Durable–Goods Monopoly with Varying Cohorts

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Abstract

This paper solves for the profit maximising strategy of a durable–goods monopolist when incoming demand varies over time. Each period an additional demand curve enters the market; these consumers can then choose whether and when to purchase. The potential for delay creates an asymmetry in the optimal price path which exhibits fast increases and slow declines. This asymmetry also pushes the price level above that charged by a firm facing the average level of demand. Applications of this framework include deterministic demand cycles, one–off shocks and IID demand draws. The optimal policy can be implemented by a best–price provision and outperforms renting.

1 Introduction

Each September thousands of students return to universities and colleges across Canada. Canadian Tire, a large retailer, responds to this influx of new consumers by holding a back–to–school sale, reducing their price on furniture, stationary and kitchen utensils. While the price reduction helps increase profits from the student community, it also causes customers who would have bought in July and August to delay their purchases. This paper analyses this tradeoff, solving for the optimal pricing strategy of a durable–goods monopolist when incoming demand varies over time.

Consider a monopolist selling a durable good, defined broadly to be any product where agents can choose both whether and when to purchase. Each period the market is kept alive by the entry of new consumers. These new consumers are associated with a demand curve which is

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allowed to change over time. The firm's problem is to choose a sequence of prices to maximise their profits subject to consumers optimally choosing their purchase time.

Each consumer's purchase decision can be analysed as an optimal stopping problem. Using this formulation the paper characterises the consumers' purchasing rule by a sequence of cutoffs: at any time t a consumer purchases if their valuation lies above the time–t cutoff. Mechanism design can then be utilised to describe the firm's profit as a function of these cutoffs.

Under a monotonicity condition the profit–maximising cutoffs are characterised by a myopic algorithm, where the allocation at time t only depends upon the consumers who have entered the market up to time t. Consumers are forward looking and prices depend upon the sequence of future demand; the optimal allocation, however, only depends upon past demand. The optimal prices can then be derived from the consumers' optimal purchase problem.

The optimal myopic algorithm has an intuitive interpretation. The new demand curve in any period t can be associated with a marginal revenue curve, where marginal revenue is with respect to price, not quantity. In the first round the firm sells the good to agents with positive marginal revenue (net of costs). In each period thereafter, the firm adds the marginal revenue of the old consumers who have yet to buy to that of the new agents, forming a cumulative marginal revenue function. The firm then sells the good to agents with positive cumulative marginal revenue.

Consumers' ability to delay induces an asymmetry in the optimal price path. When demand grows stronger over time, in that valuations tend to rise, the firm will want to increase their price. Agents then have no incentive to delay and the firm can discriminate between the different generations, charging the monopoly price against the incoming generation: the *myopic price*.

When demand weakens over time, in that valuations tend to fall, the firm will want to decrease their price over time. Charging the myopic price, however, will now lead to falling prices, causing some customers to delay their purchases. Anticipating this delay, the firm slows the rate at which prices fall. This decline in price is reduced so much that prices always stay above that chosen by a firm who pools all generations together and prices against the average level of demand: the *average–demand price*.

The contrast between increases and decreases in demand is stark. If there is a permanent and unanticipated increase in demand the price quickly jumps up to the new higher monopoly price. However if demand falls the price will jump down a little and slowly fall towards the lower monopoly price over time.

This asymmetry between demand increases and decreases crucially affects the firm's optimal pricing policy. When demand follows stationary cycles, it leads to sharp price increases and gentle declines. This is shown in *Figure 1*, where the lower panel describes which generations purchase in which periods.

During the first quarter of the cycle, as demand rises from its average level to the peak of

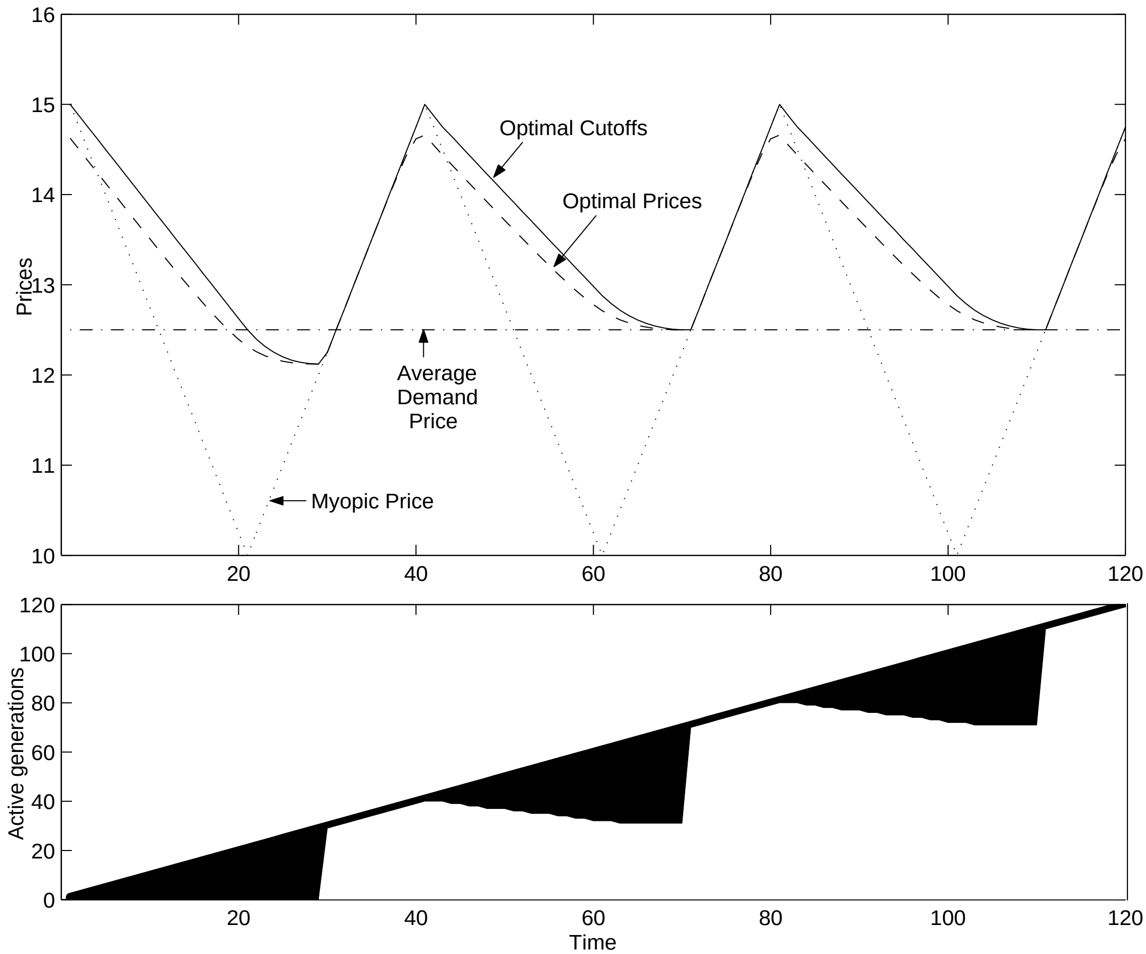

Figure 1: Price Cycles

the boom, the price rises quickly. For the other three–quarters of the cycle, as demand falls and then returns to its average position, the price slowly falls. These price cycles are stationary, showing no decline in amplitude no matter how long the market has existed. The price is also minimized not in the period of lowest demand, but in the last period of the slump, just before demand returns to its average position. The asymmetry between increases and decreases in demand also raises the price level: price always exceeds the average–demand price. This means introducing variation in demand leads to an increase in all prices. In other words, when the firm has more information about demand cycles *all* consumers are made worse off and social welfare is reduced.

The basic model makes two assumptions of note.

First, there is no resale. There are many goods where this is the right assumption: one–time experiences, (e.g. a trip to Disneyland), intermediate products (e.g. aluminium), regulated markets (e.g. plutonium), potential lemons (e.g. computers), goods with high transactions costs (e.g. fridges) or those with emotional attachment (e.g. diamonds). However such an assumption is not innocuous. With perfect resale price variations decrease in amplitude as the market gets older; without resale the market fluctuations never abate. With resale price responds symmetrically to changes in demand; without resale price movements are highly asymmetric. Introducing resale has no effect on allocations if and only if new demand falls over time; otherwise the presence of resale lowers the monopolist's profits. Since renting is identical to commitment pricing with resale, renting attains maximal profits if and only if demand is declining.

Second, it is assumed that the monopolist can perfectly commit to a sequence of prices. With homogenous demand and no commitment there are many equilibria ranging from the those that are very bad for the firm to others which are close to the full–commitment outcome (Sobel (1991)). This paper should thus be viewed as establishing the best possible outcome for the firm. This seems particularly reasonable if the firm is concerned about its reputation across several durable–goods markets. In addition, there are contractual solutions to the commitment problem. We extend the result of Butz (1990) by showing if the firm can use a best–price provision they can implement the optimal scheme without requiring the pre–commitment.

The starting point for this paper are the models of Stokey (1979) and Conlisk, Gerstner, and Sobel (1984), as examined in Section 3.2. Other authors introduce dynamics into durable goods models in different ways. Conlisk (1984), Laffont and Tirole (1996), Biehl (2001), and Board (2004) have stochastic valuations. Cost variations have been analysed by Stokey (1979), and Levhari and Pindyck (1981). When consumers and the firm have different discount rates the optimal price may fall over time as examined by Sobel and Takahashi (1983), Landsberger and Meilijson (1985), and Wang (2001). Rust (1985, 1986), Waldman (1996) and Hendel and Lizzeri (1999) allow the good to depreciate and for consumers to scrap their the product.

The paper is organised as follows: Section 2 contains a description of the model and a derivation of the firm's problem. Section 3 derives optimal control problem which is solved in Section 4. Section 5 discusses applications including monotone demand paths, one–off shocks, demand cycles and IID demand. Section 6 examines the effect of information structures, discount rates and best–price schemes. Section 7 analyses resale and renting, while Section 8 concludes. Omitted proofs are contained in the Appendix.

2 Model

Time is discrete, $t \in \{1, \dots, T\}$, where we allow $T = \infty$. Demand and the discount rate are allowed to be uncertain, depending on the state of the world $\omega \in \Omega$. The information possessed by consumer and the firm is described by a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$, where $\mathcal{F}$ are the measurable sets, $\{\mathcal{F}_t\}$ is the information partition at time t, which grows finer over time, and Q is the probability measure. The common discount rate, $\delta_t \in (\underline{\delta}, \overline{\delta}) \subset (0,1)$, is $\mathcal{F}_t$–measurable, i.e. $\{\omega : \delta_t \leq x\} \in \mathcal{F}_t$ for $x \in (0,1)$. This means that at time t all agents know δ_t. Let the discounting up to time t be $\Delta_t = \prod_{s=1}^{t} \delta_s$, where Δ_∞=0.

A consumer with valuation $\theta \in [\underline{\theta}, \overline{\theta}]$ who purchases at time $t \in \{1, \dots, T, \infty\}$ at price p_t obtains utility

$$(\theta - p_t)\Delta_t$$

A consumer always has the option not to purchase, in which case they have zero utility and are said to buy at time $t = \infty$.

Each period consumers of measure $f_t(\theta)$ enter the market. Let $F_t(\theta)$ be the distribution function, where $F_t(\overline{\theta})$ is the total number of agents (and not necessarily equal to one), and denote the survival function by $\overline{F}_t(\theta) := F_t(\overline{\theta}) - F_t(\theta)$. The time–$t$ demand function, $f_t(\theta)$, is $\mathcal{F}_t$–adapted, so while agents may not know future demand, they do know current demand. This avoids issues to demand experimentation and is trivially satisfied if demand is deterministic.

Consider a consumer of type (θ, t) with valuation θ who enters in period t. Given a sequence of $\mathcal{F}_t$–adapted prices, $\{p_t\}$, they have the problem of choosing a purchasing time $\tau(\theta, t) \geq t$ to maximise expected utility,

$$u_t(\theta) = \mathcal{E}[(\theta - p_\tau)\Delta_\tau] \tag{2.1}$$

where "$\mathcal{E}$" is the expectation over Ω. The purchasing time is a random variable taking values in $\{t, \dots, T, \infty\}$, where the decision to buy at time t can only depend on information available at time t, i.e. $\{\omega : \tau \leq t\} \in \mathcal{F}_t$ $(\forall t)$. Let $\tau^*(\theta, t)$ be the earliest solution to this problem which exists by Lemma 1 in Section 3.3.

The firm's problem is to choose $\mathcal{F}_t$–adapted prices $\{p_t\}$ to maximise profit. Assuming marginal cost is constant, it can be normalised to zero, yielding expected profit,

$$\Pi = \mathcal{E}\left[\sum_{t=1}^{T} \int_{\underline{\theta}}^{\overline{\theta}} \Delta_{\tau(\theta,t)} p_{\tau^*(\theta,t)}\, dF_t\right] \tag{2.2}$$

where $\tau^*(\theta, t)$ maximises the consumer's utility (2.1). In Appendix A.2 it is argued that the restriction to a price mechanism is without loss.

3 Solution Technique

3.1 Firm's Problem

Since the purchase time is chosen optimally, we can apply the envelope theorem to the consumer's utility maximisation problem (2.1). The space of stopping times is complex so we will use the generalised envelope theorem of Milgrom and Segal (2002).[1] This yields utility,

$$u_t(\theta) = \mathcal{E}\left[\int_{\underline{\theta}}^{\theta} \Delta_{\tau^*(x,t)}\, dx + u(\underline{\theta}, t)\right] \tag{3.1}$$

Since the seller will always choose prices $p_t \geq \underline{\theta}$ $(\forall t)$ it will be the case that $u(\underline{\theta}, t) = 0$ $(\forall t)$. Integrating by parts, *consumer surplus* from generation t is

$$\int_{\underline{\theta}}^{\overline{\theta}} u_t(\theta)\, dF_t = \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \Delta_{\tau^*(\theta,t)} \overline{F}_t(\theta)\, d\theta \;\middle|\; \mathcal{F}_t\right] \tag{3.2}$$

Expected *welfare* from generation t is defined by

$$W_t := \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \Delta_{\tau^*(\theta,t)} \theta\, dF_t\right] \tag{3.3}$$

with total welfare $W = \sum_t W_t$. Since costs are zero, the welfare maximising pricing scheme is to set all prices equal to zero. Expected profit equals welfare (3.3) minus consumer surplus (3.2),

$$\begin{aligned} \Pi &= \mathcal{E}\left[\sum_{t=1}^{T} \int_{\underline{\theta}}^{\overline{\theta}} \left[\Delta_{\tau^*(\theta,t)}\theta - u_t(\theta)\right]\, dF_t\right] \\ &= \mathcal{E}\left[\sum_{t=1}^{T} \int_{\underline{\theta}}^{\overline{\theta}} \Delta_{\tau^*(\theta,t)} m_t(\theta)\, d\theta\right] \end{aligned} \tag{3.4}$$

where $m_t(\theta) := \theta f_t(\theta) - \overline{F}_t(\theta)$ is *marginal revenue* with respect to price.[2]

Profit is thus the discounted sum of marginal revenues. Notice how the marginal revenue gained from agent (θ, t) is the same no matter when they choose to buy. That is, an agent's marginal revenue is determined when they are born, and sticks to them forever.

The *firm's problem* is to choose prices $\{p_t\}$ to maximise profit (3.4) subject to consumers choosing their purchasing time $\tau^*(\theta, t)$ to maximise utility (2.1).

3.2 Special Cases

We now consider three special cases that will be useful benchmarks for what follows. Example 1 comes from Stokey (1979), where there is a single demand curve of consumers and entry

[1] This theorem requires that the set of optimal purchasing times is nonempty, which is true by Lemma 1 in Section 3.3.

[2] It is more common to use marginal revenue with respect to quantity, $MR_t(\theta) := m_t(\theta)/f_t(\theta)$. However, since we will be adding demand across generations, summing demand curves horizontally, it is easier to work with marginal revenue with respect to price. In any case, both $m_t(\theta)$ and $MR_t(\theta)$ have the same roots, and are thus interchangeable in pricing formulae.

never occurs. Example 2 supposes the monopolist can set a different price schedule for each generation. Example 3 is the homogenous entry model of Conlisk, Gerstner, and Sobel (1984).

Example 1 (Single Generation). If $f_t(\theta) = 0$ for $t \geq 2$ the profit (3.4) reduces to

$$\Pi = \int_{\underline{\theta}}^{\overline{\theta}} \Delta_{\tau^*(\theta,t)} m_1(\theta)\, d\theta \tag{3.5}$$

To maximise (3.5) the firm would like to set purchasing times as follows:

$$\begin{aligned} \tau^*(\theta, 1) &= 1 \quad \text{if} \quad m_1(\theta) \geq 0 \\ \tau^*(\theta, 1) &= \infty \quad \text{if} \quad m_1(\theta) < 0 \end{aligned}$$

That is, the firm would like positive marginal revenue customers to purchase immediately, and the rest to never buy. If $m_1(\theta)$ is increasing this optimal policy can be implemented by setting

$$\begin{aligned} p_1 &= m_1^{-1}(0) \\ p_t &\geq m_1^{-1}(0) \quad \text{if} \quad t \geq 2 \end{aligned}$$

Since the price is increasing, consumers buy in period 1 or never buy at all. A consumer then buys in period 1 if and only if $\theta \geq m_1^{-1}(0)$, which is the same as $m_1(\theta) \geq 0$. $\triangle$

Example 2 (Discrimination between Generations). Next suppose the firm could tell the different cohorts apart and set a price p_t^s for generation s in time t. They would then implement Stokey's solution for each cohort. That is,

$$\begin{aligned} p_t^s &= m_s^{-1}(0) \quad \text{if} \quad t = s \\ p_t^s &\geq m_s^{-1}(0) \quad \text{if} \quad t \geq s+1 \end{aligned}$$

Hence if the myopic monopoly price grows over time, $m_t^{-1}(0) \geq m_{t-1}^{-1}(0)$, the seller can simply charge the myopic price, $p_t^* = m_t^{-1}(0)$, and need not discriminate. Example 3 amounts to a special case of this observation. $\triangle$

Example 3 (Homogenous Demand). Finally suppose demand is identical in each period, $f_t(\theta) = f_0(\theta)\ \forall t$. The firm can implement the discriminatory optimum from Example 2 by setting $p_t^* = m_0^{-1}(0)\ \forall t$. $\triangle$

3.3 The Cutoff Approach

As suggested by Examples 1–3, rather than solving for prices directly it is easier to solve for the optimal purchasing rule and back out prices. This approach works since prices $\{p_t\}$ only enter into profits (3.4) via the purchasing rule $\tau^*(\theta, t)$—a standard feature of quasi–linear mechanism design problems. This is analogous to solving a standard monopoly model in quantities and using the demand curve to derive prices.

Lemma 1. *The earliest purchasing rule, $\tau^*(\theta, t)$, has the following properties:*
[existence] $\tau^*(\theta, t)$ exists.
[θ–monotonicity] $\tau^*(\theta, t)$ is decreasing in θ.
[non–discrimination] If $\tau^*(\theta, t_L) \geq t_H$ then $\tau^*(\theta, t_L) = \tau^*(\theta, t_H)$, for $t_H \geq t_L$.
[right–continuity] $\{\theta : \tau^*(\theta, t_L) \leq t_H\}$ is closed, for $t_H \geq t_L$.

Proof. [existence], [θ–monotonicity], [non–discrimination] follow from Lemma 4 which describes properties of the set of optimal stopping rules. These properties also apply to the least element by Topkis (1998, Theorem 2.4.3). [right–continuity] follows from the continuity of $u_t(\theta)$ in θ. □

Lemma 1 implies the optimal stopping rule $\tau^*(\theta, t)$ can be characterised by a sequence of cutoffs. The *cutoff* θ_t^* is the lowest type who purchases in period t,

$$\theta_t^* := \min\{\theta : \tau^*(\theta, t) = t\}.$$

That is, consumers in the market in period t will buy if their valuation exceeds θ_t^*. If demand is uncertain, these cutoffs are $\mathcal{F}_t$–adapted random variables. For generation $t \leq t'$ the *updated cutoff*, $\theta^*(t'; t)$, is the lowest type from generation t who buys by time t',

$$\theta^*(t'; t) := \min_{t \geq s \geq t'} \theta_s^* \tag{3.6}$$

If $t' < t$ then set $\theta^*(t'; t) = \infty$. Agents from generation t will then buy in period t' if

$$\theta \in [\theta^*(t'; t), \theta^*(t' - 1; t))$$

A simple three–period example is shown in *Figure 2*, where we suppose $\theta_1^* > \theta_3^* > \theta_2^*$.

The *firm's problem* is then to choose cutoffs $\{\theta_t^*\}$ to maximise profit (3.4).

Optimal prices can then be backed out from the optimal sequence of cutoffs. First suppose T is finite. In the last period type θ_T^* is indifferent between buying and not, so the firm sets $p_T^* = \theta_T^*$. In earlier periods an agent with value θ_t^* should be indifferent between buying in

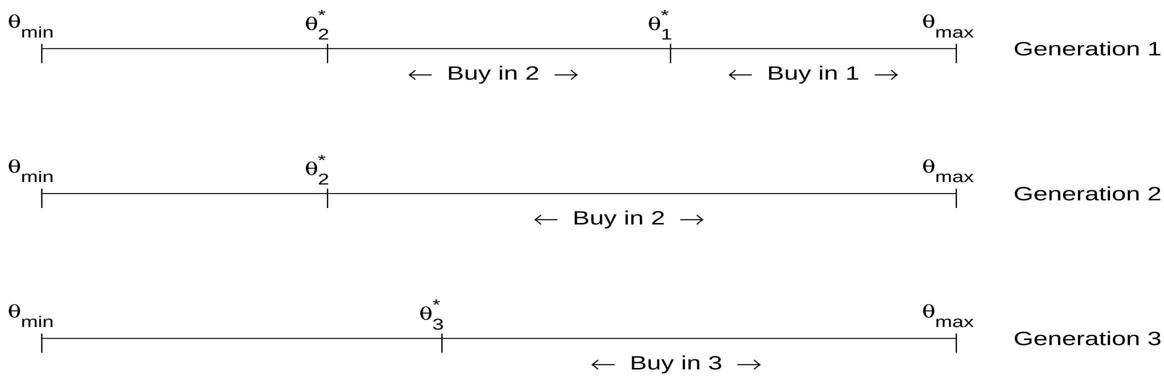

Figure 2: Cutoffs

period t and waiting. Hence prices are determined by the following algorithm:[3]

$$\begin{aligned} \Delta_t(\theta_t^* - p_t^*) &= \max_{\tau \geq t+1} \mathcal{E}\left[(\theta_t^* - p_\tau^*)\Delta_\tau | \mathcal{F}_t\right] \\ &= \mathcal{E}\left[(\theta_t^* - p^*_{\tau(\theta_t^*, t+1)})\Delta_{\tau(\theta_t^*, t+1)} | \mathcal{F}_t\right] \end{aligned} \tag{3.7}$$

where $\tau(\theta_t^*, t+1) = \min\{\tau \geq t+1 : \theta_t^* \geq \theta_\tau^*\}$. When T in infinite there is no last period, but we can still use equation (3.7). One can also truncate the problem, calculate prices for a finite T and let $T \to \infty$, as shown in Chow, Robbins, and Siegmund (1971, Theorems 4.1 and 4.3).

4 Optimal Pricing

4.1 Ordering Demand Functions

It will be useful to consider a method to rank demand curves. Period H is said to have *higher demand* than period L if $m_H^{-1}(0) \geq m_L^{-1}(0)$, so the optimal static monopoly price is higher under $F_H(\theta)$ than $F_L(\theta)$. There may, however, be more people under the "low" demand, i.e. $F_H(\overline{\theta}) \leq F_L(\overline{\theta})$, as is the case in the "back–to–school" example in the Introduction.

A sufficient condition for this is that F_H is larger than F_L in hazard order, $\overline{F}_H(\theta)/f_H(\theta) \geq \overline{F}_L(\theta)/f_L(\theta)$. Suppose θ_L is distributed according to $F_L(\theta)$ which is log–concave. Then $F_H(\theta)$ is larger than $F_L(\theta)$ in hazard order, and consequently $m_H^{-1}(0) \geq m_L^{-1}(0)$, in the following examples:

(a) Upwards Shift. $\theta_H := \theta_L + \epsilon$ for some constant $\epsilon > 0$.

[3]If the cutoffs lie in $(\underline{\theta}, \overline{\theta})$ and there is entry each period these prices are unique. An equivalent way to obtain prices is to equate (2.1) and (2.3) under the optimal purchasing time $\tau(\theta, t)$.

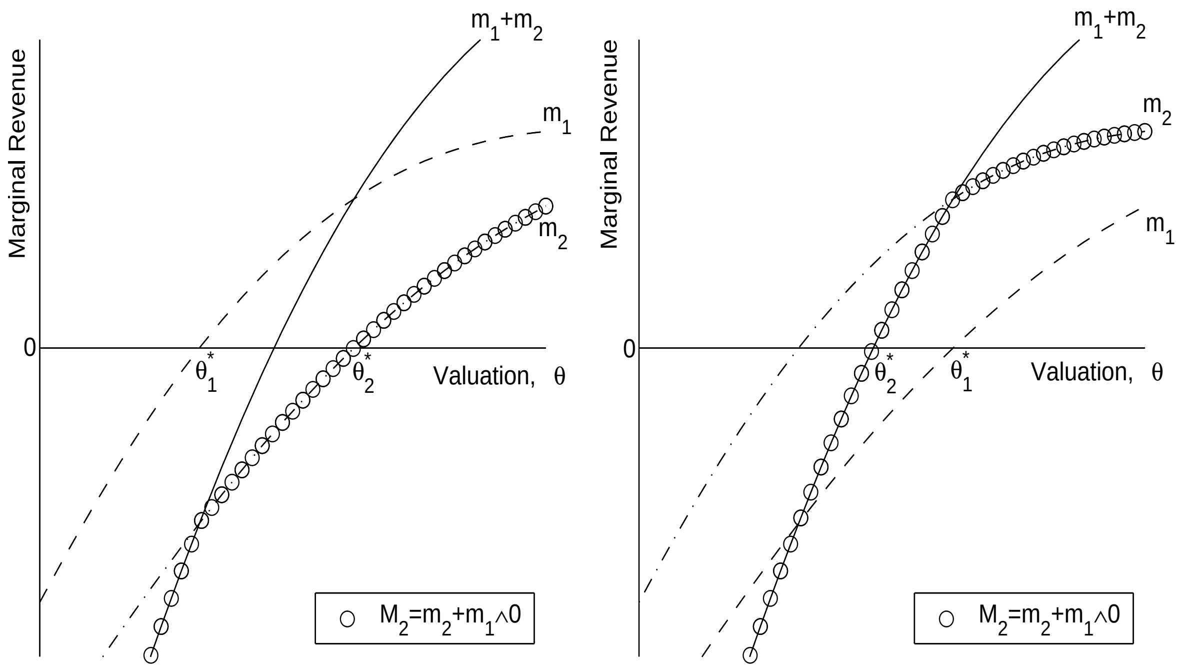

Figure 3: Two–Period Pictures

(b) Upwards Pivot. $\theta_H := \alpha\theta_L$ for $\alpha > 1$.

(c) Outwards Shift. $f_H(\theta) := \alpha f_L(\theta/\alpha)$ for $\alpha > 1$.

Let the marginal revenue from a set of generations $A \subset \{1, \ldots, T\}$ be denoted $m_A(\theta) := \sum_{s \in A} m_s(\theta)$. Similarly, let $m_{\leq t}(\theta) := \sum_{k \leq t} m_k(\theta)$ be total marginal revenue of consumers who have already entered.

4.2 Deterministic Two–Period Model

To gain some intuition behind the solution, consider two–period model, where demand is deterministic and the discount rate δ is constant. In this case, profit (3.4) reduces to,

$$\Pi = \int_{\theta_1^*}^{\overline{\theta}} m_1(\theta)\, d\theta + \delta \int_{\theta_2^*}^{\overline{\theta}} \left[m_2(\theta) + 1_{\{\theta < \theta_1^*\}} m_1(\theta) \right]\, d\theta$$

Assume that the marginal revenue functions, $m_t(\theta)$, are increasing. One can now derive the optimal policy via calculus, however the approach is not particularly illuminating. In contrast, the following argument is easy to generalise.

First consider the increasing demand case, $m_2^{-1}(0) \geq m_1^{-1}(0)$, as shown in *Figure 3A*.[4]

[4] *Figure 3* is created using the Weibull distribution.

Example 2 demonstrates that the optimal rule is the myopic policy $\theta_t^* = m_t^{-1}(0)$. To verify this consider fixing the second period cutoff and choosing θ_1^*. If the firm sells to type $(\theta, 1)$ in period 1 they will obtain profit of $m_1(\theta)$. If they do not sell to this agent in period 1 they may end up selling to them in period 2, yielding profit $\delta m_1(\theta)$, or may never sell to the agent, yielding profit 0. If $m_1(\theta) \geq 0$ then $m_1(\theta) \geq \max\{\delta m_1(\theta), 0\}$, so the firm is always better of selling now, independent of future cutoffs. Conversely, if $m_1(\theta) < 0$ then $m_1(\theta) < \min\{\delta m_1(\theta), 0\}$ and the firm is always better not to sell. So the profit–maximising rule is simple: sell to an agent of type $(\theta, 1)$ in period 1 if and only if $m_1(\theta) \geq 0$.

In period 2 selling to a type $\theta \in \left[m_1^{-1}(0), \overline{\theta}\right]$ yields a marginal revenue of $m_2(\theta)$. For types $\theta \in \left[\underline{\theta}, m_1^{-1}(0)\right)$ the firm also sells to first generation agents and marginal revenue is $m_1(\theta) + m_2(\theta)$. Hence the firm faces the cumulative marginal revenue, $M_2 := m_2(\theta) + \min\{m_1(\theta), 0\}$. Since demand is increasing the firm can sell to all generation 2 agents with positive marginal revenue, $m_2(\theta) \geq 0$, without having to sell to any more generation 1 agents. One can imagine the cutoff θ_2^* being slowly reduced, including more and more agents. The firm then stops at $m_2^{-1}(0)$, since going further will include negative marginal revenue agents from generation 2 and will eventually include negative marginal revenue agents from generation 1. This yields the cutoffs $\theta_t^* = m_t^{-1}(0)$ for $t = 1, 2$.

Second, consider the decreasing demand case, $m_2^{-1}(0) \leq m_1^{-1}(0)$, as shown in *Figure 3B*. As in the increasing demand case, the firm should sell to an agent in period 1 if and only if $m_1(\theta) \geq 0$. However, the second period is different. If the firm were to sell to all the generation 2 agents with positive marginal revenue, $\theta \geq m_2^{-1}(0)$, they would include some generation 1 agents with negative marginal revenue who did not buy in the first round. Thus the firm increases the cutoff until the total marginal from both generations, $m_{\leq 2}(\theta)$, equals zero. This yields the cutoffs $\theta_t^* = m_{\leq t}^{-1}(0)$ for $t = 1, 2$.

When demand is increasing only generation 2 buys in period 2. Hence the firm only cares about the marginal revenue from the second generation when choosing its cutoff. In comparison, when demand is decreasing both generations are active in period 2. Hence the firm cares about the total marginal revenue from both generations when choosing its cutoff.

4.3 General Solution

Definition 1. *Cumulative marginal revenue equals* $M_t(\theta) := m_t(\theta) + \min\{M_{t-1}(\theta), 0\}$, *where* $M_1(\theta) := m_1(\theta)$.

Assumption (A1). $M_t(\theta)$ is quasi–increasing $(\forall t)$.[5]

[5] A function $M(\theta)$ is (strictly) quasi–increasing if $M(\theta_L) \geq 0 \implies M(\theta_H) \geq (>) 0$ for $\theta_H > \theta_L$. Define the root of a quasi–increasing function by $M^{-1}(0) := \sup\{\theta : M_t(\theta) < 0\}$. If $M(\theta) < 0$ $(\forall\theta)$ then $M^{-1}(0) := \overline{\theta}$. If $M(\theta) \geq 0$ $(\forall\theta)$ then $M^{-1}(0) := \underline{\theta}$.

A common assumption in mechanism design is that marginal revenue with respect to quantity, $m_t(\theta)/f_t(\theta)$, is quasi–increasing. This implies that $m_t(\theta)$ is quasi–increasing. If demand jumps are not too large then $M_t(\theta)$ will also be quasi–increasing.

Theorem 1. *Under A1, the optimal cutoffs are given by* $\theta_t^* = M_t^{-1}(0)$.

Proof. Let Π_t be expected profit from generation t, and $\Pi_{\geq t}$ be the profit from generations $\{t, \dots, T\}$. Denote the positive and negative components by $M_t^+(\theta) := \max\{M_t(\theta), 0\}$ and $M_t^-(\theta) := \min\{M_t(\theta), 0\}$. The proof will proceed by induction, starting with period $t = 1$.

Fix $\{\tau(\theta, t')\}$ for $t' \geq 2$, and consider the optimal choice of $\tau(\theta, 1)$. Notice that the [non–discrimination] condition in Lemma 1 implies $\tau(\theta, 1) \in \{1, \tau(\theta, 2)\}$. Splitting up the profit equation, $\Pi = \Pi_1 + \Pi_{\geq 2}$, and applying equation (3.4),

$$\begin{aligned} \Pi &= \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \mathcal{E}\left[\Delta_{\tau(\theta,1)} M_1^+(\theta) + \Delta_{\tau(\theta,1)} M_1^-(\theta) \;\mid \mathcal{F}_1\right] d\theta\right] + \Pi_{\geq 2} \\ &\leq \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \mathcal{E}\left[\Delta_1 M_1^+(\theta) + \Delta_{\tau(\theta,2)} M_1^-(\theta) \;\mid \mathcal{F}_1\right] d\theta\right] + \Pi_{\geq 2} \end{aligned}$$

The second line solves for the optimal choice of $\tau(\theta, 1)$. Since $M_1(\theta)$ is measurable with respect to $\mathcal{F}_1$, the optimal choice is $\tau^*(\theta, 1) = 1$ if $M_1(\theta) \geq 0$ and $\tau^*(\theta, 1) = \tau(\theta, 2)$ if $M_1(\theta) < 0$. This is independent of the choice of $\tau(\theta, 2)$. If $M_1(\theta)$ is quasi–increasing this purchasing rule can be implemented by setting $\theta_1^* = M_1^{-1}(0)$.

Continuing by induction, consider period t. Suppose $\theta_s^* = M_s^{-1}(0)$ for $s < t$. Fix $\{\tau(\theta, t')\}$ for $t' \geq t$, and consider the optimal choice of $\tau(\theta, t)$. The [non–discrimination] condition in Lemma 1 implies $\tau(\theta, t) \in \{t, \tau(\theta, t+1)\}$. Splitting the profit equation, $\Pi = \Pi_{\leq t-1} + \Pi_t + \Pi_{\geq t+1}$,

$$\begin{aligned} \Pi &= \mathcal{E}\left[\sum_{s=1}^{t-1} \int_{\underline{\theta}}^{\overline{\theta}} \Delta_s M_s^+(\theta)\, d\theta\right] + \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \mathcal{E}\left[\Delta_{\tau(\theta,t)} M_t^+(\theta) + \Delta_{\tau(\theta,t)} M_t^-(\theta) \;\mid \mathcal{F}_t\right] d\theta\right] + \Pi_{\geq t+1} \\ &\leq \mathcal{E}\left[\sum_{s=1}^{t-1} \int_{\underline{\theta}}^{\overline{\theta}} \Delta_s M_s^+(\theta)\, d\theta\right] + \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \mathcal{E}\left[\Delta_t M_t^+(\theta) + \Delta_{\tau(\theta,t+1)} M_t^-(\theta) \;\mid \mathcal{F}_t\right] d\theta\right] + \Pi_{\geq t+1} \end{aligned}$$

The optimal choice of stopping rule is $\tau^*(\theta, t) = t$ if $M_t(\theta) \geq 0$ and $\tau^*(\theta, t) = \tau(\theta, t+1)$ if $M_t(\theta) < 0$. If $M_t(\theta)$ is quasi–increasing this stopping rule can be implemented by setting $\theta_1^* = M_1^{-1}(0)$. □

In the first period the monopolist can either sell to consumer θ and gain $\Delta_1 m_1(\theta)$, or not sell to the consumer and gain $\Delta_t m_1(\theta)$ if they eventually buy in period t. Since $\Delta_t < \Delta_1$ the monopolist should sell to the agent if and only if $m_1(\theta) \geq 0$. In period 2, and every subsequent period, the firm sums the marginal revenue of the new consumers and that of old agents who have yet to buy. This cumulative marginal revenue is given by $M_t(\theta) = m_t(\theta) + M_1^-(\theta)$, whereupon the monopolist again sells to an agent with valuation θ if and only if their marginal revenue is positive, $M_t(\theta) \geq 0$.

This algorithm is completely myopic: It says the optimal cutoff point at period t only depends upon the consumers who have entered by time t. That is, the optimal cutoff at time t is independent of future demand and the discount rate.

4.4 Active Generations

The optimal policy has an alternative interpretation. At time t_H a generation t_L is active if some members of generation t_L purchase in period t_H.

Definition 2. *The upper active set is* $\overline{A}(t_H) := \{t_L \leq t_H : \theta_t^* \leq \theta^*(t_H - 1; t_L)\}$.

Lemma 2. *The upper active set has the following properties:*
(a) $\overline{A}(t)$ *is connected and contains* $\{t\}$.
(b) $\overline{A}(t) \supset \overline{A}(t-1)$ *or* $\overline{A}(t) = \{t\}$.

Proof. (a) $t \in \overline{A}(t)$ since $M_t^{-1}(0) \in [\underline{\theta}, \overline{\theta}] < \infty$. $\overline{A}(t)$ is connected since $\theta^*(t_H - 1; t_L)$ is increasing in t_L. (b) If $t' \in \overline{A}(t-1)$ then $\theta^*(t-1; t') = \theta^*(t-1; t-1)$. If $\{t-1\} \in \overline{A}(t)$ then $t' \in \overline{A}(t)$. $\square$

Lemma 2(a) says the current generation is always active, and the set of active generations is connected. Lemma 2(b) says that once two generations are pooled they are never separated. Define $\mathcal{A}(t) := \{\{a, \ldots, t\} : a \leq t\}$ as the set of potentially active generations at time t.

Theorem 2. *Suppose A1 holds. Then* $M_t(\theta)$ *is the lower envelope of* $\{m_A(\theta) : A \in \mathcal{A}(t)\}$ *and the optimal cutoffs are given by*

$$\theta_t^* = \max_{A \in \mathcal{A}(t)} m_A^{-1}(0) \tag{4.1}$$

When $A = \overline{A}(t)$ *this maximum is obtained. Moreover, if* $M_t(\theta)$ *is strictly quasi–increasing and continuous then* $\overline{A}(t)$ *is the maximal set in* $\mathcal{A}(t)$ *such that* $m_A^{-1}(0) = \theta_t^*$.

Proof. Fix t and pick an arbitrary $A \in \mathcal{A}(t)$. That is, $A = \{s, \ldots, t\}$ for some $s \leq t$. By construction,

$$\sum_{i=s}^{t} m_i(\theta) = M_t(\theta) + \sum_{i=s}^{t-1} M_i^+(\theta) - M_{s-1}^-(\theta) \tag{4.2}$$

Hence $M_t(\theta) \leq m_A(\theta)$. Cumulative marginal revenue can also be written as

$$M_t(\theta) = \sum_{s \leq t} m_s(\theta) \mathbf{1}_{\theta < \theta^*(t-1; s)} \tag{4.3}$$

so for any θ, $\exists A \in \mathcal{A}(t)$ such that $M_t(\theta) = m_A(\theta)$. That is, $M_t(\theta) = \min\{m_A(\theta) : A \in \mathcal{A}(t)\}$.

Since $M_t(\theta) \leq m_A(\theta)$, if $M_t(\theta) \geq 0$ then $m_A(\theta) \geq 0$, $\forall A \in \mathcal{A}(t)$. That is, $M_t^{-1}(0) \geq \max_{A \in \mathcal{A}(t)} m_A^{-1}(0)$. To obtain the reverse inequality, (4.3) implies that for small $\epsilon > 0$, $M_t(\theta_t^* - \epsilon) = m_{\overline{A}(t)}(\theta_t^* - \epsilon)$. Hence $M_t^{-1}(0) \leq m_{\overline{A}(t)}^{-1}(0)$. Putting this together, $M_t^{-1}(0) = m_{\overline{A}(t)}^{-1}(0)$.

Fix t and define $A^*(t)$ to be the maximal set such that $m_{A^*(t)}^{-1}(0) = M_t^{-1}(0)$. Since $m_{\overline{A}(t)}^{-1}(0) = M_t^{-1}(0)$ it must be that $\overline{A}(t) \subset A^*(t)$. In order to obtain a contradiction, suppose $A^*(t) = \overline{A}(t) \cup B$ for some nonempty set B, where $b = \max\{t : t \in B\}$. Since $b \notin \overline{A}(t)$, $\theta_b^* < \theta_t^*$. $M_b(\theta)$ lies below $m_B(\theta)$ and is strictly quasi–increasing so there is a small $\epsilon > 0$ such that $m_B(\theta_t^* - \epsilon) \geq M_b(\theta_t^* - \epsilon) > \epsilon$. Moreover, $M_t(\theta)$ is continuous so ϵ can be chosen sufficiently small such that $m_{\overline{A}(t)}(\theta_t^* - \epsilon) = M_t(\theta_t^* - \epsilon) \in (-\epsilon, 0)$. Hence $m_{A^*(t)}(\theta_t^* - \epsilon) = m_{\overline{A}(t)}(\theta_t^* - \epsilon) + m_B(\theta_t^* - \epsilon) > 0$, and $m_{\overline{A}(t)}^{-1}(0) > m_{A^*(t)}^{-1}(0)$, contradicting the assumption that $m_{A^*(t)}^{-1}(0) = M_t^{-1}(0)$. □

Theorem 2 says the optimal cutoffs are determined by the marginal revenue of the active generations. Let $\overline{a}(t) = \min \overline{A}(t)$. Theorem 2 induces the first order condition:

Proposition 1. *Suppose A1 holds. Then*

$$m_{\overline{a}(t)-1}^{-1}(0) < m_{\overline{A}(t)}^{-1}(0) \leq m_{\overline{a}(t)}^{-1}(0) \tag{4.4}$$

Proof. First, $\overline{a}(t) - 1 \notin \overline{A}(t)$, so $m_{\overline{A}(t)}^{-1}(0) = \theta_t^* > \theta_{\overline{a}(t)-1}^* \geq m_{\overline{a}(t)-1}^{-1}(0)$. The last inequality comes from Theorem 2. Second, $\overline{a}(t) \in \overline{A}(t)$, so $m_{\overline{A}(t)}^{-1}(0) = \theta_t^* \leq \theta_{\overline{a}(t)}^* = m_{\overline{a}(t)}^{-1}(0)$. The last equality follows from Lemma 2(b) which implies $\overline{A}(\overline{a}(t)) = \overline{a}(t)$. □

If time is continuous and the monopoly price, $m_t^{-1}(0)$, varies continuously in t, then equation (4.4) reduces to $m_{\overline{A}(t)}^{-1}(0) = m_{\overline{a}(t)}^{-1}(0)$. This provides a fixed–point interpretation: the monopolist serves enough agents so that the monopoly price against the marginal agent equals the monopoly price against the average agent. One must, however, be careful with this first order approach since the fixed point may not be unique.

4.5 Ironing

Theorem 1 assumes the cumulative marginal revenue M_t is quasi–increasing (A1). If this fails in a one–period model one can calculate the ironed marginal revenue $\overline{M}_1(\theta)$ (e.g. Myerson (1981)). The firm sells to an agent if and only if $\overline{M}_1(\theta) \geq 0$. Unfortunately in the multi–period model it is not possible to use the myopic policy of ironing each M_t individually, as the following example shows.

Example 4. Suppose $T = 2$, δ constant, $[\underline{\theta}, \overline{\theta}] = [0, 1]$ with $m_1(\theta) = 1 - 4\mathbf{1}_{[1/2,1]}$ and $m_2(\theta) = -10 + 20\mathbf{1}_{[1/2,1]}$. This yields $\overline{M}_1(\theta) = -1$ so a myopic policy suggests not awarding the good to any agent. In period 2, $M_2(\theta) = -9 + 16\mathbf{1}_{[1/2,1]}$ which is quasi–increasing with $M_2^{-1}(0) = 1/2$ yielding revenue $7\delta/2$. However if the firm sells to consumers $[0, 1]$ in period 1 and $[1/2, 1]$ in period 2 then revenue is $-1 + 10\delta/2$ which is preferable if $\delta \geq 2/3$. $\triangle$

The existence of negative marginal revenue agents may stop the monopolist selling to all positive marginal revenue consumers. However if these negative consumers end up buying anyway, the firm should take this into account in its ironing calculation. Since the firm now has to be forward looking the simple myopic policy in Theorem 1 no longer holds.

5 Applications

Assumption (A2). $m_t(\theta)$ is strictly increasing and continuous in θ and $m_t^{-1}(0) \in (\underline{\theta}, \overline{\theta})$ $(\forall t)$.

This Section (as well as Section 7) uses A2 rather than A1. This stricter monotonicity assumption simplifies proofs and helps provide a cleaner characterisation of demand cycles (Section 5.4). Versions of many of these results, however, extend to A1.[6]

Lemma 3. *Suppose A2 holds. Then*
(a) $m_1^{-1}(0) > m_2^{-1}(0)$ *implies* $m_{\{1,2\}}^{-1}(0) \in (m_2^{-1}(0), m_1^{-1}(0))$
(b) $m_1^{-1}(0) \geq m_2^{-1}(0)$ *implies* $m_{\{1,2\}}^{-1}(0) \in [m_2^{-1}(0), m_1^{-1}(0)]$

Proof. (a) On $[m_1^{-1}(0), \overline{\theta}]$, $m_1(\theta) \geq 0$, $m_2(\theta) > 0$ (by strict monotonicity) and $m_{\{1,2\}}(\theta) > 0$. Continuity implies $m_{\{1,2\}}^{-1}(0) < m_1^{-1}(0)$. On $[\underline{\theta}, m_2^{-1}(0)]$, $m_1(\theta) < 0$ (by monotonicity), $m_2(\theta) \leq 0$ (by continuity and monotonicity) and $m_{\{1,2\}}(\theta) < 0$. Continuity implies $m_{\{1,2\}}^{-1}(0) > m_2^{-1}(0)$.

(b) On $[m_1^{-1}(0), \overline{\theta}]$, $m_1(\theta) \geq 0$, $m_2(\theta) \geq 0$ and $m_{\{1,2\}}(\theta) \geq 0$. Hence $m_{\{1,2\}}^{-1}(0) \leq m_1^{-1}(0)$. On $[\underline{\theta}, m_2^{-1}(0)]$, $m_1(\theta) \leq 0$ (by monotonicity and continuity), $m_2(\theta) \leq 0$ and $m_{\{1,2\}}(\theta) \leq 0$. Hence $m_{\{1,2\}}^{-1}(0) \geq m_2^{-1}(0)$. $\square$

5.1 Lower Active Set

Definition 3. *The lower active set is* $\underline{A}(t_H) := \{t_L \leq t_H : \theta_t^* < \theta^*(t_H - 1; t_L)\}$.

Lemma 2 also applies to the lower active set.

Proposition 2. *Under A2,* $m_{\underline{A}(t)}^{-1}(0) = \theta_t^*$. *Moreover,* $\underline{A}(t)$ *is the minimal set is* $\mathcal{A}(t)$ *such that* $m_{\mathcal{A}}^{-1}(0) = \theta_t^*$.

[6] To illustrate, A1 is sufficient for the "if" part of the monotone demand characterisation (Proposition 3). Similarly, when demand cycles, cutoffs are stationary (Proposition 4) and, using Theorem 2, cutoffs and prices exceed the average–demand price (Proposition 5).

Proof. Using (4.3) and A2 $M_t(\theta_t^*) = m_{\underline{A}(t)}(\theta_t^*) = 0$. By A2, this uniquely defines $\theta_t^* = m_{\underline{A}(t)}^{-1}(0)$.

Fix t and let $A^*(t)$ be the minimal set such that $m_{A^*(t)}^{-1}(0) = \theta_t^*$. In order to attain a contradiction suppose $\underline{A} = A^* \cup B$ for some nonempty set B, where $b = \max\{t : t \in B\}$. Since $b \in \underline{A}(t)$, $\theta_b^* > \theta_t^*$ and $M_b^-(\theta_t^*) < 0$. Using equation (4.2), $0 = M_t(\theta_t^*) \leq m_{A^*(t)}(\theta_t^*) + M_b^-(\theta_t^*) < 0$, yielding a contradiction. □

5.2 Monotone Deterministic Demand: Fast Rises and Slow Falls

Definition 4. *Demand is increasing if $m_{t+1}^{-1}(0) \geq m_t^{-1}(0)$. Demand is weakly decreasing if $m_{\leq t}^{-1}(0) \geq m_{t+1}^{-1}(0)$.*

Increasing demand means that the myopic monopoly price against the incoming generation rises over time. Weakly decreasing demand means the myopic monopoly price against the incoming generation is lower than the monopoly price against the sum of the previous demands.

Proposition 3 characterises the optimal cutoffs and prices when demand is growing or falling. These results are simple extensions of the two–period solution in Section 4.2.

Proposition 3. *Suppose demand is deterministic and A2 holds. Optimal cutoffs are given by $\theta_t^* = m_t^{-1}(0)$ ($\forall t$) if and only if demand is increasing. This can be implemented by prices*

$$p_t^* = m_t^{-1}(0)$$

Optimal cutoffs are given by $\theta_t^* = m_{\leq t}^{-1}(0)$ if and only if demand is weakly decreasing. This can be implemented by prices

$$p_t^* = \sum_{s=t}^{T} \mathcal{E}\left[\left(\frac{\Delta_s}{\Delta_t} - \frac{\Delta_{s+1}}{\Delta_t}\right) m_{\leq s}^{-1}(0) \,\middle|\, \mathcal{F}_t\right] \tag{5.1}$$

where $\Delta_{T+1} := 0$.

Proof. [If]. Increasing demand case. For $t = 1$, $\theta_1^* = m_1^{-1}(0)$. Suppose $\theta_s^* = m_s^{-1}(0)$ for $s < t$ and consider period t. $M_t(\theta) = m_t(\theta)$ on $[\theta_{t-1}^*, \overline{\theta}] = [m_{t-1}^{-1}(0), \overline{\theta}]$, using the induction hypothesis. Demand is increasing so $M_t^{-1}(0) = m_t^{-1}(0)$.

Decreasing demand case. For $t = 1$, $\theta_1^* = m_{\leq 1}^{-1}(0)$. Suppose $\theta_s^* = m_{\leq s}^{-1}(0)$ for $s < t$ and consider period t.

$M_t(\theta) = m_{\leq t}(\theta)$ on $[\underline{\theta}, \min\{\theta_1^*, \ldots, \theta_{t-1}^*\}] = [\underline{\theta}, m_{\leq t-1}^{-1}(0)]$, using the induction hypothesis. Demand is decreasing so $M_t^{-1}(0) = m_{\leq t}^{-1}(0)$.

[Only If]. Increasing demand case. Applying the contrapositive, suppose $m_t^{-1}(0) < m_{t-1}^{-1}(0)$. Theorem 2 means $\theta_t^* \geq m_{\{t-1,t\}}^{-1}(0) > m_t^{-1}(0)$ using Lemma 3.

Decreasing demand case. Applying the contrapositive, suppose $m_t^{-1}(0) > m_{\leq t-1}^{-1}(0)$. Theorem 2 means $\theta_t^* \geq m_t^{-1}(0) > m_{\leq t}^{-1}(0)$ using Lemma 3.

Prices can then be derived from equation (3.7). With increasing demand this is immediate. With weakly decreasing demand prices obey the AR(1) equation $(\theta_t^* - p_t^*) = \mathcal{E}[(\theta_t^* - p_{t+1}^*)\delta_{t+1}|\mathcal{F}_t]$ □

When demand is increasing over time the firm can charge the optimal myopic price, $p_t^* = m_t^{-1}(0)$. Since the price is increasing, no consumers will delay their purchases, and the problem can be broken into T disjoint sub–problems (see Example 2 in Section 3.2). In contrast, if the firm charges the myopic price when demand is decreasing then there will be much delay. The firm takes this into account and chooses the cutoff points so that at time t agent θ buys if the "past average" marginal revenue, $m_{\leq t}(\theta)$, is positive. Prices are then given by a geometric sum of future "past averages".

Two price paths, mentioned in the introduction, will be useful benchmarks.

Definition 5. *The myopic price is* $p_t^M := m_t^{-1}(0)$*. The average–demand price is* $p^A := \lim_{t \to T} m_{\leq t}^{-1}(0)$*, assuming the limit exists.*

The myopic price would be the price charged by a monopolist who only takes current generation of consumers into account, ignoring the previous ones. By Example 2, this equals the optimal price when the monopolist can discriminate between generations. It is also the optimal price if consumers are banned from delaying consumption. The average–demand price is charged by a monopolist who faces average demand $\frac{1}{T}\sum_{t=1}^{t} F_t(\theta)$ each period. This is also the price charged by an uniformed firm who knows total demand over the T periods, but does not know the demand each period.[7]

Figure 4A compares different price paths under increasing demand.[8] As can be seen, $p_t^* = p_t^M$ is increasing. Since agents never delay their purchases under increasing demand, the optimal price path is independent of the discount factor.

This can be contrasted to decreasing demand, as shown in *Figure 4B*. Here p_t^* is decreasing, starting off below the myopic price and ending above it. The optimal price p_t^* converges to the average–demand price from above as $t \to T$. That is,

$$\lim_{t \to T} [p_t^R - p^A] = \sum_{s=t}^{T} \mathcal{E}\left[\left(\frac{\Delta_s}{\Delta_t} - \frac{\Delta_{s+1}}{\Delta_t} \right) (m_{\leq s}^{-1}(0) - m_{\leq t}^{-1}(0)) \,\middle|\, \mathcal{F}_t \right] = 0$$

since every convergent sequence is cauchy. The discount factor is also relevant when demand decreases: price fall towards the average demand price as agents become more patient. For more on discounting see Section 6.2.

[7]The uncertainty interpretation assumes the firm cannot update their strategy over time, perhaps because because sales are unobservable in the short term.

[8]*Figures 1, 4* and *5* are generated by linear demand curves. In particular, types have measure 1 on $[0, b_t]$ where $b_t \in [20, 30]$. The discount rate is $\delta = 0.9$ in *Figures 4–5* and $\delta = 0.75$ in *Figure 1.* Example 4 in Section 4.3 further analyses the linear demand specification.

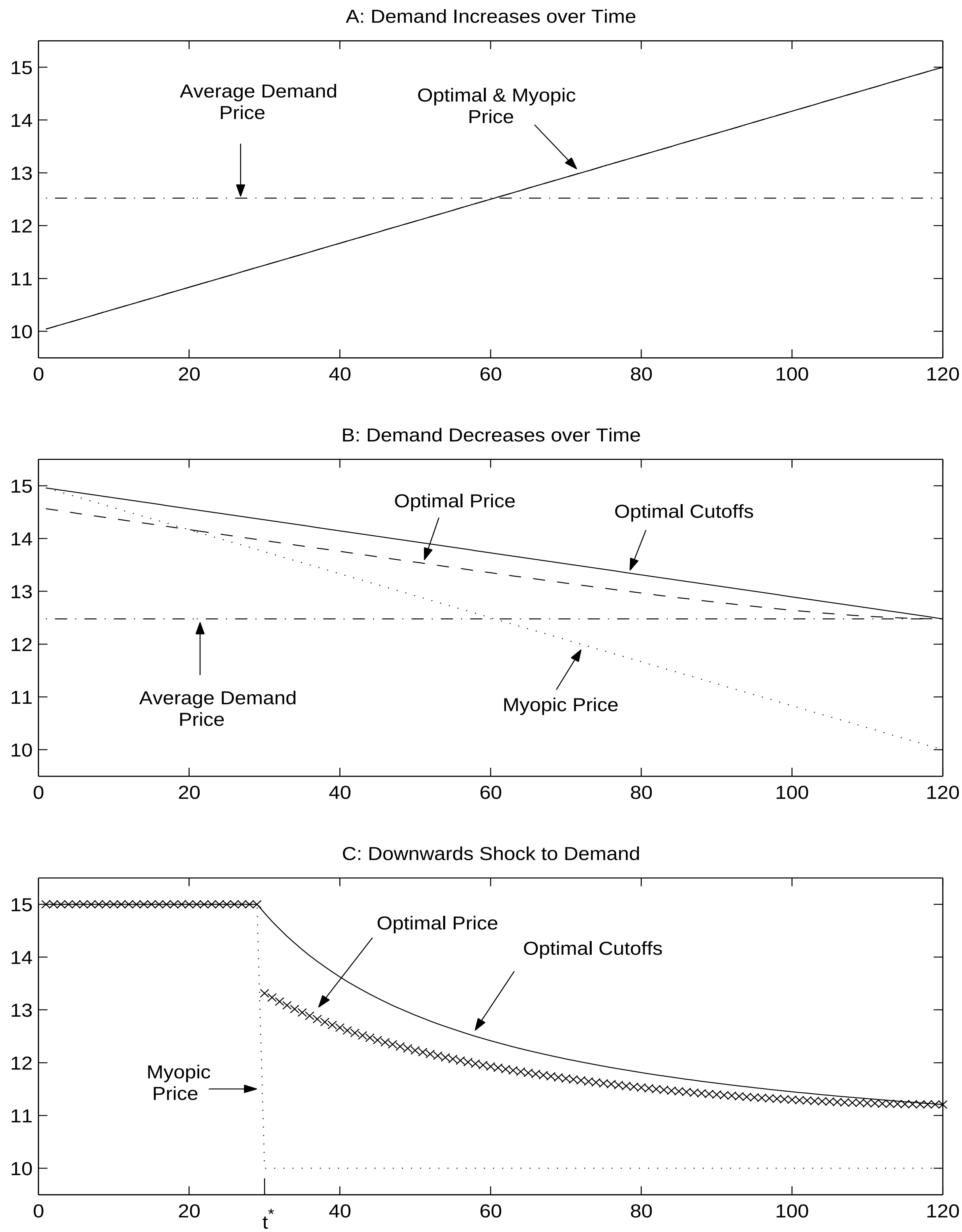

Figure 4: Monotone Demand Paths

5.3 Permanent Shocks to Demand

The model can be used to analyse the effect of permanent shocks to demand. Moreover, since demand curves are allowed to be uncertain, we can examine the price paths when this shock is anticipated or unanticipated.

For the first $t'-1$ periods demand is constant, $m_t(\theta) = m_\alpha(\theta)$ for $t \in \{1, \dots, t'-1\}$. There are two states of the world. In state ω_α demand stays at $m_t(\theta) = m_\alpha(\theta)$ for $t \in \{t', \dots, T\}$. In state ω_β demand shifts to $m_t(\theta) = m_\beta(\theta)$ for $t \in \{t', \dots, T\}$. To complete the description of the world, suppose the state of the world is realised in period $t'' \leq t'$ and let the probability of state ω_α be α.

First consider an upwards jump in demand, $m_\beta^{-1}(0) \geq m_\alpha^{-1}(0)$. Proposition 3 says the optimal cutoffs are given by $\theta_t^* = m_t(\theta)$. This allocation is implemented by prices $p_t^* = m_t(\theta)$. In state ω_α, cutoffs and prices remain constant over time, $p_t^* = \theta_t^* = m_\alpha^{-1}(0)$ $(\forall t)$. In state ω_β, prices stay constant until time t' and then jump upwards.

Next, consider a downwards jump in demand, $m_\beta^{-1}(0) \leq m_\alpha^{-1}(0)$. Proposition 3 says the optimal cutoffs are given by $\theta_t^* = m_{\leq t}(\theta)$. In state ω_α, these cutoffs remain constant over time, $\theta_t^* = m_\alpha^{-1}(0)$ $(\forall t)$. In state ω_β, these cutoffs slowly decline after time t'.

Prices are trickier in the downwards jump case. Before period $t''-1$ prices slowly fall in accordance with AR(1) equation $p_{t-1}^* = (1-\delta_t)m_\alpha^{-1}(0) + \delta_t p_t^*$ with boundary condition

$$p_{t''-1}^* = (1-(1-\alpha)\delta_{t''})m_\alpha^{-1}(0) + (1-\alpha)\delta_{t''}p_{t''}^*(\omega_\beta)$$

At time t'' the state is revealed. In state ω_α the price jumps upwards to $p_t^*(\omega_\alpha) = m_\alpha^{-1}(0)$ for $t \geq t''$. In state ω_β, the price $p_t^*(\omega_\beta)$ jumps downwards and slowly converges to $m_\alpha^{-1}(0)$ according to equation (5.1). If the shock is "unexpected" $(\alpha \to 1)$ then $p_t^* \to m_\alpha^{-1}(0)$ for $t < t''$. This is shown in *Figure 4C*, where $t'' = t'$.

There are two further points worth noting. First, the time t'' information is revealed only affects prices in the case of a decline in demand. However, even then it does not affect allocations or utility (see Section 6.1). Second, an increase in today's price not associated with an increase in demand does *not* mean future demand will increase; rather it means an anticipated demand fall is no longer expected.

5.4 Deterministic Demand Cycles

This section examines the implications of demand that follows deterministic cycles, where the sequence of demand functions is described by K repetitions of $\{F_1, \dots, F_T\}$, where $T < \infty$ but we allow $K = \infty$. Denote the period t of cycle k by t_k. An example of this was seen in *Figure 1* in the Introduction which also illustrates the set of active agents, $\underline{A}(t)$. One can see the pattern of sharp price increases, and slow declines, which intuitively follow from Proposition 3.

When new demand is growing the price rises quickly along with the myopic price and there is no delay. When new demand is falling agents delay their purchases and the price falls much more slowly. The picture also illustrates other regularities:

1. After the first cycle, cutoffs and prices follow a regular pattern.
2. After the first cycle, prices always lie above the average–demand price.
3. The lowest price occurs in the last period of the slump.

The next three propositions correspond to these results.

Proposition 4. *Suppose demand follows deterministic cycles and A2 holds. Then* $|\underline{A}(t)| \leq T$*. Hence if* $k \geq 2$*, the cycles are stationary,* $\theta^*_{t_k} = \theta^*_{t_2}$*.*

Proof. Suppose $|\underline{A}(t)| > T$. Define the set A such that $\underline{A}(t) = A \cup B$ where B consists of the union of sets of the form $\{1, \ldots, T\}$ and $|A| \leq T$. Then $m^{-1}_{\underline{A}(t)}(0) \leq \max\{m^{-1}_A(0), m^{-1}_{\{1,\ldots,T\}}(0)\}$ by Lemma 3, contradicting the fact that $\underline{A}(t)$ is the smallest set to achieve the maximum in Theorem 2. □

Proposition 4 means the cutoffs will be the same for each cycle $k \geq 2$. This substantially simplifies analysis: when $k \geq 2$ we can use modular arithmetic to write the sets of potentially active generations $\mathcal{A}(t)$ as:

$$\mathcal{A}_K(t) = \{\{a, \ldots, t\} : a \in \{1, \ldots, T\}\} \quad (\text{mod–}T)$$

The cutoff and price are minimised at time

$$\underline{t} := \min\{\text{argmin}_{t \in \{1,\ldots,T\}}\{\theta^*_{t_k} : k \geq 2\}\}$$

The market is effectively cleared out in period $\underline{t}$ and everything resets. This means that the starting position of the cycle only matters for the first $\underline{t}$ periods, after which the stationary cycles start.

Prices are determined by equation (3.7). For cycles $k \in \{2, \ldots, K-1\}$ the prices are stationary with boundary condition $p^*_{\underline{t}} = \theta^*_{\underline{t}} = m^{-1}_{\{1,\ldots,T\}}(0)$, using Proposition 5(a). For the final cycle, where $k = K$ and $t > \underline{t}$, the relevant boundary condition is $p^*_T = \theta^*_T$. Since there is not as much scope for delay, prices in the last cycle may be higher than in previous cycles.

Proposition 5. *Suppose demand follows cycles with* $k \geq 2$ *and A2 holds. Then*
(a) The lowest optimal cutoff equals the average–demand price. Hence agents buy later under optimal pricing.
(b) The lowest optimal price equals the average–demand price.

(c) Under the optimal price path, in comparison to average–demand pricing, profits are higher, utility is lower for every type (θ, t) and welfare is lower for every generation.

Proof. (a) If $k \geq 2$ then $\{1, \ldots, T\} \in \mathcal{A}(t)$ and Theorem 2 implies $\theta_t^* \geq m_{\{1,\ldots,T\}}^{-1}(0)$. Equation (4.2) implies

$$m_{\{1,\ldots,T\}}(\theta_{\underline{t}}^*) = M_{\underline{t}_k}(\theta_{\underline{t}}^*) + \sum_{s=\underline{t}_{k-1}+1}^{\underline{t}_k - 1} M_s^+(\theta_{\underline{t}}^*) - M_{\underline{t}_{k-1}}^-(\theta_{\underline{t}}^*) = 0$$

using (1) the definition $\theta_{\underline{t}}^*$, (2) the fact that $\theta_s^* \geq \theta_{\underline{t}_k}^*$ for $s \in \{\underline{t}_{k-1} + 1, \ldots, \underline{t}_k - 1\}$ and (3) $\theta_{\underline{t}_k}^* \geq \theta_{\underline{t}_{k-1}}^*$. Thus $\theta_{\underline{t}}^* = m_{\{1,\ldots,T\}}^{-1}(0) = p_t^A$.

(b) $p_{\underline{t}}^* = \theta_{\underline{t}}^* = p_t^A$.

(c) Profit is lower under the average–demand price regime by revealed preference. The utility of any customer (3.1) and welfare of any generation (3.3) is higher under the average–demand price since the cutoffs are always higher. □

In each period the cutoff is higher than if the monopolist faced average demand, the price is higher, and welfare and consumer surplus are lower. That is, all customers are made worse off by the ability of the monopolist to discriminate between generations. Intuitively, during a high demand period the price will be high, and only the high demand generations will be active. However during low demand periods both high and low generations will be active. Thus the high demand cohorts exert a negative externality on the low cohorts, raising the price to an average level.

With quasi–linear utility the indirect utility function is convex in prices, suggesting price variation benefits consumers because of the option value. However Proposition 5 shows that when prices are endogenous, price variation may hurt all consumers. The result can also be contrasted with the standard view that the welfare effect of third degree price discrimination is indeterminate (e.g. Tirole (1988, p.137)). Inter– and intra–temporal price discrimination can have very different properties.

Define a cycle as *simple* if $m_t^{-1}(0) - m_{t-1}^{-1}(0)$ is nonzero and has at most two changes of sign. That is, each cycle has one "boom" and one "slump".

Proposition 6. *Suppose demand follows cycles with $k \geq 2$ and A2 holds. Then $\underline{t}$ obeys*

$$m_{\underline{t}}^{-1}(0) \leq m_{\{1,\ldots,T\}}^{-1}(0) \leq m_{\underline{t}+1}^{-1}(0)$$

When the cycle is simple this uniquely defines $\underline{t}$.

Proof. First, Proposition 5(a) and Theorem 2 imply $m_{\{1,\ldots,T\}}^{-1}(0) = \theta_{\underline{t}}^* \geq m_{\underline{t}}^{-1}(0)$. Second, by the definition of $\underline{t}$, $\underline{A}(\underline{t}+1) = \{\underline{t}+1\}$. Hence $m_{\underline{t}+1}^{-1}(0) \geq m_{\{1,\ldots,T\}}^{-1}(0)$. When the cycle is simple this uniquely defines $\underline{t}$ and implies $\underline{A}(\underline{t}) = \{1, \ldots, T\}$. □

In a simple cycle $\underline{t}$ is uniquely defined as the last period of the slump, just before new demand returns to its long run average. In literary terms: The darkest hour is that before dawn.

Proposition 5 compares the optimal policy to a weak monopolist who can only price against the average demand. It is also of interest to compare the optimal policy to a strong monopolist who can completely discriminate between generations, as in Example 2. By revealed preference the strong monopolist would make greater profits. However, with linear demand curves, welfare is lower under the strong monopolist (see Example 5). In general the welfare effects are indeterminate (see Example 6). When the firm can completely discriminate between generations the stronger cohorts lose, while the weaker ones gain; the overall effect depends upon the distribution of consumers' rents.

Example 5 (Linear Demand). Suppose $f_t(\theta) = 1$ on $[0, 2b_t]$ where $b_t \in [10, 20]$.[9] Linear demand curves have a very useful property: when third degree price discrimination is introduced to a static market the average quantity sold remains unaffected (Tirole (1988, p.139)). Similarly in this model, the quantity b_t is sold each period under complete discrimination and under Theorem 1 (see the Appendix A.3). Complete discrimination thus leads to lower welfare and consumer surplus since the same quantity is allocated less efficiently. $\triangle$

Example 6. Let $T = 2$ and $\delta = 1$. (a) Suppose $\theta_1 = 2$ with mass 1 and $\theta_2 \sim U[0, 2]$. Under complete discrimination the myopic price is $(p_1^M, p_2^M) = (2, 1)$, while under Theorem 1 the seller chooses $(p_1^*, p_2^*) = (2, 2)$. Welfare is higher under complete discrimination. (b) Instead suppose $\theta_1 \sim U[0, 4]$ and $\theta_2 = 1$ with mass 1. Under complete discrimination the myopic price is $(p_1^M, p_2^M) = (2, 1)$, while under Theorem 1 the seller chooses $(p_1^*, p_2^*) = (1, 1)$. Welfare is lower under complete discrimination. $\triangle$

5.5 IID Demand

The model allows demand to be uncertain enabling us to study the effect of incoming cohorts that are independently and identically distributed (IID).[10] Each period demand is drawn from $\{m_x(\theta)\}_x$ with probability measure $\mu(x)$ for $x \in [0, 1]$, where higher indices imply higher demand: $m_{x_H}^{-1}(0) \geq m_{x_L}^{-1}(0)$.

In this stochastic setting we can define the *average–demand price* by $p^A := [\int m_x(\theta) d\mu(x)]^{-1}(0)$. As in the deterministic setting, this is the optimal price for a monopolist who is ignorant about demand. Example 7 explicitly derives the optimal price schedule in a two–period model.

[9] The bounds on b_t are required to ensure the marginal revenue functions are quasi–monotone.

[10] Other sequences of demand functions are have interesting properties. One nice example is where demand curves are linear with the intercept following a random walk. In this model, the asymmetric treatment of rises and falls in demand means the ex–ante expectation of θ_t^* is increasing in t.

Example 7 (T=2 with IID Demand). Denote the demand in the first and second period by $m_1(\theta) = m_{x_1}(\theta)$ and $m_2(\theta) = m_{x_2}(\theta)$ respectively. Theorem 2 implies $\theta_1^* = m_{x_1}^{-1}(0)$ and $\theta_2^* = \max\{m_{\{x_1,x_2\}}^{-1}(0), m_{x_2}^{-1}(0)\}$. The second period price is $p_2^* = \theta_2^*$, while the initial price is given by

$$p_1^* = \left(1 - \delta \int_0^{x_1} d\mu(x)\right) m_{x_1}^{-1}(0) + \delta \int_0^{x_1} m_{\{x_1,x\}}^{-1}(0)\, d\mu(x) \tag{5.2}$$

If demand of the first generation is low then demand is likely to increase and p_1^* is determined by the first generation's demand, $m_{x_1}^{-1}(0)$. If demand of the first generation is high then demand is likely to decrease and p_1^* is determined by both generations' demand, $m_{\{x_1,x_2\}}^{-1}(0)$. These extreme cases of $x_1 \in \{0,1\}$ are directly analogous to the monotone demand results in Proposition 3. The general principle is that p_1^* is affected by a state of the world only if the first generation is active in that state. $\triangle$

When there are more periods it becomes harder to solve the option problem required to back out prices. However one can make comparisons as the number of periods grows large. This next result is the stochastic analogue to Proposition 5.

Proposition 7. *Suppose demand is IID, $\{m_x(\theta)\}_x$ are uniformly bounded and A2 holds. Then $\lim_{t\to\infty} \theta_t^* \geq p^A$ and $\lim_{t\to\infty} p_t^* \geq p^A$ a.s..*

Proof. Define $g_t(\theta) = \frac{1}{t}\sum_{s=1}^{t} m_s(\theta)$ and $g(\theta) = \int m_x(\theta) d\mu(x)$. The strong law of large number implies that $g_t(\theta) \overset{as}{\to} g(\theta)$ pointwise $\forall \theta \in [\underline{\theta}, \overline{\theta}]$. The collection $\{m_x(\theta)\}_x$ is uniformly bounded and increasing so the Glivenko–Cantelli Theorem (Davidson (1994, Theorem 21.5)) states that $\sup_\theta |g_t(\theta) - g(\theta)| \overset{as}{\to} 0$. Since $g(\theta)$ is strictly increasing $m_{\{1,\dots,t\}}^{-1}(0) = g_t^{-1}(0) \overset{as}{\to} g^{-1}(0) = p^A$. Finally, $\{1,\dots,t\} \in \mathcal{A}(t)$ so Theorem 2 implies $\lim_{t\to\infty} \theta_t^* \geq p^A$ a.s.. $\square$

In the long run, the price should be at least as high as the average demand price. Moreover, it will often be strictly higher (e.g. if $m_t^{-1}(0) > p^A$). This means that prices increase when there is more demand variation, or when the firm has more information about the incoming demand.

6 Comparative Statics

This Section explores the properties of the optimal policy given in Theorem 1. Consequently we only require the weaker monotonicity condition, A1.

6.1 Information Structure

Theorem 1 can be used to address the effect of varying uncertainty about future demand.

Proposition 8. *Consider two information structures $\mathcal{F}_t$ and $\mathcal{F}'_t$ such that m_t is $\mathcal{F}_t \cap \mathcal{F}'_t$–adapted and suppose A1 holds. Then utilities and profit are the same under both information structures.*

Proof. Utility and profit are determined by $\tau(\theta, t)$, as shown in equations (3.1) and (3.4). By Theorem 1 this is independent of the information structure if $m_t(\theta)$ is measurable with respect to $\mathcal{F}_t$. $\square$

Proposition 8 shows that under the optimal mechanism payoffs are independent of the information, so long as the history of demands is known at time t. Prices, however, will be different under different information structures. This result very much depends upon the simple structure of the optimal solution and may fail without A1.

To consider an application of Proposition 8, recall the examples of unexpected shocks in Section 5.3. In this example, demand may receive a permanant demand shock at time t', which they find out about at time $t'' \leq t'$. Proposition 8 then says that agents should be indifferent over the announcement time, t''. For example, in the downwards shock case, when an ε–probability event occurs the price jumps down a lot, while if the shock does not occur the price rises a little. These two effects exactly cancel out.

So far we have made two assumptions about agents' information.

First, the firm and consumers possess the same information. If firms and consumers possess different information (and both know current demand) the monopolist would still like to implement the cutoffs given by Theorem 1. If firms know more than customers the prices are determined by customers' information sets and payoffs are the same as with symmetric information. However, if customers know more than firms then there may be no prices to implement the optimal cutoffs.

Second, we assume agents know time $f_t(\theta)$ at time t. Without this assumption there are two problems. First, the firm may engage in experimentation. Second, the firm may delay purchases until they have better information about demand.

6.2 Prices and Discount Rates

Proposition 9. *Suppose A1 holds. Then increasing δ_t a.e. reduces price p^*_s a.e., for $s < t$. As $\delta_t \stackrel{as}{\to} 1$ $(\forall t)$ so $(p^*_t - \min_{s \geq t} \theta^*_t) \stackrel{as}{\to} 0$. As $\delta_t \stackrel{as}{\to} 0$ $(\forall t)$ so $(p^*_t - \theta^*_t) \stackrel{as}{\to} 0$.*

Proof. (a) Pick t and consider increasing δ_t a.e.. By backwards induction, price p^*_s is independent of δ_t for $s \geq t$. Continuing by induction, pick $s < t$ and suppose future prices are (weakly) decreasing in δ_t. An increase in δ_t thus increases the utility of type θ^*_s if they choose to delay, the right hand side of equation (3.7). Thus the price p^*_s decreases.

(b) Suppose $\delta_t \stackrel{as}{\to} 1$ $(\forall t)$. Equation (3.7) implies that that agent waits for the lowest price, $p^*_t \stackrel{as}{\to} \min\{\theta^*_t, \min_{s>t}\{p^*_s\}\}$. With the boundary condition $p^*_T = \theta^*_T$ this implies $p^*_t \stackrel{as}{\to} \min_{s \geq t} \theta^*_t$.

(c) Suppose $\delta_t \overset{as}{\to} 0$ $(\forall t)$. Then $\Delta_{\tau(\theta_t^*,t+1)}/\Delta_t \overset{as}{\to} 0$ and the right hand side of equation (3.7) converges to zero. $\square$

When agents are impatient ($\delta_t \approx 0$) and the firm simply sets the price equal to the current optimal cutoff, $p_t^* = \theta_t^*$. As agents become more patient consumers find it less costly to delay, reducing the prices required to implement any sequence of cutoffs. In the limit, when agents are completely patient, they wait for the lowest price and the price is determined by the lowest cutoff in all future periods, $p_t^* = \min_{s \geq t} \theta_t^*$.

Since discount rates are allowed to be uncertain it is also easy to analyse the impact of changes in interest rates. For example, an unexpected increase in future interest rates will increase today's price. Again, this depends upon the myopic nature of the optimal policy in Theorem 1.

6.3 Best–Price Provisions and Time Consistent Pricing

Applying the revenue equivalence theorem, Proposition 10 shows that the optimal allocation (Theorem 1) can be implemented by a best–price provision. Moreover, the best–price provision is time consistent so the firm need not commit to a sequence of prices at time 0, so long as they can promise to honour the best price agreement.[11]

A *best–price provision* works as follows. In each period the firm announces a price p_t^{BP}. If a consumer buys in period t and the price then falls, they are then given a rebate equal to the difference of the prices. In each subsequent period s they are given a rebate equal to $\min\{p_t^{\text{BP}}, \ldots, p_s^{\text{BP}}\} - \min\{p_t^{\text{BP}}, \ldots, p_{s-1}^{\text{BP}}\}$. In discounted terms, the consumer purchasing in period t pays

$$\sum_{s=t}^{T} (\Delta_s - \Delta_{s+1}) \min\{p_t^{\text{BP}}, \ldots, p_s^{\text{BP}}\}$$

Proposition 10. *Suppose A1 holds. Then the firm's optimal policy under a best–price provision is to set $p_t^{BP} = M_t^{-1}(0)$ inducing the same allocation and profits as Theorem 1. This policy is time consistent.*

Proof. A customer of type (θ, t) choose $\tau(\theta, t)$ to maximise utility,

$$u_t(\theta) = \max_{\tau \geq t} \mathcal{E} \left[\sum_{s=\tau}^{T} (\Delta_s - \Delta_{s+1}) [\theta - \min\{p_\tau^{\text{BP}}, \ldots, p_s^{\text{BP}}\}] \right]$$

It is simple to verify that the agent's optimal policy is to purchase at the first date after t such that $\theta \geq p_\tau^{\text{BP}}$. Therefore consumer's purchasing rules are characterised by Lemma 1. Profits

[11] Butz (1990) reaches a similar conclusion in a model with declining demand and resale. When demand is allowed to increase, however, the introduction of resale means that the best–price provision may not be time consistent, as shown in Section 6.3.

are given by equation (3.4) so the optimal cutoffs are given by Theorem 1 and implemented by setting $p_t^{\mathrm{BP}} = \theta_t^*$.

To prove time consistency, let the time consistent cutoffs be denoted by θ_t^{BP}. At each point in time t, the firm chooses p_t^{BP} to maximise profits from period $s \geq t$,

$$\Pi_{\geq t}^{\mathrm{BP}} = \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \sum_{s=t}^{T} \sum_{r=1}^{s} \Delta_s \mathbf{1}_{\tau(\theta,r)=s} m_r(\theta)\, d\theta\right]$$

Since $\theta_t^{\mathrm{BP}} = p_t^{\mathrm{BP}}$, we can also think of the firm choosing θ_t^{BP} directly. However notice that the difference

$$\Pi - \Pi_{\geq t}^{\mathrm{BP}} = \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \sum_{s=1}^{t-1} \sum_{r=1}^{s} \Delta_s \mathbf{1}_{\tau(\theta,r)=s} m_r(\theta)\, d\theta\right]$$

is independent of θ_t^{BP}, so the choice of θ_t^{BP} also maximises total profits, Π. □

7 Resale and Renting

So far it has been assumed there is no resale. This is reasonable if the good is a one–time experience or suffers from high transactions costs. This section analyses the opposite case: perfect resale.

Each period t the consumer obtains discounted rental utility $(\Delta_t - \Delta_{t+1})\theta$ where $\Delta_{T+1} = 0$. As before, demand $f_t(\theta)$ enters the market each period, where $f_t(\theta)$ and Δ_t are $\mathcal{F}_t$–adapted. Throughout this section assume that A2 holds.

Consider two alternative policies:

1. The firm rents the good at price $\{R_t\}$ each period.

2. The firm sells the good, committing to a price schedule $\{p_t^R\}$, where perfect resale is possible and the seller is allowed to buy goods back.

Bulow (1982) and Butz (1990) show these policies induce the same profits and utilities.

Theorem 3. *Suppose A2 holds and the monopolist either rents the good or they sell the good and allow resale. Then the profit–maximising cutoffs are given by $\theta_t^R = m_{\leq t}^{-1}(0)$.*

Proof. Both policies induce allocations of the form $\{\theta : \theta \geq \theta_t^R\}$ for some $\theta_t^R \in [\underline{\theta}, \overline{\theta}]$. When renting (policy 1) each period can be treated separately, so at time t the good should be allocated to $\{\theta : m_{\leq t}(\theta) \geq 0\}$. Under A2 this is implemented by setting $\theta_t^R = m_{\leq t}^{-1}(0)$. The revenue equivalence theorem implies that the optimal cutoffs are identical when the firm commits to a sequence of sale prices and allows resale (policy 2). □

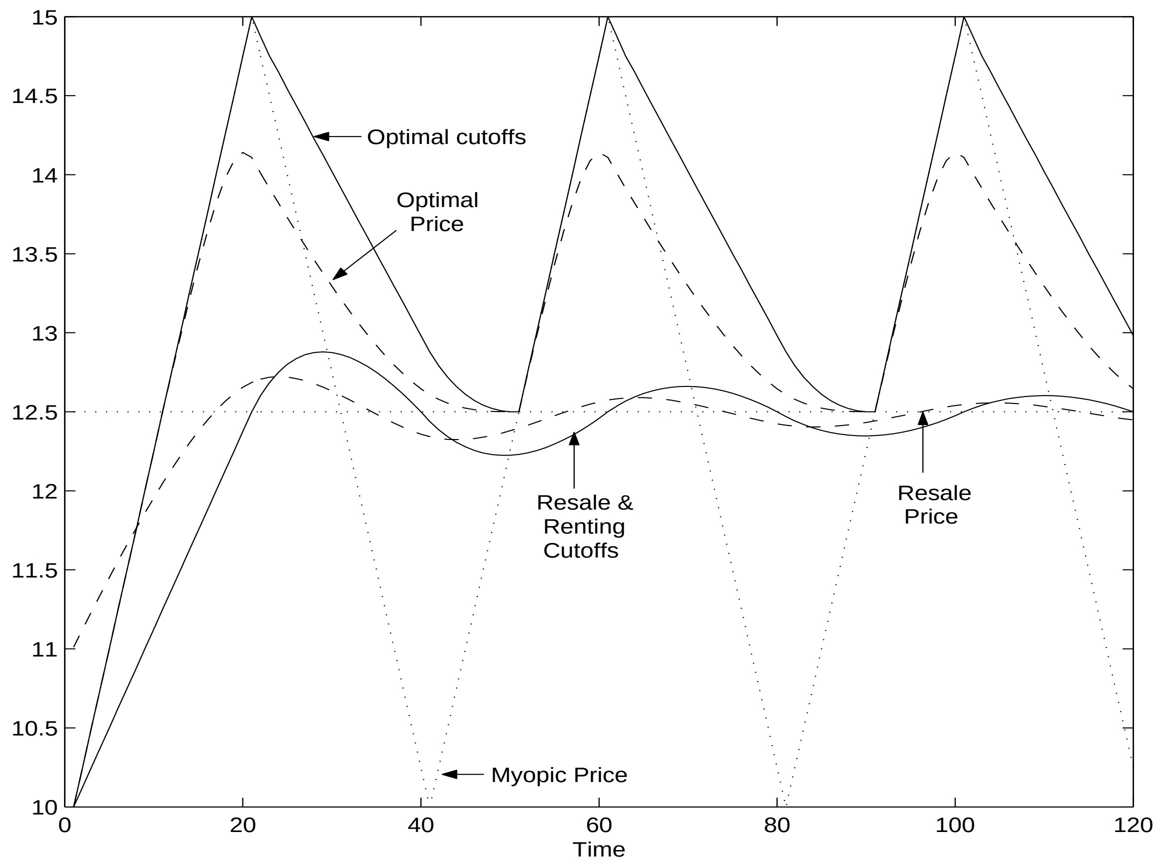

Figure 5: Resale

When the monotonicity assumption, A2, fails the firm should simply iron $m_{\leq t}(\theta)$ $(\forall t)$.

The rental price under the optimal strategy is $R_t = (1-\delta_t)m_{\leq t}^{-1}(0)$. The optimal price path with resale, p_t^R, can be derived from the AR(1) system,

$$(\theta_t^R - p_t^R) = \mathcal{E}[(\theta_t^R - p_{t+1}^R)\delta_{t+1}|\mathcal{F}_t]$$

Hence the resale price is the geometric sum of future rental values, as given by equation (5.1). Hence, if limits exist, the resale price converges to the average–demand price, $p^A := \lim_{t\to T} m_{\leq t}^{-1}(0)$. After enough time the resale market grows very large and the firm loses the ability to discriminate.

7.1 The Effect of Resale

Figure 5 can be used to assess the effect of resale. With resale, troughs and peaks are treated symmetrically, and the cycles decrease in amplitude as the size of the resale market engulfs new production. In the limit, the resale price converges to the average–demand price. Without resale, price cycles are highly asymmetric and are stationary. The reason for this difference is that with resale a low valuation consumer may buy if they anticipate the price to rise, and so all previous generations remain active. In contrast without resale only the high demand generations remain active.

Proposition 11. *Suppose A2 holds. Then cutoffs and profits are lower with resale than without resale. Moreover, resale has no effect on allocations if and only if demand is weakly decreasing with probability 1.*

Proof. (a) Since $\{1, \dots, t\} \in \mathcal{A}(t)$, $\theta_t^R \le \theta_t^*$ by Theorem 2.

(b) Using Theorem 1, profits without resale are

$$\Pi = \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \sum_{s=1}^{T} \Delta_s M_s^+(\theta) d\theta\right] = \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \sum_{t=1}^{T} (\Delta_t - \Delta_{t+1}) \sum_{s=1}^{t} M_s^+(\theta)\, d\theta\right]$$

changing the order of summation and noting $\Delta_s = \sum_{t=s}^{T}(\Delta_t - \Delta_{t+1})$. Profits with resale are

$$\begin{aligned}
\Pi^R &= \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \sum_{t=1}^{T} (\Delta_t - \Delta_{t+1}) m_{\le t}^+(\theta)\, d\theta\right] \\
&= \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \sum_{t=1}^{T} (\Delta_t - \Delta_{t+1}) \left[\max\left\{0, M_t(\theta) + \sum_{s=1}^{t-1} M_s^+(\theta)\right\}\right] d\theta\right] \\
&\le \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \sum_{t=1}^{T} (\Delta_t - \Delta_{t+1}) \left[\max\{0, M_t(\theta)\} + \sum_{s=1}^{t-1} \max\{0, M_s^+(\theta)\}\right] d\theta\right] \\
&= \mathcal{E}\left[\int_{\underline{\theta}}^{\overline{\theta}} \sum_{t=1}^{T} (\Delta_t - \Delta_{t+1}) \left[\sum_{s=1}^{t} M_s^+(\theta)\right] d\theta\right] = \Pi
\end{aligned}$$

where the second line uses equation (4.2) and the third uses Jensen's inequality.

(c) Proposition 3 shows $\theta_t^* = m_{\le t}^{-1}(0)$ if and only if $m_{\le t-1}^{-1}(0) \ge m_t^{-1}(0)$ a.e.. ☐

With resale Coase (1972) and Bulow (1982) argue that the renting achieves the same profits as selling, as shown in Theorem 3. Without resale, Proposition 11 shows renting can achieve the same profits as selling if and only if demand is weakly decreasing.

With linear demand curves (Example 5 in Section 5.4) the quantity sold each period is unaffected by resale. Resale then increases allocative efficiency and welfare (see Appendix A.3). More generally, the welfare effect of introducing resale is ambiguous, as shown by Example 8.

Example 8. Suppose $T = 2$ and δ is constant. (a) Suppose $\theta_1 = 1$ with mass 1 and $\theta_2 \sim U[0, 4]$. With resale cutoffs are $(\theta_1^R, \theta_2^R) = (1, 1)$ yielding welfare $(8 + 15\delta)/8$. Without resale cutoffs are $(\theta_1^*, \theta_2^*) = (1, 2)$ and welfare is $(8 + 12\delta)/8$, lower than with resale. (b) Suppose $\theta_1 \sim U[0, 2]$ and $\theta_2 = 2$ with mass 1. With resale cutoffs are $(\theta_1^R, \theta_2^R) = (1, 2)$ yielding welfare $(6 + 10\delta)/8$. Without resale cutoffs are $(\theta_1^*, \theta_2^*) = (1, 2)$ and welfare is $(6 + 16\delta)/8$, higher than with resale. $\triangle$

With linear demand curves, resale increases welfare, reduces profits and therefore increases consumer surplus. However, it is not the case that all consumers are better off. To see this consider a two–period model with increasing demand. The first (low–demand) cohort prefers no resale since they then avoid being pooled with the second (high–demand) cohort. For the same reason the second generation prefers resale.[12] This idea can be generalised to the following result, where $u_t(\theta)$ is utility of type (θ, t) without resale, and $u_t^R(\theta)$ the utility with resale (and when renting).

Proposition 12. *Suppose A2 holds and consider the utility of generation t.*
(a) If $m_s^{-1}(0)$ is decreasing in s a.e. for $s \le t$, then $u_t(\theta) \ge u_t^R(\theta)$ a.e..
(b)If $m_s^{-1}(0)$ is decreasing in s a.e. for $s \ge t$, then $u_t(\theta) \le u_t^R(\theta)$ a.e..

Proof. With resale utility is

$$u_t^R(\theta) = \mathcal{E}\left[\int_{\underline{\theta}}^{\theta} \sum_{s=t}^{T} (\Delta_s - \Delta_{s+1}) \mathbf{1}_{x \ge \theta_s^R} \, dx\right] \tag{7.1}$$

Without resale utility (3.1) is

$$u_t(\theta) = \mathcal{E}\left[\int_{\underline{\theta}}^{\theta} \sum_{s=t}^{T} (\Delta_s - \Delta_{s+1}) \mathbf{1}_{x \ge \theta^*(s;t)} \, dx\right] \tag{7.2}$$

(a) Consider generation t and suppose $m_s^{-1}(0)$ is decreasing in s for $s \le t$. Pick $t' \ge t$, where $\theta^*(t'; t) = \min\{m_{\overline{A}(t)}^{-1}(0), \ldots, m_{\overline{A}(t')}^{-1}(0)\}$. Decreasing demand for $s \le t$ implies that $\overline{A}(t) = \{1, \ldots, t\}$ so $\cup_{s=t}^{t'} \overline{A}(s) = \{1, \ldots, t'\}$. Using Lemma 2 pick a subset of periods $\{t_1, \ldots, t_n\} \subset \{t, \ldots, t'\}$ such that $\cap_i \overline{A}(t_i) = \emptyset$ and $\cup_i \overline{A}(t_i) = \{1, \ldots, t'\}$. Since $m_{\{1,\ldots,t'\}}(\theta) = \sum_i m_{\overline{A}(t_i)}(\theta)$, Lemma 3 implies that $\theta_{t'}^R = m_{\{1,\ldots,t'\}}^{-1}(0) \ge \min_i m_{\overline{A}(t_i)}^{-1}(0) = \theta(t'; t)$. Equations (7.1) and (7.2) imply $u_t(\theta) \ge u_t^R(\theta)$.

(b) Consider period s and $s + 1$ where $m_s^{-1}(0) \ge m_{s+1}^{-1}(0)$. If $\overline{A}(s + 1) = \{s + 1\}$ then $\theta_s^* \ge m_s^{-1}(0) \ge m_{s+1}^{-1}(0) = \theta_{s+1}^*$, contradicting the fact that $s \notin \overline{A}(s + 1)$. We must therefore have $s \in \overline{A}(s + 1)$ which implies $\theta_s^* \ge \theta_{s+1}^*$.

[12] Naive intuition might suggest that first (low–demand) generation are better off under resale since they have the option to sell to second (high–demand) generation. However this is incorrect: the firm knows the first generation will resell and raises prices in the first period. Hence the agents with relatively low valuations, who buy in period 1 and resell in period 2, exert a negative externality on the high valuation agents from the first generation who buy and never resell.

Consider generation t and suppose $m_s^{-1}(0)$ is decreasing in s for $s \geq t$. Then θ_s^* decreases in s and, for any $s \geq t$, $\theta^*(s;t) = \theta_s^* \geq \theta_s^R$. Equations (7.1) and (7.2) imply $u_t(\theta) \leq u_t^R(\theta)$. $\square$

With demand cycles Proposition 5 shows that without resale the optimal price exceeds the average–demand price. In contrast, the optimal price with resale converges to the average–demand price. Eventually all consumers must therefore prefer resale.[13]

Proposition 13. *Suppose demand follows K deterministic cycles of length $T < \infty$, and A2 holds. Then* $\lim_{t \to KT}[u_t^R(\theta) - u_t(\theta)]/\Delta_t \geq 0$ *and* $\lim_{t \to KT}[W_t^R - W_t]/\Delta_t \geq 0$.

Proof. The average–demand price is $p^A = m_{\{1,\dots,T\}}^{-1}(0)$. After the first cycle $\theta_t^s \geq p^A$. From (7.1) and (7.2) the difference in utility for type (θ, t) thus satisfies

$$\lim_{t \to KT}[u_t^R(\theta) - u_t(\theta)] \geq \lim_{t \to KT} \sum_{s=t}^{KT} \left(\frac{\Delta_s}{\Delta_t} - \frac{\Delta_{s+1}}{\Delta_t} \right) \int_{\underline{\theta}}^{\theta} \left[\mathbf{1}_{x \geq \theta_s^R} - \mathbf{1}_{x \geq p^A} \right] dx = 0$$

since $\lim_{t \to KT} \theta_t^R = p^A$ and every convergent sequence is cauchy. Then apply Proposition 5. Similarly, for the welfare calculation:

$$\lim_{t \to KT}[W_t^R - W_t] \geq \lim_{t \to KT} \sum_{s=t}^{KT} \left(\frac{\Delta_s}{\Delta_t} - \frac{\Delta_{s+1}}{\Delta_t} \right) \int_{\underline{\theta}}^{\overline{\theta}} \left[\mathbf{1}_{x \geq \theta_s^R} - \mathbf{1}_{x \geq p^A} \right] \theta \, dF_t = 0.$$

$\square$

7.2 Best–Price Provisions and Resale

Section 6.3 shows that without resale a best–price provision can achieve the same profits as committing to a sequence of prices in a time–consistent manner. Butz (1990) argues that a best–price provision is also time consistent when resale is allowed. This paper, however, assumes that prices fall over time. In contrast, Example 9 that when prices can increase a best–price scheme is *not* time consistent.

Example 9. Suppose $T = 2$, δ is constant and A2 holds. If demand is decreasing ($m_1^{-1}(0) \geq m_2^{-1}(0)$) the subgame perfect outcome is $\theta_1^* = m_1^{-1}(0)$ and $\theta_2^* = m_{\{1,2\}}^{-1}(0)$. If demand is not decreasing ($m_1^{-1}(0) \leq m_2^{-1}(0)$) the subgame perfect outcome is $\theta_1^* \geq m_1^{-1}(0)$ and $\theta_2^* < m_{\{1,2\}}^{-1}(0)$. Hence the commitment solution is time consistent under a best–price policy if and only if demand is decreasing. See Appendix A.4 for a proof. $\triangle$

When demand is increasing the commitment price p_t^R increases over time. A little extra production in the second period then lowers the price below p_2^R but does not lead to rebates. Hence the firm does not internalise the effect of this extra production on its first period self.

[13] Using discounted utility, Proposition 13 is trivial.

The firm can, however, achieve the maximal profit from Theorem 3 in a time consistent manner through renting or a price–updating policy. *Price–updating* works as follows: In period t the firm chooses a price p_t^{PU} and agents choose whether to purchase or not. In each subsequent period s an agent who owns the good is asked to pay $p_{s+1}^{\mathrm{PU}} - p_s^{\mathrm{PU}}$, so decreasing the price leads to rebates, while increasing the price leads to surcharges.

Proposition 14. *Suppose there is resale and A2 holds. The firm's optimal policy under a price–updating scheme is to set $p_t^{PU} = m_{\leq t}^{-1}(0)$ inducing the same allocation and profits as Theorem 3. This policy is time consistent.*

Proof. By purchasing the good in period t and reselling it at time $t+1$ an agent must pay $(1-\delta_{t+1})p_t^{\mathrm{PU}}$, taking into account the price updating and capital gain. The scheme is thus identical to renting with a rental price $(1-\delta_{t+1})p_t^{\mathrm{PU}}$ and is thus time consistent. □

8 Conclusion

This paper characterises the monopolist's pricing strategy under demand variation. When new demand grows stronger the price rises quickly, unhindered by the presence of old consumers. In contrast, when new demand becomes weaker, the price falls slowly as customers delay their purchases. This asymmetry between rises and falls, leads to an increase in the price level which harms consumers and reduces welfare below that induced by a monopolist who charged the optimal monopoly price against the average level of demand.

There are a number of papers that analyse how markups change over the business cycle (e.g. Rotemberg and Woodford (1999)). This paper, however, has said little about how prices and quantities relate. Example 5 in Section 5.4 shows that with linear demand curves the optimal quantity equals the myopic quantity. That is, sales are pro-cyclical and consumer's ability to delay their purchases has no affect on the path of sales. On the other hand, sales are counter-cyclical in the back–to–school example in the Introduction. Each September there is an influx of price sensitive agents, leading to a price reduction.

In this paper the movement of the markup over time is determined by marginal revenue. In this sense one can separate how markups move over time and the level of output. Hence if periods of high and low output are viewed as booms and recessions, there is no necessary link between markups and the cycle. To make predictions of this sort one by needs to further restrict the possible sequence of demand curves, either through theoretical or empirical means.

A Omitted Material

A.1 Properties of the Consumer's Maximisation Problem

Lemmas 4 establishes some properties of the agent's utility maximisation problem (2.1).

Denote the set of maximisers by $\hat{\tau}(\theta,t)$. Comparing two stopping rules, let $\tau_H \geq \tau_L$ if $\tau_H(\omega) \geq \tau_L(\omega)$ (a.e.–$\omega \in \Omega$). Comparing two sets of stopping rules, $\hat{\tau}_H \geq \hat{\tau}_L$ in strict set order if $\tau' \in \hat{\tau}_H$ and $\tau'' \in \hat{\tau}_L$ imply that $\tau' \vee \tau'' \in \hat{\tau}_H$ and $\tau' \wedge \tau'' \in \hat{\tau}_L$.

Lemma 4. *The consumer's optimal purchase decision has the following properties:*
(a) $\hat{\tau}(\theta,t)$ is a nonempty sublattice and contains a greatest and least element.
(b) Every selection from $\hat{\tau}(\theta,t)$ is decreasing in θ.
(c) If $t_H \geq t_L$ then $\hat{\tau}(\theta,t_H) = \hat{\tau}(\theta,t_L) \cap \{t_H,\ldots,T\}$, in states where the latter is nonempty. Hence $\hat{\tau}(\theta,t)$ is increasing in t in strict set order.

Proof. (a) Nonemptiness of $\hat{\tau}(\theta,t)$ follows from Klass (1973, Theorem 1). The set of optimal rules is characterised by Klass (1973, Theorem 6) and contains a least and greatest element. The least element can be found by using the rule: stop when current utility is weakly greater than the continuation utility (Chow, Robbins, and Siegmund (1971, Theorem 4.2)). Since the set of purchasing times is a lattice and $u_t(\theta)$ is modular in τ, the set of maximisers is a sublattice by Topkis (1998, Theorem 2.7.1).

(b) $u_t(\theta)$ has strictly decreasing differences in (θ,τ) since $\delta_t < 1$, and is modular in τ. Hence every selection is decreasing by Topkis (1998, Theorem 2.8.4).

(c) If in states $A \in \mathcal{F}$, $\tau \in \hat{\tau}(\theta,t_L) \cap \{t_H,\ldots,T\}$ maximises the utility of (θ,t_L) it must also maximise the utility of (θ,t_H) who has a smaller choice set. □

Let $\tau^*(\theta,t)$ be the least element from $\hat{\tau}(\theta,t)$. If $\tau \in \hat{\tau}(\theta,t)$ then $\tau^* = \tau$ (a.e.–θ) for any state ω, by Lemma 4(b). Consequently we can assume the consumer chooses purchasing rule $\tau^*(\theta,t)$ without loss of generality. Several properties of $\tau^*(\theta,t)$ are described in Lemma 1.

A.2 Optimal Mechanism

This section derives the optimal mechanism where agents have private information about their valuation θ and birth date t. The optimum can be implemented by a price mechanism and is hence the same as that analysed in the paper. For simplicity, this section assumes constant discounting and known demand.

Suppose there is an agent with type $(\theta,t) \in [\underline{\theta},\overline{\theta}] \times \{1,\ldots,T\}$ drawn from density $f(\theta,t)$. Consider the direct revelation mechanism. Agent (θ,t) reports type $(\tilde{\theta},\tilde{t})$. The mechanism is then defined by a purchase decision $\tau(\tilde{\theta},\tilde{t}) \in \{1,\ldots,T,\infty\}$ and a payment $y(\tilde{\theta},\tilde{t}) \in \Re$. Agent

(θ, t) is has utility

$$\begin{aligned} u(\tilde{\theta}, \tilde{t}|\theta, t) &= \theta \Delta_{\tau(\tilde{\theta}, \tilde{t})} - y(\tilde{\theta}, \tilde{t}) \quad \text{if} \quad \tau(\tilde{\theta}, \tilde{t}) \geq t \\ &= -\infty \qquad\qquad\qquad\qquad \text{if} \quad \tau(\tilde{\theta}, \tilde{t}) < t \end{aligned}$$

The [IR] constraint states that

$$u(\theta, t|\theta, t) \geq 0$$

The [IC] constraint states that

$$u(\theta, t|\theta, t) \geq u(\tilde{\theta}, \tilde{t}|\theta, t)$$

The analysis can be extended to N agents whose types are IID draws from $f(\theta, t)$. However the assumption of constant costs implies that each agent can be treated separately (see Segal (2003)).[14] This means there is no reason to make price at time t depend upon past sales.

Lemma 5. *A mechanism $\langle \tau, y \rangle$ is incentive compatable and individually rational if and only if the following hold:*
[$\underline{IR}$] $u(\underline{\theta}, t_L|\underline{\theta}, t_L) \geq u(\underline{\theta}, t_H|\underline{\theta}, t_H) \geq 0$. If $\tau(\underline{\theta}, t_L) \geq t_H$ then $u(\underline{\theta}, t_L|\underline{\theta}, t_L) = u(\underline{\theta}, t_H|\underline{\theta}, t_H) \geq 0$, for $t_H \geq t_L$.
[ICFOC] Utility is given by (3.1).
[positivity] $\tau(\theta, t) \geq t$.
[θ–monotonicity] $\tau(\theta, t)$ is decreasing in θ.
[non–discrimination] If $\tau(\theta, t_L) \geq t_H$ then $\tau(\theta, t_L) = \tau(\theta, t_H)$, for $t_H \geq t_L$ (θ–a.e.).

Proof. **Necessity.** Fix $t_H \geq t_L$. If $\tau(\theta, t_L) \geq t_H$ [$\underline{\text{IR}}$] follows from [IR] and the [IC] constraint for (θ, t_L) and (θ, t_H). If $\tau(\theta, t_L) < t_H$ [$\underline{\text{IR}}$] follows from [IR] and the [IC] constraint for (θ, t_L). [positivity] follows from [IR]. [ICFOC] and [θ–monotonicity] follow from [IC] with $\tilde{t} = t$.

To show [non–discrimination] suppose $t_H \geq t_L$ and $\tau(\theta, t_H) \neq \tau(\theta, t_L) \geq t_H$ for $\theta \in B$, where B has positive measure. The [IC] constraints for (θ, t_H) and (θ, t_L) imply that $u(\theta, t_H|\theta, t_H) = u(\theta, t_L|\theta, t_L)$. Pick an open interval $B' \subset B$ of positive measure (Royden (1988, p.63)). For any $\theta \in B'$ [ICFOC] implies

$$\int_{\inf(B')}^{\theta} \Delta_{\tau(x, t_L)} - \Delta_{\tau(x, t_H)} \, dx = 0$$

Therefore $\Delta_{\tau(x, t_L)} = \Delta_{\tau(x, t_H)}$ a.e. (Royden (1988, p.105)) and $\Delta_t < 1$ implies $\tau(x, t_L) = \tau(x, t_H)$ a.e.. Thus we have a contradiction.

Sufficiency. [IR] follows from [$\underline{\text{IR}}$] and [ICFOC]. To verify [IC] we consider four cases. While the analysis of each is quite similar, all four are presented for completeness.

[14] Segal (2003) also shows the assumption that there is a continuum of agents is also sufficient to treat every agent separately.

First suppose (θ, t) copies type (θ_L, t_L) where $\theta \geq \theta_L$ and $t \geq t_L$. If $\tau(\theta_L, t_L) < t$ then [$\underline{\text{IR}}$] implies [IC]. Instead suppose $\tau(\theta_L, t_L) \geq t$. [$\theta$–monotonicity] implies $\tau(x, t_L) \geq t$ for $x \leq \theta_L$. [$\underline{\text{IR}}$] implies $u(\underline{\theta}, t|\underline{\theta}, t) = u(\underline{\theta}, t_L|\underline{\theta}, t_L)$ and [non–discrimination] implies $\tau(x, t) = \tau(x, t_L)$ for $x \leq \theta_L$. From [ICFOC], $u(\theta_L, t|\theta_L, t) = u(\theta_L, t_L|\theta_L, t_L)$ and

$$
\begin{aligned}
u(\theta, t|\theta, t) &= u(\theta_L, t_L|\theta_L, t_L) + \int_{\theta_L}^{\theta} \Delta_{\tau(x,t)}\, dx \\
&\geq u(\theta_L, t_L|\theta_L, t_L) + \int_{\theta_L}^{\theta} \Delta_{\tau(\theta_L,t_L)}\, dx \\
&= u(\theta_L, t_L|\theta_L, t_L) + [u(\theta_L, t_L|\theta, t) - u(\theta_L, t_L|\theta_L, t_L)] \\
&= u(\theta_L, t_L|\theta, t)
\end{aligned}
$$

where the second line uses [θ–monotonicity] and [non–discrimination].

Second, suppose (θ, t) copies type (θ_H, t_L) where $\theta \leq \theta_H$ and $t \geq t_L$. If $\tau(\theta_H, t_L) < t$ then [$\underline{\text{IR}}$] implies [IC]. Instead suppose $\tau(\theta_H, t_L) \geq t$. [$\theta$–monotonicity] implies $\tau(x, t_L) \geq t$ for $x \leq \theta_H$. [$\underline{\text{IR}}$] implies $u(\underline{\theta}, t|\underline{\theta}, t) = u(\underline{\theta}, t_L|\underline{\theta}, t_L)$ and [non–discrimination] implies $\tau(x, t) = \tau(x, t_L)$ for $x \leq \theta_H$. From [ICFOC], $u(\theta, t|\theta, t) = u(\theta, t_L|\theta, t_L)$ and

$$
\begin{aligned}
u(\theta, t|\theta, t) &= u(\theta_H, t_L|\theta_H, t_L) - \int_{\theta}^{\theta_H} \Delta_{\tau(x,t_L)}\, dx \\
&\geq u(\theta_H, t_L|\theta_H, t_L) - \int_{\theta}^{\theta_H} \Delta_{\tau(\theta_H,t_L)}\, dx \\
&= u(\theta_H, t_L|\theta_H, t_L) - [u(\theta_H, t_L|\theta_H, t_L) - u(\theta_H, t_L|\theta, t)] \\
&= u(\theta_H, t_L|\theta, t)
\end{aligned}
$$

where the second line uses [θ–monotonicity].

Third, suppose (θ, t) copies type (θ_L, t_H) where $\theta \geq \theta_L$ and $t \leq t_H$. [non–discrimination] implies $\tau(\theta, t_L) \leq \tau(\theta, t_H)$. Hence [$\underline{\text{IR}}$] and [non–discrimination] imply that $u(\theta_L, t|\theta_L, t) \geq u(\theta_L, t_H|\theta_L, t_H)$. From [ICFOC],

$$
\begin{aligned}
u(\theta, t|\theta, t) &\geq u(\theta_L, t_H|\theta_L, t_H) + \int_{\theta_L}^{\theta} \Delta_{\tau(x,t)}\, dx \\
&\geq u(\theta_L, t_H|\theta_L, t_H) + \int_{\theta_L}^{\theta} \Delta_{\tau(\theta_L,t_H)}\, dx \\
&= u(\theta_L, t_H|\theta_L, t_H) + [u(\theta_L, t_H|\theta, t) - u(\theta_L, t_H|\theta_L, t_H)] \\
&= u(\theta_L, t_H|\theta, t)
\end{aligned}
$$

where the second line uses [θ–monotonicity] and [non–discrimination].

Fourth, suppose (θ, t) copies type (θ_H, t_H) where $\theta \leq \theta_H$ and $t \leq t_H$. [$\underline{\text{IR}}$] and [non–

discrimination] imply that $u(\theta,t|\theta,t) \geq u(\theta,t_H|\theta,t_H)$. From [ICFOC],

$$\begin{aligned} u(\theta,t|\theta,t) &\geq u(\theta_H,t_H|\theta_H,t_H) - \int_\theta^{\theta_H} \Delta_{\tau(x,t_H)}\,dx \\ &\geq u(\theta_H,t_H|\theta_H,t_H) - \int_\theta^{\theta_H} \Delta_{\tau(\theta_H,t_H)}\,dx \\ &= u(\theta_H,t_H|\theta_H,t_H) - [u(\theta_H,t_H|\theta_H,t_H) - u(\theta_H,t_H|\theta,t)] \\ &= u(\theta_H,t_H|\theta,t) \end{aligned}$$

where the second line uses [θ–monotonicity]. □

A.3 Linear Demand: Example 5

The model has particularly clean predictions when demand is linear: $f_t(\theta) = 1$ on $[0, 2b_t]$, where $b_t \in [10, 20]$. This latter constraint means that any sum of collection of marginal revenues is strictly quasi–increasing and continuous on $[0, 20]$. Consequently the results that use A2 also hold with linear demand.

Allocations, quantities and payoffs under optimal policy. Marginal revenue is $m_t(\theta) = 2(\theta - b_t)$, so Theorem 2 implies the optimal cutoff is

$$\theta_t^* = \frac{1}{|\overline{A}(t)|} \sum_{s\in\overline{A}(t)} b_s =: b_{\overline{A}(t)}$$

where $|\overline{A}(t)|$ is the number of elements in the upper active set.

To derive quantity sold let us use the following construction (which is also used in Proposition 12). Using Lemma 2 pick a subset of periods $\{t_1,\ldots,t_n\} \subset \{1,\ldots,t-1\}$ such that $\cap_i \overline{A}(t_i) = \emptyset$ and $\cup_i \overline{A}(t_i) = \{1,\ldots,t-1\}$. Then for generation $s \in \overline{A}(t_i)$, $\theta^*(t-1;s) = b_{\overline{A}(t_i)}$. By Lemma 2, we then have $\overline{A}(t) = \cap_{i=k}^n \overline{A}(t_i) \cup \{t\}$ for some $k \in \{1,\ldots,n\}$. In period t sales are made to agents born after period t_{k-1} and the quantity sold is

$$
\begin{aligned}
Q_t &= \sum_{t_{k-1}<s\leq t-1} [\theta^*(t-1;s) - \theta^*(t;s)] + [2b_t - \theta^*(t;t)] \\
&= \sum_{i=k}^{n} \sum_{s\in \overline{A}(t_i)} [b_{\overline{A}(t_i)} - b_{\overline{A}(t)}] + [2b_t - b_{\overline{A}(t)}] \\
&= \left[\sum_{i=k}^{n} \sum_{s\in \overline{A}(t_i)} b_{\overline{A}(t_i)} + b_t\right] - \left[\sum_{s\in \overline{A}(t)} b_{\overline{A}(t)}\right] + b_t \\
&= \left[\sum_{s\in \overline{A}(t)} b_s\right] - \left[\sum_{s\in \overline{A}(t)} b_s\right] + b_t = b_t
\end{aligned}
$$

Welfare from sales in period t equals,

$$\Delta_t \sum_{s\leq t} \int_{\theta^*(t;s)}^{\theta^*(t-1;s)} \theta\, d\theta = \frac{\Delta_t}{2}\left[\sum_{i=k}^{n} |A(t_i)| b_{A(t_i)}^2 + 4b_t^2 - |\overline{A}(t)| b_{\overline{A}}^2\right]$$

Allocations, quantities and payoffs under complete discrimination. Suppose the firm can completely discriminate between generations, as in Example 2. In period t an agent buys if they are from generation t and $\theta \geq b_t$. Hence quantity b_t is sold in period t. This is the same as under the optimal policy (see above).

Under complete discrimination welfare from sales in period t equals,

$$\Delta_t \int_{b_t}^{2b_t} \theta\, d\theta = \frac{3}{2}\Delta_t b_t^2$$

This is less than welfare without complete discrimination since Jensen's inequality implies

$$\sum_{i=k}^{n} \frac{|A(t_i)|}{|\overline{A}(t)|} b_{A(t_i)}^2 + \frac{1}{|\overline{A}(t)|} b_t^2 \geq b_{\overline{A}(t)}^2$$

Hence the welfare generated at each point in time is larger without discrimination, and welfare across all periods must also be larger, independent of the discount rate. Profit is larger and welfare lower under discrimination, so consumer surplus must also be lower.

Allocations, quantities and payoffs under resale/renting. Suppose the good can be resold.

Theorem 3 implies that the time t cutoff is given by

$$\theta_t^R = \frac{1}{t}\sum_{s\le t} b_s$$

The quantity sold up to period t is $\sum_{s\le t}(2b_t - \theta_t^R) = \sum_{s\le t} b_s$. Hence quantity b_t is sold in period t. This is the same as without resale (see above).

Welfare with resale in period t is

$$(\Delta_t - \Delta_{t+1})\sum_{s\le t}\int_{\theta_t^R}^{2b_t}\theta\, d\theta = (\Delta_t - \Delta_{t+1})\sum_{s\le t}\left[2b_t^2 - [\theta_t^R]^2\right]$$

In rental terms, welfare in period t without resale is

$$(\Delta_t - \Delta_{t+1})\sum_{s\le t}\int_{\theta^*(t;s)}^{2b_t}\theta\, d\theta = (\Delta_t - \Delta_{t+1})\sum_{s\le t}\left[2b_t^2 - [\theta^*(t;s)]^2\right]$$

Welfare is then greater with resale since

$$\sum_{s\le t}[\theta^*(t;s)]^2 \ge \frac{1}{t}\left[\sum_{s\le t}\theta^*(t;s)\right]^2 = \frac{1}{t}\left[\sum_{s\le t} b_s\right]^2 = \frac{1}{t}\left[t\theta_t^R\right]^2 = \sum_{s\le t}[\theta_t^R]^2$$

where the first equality uses the fact that the firm sells quantity b_t each period under Theorem 1. Intuitively, welfare is higher with resale since the same quantity is allocated more efficiently. With resale, profits are lower and welfare higher, so consumer surplus is also higher.

A.4 Best–Price Policy with Resale: Example 9

Suppose $T = 2$, δ is constant and A2 holds.[15] For convenience let us drop the "BP" superscripts.

Fix the period 1 price and cutoff (p_1, θ_1^*). In period 2 the firm chooses p_2 inducing cutoff $\theta_2^* = p_2$. p_2 is chosen to maximise

$$\Pi_2 = p_2[\overline{F}_{\{1,2\}}(p_2) - \overline{F}_1(\theta_1^*)] - \overline{F}_1(\theta_1^*)\max\{0, p_1 - p_2\}$$

The optimal choice of p_2 is then given by

$$\begin{aligned} p_2^* &= m_{\{1,2\}}^{-1}(0) \quad \text{if} \quad p_1 \ge m_{\{1,2\}}^{-1}(0) \\ &< m_{\{1,2\}}^{-1}(0) \quad \text{if} \quad p_1 < m_{\{1,2\}}^{-1}(0) \end{aligned} \tag{A.1}$$

[15] A full characterisation of the subgame perfect equilibrium can be obtained from the author.

Thus $p_2^* \leq \max\{p_1, m_{\{1,2\}}^{-1}(0)\}$.

In period 1 consumers' purchasing decision depends upon the current price p_1 and their expectation of the second period price p_2. The utility of agent θ_1^* who buys in period 1 and sells in period 2 is

$$(1-\delta)\theta_t^* - p_1 + \delta p_2 + \delta \max\{0, p_1 - p_2\} \tag{A.2}$$

Since agent θ_1^* must be indifferent between buying and not, setting (A.2) equal to zero yields the first period price

$$p_1 = (1-\delta)\theta_1^* + \delta \max\{\theta_1^*, p_2\} \tag{A.3}$$

Fix θ_1^* and consider two cases. First suppose $\theta_1 \geq m_{\{1,2\}}^{-1}(0)$. Equation (A.3) implies $p_1 \geq \theta_1^* \geq m_{\{1,2\}}^{-1}(0)$. Equation (A.1) implies $\theta_2^* = p_2^* = m_{\{1,2\}}^{-1}(0)$.

Second suppose $\theta_1^* < m_{\{1,2\}}^{-1}(0)$. Since $p_2^* \leq m_{\{1,2\}}^{-1}(0)$, equation (A.3) implies $p_1 < m_{\{1,2\}}^{-1}(0)$. Equation (A.1) implies $\theta_2^* = p_2^* < m_{\{1,2\}}^{-1}(0)$.

Now let us examine the optimal period 1 cutoffs. First, suppose demand is decreasing. The firm can choose $\theta_1^* = m_1^{-1}(0) \geq m_{\{1,2\}}^{-1}(0)$ which induces $\theta_2^* = m_{\{1,2\}}^{-1}(0)$ in period 2. This implements the commitment optimum and is therefore optimal in period 1.

Next, suppose demand is not decreasing. If the firm chooses $\theta_1^* = m_1^{-1}(0) < m_{\{1,2\}}^{-1}(0)$ the optimal period 2 choice is $p_2^* < m_{\{1,2\}}^{-1}(0)$ and the firm overproduces. Hence the commitment optimum is not implementable. It can be shown that the period 2 cutoff θ_2^* is increasing in period 1 cutoff θ_1^*. It follows that the optimal period 1 cutoff satisfies $\theta_1^* > m_1^{-1}(0)$.

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