Assortative matching with explicit search costs *

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Abstract

In this paper, I analyze a decentralized search and matching economy with transferable utility comprised of heterogeneous agents. I explore whether Becker's assortative matching result generalizes to an economy where agents engage in costly search. In an economy with additive search costs, complementarities in joint production (supermodularity of the joint production function) leads to assortative matching. This is in contrast to previous literature, which had shown that in a search economy with discounting, assortative matching may fail even when the joint production function is supermodular.

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1 Introduction and Related Literature

Many market situations are characterized by heterogeneous agents on both sides of a market trying to match with "appropriate" trading partners. Each agent might be thought of as trying to buy or sell a unit of an indivisible good. Examples of markets with these characteristics include the labor market, where workers, who differ in various dimensions, try to match with jobs; the real estate market, where apartments are matched with tenants; or the venture capital market, where projects are matched with financiers.

In a series of seminal papers, Becker (1973, 1974) showed that, in a transferable utility matching market, if there are complementarities in production (supermodularity of the joint production function), then the unique core allocation, and consequently, the unique competitive equilibrium exhibits assortative matching. In other words, individuals sort themselves into matches with alike agents. When the set of types is taken as the unit interval, equilibrium matches occur along the 45° line. In Becker's work, however, there are no search friction and thus no costs involved in finding a matching partner.

Subsequent researchers (e.g., Sattinger (1995), Lu and McAfee (1996) and Shimer and Smith (2000)) have introduced search frictions-such as time costs due to discounting. In a discounted search model, complementarities in joint production are not sufficient for assortative matching. Moreover, conditions sufficient for restoring assortative matching are quite restrictive. For example, the sufficient conditions rule out strictly increasing joint production functions.

Here, I extend Becker's assortative matching insight to a setting where finding a matching partner is costly. In my model, agents search over the infinite horizon for matching partners to maximize their undiscounted payoffs. In each period, agents pay an additive search cost and randomly match with potential partners. If two matched agents reach a mutual agreement, they split their joint surplus according to the Nash Bargaining Solution and permanently leave the market. The distribution of agents in the economy remains in steady state, since the outflow of agents is exactly balanced by an inflow of new agents. My results stand in sharp contrast with previous literature and show that in this setting with additive search costs, complementarities lead to assortative matching. Thus, in a labor market, better-qualified workers match with better jobs, and in a marriage market, more-handsome men marry more-beautiful women – and this occurs even when finding a partner is costly.

Although my paper is the first to focus on transferable utility and additive search costs, a number of previous papers have analyzed matching within a decentralized search framework.

Sattinger (1995), Lu and McAfee (1996) and Shimer and Smith (2000) present models with transferable utility and discounting; Burdett and Coles (1997) and Smith (2002) present models with non-transferable utility and discounting; and Chade (2001) presents a model with non-transferable utility and additive search costs.

Shimer and Smith's (2000) analysis is the most closely related to the present paper. As was previously mentioned, Shimer and Smith show that when the search frictions are modelled as time costs resulting from discounting, the equilibrium does not necessarily involve assortative matching even when the production function is supermodular. In the paper, they also provide sufficient conditions that restore the classical result. Assortative matching can break down in the discounted model because the cost of search is not constant across types, that is, agents of different quality have different valuations for their time.[1] A higher-quality agent has more valuable time than a lower-quality agent has. Consequently, the higher-quality agent may actually end up settling for a match with an agent of even lower quality than the partner chosen by the lower-quality agent.

In Shimer and Smith's (2000) model, the search costs are multiplicative and can differ across types, whereas in the present paper costs are additive and constant for all types. In fact, my argument for assortative matching relies on the Constant Surplus Condition, which asserts that every agent enjoys the same expected surplus in future matches. This condition emerges since the expected surplus from future matches is the benefit of additional search and must, at an optimum, equal the constant cost of search.

The paper proceeds as follows: Section 2 outlines the general model, Section 3 shows that if the joint production function is supermodular, then equilibrium matching is assortative, and Section 4 provides a discussion of results and conclusions. Proofs, that are not included in the main text, are in the Appendix.

2 The Search Economy

Agents in the economy engage in costly search for possible trading partners. Time is taken as discrete. In each period, agents incur a strictly positive, search cost $c > 0$ and meet a potential partner. If both agents agree, then a match is formed, joint production takes

[1] As Shimer and Smith (2000) note in their discussion, the break-up probability is also a "multiplicative" time cost that compounds the effect of discounting. In their model, matched agents do not permanently leave the market, but remain dormant in the economy receiving their per-period payoff until their match is broken-up by an exogenously given probability, and they start searching again. The steady state is maintained through break-up and formation of new matches. For further discussion, see Atakan (2003).

place according to the function f, and the matched agents leave the market. The unmatched agents continue searching for potential matching partners. There is transferable utility and the proceeds from joint production are divided according to the Nash Bargaining Solution.

Population of Agents. The economy is comprised of agents with publicly observable and exogenous types belonging to the unit interval. $\mathbb{P}$ denotes the steady state measure over the set of types, $[0,1]$. Agents, who leave the economy following a successful match, are immediately replaced with identical "clones" and consequently the type distribution remains in steady state. In each period, the probability of meeting an agent whose type is an element of $Z \subset [0,1]$ is just $\int_Z d\mathbb{P}$. In general, I assume that $\mathbb{P}$ is absolutely continuous with respect to the Lebesgue measure (i.e., there is a continuum of types in the economy, and the support of $\mathbb{P}$ is the whole interval). This assumption is not essential; $\mathbb{P}$ having finite support, that is, having finitely many types in the economy, would also suffice for the results.

Trade. Trade between two individuals creates total value, $f(x,y)$. Individuals do not create any output when they are unmatched. The following assumption on f is required to prove that an equilibrium exists.

Assumption 1 *The production function $f(x,y)$ is non-negative, symmetric ($f(x,y) = f(y,x)$), and Lipschitz continuous of modulus k, that is, $|f(x_1,y) - f(x_2,y)| \leq k|x_1 - x_2|$.*

It should be noted that all the requirements of Assumption 1, excluding symmetry, can be discarded if there are finitely many types. The following supermodularity assumption is employed in the proof of assortative matching.

Assumption 2 *The production function $f(x,y)$ is strictly supermodular, that is, if $x_1 > x_2$ and $y_1 > y_2$, then $f(x_1,y_1) + f(x_2,y_2) > f(x_1,y_2) + f(x_2,y_1)$.*

Agent Behavior and Strategies. Let x and y denote agents of types x and y. When x is matched with y, they split their joint surplus $f(x,y)$ according to the Nash Bargaining Solution. The disagreement points for the agents, $v(x)$ and $v(y)$, are their option values of remaining unmatched in the economy and are determined in equilibrium. Consequently, the payoff to x of matching with y is

$$\frac{f(x,y) + v(x) - v(y)}{2}. \tag{1}$$

Each agent receives his or her continuation value $v(x)$ plus one-half of the joint surplus in excess of the disagreement values that the match creates, or, $\frac{f(x,y)-v(x)-v(y)}{2}$.

A strategy for x specifies whether she will accept a match given that she has been paired with y in the current period. The paper assumes that all agents of a certain type use identical, stationary strategies. Furthermore, I assume that strategies depend only on the type of the possible matching partners and are consequently, independent of identity.

A strategy for x is represented by $A(x) \subset [0,1]$, which is the set of agents who x will accept in a match. Also define $B(x) = \{y : x \in A(y)\}$, as the types of agents who will accept x in a match and; $M(x) = A(x) \cap B(x)$, as the types with whom x will consummate a match.

For x using strategy $A(x)$, the per-period reward function, (given that she has been paired up with y who is using strategy $A(y)$), is as follows:

$$\pi(x, y, A(x), A(y)) = \begin{cases} -c \text{ if the match is rejected} \\ -c + \frac{f(x,y)+v(x)-v(y)}{2} \text{ if the match is accepted,} \end{cases} \tag{2}$$

where a match is accepted if $y \in A(x)$ and $x \in A(y)$ and rejected otherwise. For x, who has previously accepted a match, $\pi(x,\cdot,\cdot,\cdot) = 0$. Each agent solves the following optimization problem given that the other agents are using strategies $\{A(y)\}_{y\in[0,1]}$:

$$v(x) = \max_{\hat{A}} \left[\mathbb{E} \sum_{t=0}^{\infty} \pi\left(x, y_t, \hat{A}, A(y_t)\right) \right], \tag{Optimality}$$

where the y_t is drawn according to the steady state measure $\mathbb{P}$ if there was a rejection in period $t-1$, and $\pi(x,\cdot,\cdot,\cdot) = 0$, if x was involved in a match in a previous period.

Market games, such as the one considered here, admit a variety of no-trade equilibria in which x rejects y based on the belief that y intends to reject x - and vice versa. To rule out such equilibria, I assume that players do not employ weakly dominated strategies. Agent x accepts y, if and only if, the value x receives from accepting the match and leaving the game exceeds her value from remaining in the game unmatched. More precisely, if $-c+\frac{f(x,y)+v(x)-v(y)}{2} \geq -c+v(x)$, or alternatively, if $f(x,y)-v(x)-v(y) \geq 0$, then x always accepts to match with y.[2] Similarly, if this inequality holds, (i.e., if $f(x,y)-v(x)-v(y) \geq 0$), then y also will be willing to match with x and consequently the match will be consummated.

[2] Making the announcements of acceptance sequential instead of simultaneous and requiring subgame perfection would justify the assumption that agents never reject positive surplus matches. Alternatively, imposing trembling hand perfectness would also deliver the desired result.

The previous weak inequality can be replaced by a strict inequality without changing the results presented here. Nevertheless, I operate under the additional tie-breaking rule that in the case of an equality, if $f(x,y) - v(x) - v(y) = 0$, then the agent accepts the match. In other words, weakly preferred matches are always accepted. Let

$$s(x,y) = f(x,y) - v(x) - v(y) \tag{3}$$

denote the joint surplus x and y produce, if they match. The preceding discussion implies that, $y \in A(x)$ if and only if $s(x,y) \geq 0$, and $y \in A(x)$ if and only if $x \in A(y)$. Consequently, we have that $A(x) = B(x) = M(x)$.

Search Equilibrium. A search equilibrium is comprised of a function $v : [0,1] \to \mathbb{R}$, which is the maximized value for x of participation in the economy, and a strategy $A(x)$ for each $x \in [0,1]$, such that

(i) Each $A(x)$ solves the optimization problem defined by the Optimality Equation, given that all y behave according to strategy $A(y)$ and the per-period reward function, π, is defined according to v,

(ii) The value function v, satisfies the Optimality Equation for each x, given that all y behave according to strategy $A(y)$ and π is defined according to v.

The following theorem outlines conditions under which a search equilibrium exists and provides some useful properties of the equilibrium.

Theorem 1 *If f satisfies Assumption 1, then a search equilibrium exists and the equilibrium satisfies the following properties:*

(i) The matching sets $M(x) = \{y : s(x,y) \geq 0\}$ form a symmetric, non-empty and compact valued, upperhemicontinuous correspondence,

(ii) The equilibrium value function, $v(x)$, satisfies the following ***Constant Surplus Condition (CSC)*** *for all x:*[3]

$$\mathbb{E}_{y \in M(x)} \left[f(x,y) - v(x) - v(y) \right] = \mathbb{E}_{y \in M(x)} \left[s(x,y) \right] = 2c \tag{CSC}$$

[3] A word on notation: $\mathbb{E}_{y \in M(x)} g(y)$ or $\mathbb{E}_{M(x)} g(y)$ means the expectation over the set $M(x)$ not the conditional expectation given $y \in M(x)$. Specifically, $\mathbb{E}_{M(x)} g(y) = \mathbb{E}[\chi_{M(x)}(y) g(y)]$, where $\chi_{M(x)}(y)$ is the indicator function for the set $M(x)$.

Proof. The proofs for the existence of the search equilibrium and item 1 of Theorem 1 can be found in Atakan (2004) and Shimer and Smith (2000), respectively. The following is the Bellman equation for the optimal stopping problem that each agent solves:

$$v(x) = \mathbb{E}_y \max\left\{-c + \frac{f(x,y) + v(x) - v(y)}{2}, -c + v(x)\right\} = -c + v(x) + \mathbb{E}_{y \in M(x)}[\frac{s(x,y)}{2}]$$

cancelling $v(x)$ and rearranging gives us the **CSC**. ■

3 Assortative Matching

The unique core allocation of the frictionless matching model with complementarities involves perfect assortative matching, that is, in the symmetric model, type x agents match with type y agents if and only if $x = y$. This is also the case in a search model with finite types and negligible search frictions ($\delta \approx 1$ or $c \approx 0$, for an argument, see Atakan (2004).) When search is costly, we can no longer expect perfect assortative matching since agents will inevitably widen the set of agents that they will accept. The following definition, due to Shimer and Smith (2000), generalizes assortative matching to a setting in which matching sets need not be singletons.

Definition 1 (Assortative Matching) *Take $x_1 < x_2$ and $y_1 < y_2$. The matching sets are assortative if $y_1 \in M(x_2)$ and $y_2 \in M(x_1)$ implies $y_1 \in M(x_1)$ and $y_2 \in M(x_2)$, that is, the matching correspondence is a lattice in $\mathbb{R}^2$.*

The proposition below shows that Definition 1 captures the intuitive understanding of assortative matching: In equilibrium, only "similar" agents should enter into matches. Definition 1 implies that the matching sets form an increasing and connected band around the 45° line. Consequently, the definition guarantees positive correlation between the types of matched partners, for any steady state distribution.

Proposition 1 *Assume $M(x)$ is closed and non-empty and let $l(x) = \min\{y|y \in M(x)\}$ and $u(x) = \max\{y|y \in M(x)\}$. Matching is assortative if and only if $u(x)$ and $l(x)$ are non-decreasing in x, and $M(x)$ is convex for all x.*[4]

In characterizing the matching structure, we show that if the surplus function, which is symmetric and supermodular, satisfies the CSC, then matching is assortative. Since $s(x,y) =$

[4]The statement is identical to Proposition 3 in Shimer and Smith (2000), where a proof can be found.

$f(x,y) - v(x) - v(x)$, if we assume that $f(x,y)$ is symmetric and strictly supermodular, then so is the surplus function $s(x,y)$. It is worthwhile to note that the discounted search model, where $\mathbb{E}_{y\in M(x)}[s(x,y)] = 2(1-\delta)v(x)$, does not satisfy the CSC, since the lost value due to delay (cost of search), $(1-\delta)v(x)$, is not in general, constant across types.[5]

We first prove the following lemma, which is a convenient restatement of the CSC and characterizes the optimal behavior of agents in the economy. Specifically, $\mathbb{E}_z\left(s(x,z) - s(x,y)\right)^+$ represents the expected increase in x's surplus or the benefit, from making another draw from the distribution, given x's current draw of y. An agent x should accept her current partner y, if $\mathbb{E}_z\left(s(x,z) - s(x,y)\right)^+$, is less than the cost of further search $2c$.

Lemma 1 *$y \in M(x)$, if and only if $\mathbb{E}_z\left[\left(s(x,z) - s(x,y)\right)^+\right] \leq 2c$ where $\left(s(x,z) - s(x,y)\right)^+ =$ $\max\{s(x,z) - s(x,y), 0\}$.*

Proof. Assume $y \in M(x)$ and observe that

$$\mathbb{E}_z\left[\left(s(x,z) - s(x,y)\right)^+\right] = \mathbb{E}_{z\in M(x)}\left[\left(s(x,z) - s(x,y)\right)^+\right] + \mathbb{E}_{z\notin M(x)}\left[\left(s(x,z) - s(x,y)\right)^+\right].$$

$\mathbb{E}_{z\notin M(x)}\left[\left(s(x,z) - s(x,y)\right)^+\right] = 0$, because $s(x,z) < 0$, which is in turn because $z \notin M(x)$, and $s(x,y) \geq 0$ because $y \in M(x)$. For $z \in M(x)$, $\left(s(x,z) - s(x,y)\right)^+ \leq s(x,z)$ because $s(x,z) \geq 0$ and $s(x,y) \geq 0$. Consequently, if $y \in M(x)$, then $\mathbb{E}_z\left[\left(s(x,z) - s(x,y)\right)^+\right] = \mathbb{E}_{z\in M(x)}\left[\left(s(x,z) - s(x,y)\right)^+\right] \leq \mathbb{E}_{z\in M(x)}\left[s(x,z)\right] = 2c$. Now I show that if $y \notin M(x)$, then $\mathbb{E}_z\left[\left(s(x,z) - s(x,y)\right)^+\right] > 2c$. Observe, $s(x,y) < 0$, by assumption, and $s(x,z) \geq 0$ for $z \in M(x)$; hence $\left(s(x,z) - s(x,y)\right)^+ = s(x,z) - s(x,y) > s(x,z)$ for $z \in M(x)$. Consequently by the CSC, $\mathbb{E}_{z\in M(x)}\left[\left(s(x,z) - s(x,y)\right)^+\right] > 2c$. However, $\mathbb{E}_z\left[\left(s(x,z) - s(x,y)\right)^+\right] \geq \mathbb{E}_{z\in M(x)}\left[\left(s(x,z) - s(x,y)\right)^+\right]$, proving the result. ■

Given Lemma 1, there is further intuition for the assortative matching result under constant search costs, and the contrasting failure of this result with discounting. As was argued in the introduction, with discounting, high-quality agents (h) may accept low-quality agents (l) since h's search costs, $(1-\delta)v(h)$, are prohibitive. This may, in turn, break assortative matching. Type ls may now reject other ls, because they anticipate a profitable encounter with an h and because their search costs, $(1-\delta)v(l)$, are negligible. In contrast, with constant search costs, this failure cannot happen. If type ls were to reject other ls, then, by lemma 1,

[5] In the discounted search model, $c = 0$, however agents discount the future at rate $\delta \in (0,1)$. The Belman equation for this model is $v(x) = \mathbb{P}\{M(x)\}\mathbb{E}[\frac{f(x,y)+\delta v(x)-\delta v(y)}{2}|y \in M(x)] + (1 - \mathbb{P}\{M(x)\})\delta v(x)$. Rearranging gives $\mathbb{E}_{y\in M(x)}[s(x,y)] = 2(1-\delta)v(x)$ where $s(x,y) = f(x,y) - \delta v(x) - \delta v(y)$.

the expected gain in surplus from further search when matched with an l must exceed the cost, or $p(h)(s(h,l) - s(l,l)) > 2c$. However, by supermodularity, $p(h)(s(h,h) - s(h,l)) > p(h)(s(h,l) - s(l,l)) > 2c$, that is, for an h, the expected gain in surplus from further search when matched with an l must be even larger than the expected gain for an l when matched with an l. As was previously mentioned, this line of reasoning fails with discounting, since the cost of search may increase faster than the increase in surplus for h. In other words, $(1-\delta)v(h) \geq p(h)(s(h,h) - s(h,l)) > p(h)(s(h,l) - s(l,l)) > (1-\delta)v(l)$.

To prove that matching is assortative, I maintain Assumptions 1 and 2, and demonstrate that all the requirements of Proposition 1 are satisfied in any search equilibrium. I show that both $u(x)$ and $l(x)$ are nondecreasing in x (see Proposition 2), $x \in M(x)$ for all x (see Proposition 3) and all matching sets $M(x)$ are convex (see Proposition 4). Recall that in Theorem 1 the upperhemicontinuous matching correspondence, $M(x)$, was established as non-empty and compact valued.

To prove that the lower bounds of the matching sets are nondecreasing, I argue that if $y < l(x)$ (i.e., if x does not accept y), then y is not accepted by any $x' > x$ either. By Lemma 1, $y < l(x)$ implies that, for x, the increase in surplus that accrues from rejecting y and continuing to search, $\mathbb{E}_{z \in M(x)}[s(x,z) - s(x,y)]$, exceeds the cost. However, by supermodularity, $s(x,z) - s(x,y) < s(x',z) - s(x',y)$ for $z \in M(x)$ and any type $x' > x$. Consequently, for type $x' > x$, the benefit from continued search must be at least as large as type x's benefit, and hence, type x' must also reject agent y.

Proposition 2 *The upper and lower bound functions, $u(x)$ and $l(x)$, are nondecreasing in x and $l(x) \leq x \leq u(x)$.*

The following example gives an argument for assortative matching in a three type economy with constant search costs and demonstrates its failure with discounting. The method of proof developed in the example builds the basis of the arguments given subsequently in Propositions 3 and 4.

Example 1 *In a three-type economy with constant search costs matching is assortative. In particular, the failure depicted in Figure 1 cannot occur. In contrast, with the following production function and steady state distribution, the matching depicted in Figure 1 is an equilibrium in a three-type economy with discounting. Let $f(1,1) = f(3,3)$, $f(1,3) = f(2,2) = 0$, $0 < f(1,2) = f(3,2) < \frac{f(1,1)}{2}$ and $p(1) = p(3) = p$. If p is small, then the matching in Figure 1 is an equilibrium with discounting.*

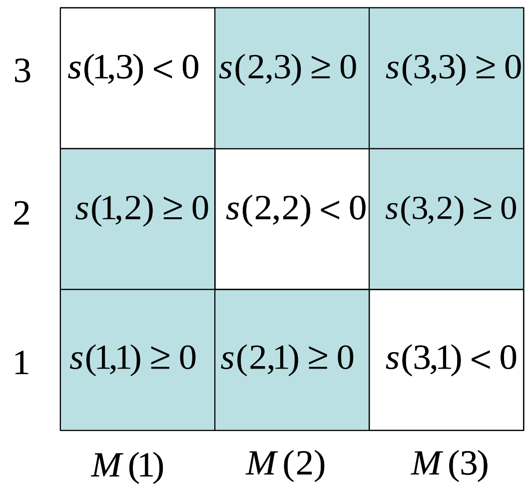

Figure 1: Matching sets for Example 1: $M(1) = \{1,2\}$, $M(2) = \{1,3\}$ and $M(3) = \{2,3\}$

Discussion. The failure of assortative matching depicted in Figure 1 is impossible with three types. Also, this is the only failure of assortative matching not ruled out by Proposition 2. The CSC implies that $\mathbb{E}_{M(3)} s(3,y) = \mathbb{E}_{M(2)} s(2,y) = \mathbb{E}_{M(1)} s(1,y)$, or alternatively $\frac{1}{2}\mathbb{E}_{M(3)} s(3,y) + \frac{1}{2}\mathbb{E}_{M(1)} s(1,y) = \mathbb{E}_{M(2)} s(2,y)$, which implies

$$\frac{1}{2}\left(p(3)\,s(3,3) + p(2)\,s(3,2)\right) + \frac{1}{2}\left(p(2)\,s(2,1) + p(1)\,s(1,1)\right) = p(3)\,s(2,3) + p(1)\,s(2,1) \tag{4}$$

However, $s(3,3) + s(2,2) > 2s(2,3)$ by supermodularity. Because $2 \notin M(2)$, $s(2,2) < 0$. Consequently, $s(3,3) > 2s(2,3)$. Likewise, $s(1,1) + s(2,2) > 2s(2,1)$, which implies that $s(1,1) > 2s(2,1)$. Now, substituting for $s(3,3)$ and $s(1,1)$ shows that Equation 4 is not satisfied (i.e., $\frac{1}{2}\left(\mathbb{E}_{M(3)} s(3,y) + \mathbb{E}_{M(1)} s(1,y)\right) > \mathbb{E}_{M(2)} s(2,y)$).

For the discounted model, assume that type 1 (and 3) only matches with other $1's$ (and $3's$). Then;

$$v(1) = p\frac{f(1,1)}{2} + \delta(1-p)\,v(1) = \frac{p}{1-\delta(1-p)}\frac{f(1,1)}{2},$$

$v(3) = v(1)$ and $v(2) = 0$. For this to be an equilibrium, Agents 1 and 2 must not match, i.e., $f(1,2) - \delta v(1) - \delta v(2) < 0$ or $f(1,2) < \frac{\delta p}{1-\delta(1-p)}\frac{f(1,1)}{2}$. For small p, however, this inequality is not satisfied. Consequently, for small p, $2 \in M(1)$. Since $2 \in M(1)$, $v(2) > 0$ and consequently $2 \notin M(2)$. Also, working through the algebra, if $f(1,2) \geq \frac{\delta p}{1-\delta(1-p)}\frac{f(1,1)}{2}$, then $M(1) = \{1,2\}$, $M(2) = \{1,3\}$ and $M(3) = \{3,2\}$ is an equilibrium. Observe that if type 1 is disregarded, then this is just a formal demonstration of the two-type example previously discussed. ■

The following proposition shows that all agents will accept to match with others of the same type, i.e., $x \in M(x)$. The method of proof is identical to the argument used in Example 1. Let y_u and y_l maximize the surplus for x, (i.e, $s(x,z)$) for $z \geq x$ and $z \leq x$, respectively. The proof shows that if $x \notin M(x)$, then either y_u or y_l must generate more surplus than type x; which contradicts the CSC.

Proposition 3 *$x \in M(x)$ for all x.*

Proof. Assume that $x \notin M(x)$, and let $y_u \in \arg\max_{y \geq x} s(x,y)$ and $y_l \in \arg\max_{y \leq x} s(x,y)$.
Claim If $y > x$, then $s(y, y_u) > 2s(x,y)$. Also, if $y < x$, then $s(y, y_l) > 2s(x,y)$.
By supermodularity, $s(y_u, y) - s(y_u, x) > s(x,y) - s(x,x)$ for $y > x$. Since $s(x,x) < 0$, then $s(y_u, y) > s(y_u, x) + s(x,y)$. However, y_u maximizes $s(x,y)$ for $y \geq x$. Consequently, $s(y_u, y) > 2s(x,y)$. Again by supermodularity, $s(y_l, y) - s(y_l, x) > s(x,y) - s(x,x)$ for $y < x$. Since $s(x,x) < 0$, then $s(y_l, y) > s(y_l, x) + s(x,y)$. However, y_l maximizes $s(x,y)$ for $y \leq x$. Consequently, $s(y_l, y) > 2s(x,y)$.

Observe that, the claim implies, if $x < z \in M(x)$, then $z \in M(y_u)$, since $s(z, y_u) > 2s(x,z) \geq 0$. Likewise, if $x > z \in M(x)$, then $z \in M(y_l)$. Consequently,

$$\frac{1}{2}\mathbb{E}_{M(y_u)} s(y_u, z) + \frac{1}{2}\mathbb{E}_{M(y_l)} s(y_l, z) \geq \frac{1}{2}\mathbb{E}_{M(x)\bigcap\{z>x\}} s(y_u, z) + \frac{1}{2}\mathbb{E}_{M(x)\bigcap\{z<x\}} s(y_l, z).$$

Hence, using the claim to substitute in for $s(y_u, z)$ and $s(y_l, z)$ gives

$$\begin{aligned}\frac{1}{2}\left(\mathbb{E}_{M(y_u)} s(y_u, z) + \mathbb{E}_{M(y_l)} s(y_l, z)\right) &> \frac{1}{2}\left(\mathbb{E}_{M(x)\bigcap\{z>x\}} 2s(x,z) + \mathbb{E}_{M(x)\bigcap\{z<x\}} 2s(x,z)\right) \\ &= \mathbb{E}_{M(x)} s(x,z).\end{aligned}$$

The CSC requires, however, that $\mathbb{E}_{M(x)} s(x,z) = \frac{1}{2}\left(\mathbb{E}_{M(y_u)} s(y_u, z) + \mathbb{E}_{M(y_l)} s(y_l, z)\right)$, which is a contradiction and thus completes the proof. ■

The argument used to prove convexity is identical to the reasoning used in Proposition 2. Assuming non-convex matching sets implies that there exists agents x and y such that $l(y) < x < y < u(x)$ with $y \notin M(x)$. I then pick agents y_u and y_l, who maximize $s(y,z)$ for $z \geq y$ and $s(x,z)$ for $z \leq x$, respectively, and proceed to show that the sum of the surplus generated by y_u and y_l, must exceed the sum generated by y and x. This again contradicts the CSC.

Proposition 4 *$M(x)$ is convex for all x.*

Proposition 2 through 4 showed that the requirements of Proposition 1 are satisfied and consequently that in any equilibrium, matching is assortative. The following Theorem summarizes the findings of the paper.

Theorem 2 *If f satisfies Assumption 1 and 2, then a search equilibrium exists and in any equilibrium matching is assortative.*

Proof. Existence follows from Theorem 1 and assortative matching follows from Proposition 2 through 4. ■

4 Discussion of Results and Summary

In this paper, I have shown that the classical assortative matching result extends to a decentralized search setting with transferable utility if the cost of search is constant across types. This result is robust to many of the modelling choices made here. In particular, in my model, the steady-state distribution of types is maintained by replacing agents that leave the market with identical clones: In some sense, the steady state distribution of types is exogenously given. A preferable modelling strategy might be to assume an exogenously fixed inflow of new agents into the market each period balanced by the outflow of agent following successful matches. This results in an endogenous determined steady-state type distribution. Since the arguments concerning assortative matching hold for any steady state distribution, these results would also hold under this alternative modelling choice.[6]

As may be apparent, it may not be individually rational for some types to participate in this market game because the model does not guarantee that the value for each type is non-negative. Making participation voluntary and then working with the resulting type distribution would address this concern. Since the model is stationary, this would be equivalent to a game in which each individual could opt out of the market in each period. Again, since all our characterization results are distribution independent, they would continue to hold without alteration.

[6] The alternative modelling choice, however, complicates the argument for equilibrium existence.

A Appendix

Proof of Proposition 2. First, I show, if $y < l(x)$ and $x' > x$, then $y \notin M(x')$, and consequently $l(x') \geq l(x)$. If $z \in M(x)$, then $z > y$. For $z \in M(x)$,

$$0 \leq s(x,z) < s(x,z) - s(x,y) < s(x',z) - s(x',y).$$

The first (weak) inequality follows because $z \in M(x)$. The second because $y \notin M(x)$ implies $s(x,y) < 0$; and the third because of the supermodularity of $s(\cdot)$ in conjunction with $x' > x$ and $z > y$. Taking expectations over $M(x)$ gives

$$2c = \mathbb{E}_{M(x)} s(x,z) < \mathbb{E}_{M(x)} (s(x',z) - s(x',y)) \leq \mathbb{E}_z (s(x',z) - s(x',y))^+.$$

Hence, by Lemma 1, $y \notin M(x')$.

Second, if $y > u(x)$ and $x' < x$, then $y \notin M(x')$ and consequently $u(x') \leq u(x)$. If $z \in M(x)$, then $z < y$. For $z \in M(x)$,

$$0 \leq s(x,z) < s(x,z) - s(x,y) < s(x',z) - s(x',y)$$

The first (weak) inequality follows because $z \in M(x)$. The second because $y \notin M(x)$ implies $s(x,y) < 0$ and the third because of the supermodularity of $s(\cdot)$ in conjunction with $x' < x$ and $z < y$. Taking expectations over $M(x)$, gives, by Lemma 1, $y \notin M(x')$ because

$$2c = \mathbb{E}_{M(x)} s(x,z) < \mathbb{E}_{M(x)} (s(x',z) - s(x',y)) \leq \mathbb{E}_z (s(x',z) - s(x',y))^+.$$

Third and finally, $l(x) \leq x \leq u(x)$. Assume to the contrary, $y = l(x) > x$. By symmetry, $y \in M(x)$ implies that $x \in M(y)$. However, this shows that $l(y) \leq x$, which contradicts $l(x) \leq l(y)$. The argument for the upper bound is identical. ■

Proof of Proposition 4. Assume, without loss of generality, that there exists $y > x$ such that $l(x) < y < u(x)$ and $y \notin M(x)$. Define the following sets:

$$\begin{array}{ll} U(y) = \{z \in M(x) : z \geq y\} & L(y) = \{z \in M(x) : z < y\} \\ U(x) = \{z \in M(y) : z > x\} & L(x) = \{z \in M(y) : z \leq x\} \end{array}$$

Let $y_u \in \arg\max_{z \in U(y)} s(z,y)$ and $y_l \in \arg\max_{z \in L(x)} s(z,x)$. The maximums are well defined since the sets are compact and $U(y) \neq \varnothing$, because $u(x) \in U(x)$. To show $L(x) \neq \varnothing$, observe

$u(x) \in M(x)$, which implies $l(u(x)) \leq x$, however, since $u(x) > y$, monotonicity of $l(\cdot)$ implies that, $l(y) < x$, which shows that $l(y) \in L(x)$.

Claim *For all $z \in U(y)$, $s(y,z) > 0$ and $s(z, y_u) > s(y,z) + s(x,z)$. Also, for all $z \in L(x)$, $s(x,z) > 0$ and $s(z, y_l) > s(x,z) + s(y,z)$.*

For $z \in U(y)$, $s(z, y_u) + s(y,y) > s(y,z) + s(y, y_u) \geq 2s(y,z)$, where the first equality follows from supermodularity and the second (weak) inequality follows because $s(y, y_u) \geq s(y,z)$ since $y_u \in \arg\max_{z \in U(y)} s(y,z)$. Also, by supermodularity, $s(y,z) + s(x,y) > s(x,z) + s(y,y)$. However, $y \notin M(x)$ implies that $s(x,y) < 0$. $z \in M(x)$ implies that $s(x,z) \geq 0$; and $y \in M(y)$ implies that $s(y,y) \geq 0$. Consequently, $s(y,z) > s(x,z) + s(y,y) \geq 0$. Combining $s(z, y_u) + s(y,y) > 2s(y,z)$ and $s(y,z) > s(x,z) + s(y,y)$ provides $s(z, y_u) + s(y,y) > s(y,z) + s(x,z) + s(y,y)$. This implies that $s(z, y_u) > s(y,z) + s(x,z)$. The argument for $z \in L(x)$, is identical: $s(z, y_l) + s(x,x) > s(z,x) + s(x, y_l) > 2s(x,z)$. Also, $s(x,z) + s(x,y) > s(x,x) + s(y,z)$. Observing that $s(x,y) < 0$ and combining the inequalities delivers the result.

Claim *For all $z \in U(x)$, $s(z, y_u) > s(z,y)$ and for all $z \in L(y)$, $s(z, y_l) > s(z,x)$.*

For $z \in U(x)$, $s(z, y_u) + s(x,y) > s(z,y) + s(x, y_u)$ by supermodularity. However, $s(x,y) < 0$ and $s(y_u, x) \geq 0$ because $y_u \in M(x)$. Consequently, $s(z, y_u) > s(z,y)$. For $z \in L(y)$, $s(z, y_l) + s(x,y) > s(z,x) + s(y, y_l)$ by supermodularity. However, $s(x,y) < 0$ and $s(y, y_l) > 0$. Consequently, $s(z, y_l) > s(z,x)$.

The two claims imply that $U(y) \subset U(x) \subset M(y_u)$ because, $y > x$ by assumption, $s(y,z) > 0$ for $z \in U(y)$ and $s(z, y_u) > 0$ for $z \in U(x)$. Likewise, $L(x) \subset L(y) \subset M(y_l)$. See Figure 2 for a depiction of these sets.

Given the two claims proved above, the following inequalities hold

$$\begin{aligned}
\mathbb{E}_{M(y_u)} s(z, y_u) &\geq \mathbb{E}_{U(y)} s(z, y_u) + \mathbb{E}_{U(x)\backslash U(y)} s(z, y_u) \\
&> \mathbb{E}_{U(y)} \left(s(y,z) + s(x,z)\right) + \mathbb{E}_{U(x)\backslash U(y)} s(y,z) \\
&> \mathbb{E}_{U(y)} s(x,z) + \mathbb{E}_{U(x)} s(y,z) \\
\mathbb{E}_{M(y_l)} s(z, y_l) &\geq \mathbb{E}_{L(x)} s(z, y_l) + \mathbb{E}_{L(y)\backslash L(x)} s(z, y_l) \\
&> \mathbb{E}_{L(x)} \left(s(x,z) + s(y,z)\right) + \mathbb{E}_{L(y)\backslash L(x)} s(x,z) \\
&> \mathbb{E}_{L(x)} s(y,z) + \mathbb{E}_{L(y)} s(x,z)
\end{aligned}$$

Summing up the previous two inequalities gives

$$\mathbb{E}_{M(y_u)} s(z, y_u) + \mathbb{E}_{M(y_l)} s(z, y_l) > \mathbb{E}_{U(y)} s(x,z) + \mathbb{E}_{U(x)} s(y,z) + \mathbb{E}_{L(x)} s(y,z) + \mathbb{E}_{L(y)} s(x,z).$$

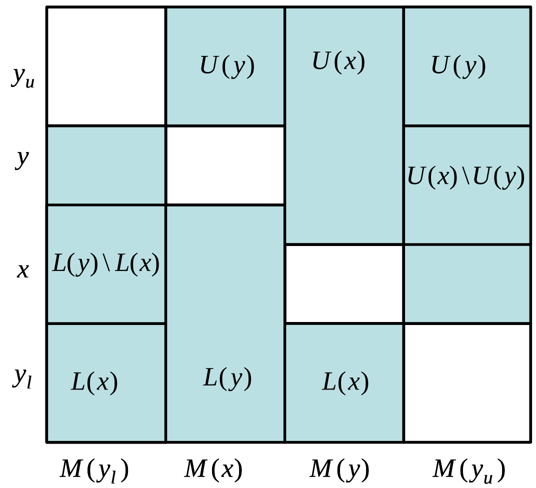

Figure 2: Sets Defined for Proposition 4

Consequently, $\mathbb{E}_{M(y_u)} s(z, y_u) + \mathbb{E}_{M(y_l)} s(z, y_l) > \mathbb{E}_{M(x)} s(z, x) + \mathbb{E}_{M(y)} s(z, y)$, which contradicts the CSC and completes the proof. ■

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