Termination of Dynamic Contracts in an Equilibrium Labor Market Model

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Abstract

I construct an equilibrium model of the labor market where workers and firms enter into dynamic contracts that can potentially last forever but are subject to optimal terminations. Upon a termination, the firm hires a new worker, the worker who is terminated receives a termination compensation from the firm and is then free to go back to the labor market to seek new employment opportunities and enter into new dynamic contracts. The model permits only two types of equilibrium terminations that resemble respectively the two typical kinds of labor market separations observed in practice: involuntary layoffs and voluntary retirements. The model thus allows simultaneous determination of its equilibrium turnover, unemployment, retirement, as well as the expected utility of the new labor market entrants.

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1 Introduction

I construct an equilibrium model of the labor market where workers and firms enter into dynamic contracts that can potentially last forever but are subject to optimal terminations. Moral hazard is the underlining information friction, that contracts are dynamic and terminations are optimal are both driven solely by incentive considerations. Upon termination of a contract, the firm hires a new worker, the worker who is terminated receives a termination compensation from the firm and is then free to go back to the labor market to seek new employment opportunities and enter into new dynamic contracts.

Despite the potentially complex interactions that can take place between the workers and firms in the model, the equilibrium of the model has a simple structure. The model permits only two types of equilibrium terminations that resemble respectively the two kinds of labor market separations that are observed in practice: involuntary layoffs and voluntary retirements. When an involuntary layoff occurs, the firm promises no future payments to the worker, and the expected utility of the worker is strictly lower than the new worker the firm hires to replace him. When a voluntary retirement occurs, the worker leaves the firm with a termination compensation that is equal to a sequence of constant payments, and he never goes back to the labor market to seek new employment again. The model thus allows simultaneous determination of its equilibrium turnover, unemployment, retirement, as well as the equilibrium expected utility of the new labor market entrants.

Unemployment is involuntary in my model, as in the models of efficiency wages (e.g., Shapiro and Stiglitz 1984). Compared to the existing models of efficiency wages though, my model offers three advantages. First, efficiency wage models are often criticized because the employment contracts in these models are not fully optimal. In Shapiro and Stiglitz for example, because wages are constant, termination (lay-off) is the only incentive device that firms have available to prevent workers from shirking. In the model here, workers and firms enter into fully dynamic contracts where wages vary optimally with the worker's performance history. Second, in the existing models of efficiency wages, in equilibrium *no* workers are actually fired because of shirking (the contract makes effort-making incentive compatible so no one shirks), and the unemployed are a rotating pool of workers who quit for reasons that are not modelled. In the model here, workers are actually fired involuntarily from their jobs: firing is part of the model's equilibrium path. Third, my model permits simultaneously involuntary unemployment and voluntary retirement as its equilibrium outcome.

This paper also extends the existing theories of dynamic contract following Green (1987) and Srivastava (1987). What this paper does is to put fully dynamic contracts with endogenous terminations into an equilibrium framework where agents can enter into contracting relationships multiple times. This has not been done in the existing literature. The paper that is closest to the current paper is Spear and Wang (2001). [1] Spear and Wang adds an exogenous external labor market to the otherwise standard model of repeated moral hazard. This external labor market allows the firm to fire the worker and replace him with a new worker. Spear and Wang is a partial equilibrium setup where the unemployed workers' reservation utilities are exogenously given, and it is imposed that workers who are terminated are never employed again. In the current paper, workers who are terminated are allowed to go back to the labor market to seek new employment opportunities, and the model makes clear predictions about who actually choose to go back to the labor market and who choose to stay out of the labor market permanently. Being an equilibrium setup, the model here allows me to determine simultaneously the equilibrium aggregate unemployment and retirement, as well as the model's other aggregate variables, including the equilibrium labor turnover and the expected utility of the new labor market entrants. Termination of dynamic contracts is also studied by DeMarzo and Fishman (2003) in a partial equilibrium model of corporate finance with privately observed cash flows. Stiglitz and Weiss (1983) model the incentive effects of termination in a two period environment where there is only one worker and one firm.

A notable feature of the dynamic contracts in this paper is that they are required to be renegotiation proof. This plays a key role for simplifying the model's equilibrium structure, allowing me to avoid heterogeneity among the unemployed. Since all workers are identical in ability, that contracts must be renegotiation proof implies that the termination compensation of an involuntarily terminated worker (who after termination goes back to the labor market to seek new employment) must be zero. Otherwise a renegotiation between the firm and the worker can make both parties strictly better off. This renegotiation simply requires that the worker gives back the termination compensation and the firm hires back the worker.

Section 2 describes the model. Section 3 defines the contracts and labor market equilibrium. Section 4 characterizes the voluntary retirement involuntary layoff. Section 5 concludes the paper.

[1] A subset of Spear and Wang (2001) is published in Spear and Wang (2005) where the analysis is restricted to a two-period setting.

2 Model

Time is discrete and lasts forever. There is one perishable consumption good in each period. The economy is populated by a sequence of overlapping generations, each of which contains a continuum of workers. The total measure of workers in the economy is equal to one. Each worker faces a time-invariant probability Δ of surviving into the next period. Each new generation has measure $1-\Delta$, so the number of births and the number of deaths are equal in each period. [2]An individual who is born at time τ has the following preferences:

$$E_{\tau-0}\sum_{t=\tau}^{\infty}(\beta\Delta)^{t-\tau}\,H(c_t,a_t),$$

where $E_{\tau-0}$ denotes expectation taken at the beginning of period τ, c_t denotes period t consumption, a_t denotes period t effort, $H(c_t,a_t)$ denotes period t utility, and $\beta\in[0,1)$ is the discount factor. Assume $H(c,a)=u(c)-\phi(a)$, for $c\in\mathsf{R}_+$, $a\in\{0\}\bigcup\mathsf{A}$, where A is the individual's compact set of all feasible effort levels when he is employed. The individual's effort takes the value 0 if he is not employed. Let $\underline{a}\equiv\min\{a\in\mathsf{A}\}>\mathsf{0}$. Finally, the functions u is strictly increasing and concave in c, and function ϕ is strictly increasing in a with $\phi(0)=0$.

There are a measure of $\eta\in(0,1)$ units of firms. Firms live forever and maximize expected discounted net profits. For convenience, I assume in each period, each firm needs to employ only one worker. [3] The worker's effort is the only input in the firm's production function, and the worker's effort is observed by himself only. By choosing effort a_t in period t, the worker produces a random output in period t that is a function of a_t. Let θ^t denote the realization of this random output. Assume $\theta^t\in\Theta$, where $\Theta=\{\theta_1,\theta_2,...,\theta_n\}$ with $\theta_i<\theta_j$ for $i<j$. Let $X_i(a)=Prob\{\theta^t=\theta_i|a_t=a\}$, for all $\theta_i\in\Theta$, all $a\in\mathsf{A}$ and all t.

The firm and a newly hired worker can enter into a labor contract that is fully dynamic. A component of this dynamic contract is a history dependent plan that specifies whether the worker is terminated at the end of each date. If the worker is terminated, he is free to go immediately back to the labor market to seek new employment opportunities, and the firm then hires a new worker to replace him. For convenience I assume the process of termination

[2]The OLG structure is needed here in order for me to model stationary equilibria with voluntary retirements.

[3]It would not make a difference if I allow firms to employ more workers, as long as they operate independent production technologies.

and replacement involves no physical costs to both the firm and the worker. [4]

As part of the model's physical environment, I make the following assumptions about the contracts that are feasible between the worker and the firm. First, contracts are subject to a non-negativity constraint which requires that all compensation payments that the worker receives from the firm be non-negative.

Second, contracts are subject to renegotiations, provided that the renegotiations are mutually beneficial and *strictly* beneficial to the firm. This assumption puts a restriction on the structure of the dynamic contract that can be signed between the firm and the worker: the contract must be renegotiation-proof (RP). Note that in order for renegotiations to take place, I require that they be *strictly* beneficial to the firm. That is, the firm can commit to carrying out the continuation of a dynamic contract if renegotiations can only benefit the worker, leaving the firm indifferent. As will become clear later in the analysis of the model, since workers are identical, the requirement that the firm be strictly better off in a renegotiation is needed in order to make involuntary terminations occur in equilibrium. Without this requirement, involuntary terminations can not be part of the equilibrium RP contract. [5]

Third, it is feasible for the firm to continue to make compensation payments to the worker even after the worker is terminated from the firm (i.e., he is replaced by a new worker). But there is a restriction. Post-termination compensations cannot be contingent on the worker's performance and compensation at the new firm the worker works in the future, although these compensations can be made a function of the worker's future employment status. In other words, post termination compensations must be a step function of the worker's employment status after his separation from the current employer. [6]

[4] An interesting extension of this current work is to study the effects of a cost of termination which may be imposed by a policy maker.

[5] See Wang (2000) and Zhao (2004) for an existing analysis of renegotiation-proof contracts in the model of dynamic moral hazard. In Zhao (2004), a RP contract under the qualification that renegotiations be strictly beneficial to the principal is called principal RP. Zhao used this concept for a different reason than mine.

[6] This assumption saves me from the difficulty of modelling a potentially complicated dynamic game that can be played between the worker's former and current employers.

3 Contracts and Equilibrium

In this section, I first define a dynamic contract, taking as given the labor market in which this contract must operate; I then define what a labor market equilibrium is by requiring the market be consistent with individual firms' optimal contracts.

I take a guess and verify approach to find the model's equilibrium. Specifically, when defining the optimal contract, I take as given that in equilibrium the labor market has the characteristic that all unemployed workers (those who are not employed and looking for jobs) either were never employed (including the new labor market entrants) or are not entitled to any post termination compensations from their former employers. I will then verify that this indeed is part of the labor market equilibrium.

3.1 Contracts

Let σ denote a contract between a firm and its newly hired worker. For convenience I now use $t = 0$ to denote the time the contract is signed. Let $t(\geq 1)$ denote the tth period into the contract. ($t = 1$ is the first period the worker is hired to work for the firm and so on.) Then σ takes the following extensive form: $\{[a_t(h_{t-1}), c_t(h_t)],\ I_t(h_t);\ g_t(h_t)\}_{t=1}^{\infty}$. Here, for each $t \geq 1$, $h_t = \{\theta^1, ..., \theta^t\} \in H_t \equiv \Theta^t$ denotes a history of output up to the end of period t, with $h_0 = \emptyset$. At the beginning of period t, history is h_{t-1}. The function $a_t : H_{t-1} \to \mathsf{A}$ specifies the level of effort the firm wants the worker to make in period t. After the worker makes his effort, the firm's output is realized, history is then updated to become h_t. The function $c_t : H_t \to \mathsf{R}_+$ then specifies the worker's compensation in period t. The function $I_t : H_t \to \{0, 1\}$ is the termination function. If $I_t(h_t) = 1$, then the contract continues into period $t+1$. If $I_t(h_t) = 0$, then the contract is terminated on history h_t. Finally, upon termination of the contract on history h_t, the worker receives a "termination contract" $g_t(h_t)$ that specifies the future payments he may receive from the firm. [7]

Remember the termination contract, which is a subcontract of σ, must be a step function of the worker's employment status after termination. As I

[7]Clearly, the function a_t need not be defined on the whole space H_{t-1}, and c_t and g_t need not be defined on the whole space H_t. Let $H_0 = \emptyset$, $H_t = \Theta^t$ for $t \geq 1$. Let $\tilde{H}_t \equiv \{h_t : h_t \in H_t,$ and $I_t(h_t) = 1\}$. Then $a_t : \tilde{H}_{t-1} \to \mathsf{A}$, $c_t : \tilde{H}_{t-1} \times \Theta \to R_+$. And the termination contract g_t is defined on the set $H_t - \tilde{H}_t$. Here I impose that if $I(h_t) = 0$, then $I(\{h_t, \theta\}) = 0$ for all $\theta \in \Theta$. This ensures that $I(h_t) = 0$ means the termination of the worker.

will show, in equilibrium $g_t(h_t)$ takes an even simpler form: it is a stream of non-negative constant compensation that the worker receives from the firm after termination.

As in the literature of dynamic contracting following Green (1987) and Spear and Srivastava (1987), it can be shown that the contract σ can be written recursively to take the following form:

$$\sigma = \left\{ \begin{array}{c} \Phi = \Phi_r \cup \Phi_f \\ g(w), w \in \Phi_f \\ (a(w),\, c_i(w), w_i(w)),\, w \in \Phi_r \end{array} \right\}$$

Here, w denotes the worker's beginning of period expected utility, the set $\Phi \subseteq R$ being the domain of w. Φ is partitioned into two subsets Φ_r and Φ_f with $\Phi_r \cap \Phi_f = \emptyset$. This partition of Φ is constructed according to the following definition: If $w \in \Phi_f$, then the worker is terminated; if $w \in \Phi_r$, the worker continues. Next, $g(w)$ denotes the termination contract that the worker receives from the firm in the termination state $w \in \Phi_f$. Finally, conditional on $w \in \Phi_r$, $a(w)$ is the worker's recommended effort in the current period; $c_i(w), w_i(w)$ are, respectively, the worker's compensation in the current period and promised expected utility at the beginning of the next period, conditional on the worker's current period output being θ_i.

The contract σ is said to be feasible if for all $w \in \Phi_r$, $a(w) \in \mathsf{A}$, $c_i(w) \geq 0$, $w_i(w) \in \Phi$; and that for all $w \in \Phi_f$, all post termination compensation payments to the worker that are dictated by the termination contract $g(w)$ are all non-negative.

The contract must satisfy a promise-keeping constraint. This constraint requires that the structure of σ be consistent with the definition of w being the worker's expected utility at the beginning of a given period, for all $w \in \Phi$. In particular, the termination contract $g(w)$ must be designed to guarantee that the worker who leaves the firm with an expected utility entitlement w is indeed to receive expected utility equal to w. That is, given $g(w)$, and given what the market has to offer to the worker after termination, the worker's expected utility must be equal to w when he leaves the firm. Thus the promise-keeping constraint can be formulated as:

$$w = \sum_i X_i(w)[H(c_i(w), a(w)) + \beta \Delta w_i(w)],\ \forall w \in \Phi_r, \tag{1}$$

$$M[g(w)] = w,\ \forall w \in \Phi_f. \tag{2}$$

In equation (1), because the worker by assumption is not entitled to any post termination compensation from his previous employers (if any), $c_i(w)$ is just the worker's current period consumption. In equation (2), I use $M(x)$ to denote the value of the expected utility that an arbitrary termination contract x delivers to the worker, given the market that x takes as given. That is, the worker's expected utility is $M(x)$ if he leaves the firm with termination contract x. At this stage, M is taken as given.

A contract σ is called incentive compatible if

$$\begin{aligned} &\sum_i X_i(a(w))[H(c_i, a(w)) + \beta\Delta w_i(w)] \\ \geq &\sum_i X_i(a')[H(c_i(w), a') + \beta\Delta w_i(w)],\ \forall w \in \Phi_r,\ \forall a' \in \mathsf{A}. \end{aligned} \tag{3}$$

Notice that the promise-keeping constraint is defined for all $w \in \Phi$, whereas the incentive constraint need only be defined for all $w \in \Phi_r$.

Given σ, and given the *market* (i.e., what happens to the worker after termination) that the contract takes as given, I can calculate the firm's expected utility $U(w)$ for each $w \in \Phi$. I then refer to $U : \Phi \to R$ as the value function of the contract σ (conditional on the market that σ takes as given).

I am now in a position to define renegotiation-proof (RP) contracts. I call a contract σ RP if it supports a value function that is RP. Here I emphasize that, as the definition of the contract σ, the definition of the RP-ness of σ is also conditional on the *market* that σ takes as given. In the following, I first define what it means to say that a value function is RP. I then define what it means to say that a contract supports a RP value function.

An important component of the market that a contract must take as given is the expected utility of a new labor market entrant which I denote by w_*. Obviously, w_* is also the expected utility of a worker who either was never employed, or he was employed but is not entitled to any post termination compensation payments from his previous employers. These qualifications make him essentially the same as a new labor market entrant.

Let $\Phi \subseteq [w_*, \infty)$ and $\Phi = \Phi_r \bigcup \Phi_f$, where Φ_r, $\Phi_f \subseteq R$ and $\Phi_r \cap \Phi_f = \emptyset$. Let $\mathbf{B}$ denote the space of all value functions that map from Φ to $\mathbf{R}$. So the value functions that I consider will each have two components to its domain: one associated with the continuation of the contract (Φ_r), one associated with the termination of the contract (Φ_f). Notice also that the value functions that I consider will not be defined for expected utility levels that are lower than w_*. [8]

[8]As will become clear later, w_* is the lowest expected utility of the worker that a RP contract can implement.

Let $U \in \mathbf{B}$. U is said to be (internally) renegotiation-proof if it satisfies the following functional equation:

$$U = PTU, \tag{4}$$

where T and P, to be defined in the following, are both operators that map from $\mathbf{B}$ to $\mathbf{B}$.

Equation (4) is based on Ray (1994) where the operator T gives the set of all optimal expected utility pairs that are generated by U, and P then gives the subset of the graph of TU that are not Pareto dominated by any other utility pair in the graph of TU. [9]

I first define the operator T. Let $U \in \mathbf{B}$. Then $TU : [w_*, \infty) = \Phi'_r \bigcup \Phi'_f \to R$, where for each $w \in [w_*, \infty)$, the value of $TU(w)$ is defined by

$$TU(w) = \max\{U_r(w), U_f(w)\}, \tag{5}$$

where the functions U_r and U_f are to be given shortly, the sets Φ'_r and Φ'_f are defined by

$$\Phi'_r = \{w \in [w_*, \infty) : \ U_r(w) \geq U_f(w)\}, \tag{6}$$

$$\Phi'_f = \{w \in [w_*, \infty) : \ U_r(w) < U_f(w)\}. \tag{7}$$

In the above, $U_r(w)$ is the value of the firm if the the worker, who has an expected utility entitlement of w, is retained; and $U_f(w)$ is the value of the firm if the worker is terminated. Equation (5) says that the firm chooses to retain or fire the worker depending on which action gives the firm a better value. I now define the value functions $U_r(w)$ and $U_f(w)$.

To define $U_r(w)$, I first let $\tilde{\Phi}_r$ denote the set of all w such that there exists $\{a, c_i, w_i\}$ that satisfies the following constraints:

$$a \in \mathsf{A}; \ c_i \geq 0, \ w_i \in \Phi, \ \forall i, \tag{8}$$

$$\sum_i X_i(a)[H(c_i, a) + \beta \Delta w_i] \geq \sum_i X_i(a')[H(c_i, a') + \beta \Delta w_i], \ \forall a' \in \mathsf{A}, \tag{9}$$

$$w = \sum_i X_i(a)[H(c_i, a) + \beta \Delta w_i]. \tag{10}$$

[9] There are several other ways to define the sets of renegotiation-proof payoffs for infinitely repeated games. Ray's is a natural extension of the concept of renegotiation-proof payoff sets in finitely repeated games to infinitely repeated games. Ray's concept was used by Zhao (2004) to study renegotiation-proof dynamic contracts with moral hazard.

Then for each $w \in \tilde{\Phi}_r$, let

$$U_r(w) \equiv \max_{\{a,c_i,w_i\}} \sum_i X_i(a)[\theta_i - c_i + \beta\Delta U(w_i)] + \beta(1-\Delta) \max_{w' \in \Phi_r, w' \geq w_*} U(w') \quad (11)$$

subject to (8),(9),(10). Finally, extend U_r from $\tilde{\Phi}_r$ to the domain $[w_*, \infty)$ by letting $U_r(w) = -\infty$ for all $w \in [w_*, \infty) - \tilde{\Phi}_r$.

The function $U_f : [w_*, \infty) \to R$ is defined by

$$U_f(w) \equiv \max_{g \in \mathcal{G}} \left\{ -C(g) + \max_{w' \in \Phi_r,\, w' \geq w_*} U_r(w') \right\} \quad (12)$$

subject to

$$M(g) = w, \quad (13)$$

where $\mathcal{G}$ denotes the space of all feasible termination contracts.

Equation (9) is the incentive constraint, (10) is the promise-keeping constraint. Equation (11) reflects the fact that with probability $(1 - \Delta)$ the existing worker will die, in which case the firm must go back to the labor market to hire a new worker. This new worker has a reservation utility equal to w_*.

Equations (12)-(13) give the value of the firm conditional on the firm terminating the worker with an utility entitlement of w. This utility entitlement cannot be lower than w_* which is the worker's reservation utility. Equation (13) is promise-keeping. It says that the termination contract $g(w)$ must be such that, upon leaving the firm, the worker's expected utility is indeed equal to w. Here, I use $C[g(w)]$ to denote the cost of the termination contract $g(w)$ to the firm. This is essentially the expected discounted payment that the firm makes to the worker after termination. Again I use $M[g]$ to denote the value of the termination contract g to the worker. When this value is equal to w, then g delivers expected utility w to the worker. Obviously, the functions C and M depend on what is out there for a terminated worker in the *market*: given $g(w)$, the cost of $g(w)$ to the firm, as well as the expected utility of the worker is determined by the parameters of the market, including when the worker will find new employment and what the terms of the new contract will be.

I now move on to define the operator P. I say that a pair of expected utilities (w, u) is Pareto dominated by another pair of expected utilities (w', u') , denoted $(w', u') >_p (w, u)$, if $w' \geq w,\ u' > u$. Here, w and w' are expected utilities of the worker, u and u' are expected utilities of the firm.

Again, let $U : \Phi = \Phi_r \cup \Phi_f \to R$. Then $PU : \Phi'_r \cup \Phi'_f \to R$ is defined by

$$\Phi'_k = \{w \in \Phi_k : \nexists w' \in \Phi \text{ such that } (w', U(w')) >_p (w, U(w))\},$$

for $k = r, f$, and

$$PU(w) = U(w), \ \forall w \in \Phi'_r \cup \Phi'_f.$$

This finishes defining the RP-ness of a value function U.

Let $U : \Phi(= \Phi_r \cup \Phi_f) \to R$ be a RP value function. I say that contract $\sigma = \{(a(w), c_i(w), w_i(w)),\ w \in \Phi_r;\ g(w),\ w \in \Phi_f\}$ supports value function U (and is hence RP) if:

(i) for all $w \in \Phi_r$, $\{a(w), c_i(w), w_i(w)\}$ is a solution to the maximization problem (11) and $g(w)$ is a solution to the maximization problem (12) for all $w \in \Phi_f$; and

(ii) $w \in \Phi_r$ if and only if $U_r(w) \geq U_f(w)$.

By definition, if a value function is RP, then it is weakly decreasing. Now a problem with the concept of the RP-ness of dynamic contracts is that it is difficult to guarantee uniqueness. [10] To cope with this difficulty, I define the following notation of optimality.

Let Σ denote the set of all RP contracts. Let $\sigma \in \Sigma$. Let $\{U_r^\sigma(w), w \in \Phi_r^\sigma;\ U_f^\sigma(w), w \in \Phi_f^\sigma\}$ denote the value function that σ supports. A contract $\sigma^* \in \Sigma$ is said to be optimal if

$$\sigma^* \in \arg\max_{\sigma\in\Sigma} \left\{ \max_{w\in\Phi_r^\sigma} U_r^\sigma(w) \right\}.$$

In other words, a RP contract σ^* is optimal if allows the firm to achieve the highest possible firm value.

Now given the optimal contract σ^*, suppose the firm has just hired a new worker, and suppose the firm is free to choose a level of expected utility to be promised to this new worker to maximize the value of the firm. Then the firm's optimization problem is

$$\max_{w\in\Phi_r^{\sigma^*}} U_r^{\sigma^*}(w). \tag{14}$$

Assumption 1 Problem (14) has a unique solution.

Let $\underline{w}$ denote this solution. That is, $\underline{w}$ is the expected utility of the new worker that can give the firm the highest value. In fact, $\underline{w}$ is the expected

[10]See Pearce (1995) for a discussion of the issue of the non-unique RP value functions in dynamic games.

utility of the worker at which the firm can achieve its highest value across all levels of w that are feasible under a RP contract. Now suppose $\underline{w} > w_*$, which will be shown to be the case in the model's equilibrium (Proposition 3). Then it is feasible for the firm to start a new worker with $\underline{w}$. Then $\underline{w}$ denotes the unique starting expected utility of a new worker that maximizes the firm's value. Assumption 1 offers an obvious technical convenience. Suppose Assumption 1 is not satisfied, that is, suppose the firm's value function $U_r^{\sigma^*}(w)$ is constant over an interval of w. Then it would be natural to assume that the firm starts the worker with the highest expected utility in Φ_r that maximizes $U_r(w)$.

3.2 Market and Equilibrium

I am now ready to describe the market and then define what constitutes an equilibrium of the market.

Workers in the model are divided into three groups at the beginning of any period: those who are currently employed, those who are unemployed (not employed and looking for employment, including the new labor market entrants), and those who are retired (not employed but not looking for employment either). As the economy moves into the middle of the period, some of the unemployed will become employed. Then when the period ends, a fraction of the employed will be terminated to become either unemployed or retired.

Terminations are divided into two types. A termination is called *involuntary* if the worker's expected utility is strictly below $\underline{w}$ upon termination, i.e., $w \in \Phi_f$ and $w < \underline{w}$. A termination is called *voluntary* if it is not involuntary, that is, $w \in \Phi_f$ and $w > \underline{w}$. Note that $\underline{w} \notin \Phi_f$. Thus, if an involuntary termination occurs, the worker who is terminated would like to work for a lower expected utility than what is offered by the contract of the new worker the firm hires to replace him. This is not the case in a voluntary termination.

Proposition 1 If $w \in \Phi_f$ and $w < \underline{w}$, then $C[g(w)] = 0$.

Proof. Suppose $C[g(w)] > 0$ for some w that satisfies $w \in \Phi_f$ and $w < \underline{w}$. Then

$$U(w) = U_f(w) = U_r(\underline{w}) - C[g(w)] < U_r(\underline{w}). \tag{15}$$

This implies $(\underline{w}, U_r(\underline{w})) >_p (w, U(w))$ and so the contract is not renegotiation-proof. A contradiction. Q.E.D.

In equation (15), the left hand side of the inequality is the firm's expected value if the worker is involuntarily terminated; the right hand side is the expected value of the firm if the firm retains the worker, promising him expected utility $\underline{w}$, and not granting him the termination contract $g(w)$. So the firm and the worker can both do strictly better by moving the worker's utility from w to $\underline{w}$. Thus the contract is not RP.

Because all termination contracts must specify non-negative payments from the firm to the workers in all periods, $C[g(w)] = 0$ holds if and only if the worker receives zero payments from the firm in all future periods after termination. In turn, this implies that upon an involuntary termination, the worker's utility must be equal to w_*. That is,

Corollary 1 If $w \in \Phi_f$ and $w < \underline{w}$, then $w = w_*$.

Proposition 1 confirms the conjecture that in equilibrium, all involuntarily unemployed workers in the labor market are entitled to zero compensation payments as long as they remain unemployed. Thus in the forward looking sense, all involuntarily unemployed workers (including the new labor market entrants, workers who were never employed, and workers who were involuntarily terminated) are essentially the same. They each have expected utility w_*, would like to obtain employment, and will be employed in any given period with the same probability and with the same contract.

Let $\pi \in [0,1]$ denote the equilibrium probability with which an unemployed worker gets employed in a given period (the rate of hiring out of the pool of the unemployed).

Proposition 2 If $\pi < 1$, then all the voluntarily terminated workers are never re-employed.

Proof. Let w denote a voluntarily terminated worker's expected utility. That the worker was voluntarily terminated implies $U_f(w) = U_r(\underline{w}) - C[g(w)] > U_r(w)$, or $U_r(\underline{w}) > U_r(w) + C[g(w)]$. That is, the firm is strictly better off hiring an involuntarily terminated worker than hiring a retired worker and taking his $g(w)$. The firm would never hire the voluntarily terminated. Q.E.D.

Propositions 1 and 2 greatly simplify the structure of the termination contract and dictate the following termination conditions for the firm:

$$\text{If } w \in \Phi_f, \text{ then } U(w) = U_r(\underline{w}) - C[g(w)], \tag{16}$$

where

$$C[g(w)] = \begin{cases} 0, & w < \underline{w}, \\ \frac{u^{-1}[(1-\beta\Delta)w]}{1-\beta\Delta}, & w \geq \underline{w}. \end{cases} \tag{17}$$

Propositions 1 and 2 also allow me to specify the function $M(g)$. By the propositions, I need only focus on termination contracts that take the form of a constant stream of compensation pay after termination, denote this stream by $\{c_g\}$ for the termination contract g. Then I have

$$M(g) = \begin{cases} w_*, & c_g = 0, \\ H(c_g, 0)/(1-\beta), & c_g > 0. \end{cases} \tag{18}$$

Notice that for all $w \in \Phi_f$ and $w \in \underline{w}$, $U_f(w) = U_r(\underline{w})$. That is, each time a worker is involuntarily terminated, the firm is indifferent between firing him (so the worker will receive expected utility of w_*.) and retaining him and to restart him with a promised utility equal to $\underline{w}$. This is the reason why the model requires that renegotiations be strictly beneficial to the firm in order for them to happen. Otherwise the firm will be facing a dilemma which is beyond what I can address in the current paper. Note that this is not a problem in the case of a voluntary termination where the firm is always strictly better off starting up with a new worker than staying with the old worker.

To summarize, if a worker is terminated involuntarily, then he will get no payments from the firm after termination and hence his expected utility must be equal to w_*. If the termination is voluntary, then the worker will receive in each future period from the firm a constant payment equal to $u^{-1}[(1-\beta\Delta)w]$ and he never goes back to the market again. Propositions 1 and 2 also imply that if $\pi < 1$, then all new hires will start with the same expected utility $\underline{w}$. These results greatly simplify the structure of the market for contracts, making it ready now for me to give the definition of equilibrium.

In this paper, I will focus on the model's stationary equilibria. The first equilibrium condition is the following stationarity condition for w_*:

$$w_* = \pi\underline{w} + (1-\pi)[H(0,0) + \beta\Delta w_*]$$

or

$$w_* = \frac{\pi\underline{w} + (1-\pi)H(0,0)}{1-(1-\pi)\beta\Delta} \tag{19}$$

Let μ_V denote the measure of the voluntarily terminated workers at the beginning of each period. Let μ_I denote the measure of the involuntarily terminated workers at the beginning of each period. Note that μ_I includes workers who have never been employed and workers who were terminated with $C[g(w)] = 0$. Finally, let $\mu_E : \Phi_r \to [0, 1]$ denote the distribution of the beginning of period expected utilities of the employed workers: $\int_{\Phi_r} d\mu_E(w) = 1$.

Each period, the aggregate turnover of the labor market, that is, the total number of workers who are employed in the current period ($w \in \Phi_r$) but will become unemployed next period ($w' \in \Phi_f$) (note this is also the total number of workers newly employed in a period) is equal to

$$\xi \equiv \int_{\Phi_r} \sum_{\{i:\, \theta_i \in \Omega(w)\}} X_i(a(w)) d\mu_E(w)$$

where $\Omega(w) \equiv \{\theta_i :\ w_i(w) \in \Phi_f\}$ ($w \in \Phi_r$) is the set of all realizations of the current state of output θ in which a currently employed worker with expected utility w will be terminated. In addition, let $\Omega_I(w) = \{\theta_i :\ w_i(w) \in \Phi_f,\ w_i(w) < \underline{w}\}$ and $\Omega_V(w) = \{\theta_i :\ w_i(w) \in \Phi_f,\ w_i(w) > \underline{w}\}$. So $\Omega_I(w)$ is the set of the realization of θ for which the worker is terminated involuntarily, and $\Omega_V(w)$ is the set of all realizations of θ upon which the worker is terminated voluntarily. Finally, let

$$\xi_I \equiv \int_{\Phi_r} \sum_{\{i:\theta_i \in \Omega_I(w)\}} X_i(a(w)) d\mu_E(w),$$

$$\xi_V \equiv \int_{\Phi_r} \sum_{\{i:\theta_i \in \Omega_V(w)\}} X_i(a(w)) d\mu_E(w).$$

That is, $\xi_I(\xi_V)$ is the measure of the workers to transit from employment to involuntary (voluntary) unemployment in each period. I have $\xi = \xi_I + \xi_V$, and ξ/η is the economy's aggregate turnover rate.

Definition 1 A stationary equilibrium of the model is a vector

$$\{\pi,\ w_*,\ \underline{w},\ \sigma^*,\ (\mu_E, \mu_V, \mu_I)\}$$

where

(i) σ^* is an optimal contract, given $\pi,\ w_*,\ \underline{w}$, and (μ_E, μ_V, μ_I),

(ii) $\underline{w}$ is the solution to equation (14),

(iii) w_* is given by (19),

(iv) π is given by

$$\pi = \xi/\mu_I,$$

(v) (μ_E, μ_V, μ_I) satisfy the following stationarity conditions:

$$\mu_I = (1-\Delta) + \Delta(1-\pi)\mu_I + \Delta(1-\mu_I-\mu_V)\xi_I, \tag{20}$$

$$\mu_V = \Delta\mu_V + (1-\mu_I-\mu_V)\xi_V, \tag{21}$$

$$\mu_E = \Gamma(\mu_E), \tag{22}$$

where the operator Γ maps the distribution of the expected utilities of the employed workers in the current period into that in the next period, as dictated by the law of motion for $w \in \Phi_r$ (*i.e.*, $\{w_i^*(w),\ w \in \Phi_r\}$), the equilibrium starting expected utility $\underline{w}$, and the death rate Δ.

Note μ_I is the model's equilibrium unemployment measured at the beginning of the period. The model's equilibrium aggregate unemployment measured between the beginning and the end of the period should then be $\mu_I - \xi$.

4 Voluntary and Involuntary Terminations

A necessary condition for the existence of equilibrium involuntary termination and involuntary unemployment is $w_* < \underline{w}$. In addition, if this condition holds, then all the unemployed (if any) are involuntarily unemployed.

Proposition 3 Suppose in equilibrium there is unemployment. Suppose the equilibrium is not degenerate. That is, suppose $a_t^*(h_{t-1}) > 0$ for some t and h_{t-1} with the equilibrium contract. Then

$$\underline{w} > w_*. \tag{23}$$

proof. To show $\underline{w} > w_*$ is to show that

$$\underline{w} > \frac{\pi\underline{w} + (1-\pi)H(0,0)}{1-(1-\pi)\beta\Delta},$$

or

$$\underline{w} > H(0,0)/(1-\beta\Delta) \equiv w_0.$$

To show $\underline{w} > w_0$, I take two steps.

Step 1. I show $\underline{w} \geq w_0$. In fact, w_0 is the minimum expected utility that can be attained by a feasible and incentive compatible contract. This is easy to see. Given whatever compensation scheme $\{c_t(h_t)\}$, because $c_t \geq 0$ for all t, the worker can always guarantee for himself expected utility $H(0,0)/(1-\beta\Delta)$ by following the effort plan $\{a_t = 0\}_{t=1}^{\infty}$.

Step 2. I show $\underline{w} > w_0$ by showing that w_0 is not a RP expected utility, and therefore $\underline{w}$, being a RP expected utility, must be strictly greater than w_0.

There is a unique incentive compatible contract that delivers w_0 to the worker. To show this, notice first that if an incentive compatible contract delivers expected utility w_0 to the worker, then it must hold that $c_t = 0$ for all t. For otherwise the worker can always choose the action profile $\{a_t = 0, \ \forall t\}$ to do strictly better than w_0. Next, given $c_t = 0$ for all t, clearly the only action profile that is incentive compatible is $a_t = 0$ for all t, and it then follows that $w_t = w_0$ for all $t \geq 1$.

So if w_0 is RP, then all newly employed workers will stay at $\underline{w} = 0$, and the equilibrium is degenerate. Q.E.D.

Because the expected utilities of all the unemployed are equal to w_*, Proposition 3 states that all the unemployed workers are involuntarily unemployed if and only the equilibrium is not degenerate.

I now proceed to show that involuntary termination is indeed an equilibrium phenomenon: it does occur in equilibrium. More specifically, Proposition 4 shows the equilibrium contract has $w_i^*(\underline{w}) = w_*$ for at leat some i. That is, the newly higher worker will be terminated in at least some state of the world. I start with a definition and then a lemma.

Definition 2 Let $U : \Phi(= \Phi_r \cup \Phi_f) \to R$. A utility pair (w, u) is said to be generated by U if either there exists $\{a, (c_i, w_i)\}$ that satisfies (8),(9),(10), and

$$u = \sum_i X_i(a)[\theta_i - c_i + \beta\Delta U(w_i)] + \beta(1-\Delta) \max_{w \in \Phi_r} U(w);$$

or there exists $g \in \mathcal{G}$ that satisfies equation (13) and

$$u = -C(g) + \max_{w' \in \Phi_r} U(w').$$

Finally, let $G(U)$ denote the set of all utility pairs (w, u) that can be generated by U, let $Graph(U)$ denote the graph of U.

Lemma 1 Let $U : \Phi \to R$. The U is not RP if there exists $(w, u) \in G(U)$ such that $(w, u) \notin Graph(U)$ and (w, u) is not Pareto dominated by any $(w', u') \in Graph(U)$.

The proof of Lemma 1 is in the appendix. Lemma 1 provides a sufficient condition, which is relatively easy to verify, for the non-RP-ness of a contract. With this sufficient condition, I can now prove[11]

Proposition 4 Let $\Theta = \{\theta_1, \theta_2\}$. $\Omega_I(\underline{w}) \neq \emptyset$.

Proof. Suppose $\Omega_I(\underline{w}) = \emptyset$. Let $\sigma = \{a(w);\ (c_i(w), w_i(w)),\ w \in \Phi\}$ denote the optimal RP contract. I have for all $w \in \Phi_r$,

$$c_i(w) \in \mathsf{C},\ w_i(w) \geq \underline{w},$$

$$\begin{aligned} w = & \quad (1 - X(a))[u(c_1(w)) + \beta\Delta w_1(w)] \\ & + \ X(a)[u(c_2(w)) + \beta\Delta w_2(w)] - \phi(a(w)), \end{aligned}$$

$$\begin{aligned} U_r(w) = & \quad (1 - X(a(w)))[\theta_1 - c_1(w) + \beta\Delta U(w_1(w))] \\ & + \ X(a)[\theta_2 - c_2(w) + \beta\Delta U(w_2(w))] + \beta\Delta U_r(\underline{w}). \end{aligned}$$

In the following, I derive a contradiction by constructing a new contract $\hat{\sigma} = \{\hat{a}(w);\ (\hat{c}_i(w), \hat{w}_i(w)),\ w \in \hat{\Phi}\}$ which can generate expected utilities $\hat{w}$ and $U(\hat{w})$ such that $(\hat{w}, U(\hat{w})) \in PT\Phi$ but $(\hat{w}, U(\hat{w})) \notin \Phi$, and hence by Lemma 1 the value function U is not renegation-proof.

I first set

$$\hat{\Phi} = \Phi \bigcup \{\hat{w}\},$$

$$\hat{a}(w) = a(w);\ \hat{c}_i(w) = c_i(w),\ \hat{w}_i(w)) = w_i(w),\ \forall w \in \Phi.$$

The rest of the elements in $\hat{\sigma}$ are then giving depending on I am in which of the following two cases.

Case (1). Suppose $c_2(\underline{w}) > \underline{c}$. Then set

$$\hat{c}_1 = c_1(\underline{w}),\ \hat{c}_2 = c_2(\underline{w}) - \epsilon,\ \hat{w}_1 = w_*,\ \hat{w}_2 = w_2(\underline{w}),$$

where ϵ is chosen to be sufficiently small so that $\hat{c}_2 \geq \underline{c}$ and the following holds:

$$[u(\hat{c}_2) + \beta\Delta\hat{w}_2] - [u(\hat{c}_1) + \beta\Delta\hat{w}_1] \geq [u(c_2) + \beta\Delta w_2] - [u(c_1) + \beta\Delta w_1]. \quad (24)$$

Note that the above equation ensures that $\hat{\sigma}$ implements a_2. Let

$$\hat{w} = (1 - X(a))[u(\hat{c}_1) + \beta\Delta\hat{w}_1] + X(a)[u(\hat{c}_2) + \beta\Delta\hat{w}_2].$$

[11] It can be seen from the proof of Lemma 1 that Lemma 1 is more general than being useful to prove a specific result in this paper, though elaborating on the significance of Lemma 1 is not the task of the current paper.

We have

$$\hat{w} < \underline{w}. \tag{25}$$

Meanwhile, because $U(w_*) = U(\underline{w})$ and $\hat{c}_2 < c_2(\underline{w})$, we have

$$\begin{aligned} U(\hat{w}) &= (1 - X(a_H))[\theta_1 - \hat{c}_1 + \beta\Delta U(w_*)] \\ &+ X(a_H)[\theta_2 - \hat{c}_2 + \beta\Delta U(\hat{w}_2)] + \beta\Delta U_r(\underline{w}) \\ &> U(\underline{w}). \end{aligned}$$

So $(\hat{w}, U(\hat{w})) \in G(U)$ but $(\hat{w}, U(\hat{w})) \notin \mathit{Graph}(U)$. A contradiction.

Case (2). Suppose $c_2(\underline{w}) = \underline{c} \leq c_1(\underline{w})$. There are two sub-cases here: $c_1(\underline{w}) > \underline{c}$ and $c_1(\underline{w}) = \underline{c}$.

(2i) Suppose $c_1(\underline{w}) > \underline{c}$. Then set

$$\hat{c}_1 = c_1(\underline{w}) - \epsilon,\ \hat{c}_2 = c_2(\underline{w}),\ \hat{w}_1 = w_*,\ \hat{w}_2 = w_2(\underline{w}).$$

For ϵ sufficiently small, $\hat{c}_1 \geq \underline{c}$ and (24),(25), (**??**) are all satisfied. A contradiction.

(2ii) Suppose $c_1(\underline{w}) = \underline{c}$. Then because σ implements $a = a_H$ at $w = \underline{w}$, incentive compatibility requires $w_2(\underline{w}) > w_1(\underline{w}) \geq \underline{w}$. Therefore, we can set

$$\hat{c}_1 = c_1(\underline{w}),\ \hat{c}_2 = c_2(\underline{w}),\ \hat{w}_1 = w_*,\ \hat{w}_2 = w_2(\underline{w}) - \epsilon$$

where ϵ is chose to be sufficiently small to make $\hat{w}_2 \geq \underline{w}$ hold and to satisfy equation (24), and so the incentive constraint holds at $w = \hat{w}$. [12] Clearly, $\hat{w} < \underline{w}$. Also, since U is a weakly decreasing function, it holds that $U(\hat{w}) \geq U(\underline{w})$.

Now $(\hat{w}, U(\hat{w})) \in G(U)$ but $(\hat{w}, U(\hat{w})) \notin \mathit{Graph}(U)$. [13] This is true because if $(\hat{w}, U(\hat{w})) \in \mathit{Graph}(U)$, then $\underline{w}$ would not be the unique solution to the optimization problem in equation (13). By Lemma 1, U is not RP, a contradiction. Q.E.D.

My next proposition gives a sufficient condition for when a voluntary termination occurs. Proposition 5 states that voluntary termination should occur when the worker becomes too "rich": his expected utility becomes too high. The idea is that when the worker's expected utility becomes sufficiently high, his effort becomes too expensive for the firm to compensate for, and

[12] To satisfy the incentive constraint, it is sufficient to require $\hat{w}_2 - \hat{w}_1 \geq w_2 - w_1$, or $\epsilon \leq w_1 - w_* > 0$. To summarize, we need only make sure $0 \leq \epsilon \leq \min\{w_2 - \underline{w},\ w_1 - w_*\}$.

[13] In fact, by varying ϵ, I can obtain a continuum of such pairs.

the firm is then better off replacing him with a new worker whose expected utility is lower and so his efforts are less expensive.

The basic intuition of the idea that a richer worker's efforts are more expensive can be seen by looking at a simple static compensation problem with no information frictions and uncertainties. Let c_0 denote the worker's existing consumption. Let c denote the compensation that a firm pays in order to make him willing to exert a fixed amount of effort $a > 0$. Suppose this worker was initial making zero effort. Then it is clear that c must satisfy $u(c_0 + c) - \phi(a) > u(c_0) - \phi(0)$, or

$$u(c_0 + c) - u(c_0) \geq \phi(a) - \phi(0),$$

Clearly, c increases as c_0 increases, simply because the left hand side of the equation is constant and the left hand side is increasing in c but decreasing in c_0.

Proposition 5 Assume $(u^{-1})'(x) \to \infty$ as $x \to \infty$. Then there exists $\overline{w} \in (\underline{w}, \infty)$ such that $I^*(w) = 0$ for all $w \geq \overline{w}$.

Proof. To prove the proposition it is equivalent to show that $U_f(w) > U_r(w)$ for w sufficiently large. Suppose otherwise. That is, suppose $U_r(w) \geq U_f(w)$ for all $w > \underline{w}$. I now derive a contradiction.

I first define a function $U_r^{fb}(w),\ w > \underline{w}$. Fix w, which is the expected utility the worker is entitled to at the beginning of a period. Suppose this w satisfies $w > \underline{w}$ and so $U_r(w) \geq U_f(w)$.

Now imagine the following scenario: suppose, starting the current period, there will be no moral hazard as long as the current worker remains employed; suppose moral hazard resumed when a new worker is employed. Calculate the value of the firm and denote it $U_r^{fb}(w)$. Now since the worker is retained in the case of moral hazard, it is certainly retained in the case of no moral hazard, and indeed this worker should be retained until he dies. This implies $U_r^{fb}(w)$ must satisfy

$$U_r^{fb}(w) = \overline{\theta} - c_r^{fb}(w) + \beta\Delta U_r^{fb}(w) + \beta(1-\Delta)U(\underline{w}),$$

or

$$U_r^{fb}(w) = \frac{\overline{\theta} - c_r^{fb}(w)}{1 - \beta\Delta} + \frac{\beta(1-\Delta)U(\underline{w})}{1 - \beta\Delta},$$

where

$$c_r^{fb}(w) = u^{-1}[(1 - \beta\Delta)w + \phi(a)],$$

where $a^*(w)$ denotes the optimal level of effort and $\overline{\theta}(a^*(w))$ denote the expected output conditional on $a^*(w)$. Now

$$U_f(w) = -g(w) + U(\underline{w}) = \frac{-u^{-1}[(1-\beta\Delta)w]}{1-\beta\Delta} + U(\underline{w}).$$

Therefore,

$$U_f(w) - U_r^{fb}(w) = \frac{K(w) - \overline{\theta}(a^*(w))}{1-\beta\Delta} + A$$

where A is constant in w and

$$K(w) \equiv u^{-1}[(1-\beta\Delta)w + \phi(a^*(w))] - u^{-1}[(1-\beta\Delta)w].$$

Since $U_r^{fb}(w) \geq U_r(w)$, If we can show $U_f(w) > U_r^{fb}(w)$ for w sufficiently large, then we have $U_f(w) > U_r(w)$ for w sufficiently large and hence we have a contradiction. Given that the value of $\overline{\theta}(a^*(w))$ is bounded, thus in order to prove the proposition we need only show that

$$K(w) \to \infty \text{ as } w \to \infty. \tag{26}$$

But

$$K(w) = \phi(a^*(w))(u^{-1})'[(1-\beta\Delta)w + \xi]$$

where $\xi \in [0, \phi(a^*(w))]$. Since $\phi(a^*(w)) > 0$, equation (26) holds if

$$(u^{-1})'(w) \to \infty \text{ as } w \to \infty.$$

This proves the proposition. Q.E.D.

Mathematically, Proposition 5 essentially shows that with the equilibrium contract, the equilibrium value functions $U_r(w)$ and $U_f(w)$ must cross at some $w > \underline{w}$. However, the proposition does not necessarily imply that voluntary termination occurs in equilibrium. Put differently, if I follow a new worker who starts out with expected utility $\underline{w}$, Proposition 5 does not tell me that the worker will cross w to become voluntarily terminated with a positive probability. To prove such a result seems difficult. A natural alternative is numerical methods. But this remans a limitation of the current paper.

5 Conclusion

In this paper, I built an equilibrium model of the labor market where labor contracts are fully dynamic, job turnover is endogenous, workers separated from their current employers are free to go back to the labor market to obtain new employment. At the heart of the model is an optimal termination mechanism that governs the timing and the type of the separations of workers and firms. In equilibrium, this optimal termination mechanism appears in two different faces, one resembles the so-called involuntary layoff in which the terminated worker does go back to the labor market to seek new employment after termination, the other resembles voluntary retirement where the worker chooses to stay out of the labor market permanently.

6 Appendix: Proof of Lemma 1

Suppose U is RP. I take the following steps to construct a contradiction.

1. Because U is RP, I have $Graph(U) = Graph(PTU) = Graph(TU)$.

2. Notice that it is without loss of generality to assume that $(w, u) \in Graph(TU)$. To show this, let $\hat{u} = \max\{u : (w, u) \in G(U)\}$. Then $(w, \hat{u}) \in Graph(TU)$, $(w, \hat{u}) \notin Graph(U)$ and $(w, \hat{u})$ is not Pareto dominated by any $(w', u') \in Graph(U)$. $(w, \hat{u}) \in Graph(TU)$ because if $(w, \hat{u}) \in Graph(U)$, then (w, u) is not Pareto dominated by $(w, \hat{u}) \in Graph(U)$, a contradiction. And, because $(w, \hat{u})$ Pareto dominates (w, u) and the latter is not Pareto dominated by any $(w', u') \in Graph(U)$, $(w, \hat{u})$ is not Pareto dominated by any $(w', u') \in Graph(U)$.

3. Because $(w, u) \notin Graph(PTU) = Grapg(U)$, (w, u) must be dominated by some $(\tilde{w}, \tilde{u}) \in Graph(TU)$. But since (w, u) is not Pareto dominated by any $(w', u') \in Graph(U)$, it must be that $(\tilde{w}, \tilde{u}) \in Graph(TU) - Graph(U) \neq \emptyset$.

4. Let

$$w^* \equiv \sup\{\tilde{w} : (\tilde{w}, \tilde{u}) \in Graph(TU) - Graph(U),\ (\tilde{w}, \tilde{u}) >_p (w, u)\}$$

5. w^* belongs to the domain of the function TU, i.e., $w^* \in [w_*, \infty)$. This is straightforward to show. By the definition of w_*, there is a sequence $\{w_n, u_n\} \subseteq Graph(TU)$ such that $w_n \to w_*$ as $n \to \infty$. But $w_n \in [w_*, \infty)$ (the domain of TU), which is a closed set, so $w^* \in [w_*, \infty)$.

6. I can then define $u^* \equiv TU(w^*)$ and it follows that $(w^*, u^*) \in Graph(TU)$. So either $(w^*, u^*) \in Graph(TU) - Graph(U)$ or $(w^*, u^*) \in Graph(U)$

7. Notice that $(w^*, u^*) \geq_p (w, u)$. (That is, $w' \geq w$, $u' \geq u$.) This holds because for each n, $w_n \geq w$, $u_n > u$, and so $w^* \geq w$, $u^* \geq u$.

8. Suppose $(w^*, u^*) \in Graph(TU) - Graph(U)$. Notice first that

$$(w^*, u^*) \geq_p (w, u) >_p (w', u'),\ \forall (w', u') \in Graph(U).$$

That is, (w^*, u^*) is not dominated by any $(w', u') \in Graph(U)$. Second, suppose there exists $(w', u') \in Graph(TU) - Graph(U)$ such that $(w', u') >_p (w^*, u^*)$. Then because $(w^*, u^*) \geq p(w, u)$, I have $(w', u') >_p (w, u)$. Now by the definition of w^*, it holds that $w' \leq w_*$. But $(w', u') >_p (w^*, u^*)$ implies $w' \geq w_*$. So it must hold that $w' = w_*$. Therefore

$$u' = TU(w') = TU(w^*) = u^*.$$

This is a contradiction to $(w', u') >_p (w^*, u^*)$.

9. Suppose $(w^*, u^*) \in Graph(U)$. Then (w, u) is Pareto dominated by $(w_*, u^*) \in Graph(U)$. Again a contradition. Q.E.D.

References

[1] Demarzo, P.M., and M. Fishman (2003) "Optimal Long-Term Financial Contracting with Privately Observed Cash Flows," manuscript.

[2] Green, E. (1987), Lending and the smoothing of uninsurable income, in *Contractual Arrangements for Intertemporal Trade*, University of Minnesota Press, Minneapolis, MN (E.C. Prescott and N. Wallace, eds.)

[3] Ray, D.(1994), Internally renegotiation proof equilibrium sets: limit behavior with low discounting, *Games and Economic Behavior*, 6(1), 162-177.

[4] Shapiro, C. and J.E. Stiglitz (1984), Equilibrium unemployment as a worker-discipline device, *American Economic Review*, 74, 433-444.

[5] Spear, S.E. and S. Srivastava (1987), On repeated moral hazard with discounting, *Review of Economics Studies*, 54, 599-617.

[6] Stiglitz, J.E. and A. Weiss(1983), Incentive effects of termination: applications to the credit and labor markets *American Economic Review* 73, 912-927.

[7] Thomas, J. and R. Worrall (1990), Income fluctuations and asymmetric information: an example of the principle-agent problem, *Journal of Economic Theory*, 51, 367-390.

[8] Spear S. and Wang, C. (2005), When to fire a CEO: Optimal termination in dynamic contracts, *Journal of Economic Theory.*

[9] Wang, C. (2000), Renegotiation-proof dynamic contracts with private information, *Review of Economic Dynamics*, 3(July), 396-422.

[10] Zhao, R. (2004), On Renegotiation-Proof Contracts in Repeated Agency", working paper, Department of Economics, SUNY-Albany.