that the agent may not be fully aware, says the agent knows those logical consequences of his knowledge *of which he is aware*.

Lemma 3 *The product model $\{\Omega^*, W^*, P^*\}$ satisfies $U0^*$, $U1'$, $U2^*$, $U3'$ and $K1^* - 3^*$.*

To see the connection between lemma 3 and DLR's impossibility results, first notice that by (3.7),

$$K(\Omega^*) = \{\omega^* \in \Omega^* : \mathcal{D}^* \subseteq W^*(\omega^*), P^*(\omega^*) \subseteq \Omega^*\} = \{\omega^* \in \Omega^* : \mathcal{D}^* = W^*(\omega^*)\}$$

That is, $K(\Omega^*) = \Omega^* \Leftrightarrow W^*(\omega^*) = \mathcal{D}^*$ for all $\omega^* \in \Omega^*$. In other words, necessitation holds if and only if the agent is fully aware in every full state. But this implies $\mathcal{D}_E \subseteq W^*(\omega^*)$ for all $\omega^* \in \Omega^*$ and $E \in \mathcal{E}$, hence $U(E) = \emptyset_{\Omega^*}$, which is DLR's first impossibility result.

Secondly, observe that generalized monotonicity and monotonicity differ in that generalized monotonicity does not require knowledge to be monotonic when $\mathcal{D}_E \nsubseteq \mathcal{D}_F$, while in the standard model, one necessarily has $\mathcal{D}_E = \mathcal{D}_F$. Let $E \subseteq_* F$ and $K(E) \subseteq K(F)$. By (3.7), $\{\omega^* \in \Omega^* : \mathcal{D}_E \subseteq W^*(\omega^*)\} \subseteq \{\omega^* \in \Omega^* : \mathcal{D}_F \subseteq W^*(\omega^*)\}$. This implies $U(F) \subseteq U(E)$, which says whenever the agent is unaware of F, he is unaware of E. By (3.2), for any G such that $\mathcal{D}_G = \mathcal{D}_E$,

$$U(F) \subseteq U(G) = U(E)$$

That is, whenever the agent is unaware of F, he is unaware of any event that contains the same awareness information as E. By strong plausibility, he cannot know any of them, which is DLR's second impossibility result.

Lemma 3 does not require (W^*, P^*) to be rational. This is because analogous to the standard model, rationality in information processing mainly has implications for higher-order knowledge, while none of the above properties involves higher-order knowledge. It is the generalization of information structure from merely facts to both

[21]Subjective necessitation is equivalent to the "weak necessitation" property DLR discussed in the context of propositional models.

awareness and facts, rather than specific structure of the possibility correspondence or awareness function that captures the essence of nontrivial unawareness.

Consider the following stronger properties of unawareness and knowledge:

U1* *UUU (Unawareness = unknown unknowns)*: $U(E) = \bigcap_{n=1}^{\infty} (\neg K)_n(E)$

U3* *KU introspection*: $KU(E) = \emptyset_{\Omega^*}$

K4* *The axiom of knowledge*: $K(E) \subseteq_* E$

K5* *The axiom of transparency*: $K(E) \subseteq K_2(E)$

K6* *The axiom of limited wisdom*: $\neg K(E) \cap \neg U(E) \subseteq K\neg K(E)$

Lemma 4 *The product model* (Ω^*, W^*, P^*) *satisfies* U1**,* U3* *and* K4* $-$ 6* *if the generalized information structure* (W^*, P^*) *is rational.*

The axiom of knowledge and the axiom of transparency have the same interpretation as their counterparts in the standard model. The axiom of limited wisdom extends the axiom of wisdom to an environment with unawareness by only requiring the agent to know that he does not know when he *is aware of the involved event*. UUU says the agent is unaware of an event *if and only if* he does not know it, he does not know that he does not know it, and so on. The extra strength added to strong plausibility is due to the axiom of limited wisdom: if the agent always knows that he does not know if he is aware of the event, then obviously the only circumstance where he does not know that he does not know is that he is unaware of it. Lastly, the axiom of knowledge says the agent can never have false knowledge, which, combined with weak KU introspection, yields KU introspection.

Remark 2

Information has more dramatic effects on the agent's knowledge hierarchy when the agent has nontrivial unawareness than when he does not. Upon receipt of new information, the agent updates his subjective state space as well as his subjective factual information. Formally, given ω^*, let the agent's initial information be $(W_0^*(\omega^*), P_0^*(\omega^*))$. The agent has subjective factual information:

$$\mathbb{P}^{\times W_0^*(\omega^*)}[P_0^*(\omega^*)]$$

Upon receipt of new information $(W_1^*(\omega^*), P_1^*(\omega^*))$, the agent updates his subjective factual information to

$$\mathbb{P}^{\times [W_0^*(\omega^*) \cup W_1^*(\omega^*)]}[P_0^*(\omega^*) \cap P_1^*(\omega^*)]$$

As long as $W_1^*(\omega^*) \setminus W_0^*(\omega^*) \neq \emptyset$, the agent gains new knowledge. In particular, if $P_0^*(\omega^*)$ is not an elaboration of $\mathbb{P}^{\times W_0^*(\omega^*)}[P_0^*(\omega^*)]$, that is, if it contains factual information about uncertainties beyond $W_0^*(\omega^*)$, then the agent may learn new facts even if $P_1^*(\omega^*) = \Omega^*$, i.e. the new information does not reveal any new facts at all.

For example, recall the Watson story from the introduction. The true full state is "there was no intruder"$=b$. Watson's initial information is the pair $(W_0^*(b), P_0^*(b)) = (\{\Delta\}, \{b\})$. His subjective factual information is $\mathbb{P}^{\{\Delta\}}\{b\} = \{\Delta\}$. The question "Could there have been an intruder?" is represented by the generalized information $(W_1^*(b), P_1^*(b)) = (\{\{a', b'\}\}, \{a, b\})$. Upon being asked the question, Watson updates his subjective state space and recognizes the factual information he has had all along but has been neglected by him: $\mathbb{P}^{\times(\{\Delta\}\cup\{a',b'\})}(\{b\} \cap \{a, b\}) = \{b\}$.

A useful observation is that with rational generalized information, the agent reasons just like a standard agent within his subjective state space. In a loose sense, the agent is "locally" Bayesian. This observation, combined with the dramatic learning effects, suggests that this model may be used to model situations involving "surprises." For example, consider a bidder who bids on a good with unknown quality. The bidder is unaware of another relevant uncertainty which is whether the seller legally owns the good. If the seller does not legally own the good, any transaction is not respected by the court and both parties incur legal costs. Intuitively, the bidder's behavior can change dramatically when he becomes aware of the legal ownership issue compared to when he is unaware of it. To model this situation, one can let the bidder have a "full prior" distribution over the full state space which specifies both the good's quality and whether the seller legally owns the good. When the bidder is unaware of the uncertainty of whether the seller legally owns the good, in his frame of mind, he has only a prior distribution over the space of quality. Each subjective state, which specifies the good's quality, corresponds to an event in the full state space which includes all full states in which the good has this quality. In other words, the bidder's prior over the quality space is the marginal distribution of his "full prior" over the full state space. When he becomes aware of the legal ownership issue, he updates the whole *probability space* and his *prior distribution*. Even if the bidder assigns high probability on the good having high quality, the "full prior" could have all the mass on the good being illegally acquired by the seller, which can lead the bidder to change his bids dramatically.

5 The Multi-agent Model

Consider again the Watson story. Suppose another relevant uncertainty is, who had seen the horse after it was let out of the stable, the keeper, the neighbor, or both. Suppose the police told Holmes and Watson, "The neighbor saw the horse the next morning." Apparently, the event "the neighbor saw the horse" is common knowledge between Holmes and Watson, in the sense that both of them know it, both of them know that both of them know it, and so on.

Let a, b, k, n, kn denote "there was an intruder," "there was no intruder," "the keeper saw the horse," "the neighbor saw the horse," and "both the keeper and the neighbor saw the horse" respectively. Let (W_h^*, P_h^*) and (W_w^*, P_w^*) denote Holmes and Watson's information structure respectively. Then:

$\mathcal{D}^* = \{\{a,b\}, \{k,n,kn\}\}$
$\Omega^* = \{a,b\} \times \{k,n,kn\}$
$W_h^*(\omega^*) = \mathcal{D}^*$
$W_w^*((a,\cdot)) = \mathcal{D}^*$; $W_w^*((b,\cdot)) = \{\{a,b\}\}$
$P_h^* = P_w^*$ and it induces the following information partition:
$\{\{(a,n),(a,kn)\}, \{(a,k)\}, \{(b,n),(b,kn)\}, \{(b,k)\}\}$

In the single agent model, I emphasize the distinction between subjective knowledge and objective knowledge when characterizing the knowledge hierarchy. Similarly, in the multi-agent model, the key is to track down the correct version of knowledge when characterizing interactive knowledge hierarchy. Arguably, when Watson reasons to himself, "does Holmes know that the neighbor saw the horse?" The event "Holmes knows the neighbor saw the horse" is neither the objective knowledge nor Holmes' subjective knowledge, but *Holmes' knowledge in Watson's frame of mind.* At (b,kn), the event "I know the neighbor saw the horse" in Holmes' model is $\{(b,n),(b,kn)\}$. However, restricted by his frame of mind, from Watson's perspective, Holmes' model appears to be identical to his own: $\{\{k\}, \{n,kn\}\}$. Watson's version of Holmes' subjective knowledge "he knows the neighbor saw the horse" is characterized by $\{n,kn\}$, which he does know at (b,kn). That is, at (b,kn), in Watson's frame of mind, he knows that Holmes knows the neighbor saw the horse.

The example illustrates that, in interactive knowledge "i knows that j knows E," the event "j knows E" refers to the version of j's subjective knowledge "I know E" in i's frame of mind; in interactive knowledge "i knows that j knows k knows E," the event "j knows k knows E" refers to i's version of j's subjective knowledge "I know that k knows E" where "k knows E" is j's version of k's subjective knowledge "I know E," and so on. An event E is common knowledge if everybody knows E, everybody knows that *everybody knows* E, everybody knows that *everybody knows everybody knows* E, and so on, whereas those *everybody knows*' refer to *each person's own version of interactive knowledge.* In sum, to characterize common knowledge with nontrivial unawareness, one needs to characterize an infinite hierarchy of *subjective interactive models* in each agent's frame of mind.

It turns out that if every agent's generalized information structure is sufficiently nice, common knowledge has an intuitive and simple characterization parallel to that in the standard model. In the single-agent case, only subjective information in *possible* subjective states is relevant for characterizing the agent's knowledge hierarchy, and hence the agent's subjective model being a locally information partition model suffices for a nice knowledge hierarchy. This is no longer true in the multi-agent setting. Intuitively, even if i knows state a is false, as long as i knows j considers a possible, and hence would reason about i's knowledge in a, what i would have known in a matters for i's reasoning about j's reasoning about i's knowledge. Therefore, in interactive information processing, agent i's subjective information in all subjective states $\omega \in \Omega_i(\omega^*)$ *reachable from i's true subjective state $s_i(\omega^*)$* in the sense of Aumann (1976) is relevant in characterizing i's reasoning about other agents' knowledge, which potentially extends to all subjective states in i's subjective state space. Consequently, rationality in interactive information processing requires more structure on generalized information than that in the single-agent case: the subjective factual information needs to be partitional over the set of all reachable subjective stats. A sufficient condition is the following:

Definition 4 *In the single-agent product model (Ω^*, W^*, P^*), P^* induces a* **nice partition** *over Ω^* if:*

1. *P^* induces an information partition over Ω^*;*
2. *If $\omega_1^* \in P^*(\omega^*)$, then for any $\Omega \in \mathcal{S}$ and $\omega_2^* \in \Omega^*$ such that $\mathbb{P}^\Omega(\omega_2^*) = \mathbb{P}^\Omega(\omega_1^*)$, $\omega_2^* \in P^*(\omega^{*\prime})$ for some $\omega^{*\prime}$ such that $\mathbb{P}^\Omega(\omega^*) = \mathbb{P}^\Omega(\omega^{*\prime})$.*

In words, this condition says if the agent cannot exclude a subjective state in some subjective state space at a full state, then in any other full states where he has the same subjective state space, he again must not exclude this subjective state. Mathematically, nice partition restricts the agent's factual information to nice cylinder events. It guarantees that the projection of the full possibility correspondence on any subjective state space induces an information partition over that space:

Lemma 5 *Nice partition implies that given any Ω, if $\omega_1^* \in P^*(\omega_2^*)$, then:*

$$\mathbb{P}^\Omega\Big(\bigcup_{\left\{\omega_3^* : \mathbb{P}^\Omega(\omega_3^*) = \mathbb{P}^\Omega(\omega_1^*)\right\}} P^*(\omega_3^*)\Big) = \mathbb{P}^\Omega\Big(\bigcup_{\left\{\omega_4^* : \mathbb{P}^\Omega(\omega_4^*) = \mathbb{P}^\Omega(\omega_2^*)\right\}} P^*(\omega_4^*)\Big)$$

Definition 5 *The generalized information structure (W^*, P^*) is* **strongly rational** *if it satisfies nice partition and rational awareness.*

Let $N = \{1, \cdots, n\}$ be the set of agents, and $\mathbf{W}^* = (W_1^*, \cdots, W_n^*)$ and $\mathbf{P}^* = (P_1^*, \cdots, P_n^*)$ denote the vector of awareness information and factual information respectively. Let $\cap \mathbf{W}^*(\omega^*)$ denote the intersection of all agents' awareness information at ω^*. Let $\wedge \mathbf{P}$ denote the meet of information partitions generated by the agents' possibility correspondences, and $\wedge \mathbf{P}(\omega^*)$ denote the partition element in $\wedge \mathbf{P}$ that contains ω^*.

Theorem 6 *In the multi-agent product model $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$, suppose (W_i^*, P_i^*) is strongly rational for every $i \in N$. For any $E \in \mathcal{E}^p$, the event "E is common knowledge," denoted by $CK(E)$, is characterized as follows:*

$$CK(E) = \left\{ \omega^* \in \Omega^* : \mathcal{D}_E \subseteq \bigcap_{\left\{\omega_1^* :\, \omega_1^* \in\, \wedge \mathbf{P}^*(\omega^*)\right\}} \mathbf{W}^*(\omega_1^*),\ \wedge \mathbf{P}^*(\omega^*) \subseteq E_{\Omega^*} \right\} \tag{5.1}$$

Theorem 6 says if every agent's generalized information structure is sufficiently nice, then there is no extra complication in characterizing common knowledge in the presence of nontrivial unawareness besides that in characterizing single agent's knowledge. The condition $\mathcal{D}_E \subseteq \bigcap_{\{\omega_1^*:\, \omega_1^* \in \wedge \mathbf{P}^*(\omega^*)\}} \mathbf{W}^*(\omega_1^*)$ delivers "common knowledge of awareness," which is trivially satisfied in the standard model: everyone is aware of E, everyone *knows* that everyone is aware of E, everyone *knows* that everyone *knows* that everyone is aware of E, and so on.[22]

For detailed formulation of the multi-agent model and proofs of results in this section, see Appendix B.

6 Comments on the Modelling Approach

In the product model (Ω^*, W^*, P^*), each full state is to be interpreted as a *state of the world* in the sense of Aumann (1976) in an environment with nontrivial unawareness: besides the resolution of all external uncertainties, a full state also specifies, by means of the knowledge and unawareness operators, what the agent is or is not aware, and for those events of which he is aware, what he knows and does not know. The full state space is a natural generalization of the standard state space, in both the single-agent case and the multi-agent case.

It is the use of the pair (W^*, P^*) to capture the information structure in this environment that makes the above interpretation of the full state space meaningful. By modelling the "objective" generalized information, I separate two different forms of information, namely awareness information and factual information, and hence separate the full (or objective) model from the subjective models *derived* from the objective one due to awareness constraints. This approach highlights the new issue involved in an environment with nontrivial unawareness, and henceforth has several advantages.

First of all, Theorem 2 and Theorem 6 show that the pair (W^*, P^*) provides a succinct characterization of the (interactive) knowledge hierarchy of interest, without

[22]Let $R(\omega^*) \equiv \left\{ \omega_1^* : \mathbb{P}^\Omega(\omega_1^*) = \mathbb{P}^\Omega(\omega_2^*) \text{ where } \Omega = \bigcap_{\{\omega_2^*:\, \omega_2^* \in \wedge \mathbf{P}^*(\omega^*)\}} \mathbf{W}^*(\omega_2^*) \text{ and } \omega_2^* \in \wedge \mathbf{P}^*(\omega^*) \right\}$ be the set of reachable full states from ω^*. Collect all sets in the form $W_i^*(\omega_2^*)$, $i \in N$, $\omega_2^* \in R(\omega^*)$, close it by intersection, and call the resulting set $RW(\omega^*)$. Let $\mathcal{S}(\omega^*) = \{\Omega : \Omega = \times A,\ A \in RW(\omega^*)\}$ be the set of perceivable subjective state spaces at ω^*. Modify the second part of nice partition to read:

> If $\omega_1^* \in P_i^*(\omega^*)$, then for any $\Omega \in \mathcal{S}(\omega^*)$ and $\omega_2^* \in R(\omega^*)$ such that $\mathbb{P}^\Omega(\omega_2^*) = \mathbb{P}^\Omega(\omega_1^*)$, $\omega_2^* \in P^*(\omega^{*\prime})$ for some $\omega^{*\prime}$ such that $\mathbb{P}^\Omega(\omega^*) = \mathbb{P}^\Omega(\omega^{*\prime})$.

This condition weakens nice partition by restricting attention to only relevant states and subjective state spaces. It ensures every subjective model is partitional over the set of reachable subjective states. Theorem 6 continues to hold under rational generalized information and the above weakening of nice partition condition.

the need of explicitly referring to subjective models. Secondly, the concept of rational generalized information and strongly rational generalized information crystalize what have been implicitly assumed in the standard model. Thirdly, the distinction between the agent's subjective models and the full model addresses DLR's impossibility results. Lastly and perhaps the most interestingly, the generalized information makes it clear that the awareness information is a distinctly different type of information than the usual factual information economists have been dealing with. The nature of such information is not well understood and very much worth exploring, especially in the study of bounded rationality. For example, there seems to be a close connection between awareness information and learning. By formulating this concept in a model closely related to the standard model, this model provides a convenient starting point for further research.

It should be pointed out that the interpretation of P^* is less transparent than that in the standard model. When the agent has unawareness problem, P^* is beyond his understanding. One interpretation is that P^* is the factual information the agent would have were he fully aware. In a loose sense, one can also view P^* as the signal structure.

7 Concluding remarks

In this paper, I construct a set-theoretic model of both single-agent and interactive information processing with unawareness. The main idea is that a model dealing with nontrivial unawareness needs to incorporate a new type of information, i.e. awareness information, in addition to the usual factual information in the standard model. The resulting model is a natural generalization of the standard information partition model due to Aumann (1976), and hence well connecting to the existing literature.

The idea of capturing unawareness by introducing subjective state spaces that represent the agent's frames of mind is not new in the literature. Modica and Rustichini (1999) explore a similar idea to study the logical system of single-agent information processing with unawareness. They also introduce the concept of a set-theoretic model with the underlying logical system. In an independently conceived work, Schipper (2002) also gives a set-theoretic model of unawareness. Heifetz, Meier and Schipper (2003) further extend Schipper (2002) to the multi-agent setting and provide a characterization of common knowledge. All of the above papers have, as the primitive, a possibility correspondence defined over the set of all full states *and all subjective states*, which they use to characterize the agent's knowledge hierarchy. Then they define unawareness in terms of knowledge (or ignorance). In other words, they take the agent's subjective knowledge as the primitive and "infer" the agent's awareness from it. Unlike them, I take awareness information as the primitive, and "derive" the agent's subjective model and subjective knowledge. Consequently, as the previous section summarizes, the product model extends the standard model in a more intuitive way and has parallel structure. In the context of the Watson example, Ely (1998) proposes a framework where the agent's subjective model can also be seen as derived from the full model and the subjective state spaces are represented by partitions over the full state space. It is possible to construct an unawareness model along this line, and it can be shown that such a model is isomorphic to the product model.[23]

There are a number of exciting topics one can explore using this model. The anticipation of unforeseen contingencies has a significant impact in real-life decision processes. Many people keep some personal funds for unspecified emergencies. At a collective level, the scale can be quite impressive. For instance, "the City of New York's Five-Year Financial Plan" includes a "general reserve for *unforeseen contingencies* of \$42 million in FY 2001 and reserves of \$545 million in FY 2002."[24] However, decision-making under unforeseen contingencies is not well-understood. The standard Savage framework assumes away unforeseen contingencies. Research in this field has been focused on axiomatization of preferences over menus of items, which provides the dynamic structure intrinsically related to unforeseen contingencies. This model sheds light on how a more direct approach exploring the generalized information structure might work. Contractual incompleteness is a particularly interesting and important economic phenomenon where anticipation of unforeseen contingencies seems to play an important role. It is not clear how to apply research in decision-making with unforeseen contingencies to the contractual environment without an explicit account for parties' information structure in this environment. This model provides a simple tool for that purpose.

8 Appendices

A Proofs

PROOF OF LEMMA 3:

U0* *Symmetry*: $U(E) = U(\neg E)$

Follows from the fact that $\mathcal{D}_E = \mathcal{D}_{\neg E}$.

U1′ *Strong plausibility*: $U(E) \subseteq \bigcap_{n=1}^{\infty} (\neg K)_n(E)$

Let $\omega^* \in U(E)$. Then $\mathcal{D}_E \nsubseteq W(\omega^*)$. By 3.3, $\tilde{K}_{\omega^*}(E) = \emptyset_E$. By 3.5, $\omega^* \in \neg K(E)$. Note that $\mathcal{D}_{\neg \tilde{K}_{\omega^*}(E)} = \mathcal{D}_{\tilde{K}_{\omega^*}(E)} = \mathcal{D}_{\emptyset_E} = \mathcal{D}_E \nsubseteq W(\omega^*)$, now it follows $\tilde{K}_{\omega^*} \neg \tilde{K}_{\omega^*}(E) = \emptyset_E$ and $\omega^* \in (\neg K)_2(E)$. It is easy to see that $\mathcal{D}_{\neg \tilde{K}^{n-1}_{\omega^*}(E)} = \mathcal{D}_E$ for all n, which implies $\tilde{K}_{\omega^*}(\neg \tilde{K}^{n-1}_{\omega^*}(E)) = \emptyset_E$ for all n. Thus, $\omega^* \in (\neg K)_n(E)$ for all n.

[23] Details are available in the supplementary note.

[24] The emphasis is mine. Source: "Review of the Mayor's Executive Budget for Fiscal Year 2002" by H. Carl McCall (State Comptroller), web address: http://www.osc.state.ny.us/osdc/rpt2002.pdf

U2* *AU introspection*: $U(E) \subseteq UU(E)$

Follows from the observation that $\mathcal{D}_{U(E)} = \mathcal{D}^* \supseteq \mathcal{D}_E$ for all $E \in \mathcal{E}$.

U3$'$ *Weak KU introspection*: $U(E) \cap KU(E) = \emptyset_{\Omega^*}$

Follows from plausibility and AU introspection: $U(E) \subseteq UU(E) \subseteq \neg KU(E) \Rightarrow U(E) \cap KU(E) = \emptyset_{\Omega^*}$.

K1* *Subjective necessitation*: for all $\omega^* \in \Omega^*$, $\omega^* \in K(\Omega(\omega^*))$

For any $\omega^* \in \Omega^*$, $P_{\omega^*}(s(\omega^*)) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*)) \subseteq \Omega(\omega^*)$, which implies $s(\omega^*) \in \tilde{K}_{\omega^*}(\Omega(\omega^*))$, and hence $\omega^* \in K(\Omega(\omega^*))$.

K2* *Generalized monotonicity*: $E \subseteq_* F$, $\mathcal{D}_F \subseteq \mathcal{D}_E \Rightarrow K(E) \subseteq K(F)$

This is implied by conjunction. Take E and F such that $E \subseteq_* F$, $\mathcal{D}_F \subseteq \mathcal{D}_E$. By conjunction,

$$\begin{aligned} K(E) \cap K(F) &= K(E \cap_* F) \\ &= K(E \cap F_{\times \mathcal{D}_E}) \\ &= K(E) \end{aligned}$$

It follows $K(E) \subseteq K(F)$.

K3* *Conjunction*: $K(E) \cap K(F) = K(E \cap_* F)$

Let $\omega^* \in K(E) \cap K(F)$. Then $\mathcal{D}_E \subseteq W^*(\omega^*)$, $\mathcal{D}_F \subseteq W^*(\omega^*)$ and $P^*(\omega^*) \subseteq E_{\Omega^*}$, $P^*(\omega^*) \subseteq F_{\Omega^*}$. Note that:

(1) $\mathcal{D}_E \subseteq W^*(\omega^*)$, $\mathcal{D}_F \subseteq W^*(\omega^*)$ if and only if $\mathcal{D}_E \cup \mathcal{D}_F \subseteq W^*(\omega^*)$, which implies $\mathcal{D}_{E \cap_* F} \subseteq W^*(\omega^*)$. Thus, the event $E \cap_* F$ has an elaboration in the space $\Omega(\omega^*)$.

(2) $P^*(\omega^*) \subseteq E_{\Omega^*}$, $P^*(\omega^*) \subseteq F_{\Omega^*}$ if and only if $P^*(\omega^*) \subseteq (E_{\Omega^*} \cap F_{\Omega^*})$;

Using the product structure, one has $E_{\Omega^*} = E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F))$, $F_{\Omega^*} = F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F))$. Therefore

$$\begin{aligned} E_{\Omega^*} \cap F_{\Omega^*} &= [E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F))] \cap [F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F))] \\ &= [E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \cap F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)}] \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F)) \\ &= (E \cap_* F)_{\Omega^*} \end{aligned}$$

Back to (2), one has $P^*(\omega^*) \subseteq (E \cap_* F)_{\Omega^*}$. Using (1), $P_{\omega^*}(s(\omega^*)) = \mathbb{P}^{\Omega(\omega^*)} P^*(\omega^*) \subseteq \mathbb{P}^{\Omega(\omega^*)}((E \cap_* F)_{\Omega^*}) = (E \cap_* F)_{\Omega(\omega^*)}$, hence $s(\omega^*) \in \tilde{K}_{\omega^*}((E \cap_* F)_{\Omega(\omega^*)})$, and hence $\omega^* \in K(E \cap_* F)$ $\square$

Proof of Lemma 4:

U1* *UUU*: $U(E) = \bigcap_{n=1}^{\infty} (\neg K)_n(E)$

Strong plausibility gives $\Rightarrow$; Applying De Morgan's law on the axiom of limited wisdom gives the other direction.

U3* *KU introspection*: $KU(E) = \emptyset_{\Omega^*}$

By nondelusion, $K(E) \subseteq_* E$. Thus $KU(E) \subseteq U(E)$, hence $KU(E) \cap U(E) = KU(E)$. The result follows from weak KU introspection.

Observe that when (W^*, P^*) is rational, the subjective model is a standard information partition model at the set of possible subjective states, and hence the standard results carry through. I give the details of proofs for the axiom and knowledge and the axiom of transparency and omit the proof for the axiom of limited wisdom.

K4* *The axiom of knowledge*: $K(E) \subseteq_* E$

$\omega^* \in K(E) \Rightarrow P^*(\omega^*) \subseteq E_{\Omega^*}$, by nondelusion, $\omega^* \in P^*(\omega^*) \Rightarrow \omega^* \in E_{\Omega^*}$. The result follows from $K(E) \subseteq E_{\Omega^*}$.

K5* *The axiom of transparency*: $K(E) \subseteq K_2(E)$

It is sufficient to show $s(\omega^*) \in \tilde{K}_{\omega^*}(E)$ implies $s(\omega^*) \in \tilde{K}^2_{\omega^*}(E)$.
Since (W^*, P^*) is rational, $P_{\omega^*}(\omega) = P_{\omega^*}(s(\omega^*))$ for all $\omega \in P_{\omega^*}(s(\omega^*))$. Now

$$
\begin{aligned}
& s(\omega^*) \in \tilde{K}_{\omega^*}(E) \\
\Rightarrow\ & P_{\omega^*}(s(\omega^*)) \subseteq E_{\Omega(\omega^*)} \\
\Rightarrow\ & P_{\omega^*}(\omega) \subseteq E_{\Omega(\omega^*)} \\
\Rightarrow\ & \omega \in K_{\omega^*}(E) \text{ for all } \omega \in P_{\omega^*}(s(\omega^*)) \\
\Rightarrow\ & P_{\omega^*}(s(\omega^*)) \subseteq K_{\omega^*}(E) \\
\Rightarrow\ & s(\omega^*) \in \tilde{K}^2_{\omega^*}(E)
\end{aligned}
$$

□

B Detailed formulation for the multi-agent product model

Let $W^i_{\omega^*}$ denote the agent's *subjective awareness function.* Analogous to subjective possibility correspondence, this function specifies which uncertainties i is aware of at the *subjective states* in his frame of mind at ω^*. For any $\omega \in \Omega_i(\omega^*)$, let

$$W^i_{\omega^*}(\omega) = W^*_i(\omega^*) \cap \big(\bigcup_{\left\{\omega^*_1 :\, \mathbb{P}^{\Omega_i(\omega^*)}(\omega^*_1)=\omega\right\}} W^*_i(\omega^*_1)\big) \tag{B.1}$$

(B.1) captures the idea that the agent cannot reason beyond his frame of mind. Note that the agent may be aware that he is unaware of some uncertainties in a subjective state he knows to be false.[25] His subjective model $(\Omega_i(\omega^*), W^i_{\omega^*}, P^i_{\omega^*})$ is a product model itself, with $(\Omega_i(\omega^*), P^i_{\omega^*})$ being the subjective version of the full model.

Based on the single agent's subjective product model, for any full state ω^*, one can characterize an agent's subjective multi-agent product model, *where he reasons about other agents' frames of mind within his own frame of mind.* Let $W^{ij}_{\omega^*}$ denote i's version of j's subjective awareness function at ω^*. For any $\omega \in \Omega_i(\omega^*)$, let

$$W^{ij}_{\omega^*}(\omega) = W^*_i(\omega^*) \cap \bigcup_{\left\{\omega' \in \Omega_j(\omega^*):\, \mathbb{P}^{\times(W^*_i(\omega^*)\cap W^*_j(\omega^*))}(\omega')=\mathbb{P}^{\times(W^*_i(\omega^*)\cap W^*_j(\omega^*))}(\omega)\right\}} W^j_{\omega^*}(\omega') \tag{B.2}$$

$W^{ij}_{\omega^*}(\omega)$ is i's version of j's subjective awareness function at i's subjective state $\omega \in \Omega(\omega^*)$. It is obtained by first imposing the restrictions of j's current frame of mind on j's awareness in j's subjective states, then projecting it onto i's subjective state space and imposing the restrictions of i's current frame of mind on it analogously.

i's version of j's subjective possibility correspondence is obtained by taking the projection of j's subjective factual information onto i's version of j's subjective state space. Let $\Omega^{ij}_{\omega^*}(\omega) = \times W^{ij}_{\omega^*}(\omega)$ denote i's version of j's subjective state space at i's subjective state $\omega \in \Omega_{i_1}(\omega^*)$. For any $\omega \in \Omega_i(\omega^*)$, let

$$P^{ij}_{\omega^*}(\omega) = \mathbb{P}^{\Omega^{ij}_{\omega^*}(\omega)} \bigcup_{\left\{\omega' \in \Omega_j(\omega^*):\, \mathbb{P}^{\times(W^*_i(\omega^*)\cap W^*_j(\omega^*))}(\omega')=\mathbb{P}^{\times(W^*_i(\omega^*)\cap W^*_j(\omega^*))}(\omega)\right\}} P^j_{\omega^*}(\omega') \tag{B.3}$$

Thus, at ω^*, i's version of j's subjective product model is $(\Omega_i(\omega^*), W^{ij}_{\omega^*}, P^{ij}_{\omega^*})$. Let $(\Omega(\omega^*), \mathbf{W}^i_{\omega^*}, \mathbf{P}^i_{\omega^*})$ denote i's subjective multi-agent product model, where $\mathbf{W}^i_{\omega^*} = (W^{i1}_{\omega^*}, \cdots, W^{in}_{\omega^*})$ and $\mathbf{P}^i_{\omega^*} = (P^{i1}_{\omega^*}, \cdots, P^{in}_{\omega^*})$ denote i's version of every agent's subjective awareness function and subjective possibility correspondence respectively.

[25] Rational awareness ensures $W^*_i(\omega^*) \subseteq W^i_{\omega^*}(\omega)$ for any $\omega \in P^i_{\omega^*}(s(\omega^*))$, which says the agent has the same *subjective* awareness information in all possible subjective states. This implies the agent has the same subjective model in all possible subjective states, and hence the subjective *factual* information alone suffices to characterize the agent's "local" subjective model.

Higher-order interactive models such as i's version of j's version of k's subjective product model can be characterized recursively. For instance, i's version of j's version of k's subjective awareness is obtained by first considering j's version of k's subjective awareness in j's subjective states, which is $W^{jk}_{\omega^*}$; then considering i's version of $W^{jk}_{\omega^*}$. Formally, for any m, i_1's version of i_2's version of $\cdots$ of i_m's awareness function is recursively defined by: for any $\omega \in \Omega_{i_1}(\omega^*)$,

$$W^{i_1 i_2 \cdots i_m}_{\omega^*}(\omega) = W^*_i(\omega^*) \cap \bigcup_{\left\{\omega' \in \Omega_{i_2}(\omega^*):\, \mathbb{P}^{\times(W^*_{i_1}(\omega^*) \cap W^*_{i_2}(\omega^*))}(\omega') = \mathbb{P}^{\times(W^*_{i_1}(\omega^*) \cap W^*_{i_2}(\omega^*))}(\omega)\right\}} W^{i_2 \cdots i_m}_{\omega^*}(\omega') \tag{B.4}$$

Similarly for higher-order interactive subjective possibility correspondences. For any m, let $P^{i_1 \cdots i_m}_{\omega^*}$ denote i_1's version of $\cdots i_m$'s subjective possibility correspondence at ω^*. Let $\Omega^{i_1 \cdots i_m}_{\omega^*}(\omega) = \times W^{i_1 \cdots i_m}_{\omega^*}(\omega)$ denote i_1's version of $\cdots i_m$'s subjective state space at i_1's subjective state $\omega \in \Omega_{i_1}(\omega^*)$. Define:

$$P^{i_1 \cdots i_m}_{\omega^*}(\omega) = \mathbb{P}^{\Omega^{i_1 \cdots i_m}_{\omega^*}(\omega)} \bigcup_{\left\{\omega' \in \Omega_{i_2}(\omega^*):\, \mathbb{P}^{\times(W^*_{i_1}(\omega^*) \cap W^*_{i_2}(\omega^*))}(\omega') = \mathbb{P}^{\times(W^*_{i_1}(\omega^*) \cap W^*_{i_2}(\omega^*))}(\omega)\right\}} P^{i_2 \cdots i_m}_{\omega^*}(\omega') \tag{B.5}$$

Proof of Lemma 5:

I show that nice partition implies that for any Ω, if $\mathbb{P}^{\Omega}(\omega^*_1) = \mathbb{P}^{\Omega}(\omega^*_2)$, then:

$$\mathbb{P}^{\Omega}(P^*(\omega^*_1)) = \mathbb{P}^{\Omega}(P^*(\omega^*_2)) \tag{B.6}$$

In words, equation (B.6) says the factual information in any two full states never differs along the dimensions the two full states coincide. To prove it, suppose not. Let $\omega \in \mathbb{P}^{\Omega}(P^*(\omega^*_1))$ but $\omega \notin \mathbb{P}^{\Omega}(P^*(\omega^*_2))$. It follows that:

(1) There exists ω^*_3 such that $\mathbb{P}^{\Omega}(\omega^*_3) = \omega$ and $\omega^*_3 \in P^*(\omega^*_1)$;

(2) For any ω^*_4 such that $\mathbb{P}^{\Omega}(\omega^*_4) = \omega$, $\omega^*_4 \notin P^*(\omega^*_2)$.

Since P^* induces an information partition over Ω^*, $\omega^*_1 \in P^*(\omega^*_3)$. By nice partition, ω^*_2 has to be an element of some $P^*(\omega^*_4)$ where ω^*_4 is such that $\mathbb{P}^{\Omega}(\omega^*_4) = \omega$. Contradiction. The proposition follows easily from equation (B.6). $\square$

By Lemma 5, all the interactive subjective models are product models with rational generalized information. Hence Theorem 2 applies. All interactive knowledge in an agent's frame of mind can be directly derived from the interactive generalized information. Formally, at ω^*, i_1's version of $\cdots i_m$'s subjective knowledge is:

$$\tilde{K}^{i_1 \cdots i_m}_{\omega^*}(E) = \left\{\omega \in \Omega_i(\omega^*) :\ \mathcal{D}_E \subseteq W^{i_1 \cdots i_m}_{\omega^*}(\omega),\ P^{i_1 \cdots i_m}_{\omega^*}(\omega) \subseteq E_{\Omega^{i_1 \cdots i_m}_{\omega^*}(\omega)}\right\} \tag{B.7}$$

An event E is i's *subjective common knowledge* if in i's frame of mind, everybody knows E, everybody knows everybody knows E, and so on. Note those "everybody knows" refer to *i's version of interactive knowledge*. Let $CK_i(E)$ denote the event "common knowledge of E" in i's frame of mind. Then:

$$CK_i(E) = \left\{\omega^* \in \Omega^* : \; s_i(\omega^*) \in \cap_{j=1}^n \tilde{K}^{ij}_{\omega^*}(E) \cap_{j,q=1}^n \tilde{K}^{ij}_{\omega^*}\tilde{K}^{ijq}_{\omega^*}(E) \cdots \right\} \tag{B.8}$$

$CK_i(E)$ is the set of full states in which i considers E as common knowledge among all the agents.

Proposition 7 *In the multi-agent product model, let (W_i^*, P_i^*) be strongly rational for every $i \in N$. Then,*

$$CK(E) = \cap_{i=1}^n CK_i(E) \tag{B.9}$$

That is, an event is common knowledge if and only if every agent considers it common knowledge in their frames of mind. Apparently, Proposition 7 implies Theorem 6.

Proof. It is straightforward to see that (B.9) implies (5.1) and hence the proof is omitted. I show below that (5.1) implies (B.9) under strongly rational information structure.

Lemma 8 $\mathcal{D}_E \subseteq W_i^*(\omega^*) \cap W_j^*(\omega^*)$ *implies* $s_i(\omega^*) \in \tilde{K}^{ij}_{\omega^*}(E) \Leftrightarrow s_j(\omega^*) \in \tilde{K}^{j}_{\omega^*}(E)$.

Proof. Since $s_i(\omega^*) \in \tilde{K}^{ij}_{\omega^*}(E)$, $\mathcal{D}_E \subseteq W^{ij}_{\omega^*}(s_i(\omega^*)),\; P^{ij}_{\omega^*}(s_i(\omega^*)) \subseteq E_{\Omega^{ij}_{\omega^*}(s_i(\omega^*))}$.

Observe that $P^{ij}_{\omega^*}(s_i(\omega^*))$ and $P^{j}_{\omega^*}(s_j(\omega^*))$ are just projection images of $P_j^*(\omega^*)$ on corresponding subjective state spaces. Also observe that $\mathcal{D}_E \subseteq \mathcal{D}_F$ implies that $E \subseteq \mathbb{P}^{\times \mathcal{D}_E}(F) \Leftrightarrow E_{\times \mathcal{D}_F} \subseteq F$.

Now $\mathcal{D}_E \subseteq W_i^*(\omega^*) \cap W_j^*(\omega^*) \subseteq W_j^*(\omega^*)$ implies $\times \mathcal{D}_E$ is a lower-dimensional space than both $\Omega^{ij}_{\omega^*}(s_i(\omega^*))$ and $\Omega_j(\omega^*)$. Thus $P^{ij}_{\omega^*}(s_i(\omega^*)) = \mathbb{P}^{\Omega^{ij}_{\omega^*}(s_i(\omega^*))}(P_j^*(\omega^*)) \subseteq E_{\Omega^{ij}_{\omega^*}(s_i(\omega^*))} \Leftrightarrow \mathbb{P}^{\times \mathcal{D}_E}(P_j^*(\omega^*)) \subseteq E \Leftrightarrow \mathbb{P}^{\Omega_j^*(\omega^*)}(P_j^*(\omega^*)) \subseteq E_{\Omega_j^*(\omega^*)}$. □

Lemma 8 says that concerning events of which both agents are aware, i's version of j's subjective knowledge reflects j's subjective knowledge truthfully and adequately. It is easily seen that the result extends to higher-order interactive knowledge.

Corollary 9 $\mathcal{D}_E \subseteq \bigcap_{\{\omega_1^* :\, \omega_1^* \in \wedge \mathbf{P}^*(\omega^*)\}} \mathbf{W}^*(\omega_1^*)$ *implies* $s_{i_1}(\omega^*) \in \tilde{K}^{i_1 \cdots i_m}_{\omega^*}(E) \Leftrightarrow s_{i_2}(\omega^*) \in \tilde{K}^{i_2 \cdots i_m}_{\omega^*}(E)$.

Proof. Apply Lemma 8 recursively. The condition $\mathcal{D}_E \subseteq \bigcap_{\{\omega_1^* :\, \omega_1^* \in \wedge \mathbf{P}^*(\omega^*)\}} \mathbf{W}^*(\omega_1^*)$ ensures that in i_1's frame of mind, all agents are aware of E in any subjective states reachable from $s_{i_1}(\omega^*)$. □

For any event E, let ω^* be such that $\mathcal{D}_E \subseteq \bigcap_{\{\omega_1^*:\, \omega_1^* \in \wedge \mathbf{P}^*(\omega^*)\}} \mathbf{W}^*(\omega_1^*)$ and $\wedge \mathbf{P}^*(\omega^*) \subseteq E_{\Omega^*}$. It is immediately clear that the first-order knowledge holds for every agent, that is, $s_i(\omega^*) \in \tilde{K}^i_{\omega^*}(E)$. By Lemma 8, this is reflected in every other agent's version: $s_j(\omega^*) \in \tilde{K}^{ji}_{\omega^*}(E)$. To see the second-order knowledge, note that $\wedge \mathbf{P}^*(\omega^*) \subseteq E_{\Omega^*}$ ensures $P_i^*(\omega^*) \subseteq \left\{\omega_1^* : P_j^*(\omega_1^*) \subseteq E_{\Omega^*}\right\}$. It follows that $\mathbb{P}^{\Omega_i(\omega^*)}(P_i^*(\omega^*)) \subseteq \mathbb{P}^{\Omega_i(\omega^*)}\left\{\omega^* : P_j^*(\omega^*) \subseteq E_{\Omega^*}\right\}$.

Lemma 10 $\mathbb{P}^{\Omega_i(\omega^*)}\left\{\omega_1^* : P_j^*(\omega_1^*) \subseteq E_{\Omega^*}\right\} = \left\{\omega \in \Omega_i(\omega^*) :\ \mathcal{D}_E \subseteq W^{ij}_{\omega^*}(\omega),\ P^{ij}_{\omega^*}(\omega) \subseteq E_{\Omega^{ij}_{\omega^*}(\omega)}\right\}$

Proof. (1) LHS $\supseteq$ RHS: Follows from the fact that elaborations preserve logical relations between events.

(2) LHS $\subseteq$ RHS: Let $\omega \in$ LHS. Then there exists ω_1^* such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*) = \omega$ and $P_j^*(\omega_1^*) \subseteq E_{\Omega^*}$, which implies $\mathbb{P}^{\Omega_i(\omega^*)}(P_j^*(\omega_1^*)) \subseteq E_{\Omega_i(\omega^*)}$. By equation (B.6),

$$\mathbb{P}^{\Omega_i(\omega^*)}\Big(\bigcup_{\left\{\omega_2^*:\mathbb{P}^{\Omega_i(\omega^*)}(\omega_2^*)=\omega\right\}}(P_j^*(\omega_1^*))\Big) \subseteq E_{\Omega_i(\omega^*)}$$

Project both sides of the above equation onto the subjective state space $\Omega^{ij}_{\omega^*}(\omega)$ to get $P^{ij}_{\omega^*}(\omega) \subseteq E_{\Omega^{ij}_{\omega^*}(\omega)}$ as desired. □

Hence $\mathbb{P}^{\Omega_i(\omega^*)}(P_i^*(\omega^*)) \subseteq \left\{\omega \in \Omega_i(\omega^*) :\ \mathcal{D}_E \subseteq W^{ij}_{\omega^*}(\omega),\ P^{ij}_{\omega^*}(\omega) \subseteq E_{\Omega_i(\omega^*)}\right\}$. But the left hand side of the equation is just $P_{\omega^*}(s_i(\omega^*))$ and the right hand side is $\tilde{K}^{ij}_{\omega^*}(E)$. It follows that $s_i(\omega^*) \in \tilde{K}^i_{\omega^*}\tilde{K}^{ij}_{\omega^*}(E)$. This is true for every i and j. By Corollary 9, i's interactive knowledge is reflected in every other agent's version: $s_k(\omega^*) \in \tilde{K}^{ki}_{\omega^*}\tilde{K}^{kij}_{\omega^*}(E)$. The same argument holds for all high-order interactive knowledge. □

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