# Optimal fiscal policy and the (lack of) time inconsistency problem*

Marina Azzimonti†
University of Iowa

Pierre-Daniel Sarte
Federal Reserve Bank of Richmond

Jorge Soares
George Washington University

January 2005

**Abstract**

Much work has recently been devoted to the study of optimal fiscal policy in environments where the government cannot commit to future policy actions. To this point, and without exception, this literature has abstracted either from government debt, capital, or both, and a working assumption at the outset is that the solutions with and without commitment do not coincide. In contrast, we show that once private capital and government debt are simultaneously allowed in a representative agent framework, the assumed time inconsistency of Ramsey optimal tax rates is unfounded. More specifically, we show that the high initial capital levy and zero steady state tax on capital that emerge as optimal with full government commitment also emerge as a time consistent Markov perfect equilibrium. Furthermore, this result holds either in finite or infinite horizon, and irrespective of whether the time path of government spending is determined exogenously or endogenously within the model. Finally, we discuss departures from the conventional framework that reintroduce time inconsistency as a policy concern, despite the coexistence of private capital and government debt.

*PRELIMINARY AND INCOMPLETE. We thank Huberto Ennis, Bob King, Leo Martinez, Alex Wolman, Per Krusell, Thomas Renstrom and the participats of the 2005 Midwest Macroeconomic Conference for valuable discussions.

†Corresponding author: Tel.:+1-319-335-0914; *E-mail address:* marina-azzimonti@uiowa.edu.

# 1 Introduction

Beginning with the work of Kydland and Prescott (1977, 1980), and Fisher (1980), optimal fiscal policy has come to be widely regarded as time inconsistent. In particular, their work showed that the optimal sequence of taxes depends importantly on whether the government can commit to its policies once and for all at the beginning of time. Thus, much work since then has been devoted to the study of optimal policies that do not require the government to promise a course of actions in the initial period. Some of this work uses the loss of good reputation as a way of committing the government to desirable polices (Benhabib and Rustichini, 1997; Chari and Kehoe, 1993). Another line of work, as exemplified by (Klein, Krusell, and Rios-Rull 2004**;** Azzimonti, Sarte and Soares, 2003; Krusell, 2002; Xie, 1997), has relied on Markov perfect equilibria whereby policies are constrained to be functions of the current state of the economy. Without exception, the existing literature on optimal time consistent fiscal policy has abstracted from either government debt, capital, or both, and a working assumption at the outset is that the solutions with and without commitment differ. This paper shows that when both private capital and government debt are simultaneously allowed in a representative agent framework, and factor taxes are linear, the optimal policies with and without commitment coincide. Moreover, time consistent policies throughout our analysis are Markov-perfect and, therefore, do not rely on intricate reputational mechanisms.

Following the work of Judd (1985), and Chamley (1986), it is well known that in a representative agent setting with both capital and government debt, as well as distortional taxes on capital and labor, the optimal choice of taxes with full commitment involves an initial capital levy and a zero tax on capital in the long run. Thus, a benevolent government that can credibly commit to future policies takes advantage of the inelasticity of capital in the initial period with the promise never to do so in the future. It is generally acknowledged that the government's ability to commit not to break past promises is crucial in generating this result. For instance, in the context of precisely the Ramsey problem studied by Chamley (1986), Sargent and Ljungqvist (2000, chapter 12) write that "(...) taxing the capital stock at time 0 amounts to lump-sum taxation and therefore disposes of distortionary taxation. It follows that a government without a commitment technology would be tempted in future periods to renege on its promises and levy a confiscatory tax on capital." Thus, starting at the Ramsey steady state, a government that re-optimizes with respect to taxes might decide to break its earlier promise of zero capital taxes and, once again, take advantage of the sunk nature of past investments. This conjecture turns out to be incorrect.

Contrary to conventional wisdom, we argue that sovereign debt generally helps make Ramsey taxes time consistent. In the environment studied by Chamley (1986), for instance, the presence of bonds lets the government frontload all taxes in the initial period while still allowing households to meet their desired consumption path through sovereign lending in future periods. Using the proceeds from the lump sum tax, the government can then build up enough of an asset base at time 0 and credibly promise never having to set positive taxes in the future. Specifically, we show that in this setting, Ramsey tax rates arise even when optimal policy is chosen sequentially by different governments (i.e. without commitment). We further argue that whether the time path of government spending is exogenously given or determined endogenously within the model does not matter for this result. In other words, the justification for a given sequence of public spending and the means of financing it can be studied separately.

Given the lump sum nature of optimal taxes with commitment (i.e. Ramsey taxes), existing work on optimal fiscal policy often places a restriction on the initial capital tax rate in order to restore a role for distortional taxes (Atkenson, Chari and Kehoe, 1999; Chai and Kehoe, 1999; Chari, Christiano and Kehoe, 1994; Jones, Manuelli and Rossi, 1993; Jones, Manuelli and Rossi, 1997). As we have just argued, absent this restriction, the government can potentially raise all necessary revenues through the initial tax on capital, lend the proceeds to households, and finance public expenditures over time using households' interest payments on the loan. Our analysis suggests that placing a restriction on the optimal initial capital tax arbitrarily discards a policy which, if implemented, no subsequent government would ever abandon. Thus, we discuss alternative environments that reintroduce a role for time inconsistent distortional taxes despite the coexistence of private capital and government debt.

Since the presence of capital and government debt can imply a rather extreme but time consistent tax scheme, it is natural to question the exact assumptions that underlie this result. In particular, we argue that having a representative household, whose lifespan is also that of the environment, is essential in establishing the time consistency of the initial capital levy. In a world with overlapping generations, this result breaks down as the current old, whom the government has the incentive to tax heavily because of their inelastic capital position, may not be alive to borrow from the government in subsequent periods to make up for the large initial tax. Taking this feature into account, a planner will not place the entire tax burden associated with the present value of public expenditures on the initial old generation. This fact immediately implies a role for distortional taxes as future generations must bear some of the tax burden associated with public expenditures. Furthermore, a government that is allowed to re-optimize at some date faces a non-trivial choice of how to set taxes across generations and, in particular, will always have an incentive to exploit the fact that for the old generation at that date, the capital stock is fixed. Therefore, in a world where individuals have finite lives, optimal taxes with commitment will be time inconsistent.

This paper is organized as follows. Section 2 presents a two-period version of the economy studied by Chamley (1986) as a benchmark and analyzes optimal taxation under no commitment. Section 3 shows how the results extend to the infinite horizon, while Section 4 provides alternative environments where time inconsistency may arise. Section 5 concludes.

# 2 Optimal time consistent fiscal policy in a two-period economy

This section shows that in a two-period version of the economy studied by Chamley (1986), the optimal Markov perfect path for taxes involves a capital levy in the initial period and zero capital tax rates thereafter, as well as zero taxes on labor at both dates. The two-period case is the environment in which one can perhaps most easily establish the coincidence of full commitment and time consistent optimal fiscal policies. This result easily generalizes to the case with an arbitrarily finite horizon. More generally, the fact that full commitment and discretionary policies coincide does not rely on any peculiarity associated with an infinite horizon. That said, we show in the next section that our main result continues to hold as the horizon becomes unbounded.

## 2.1 The environment

There exists a representative household who has finite life indexed by $t = 0, 1$. This household lives in a single good economy and values consumption and leisure streams, $\{c_t, l_t\}_{t=0}^{1}$, according to preferences given by

$$\sum_{t=0}^{1} \beta^t u(c_t, l_t), \tag{1}$$

$l_t = 1 - n_t$, where $n_t$ denotes labor input. The function $u$ is increasing, concave and $C^2$ in both arguments. The unique good is produced by combining labor and capital, $k_t$, using the production technology

$$F(k_t, n_t), \tag{2}$$

where $F$ is constant returns to scale with respect to its inputs. Production can be used for either private or government consumption, or to increase the capital stock,

$$c_t + k_{t+1} + g_t = F(k_t, n_t) + (1 - \delta)k_t, \tag{3}$$

where $0 < \delta < 1$ denotes the capital depreciation rate, and $\{g_t\}_{t=0}^{1}$ represents an exogenously given sequence of public expenditures.

As in Chamley (1986), the government finances its purchases using time-varying linear taxes on labor, $\tau_t^n$, and capital, $\tau_t^k$. The government can also make up for any imbalance between its revenues and expenditures by issuing one-period bonds that are perfectly substitutable with capital. We denote the level of government debt by $b_t$, where $b_t < 0$ when the government lends to the public. At each date, the government's budget constraint is given by

$$\tau_t^k r_t k_t + \tau_t^n w_t n_t + b_{t+1} = g_t + R_t b_t, \tag{4}$$

where $r_t$ and $w_t$ are the market rates of return to capital and labor, and $R_t$ denotes the gross rate of return on government bonds from $t-1$ to $t$. The left and right-hand side of equation (4) represent sources and uses of government revenue respectively.

The representative household maximizes (1) subject to the sequence of budget constraints,

$$c_t + k_{t+1} - (1 - \delta)k_t + b_{t+1} = (1 - \tau_t^n)w_t n_t + (1 - \tau_t^k)r_t k_t + R_t b_t, \tag{5}$$

from which we obtain the static equation describing households' optimal labor-leisure choice,

$$u_l(c_t, 1 - n_t) = u_c(c_t, 1 - n_t)(1 - \tau_t^n)w_t, \tag{6}$$

the conventional Euler equation,

$$u_c(c_t, 1 - n_t) = \beta u_c(c_{t+1}, 1 - n_{t+1}) \left[(1 - \tau_{t+1}^k)r_{t+1} + 1 - \delta\right], \tag{7}$$

and the asset arbitrage condition,

$$R_{t+1} = (1 - \tau_{t+1}^k)r_{t+1} + 1 - \delta. \tag{8}$$

A representative firm in this economy takes as given the sequence of prices, $\{r_t, w_t\}_{t=0}^{1}$, and maximizes profits. It follows that

$$r_t = F_k(k_t, n_t) \text{ and } w_t = F_n(k_t, n_t). \tag{9}$$

Given the environment we have just laid out, second-best Ramsey tax rates are found by maximizing welfare in (1) subject to public and private sector budget constraints (4) and (5), optimal household behavior summarized by (6), (7), and (8), as well as firms' decision rules, (9). In solving the Ramsey problem, we imagine that the government can commit to its chosen course of actions through time. We further assume for now that the capital stock at date 0 is large enough to finance the present discounted value of government purchases. Under these assumptions, it is well known that Ramsey tax rates imply a capital levy in the initial period and a zero tax on capital thereafter. Optimal fiscal policy with full commitment also implies a zero tax on labor at all dates. Because the capital stock at time 0 is fixed, this policy prescription amounts to a single initial lump sum tax and disposes of any distortional taxes. The literature on optimal fiscal policy, therefore, reasonably conjectures that without a commitment technology, a government would be tempted at future dates to renege on its promises of zero labor and capital taxes, and revert back to a confiscatory tax on capital (see Chari, Christiano, Kehoe [1995], Sargent and Ljungqvist [2000], **and others**). We show that this is in fact not the case, and that the Ramsey prescription also emerges in a time consistent Markov perfect equilibrium.

## 2.2 Markov perfect optimal tax rates

In this section, we get rid of the commitment technology implicitly assumed in the Ramsey problem. Rational households recognize that the public sector may have an incentive to deviate from the sequence of taxes they promise. Hence, setting taxes once and for all at time 0 results in policy announcements that are not necessarily credible. Such announcements do not generally constitute a subgame perfect equilibrium since, if allowed to re optimize in any period, the government will typically choose a different strategy. The objective of this section is to define a maximization problem that is associated with a time consistent tax structure in equilibrium. Attaining optimal time consistent policies has been approached by the literature in mainly one of two ways.

One approach finds the set of all possible sustainable equilibria, and characterizes the problem using reputational mechanisms that rely on trigger strategies or reversions to the worst possible equilibrium. Under this approach, payoffs are easily found but the characterization of a decentralized tax structure is not always feasible. In addition, reputational mechanisms are typically not renegotiation proof. Leading work on these issues in a public finance context is developed in Chari (1990, 1993).

Alternatively, another branch of the literature has relied on the definition of Subgame Perfect

Markov Equilibria to find optimal time consistent policies. Here, the optimal policy rule is assumed to be a function of the current states of the economy only. Policy is history independent and reputation plays no role. The outcomes that emerge in this literature do not necessarily constitute the best attainable subgame perfect equilibria, but nor do they involve any of the difficulties associated with renegotiation proofness. In this paper, we follow this second approach.

We define a stationary Subgame Perfect Markov Equilibrium along the lines of Klein, Krusell, and Rios Rull (2004). In particular, tax rates depend on the current states of the economy which, in our framework, involve the level of debt and the stock of capital.

A general feature of Ramsey policies without public debt is that if a government were given the chance to re-optimize at some date $t > 0$, it would choose to deviate from the policy sequence prescribed at $t = 0$. The question is whether alternative optimal policies exist which, when implemented, no subsequent government would ever have an incentive to abandon. To begin addressing this question within a Markov framework, consider a sequence of successive governments, each choosing labor and capital tax rates based on the state it inherits when taking office. In choosing this policy rule, each government takes as given the following government's optimal choice of taxes, given the relevant states at that date. If a policymaker can correctly infer the rule optimally used by his successor, he may then be able to ensure that whatever government follows will not deviate from the taxes he wishes for that date by creating the appropriate state through its choice of policy. For this scenario to be an equilibrium, it must be the case that all policymakers actually choose the rule anticipated by their predecessor.

In the simple two-period economy considered in this section, time consistent optimal tax rates of the kind we have just described can most easily be found by focusing on the last period and proceeding backwards.

Formally, the problem of the government at date 1 is

$$\max \ u(c_1, 1 - n_1) \tag{10}$$

subject to the government budget constraint (4), the household budget constraint (5), and the condition describing optimal labor-leisure choice (6), all evaluated at date 1. Observe that the Euler equation constraint (7) that determines savings is not relevant in the last period. The reasons are twofold. First, since the world ends with date 1, households have no incentive to save for future consumption. In particular, $k_2 = b_2 = 0$. Second, from the perspective of this government, the capital stock, $k_1$, is given and past investment decisions are sunk. One important implication is that the optimal choices of taxes, $\tau_1^k$ and $\tau_1^n$, depend only on the relevant states for that date, $\{k_1, b_1\}$, and we denote these choices by $\tau_1^k(k_1, b_1)$ and $\tau_1^n(k_1, b_1)$. Furthermore, we designate optimal allocations for consumption and labor that emerge from this maximization problem as $c_1(k_1, b_1)$ and $n_1(k_1, b_1)$ respectively.

Knowing how its successor behaves, a government at date 0 can then optimally choose how to set taxes given $\{k_0, b_0\}$. In doing so, this government takes fully into account the optimal behavior followed by the date 1 government. Formally, the maximization problem is,

$$\max \; u(c_0, 1 - n_0) + \beta V(k_1, b_1), \tag{11}$$

where $V(k_1, b_1) = u(c_1(k_1, b_1), 1 - n_1(k_1, b_1))$, subject to the government budget constraint (4), the household budget constraint (5), and the condition describing optimal labor-leisure choice (6), all evaluated at date 0, as well as the Euler equation constraint,

$$u_c(c_0, 1 - n_0) - \beta \left[ F_k(k_1, n_1(k_1, b_1))(1 - \tau_1^k(k_1, b_1)) + 1 - \delta \right] u_c(c_1(k_1, b_1), n_1(k_1, b_1)). \tag{12}$$

In this last equation, tomorrow's consumption, leisure, and taxes, are explicitly written in terms of the date 1 decision rules. By solving the backwards induction problem we have just described, we are ensuring that each government at each date responds in the best possible way to the states inherited from previous governments. Therefore, the solution obtained in this way will necessarily be time consistent, and is summarized in the following proposition.

**Proposition 1** *Optimal time consistent taxes on labor and capital are such that* $\tau_0^k \geqslant 0$, $\tau_1^k = 0$, *and* $\tau_0^n = \tau_1^n = 0$.

**Proof.** See Appendix A. ■

One can easily show that the policy described in proposition 1 is in fact identical to the one that emerges under the Ramsey problem. Under commitment, if initial private asset holdings are large enough, the policy that maximizes lifetime utility involves setting a positive capital tax rate at date 0 (which helps finance the discounted sum of future public expenditures), and no distortions to the labor decision in either period or capital taxes at date one. In principle, once date 0 has passed, the planner in power in period one might well have incentives to deviate from this path and set positive taxes. However, in choosing taxes today, the date 0 government is able anticipate the incentives facing the date 1 government and manipulate its successor's policy decisions. **Recall that ....** Hence, the initial government chooses taxes so as to leave the period 1 government with equilibrium stocks of debt and capital that exactly induce it to set taxes on capital and labor to zero. Put another way, when one allows for the presence of both capital and bonds, strategic manipulation by the initial government helps reproduce Ramsey outcomes despite the fact that choices are sequential and that the past cannot be undone.

The solution for the initial optimal tax on capital is given by[1]

$$\tau_0^k = \frac{g_0 + (F_k(k_0, n_0(k_0, b_0)) + 1 - \delta) b_0 + \frac{g_1}{F_k(k_1(k_0,b_0), n_1(k_0,b_0)) + 1 - \delta}}{F_k(k_0, n_0(k_0, b_0))(k_0 + b_0)}, \tag{13}$$

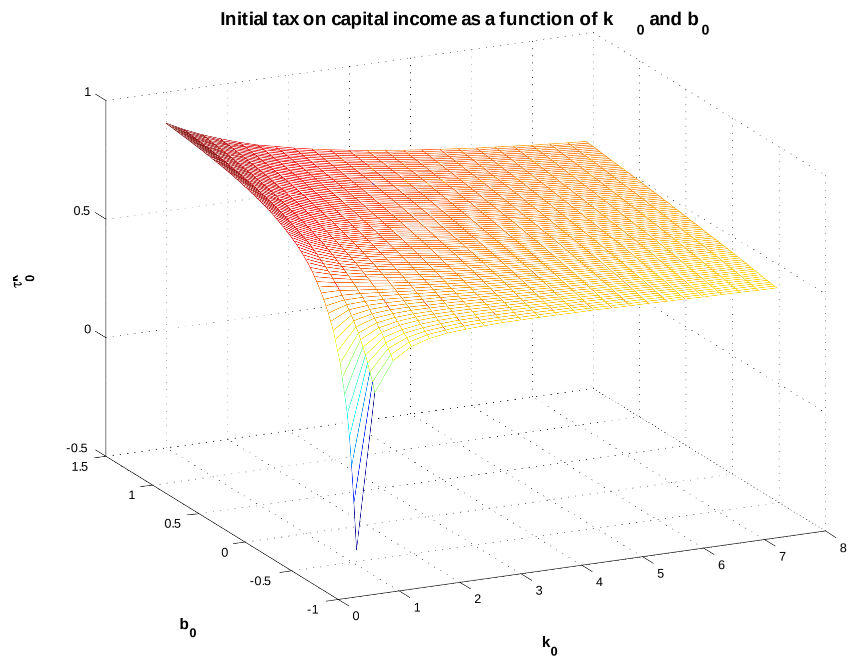
where $n_0(k_0, b_0)$, $n_1(k_0, b_0)$ and $k_1(k_0, b_0)$ denote allocations as a function of the initial states. As with the standard Ramsey solution, it is possible for taxes to be zero even in the initial period. Specifically, there exist combinations of initial capital stocks and debt, $k_0$ and $b_0$, that deliver $\tau_0^k = 0$.

Figure 1 below plots equation (13) as a function of initial asset holdings.[2] As one might expect, given a fixed value of $k_0$, as initial public debt increase, so does the tax on capital required in order to finance a given stream of expenditures, $\{g_0, g_1\}$. The relationship linking $\tau_0^k$ to private capital, however, is non-monotonic for different levels of debt. To see this, it is helpful to first identify the various effects triggered by varying initial capital, $k_0$. On the one hand, given a return to savings, $F_k(k_0, n_0)$, a large stock of capital generates high tax revenues, $\tau_0^k F_k(k_0, n_0) k_0$, which suggests that a lower value of $\tau_0^k$ may be required to meet a given sequence of public spending. On the other hand, since the return to savings itself decreases with $k_0$, it helps erode the tax base as the level of capital increases and raises the equilibrium choice of $\tau_0^k$ to maintain feasibility. In addition, since tomorrow's return to savings falls with $k_0$ – because $k_1(k_0, b_0)$ increases with initial capital – the present value of tomorrow's expenditures also rises, thus requiring a larger levy in period 0. In the end, the net effect of changes in $k_0$ on the equilibrium value of $\tau_0^k$ required to finance expenditures depends crucially on initial debt, which affects both date 0 and date 1 labor choices, $n_0(k_0, b_0)$ and $n_1(k_0, b_0)$, as well as savings in the current period, $k_1(k_0, b_0)$. It follows, therefore, that for each capital stock level, $k_0$, there exists an initial debt level, $b_0$, for which taxes of any kind never have to be levied.

The time consistency of Ramsey equilibrium taxes does not depend on the assumption of exogenous public expenditures. Consider, for instance, an economy where the government provides pure public goods – such as parks, ..., or ... – from which households directly derive utility. Assuming

[1] See Appendix A for the derivation.

[2] parameters....




that these goods can be financed with both bonds and taxes on factor income, the environment is then identical to the one analyzed above, except that government expenditures are now determined endogenously. It is still the case that the optimal policy involves high capital taxes initially with no distortions thereafter, and that this policy is time consistent. Consequently, the justification for a given sequence of public spending and the means of financing it can be studied separately.

**Proposition 2** *Consider the economy described in section 2.1, but let preferences be defined over both private and public consumption, $u(c, 1-n, g)$, where.... . Then, the optimal sequence of taxes on labor and capital under commitment are such that $\tau_0^k \geqslant 0$, $\tau_1^k = 0$, and $\tau_0^n = \tau_1^n = 0$. Furthermore, this solution is time consistent.*

The proof of this proposition is entirely straightforward and follows the exact reasoning described in Appendix A, with two additional conditions that determine the optimal size of government spending. These conditions simply equate the marginal utility derived from the consumption of private and public goods at each date.

The fact that Ramsey taxes are time consistent also generalizes to the case of an arbitrary finite horizon, $T > 0$. Under government commitment, what matters is the notion that irrespective of the time horizon, a planner never wishes to distort either the labor-leisure decision or the savings decision. The same reasoning turns out to apply when different governments, but with the same objective, make sequential decisions over time. Given that a policymaker in office in the last period chooses policy based on the states it receives $k_T$ and $b_T$, the government in period $T-1$ chooses taxes so as to leave its successor with exactly those states that induce it to set $\tau_T^k = \tau_T^n = 0$. This process repeats itself backwards until the initial period. In effect, each government acts in such a way as to provide the exact incentives needed for its successor not to introduce distortions in the economy.

# 3 Optimal time consistent fiscal policy in infinite horizon

Having laid out the basic intuition underlying the time consistency of Ramsey taxes when both capital and sovereign debt coexist simultaneously, we are now in a position to show that this intuition extends to the standard infinite horizon framework. The time horizon, therefore, is not an important determinant of outcomes in this case.

Consider the economy described in section 2 but let time range over an infinite horizon, $t = 0, 1, ..., \infty$. Define the stationary Markov-perfect equilibrium policy rules,

$$\tau^n = \theta(k, b) \text{ and } \tau^k = \psi(k, b), \tag{14}$$

so that at any point in time, taxes on labor and capital depend only on the payoff relevant state variables of the economy. As shown in Krusell and Kuruscu (2002), Markov-perfect solutions potentially allow for an infinite number of discontinuous equilibria (i.e. where the policy rules are discontinuous in the state variables). When they arise, however, these equilibria typically result from assuming an infinite horizon. For the purpose of our analysis, we shall narrow our definition of a stationary Markov perfect equilibrium and focus only on the limit of the finite horizon solution described in the previous section. Assuming that the policy functions in (14) are differentiable, we show that such an equilibrium can indeed be found. In this equilibrium, taxes on labor are always zero and taxes on capital involve a single capital levy.

Given the policy rules in (14), it is useful to write down the individual maximization problem recursively. In what follows, we denote next period's value of any variable $x$ by $x'$. Let $k' = \mathcal{H}(k, b)$ and $b' = \mathcal{B}(k, b)$ represent households' optimal capital and debt accumulation decision rules under the Markov policy functions $\theta$ and $\psi$. In other words, given aggregate states $K$ and $B$ for capital and debt respectively, $\mathcal{H}$ and $\mathcal{B}$ solve the following dynamic program,

$$W(k, b, K, B) = \max_{c,n,k',b'} \left\{ u(c, 1-n) + \beta W(k', b', K', B') \right\} \tag{P1}$$

subject to

$$c + b' + k' = [(1 - \psi(K, B))r + 1 - \delta]\, k + R^b b + (1 - \theta(K, B))wn, \tag{15}$$

where $r = F_k(K, N)$, $R^{b\prime} = (1 - \psi(k', b'))F_k(k', n') + 1 - \delta$, $w = F_n(K, N)$, and aggregate outcomes are consistent with individual optimization, $K = k$ and $B = b$.

In order that the Markov policies stated in (14) constitute a subgame perfect equilibrium, no government must ever have an incentive to deviate from $\theta(k, b)$ and $\psi(k, b)$ at any time. Therefore, in principle, one needs to consider all possible deviations from (14). As shown in **????**, however, joint deviations are never optimal and it is in fact enough to consider only one-period deviations from the conjectured policy rules at any date. Put another way, in our framework, it is sufficient to analyze the problem of a government that is allowed to "cheat" in the current period by setting tax rates $\tau^n \neq \theta(k, b)$ and $\tau^k \neq \psi(k, b)$, under the assumption that $\theta(k, b)$ and $\psi(k, b)$ are forever followed in the future. Thus, taking as given household optimal behavior summarized by (5) through (7), a government at any date that is free to set current taxes solves

$$\max_{c,n,\tau^k,\tau^n,k',b'} \; u(c, 1 - n) + \beta W(k', b') \tag{P2}$$

subject to

$$\begin{aligned} c + k' + b' &= \left[(1 - \tau^k)F_k + 1 - \delta\right](k + b) + (1 - \tau^n)F_n n \qquad (16) \\ u_l(c, 1 - n) &= u_c(c, 1 - n)(1 - \tau^n)F_n \\ u_c(c, 1 - n) &= \beta \left\{ \left[(1 - \psi(k', b'))F'_k + 1 - \delta\right] u_c(\widetilde{c}(k', b', \mathcal{H}'), 1 - n') \right\} \\ \tau^k F_k(k + b) + \tau^n F_n n &= g + (F_k + 1 - \delta)b + b', \end{aligned}$$

where

$$\widetilde{c}(k', b', \mathcal{H}') = \left(F'_k + 1 - \delta\right) k' + F'_n n' - \mathcal{H}(k', b') - g'$$

obtains from the consolidation of household and government budget constraints **under...**, $F'_i = F_i(k', n')$ with $i = k, n$, and $W(k', b')$ denotes households' continuation value when they behave optimally under the Markov policy rules $\theta$ and $\psi$. Then, in order for $\theta$ and $\psi$ to constitute time consistent equilibrium policy functions, it must be the case that the optimal choices of $\tau^n$ and $\tau^k$ that solve (P2) yield precisely $\theta$ and $\psi$ respectively. Put another way, for $\theta$ and $\psi$ to be equilibrium time consistent policies, it must be the case that a government that is allowed to deviate at any date finds it in its best interest not to do so and actually follow the rules prescribed by $\theta$ and $\psi$ in (14).

**Proposition 3** *In a Markov perfect equilibrium, $\theta$ and $\psi$ are such that $\forall k$ and $b$, $\theta(k,b) = 0$ and $\psi(\mathcal{H}(k,b), \mathcal{B}(k,b)) = 0$.*

**Proof.** See Appendix B. ■

There are two important dimensions to proposition 2. First, Markov perfect taxes on labor captured by the policy function $\tau^n = \theta(k,b)$ are always zero irrespective of the state of the economy. Second, while time consistent taxes on capital, $\tau^k = \psi(k,b)$, are generally not zero for arbitrary values of $k$ and $b$, it is the case that once we begin moving forward in the dynamic program (P1) using the decision rules $k' = \mathcal{H}(k,b)$ and $b' = \mathcal{B}(k,b)$, the Markov perfect tax on capital is always zero. Observe, for instance, that two periods hence in program (P1), the values of capital, $k''$, and debt, $b''$, are simply obtained by applying the functions $\mathcal{H}$ and $\mathcal{B}$ to states $\{k', b'\}$ respectively. Therefore, since proposition 2 holds for all values of the states $k$ and $b$, it follows that $\psi(k'', b'') = 0$. The argument, of course, holds at any stage in the household dynamic program under the policy rules $\theta$ and $\psi$. The fact that $\psi(k,b)$ is generally not zero for *arbitrary* values of $k$ and $b$ simply reflects the fact that taxes have to be levied at some stage to finance the present discounted value of government expenditures. However, given arbitrary states $k$ and $b$, the functions $\mathcal{H}$ and $\mathcal{B}$ map these states, as well as all following states, into a region of the asset space where $\psi$ is always zero. Analogously to the finite horizon problem, the planner at any date chooses policy such that the resulting equilibrium asset holdings inherited by its successor place him on the surface depicted in Figure 1 where zero taxes are optimal.

Because optimal Ramsey taxes are not observed in practice, several papers in the literature have tried to develop models that generate positive distortional taxes in equilibrium. Two types of restrictions are typically called on to obtain this result. Lansing (19xx), Klein, Krusell and Rios Rull (2004), and Azzimonti, Sarte and Soares (2003), impose that the government balance its budget every period. With this restriction in place, a benevolent planner is forced to tax capital income even in the steady state. Moreover, since the absence of bonds prohibits households from undoing any initial capital levy in future periods, such a levy is not used by the planner in equilibrium. Although a balanced-budget restriction is somewhat arbitrary, it is nevertheless helpful in producing environments where one can investigate the effects of distortional taxes in equilibrium. However, this constraint also leads to sequences of capital and labor income taxes from which the government has incentives to deviate at each date. The other important restriction imposed in the literature on optimal taxation involves fixing the initial tax on capital income at some pre-specified level. This immediately rules out the possibility of using a full capital levy in period 0. Papers that follow this approach include Christiano, Chari and Kehoe (19xx), as well as Chari and Kehoe (9xx) among others. While this modeling strategy does lead to zero taxes in the steady state (in contrast to work that assumes a balanced budget restriction), the economy may experience greater distortions in the short run.

In an effort to find a tax scheme that implies explicit distortions, different authors have thus imposed alternative restrictions that rule out the capital levy suggested by the first best solution. However, our findings above indicate that, were this solution to be implemented, no government would ever choose to abandon it. Put another way, the cost – often ignored – associated with generating environments in which distortional taxes are optimal is that only time-inconsistent solutions become feasible.

Atkinson, Chari, and Kehoe (1999) analyze a case where the after tax rate on capital is bounded above by $1 - \delta < 1$ in any period. They show that the restriction binds at most for a finite number of periods, say $n > 0$, after which taxes take on an intermediate value for one period and are zero thereafter. In that world, therefore, a capital levy of sorts is not ruled out by assumption and emerges over the first few periods. Whether this solution is time-inconsistent is an open question. However, we conjecture that the commitment and Markov-perfect solutions coincide in this case. The reasoning is as follows: under commitment, a Ramsey planner wishes to set the highest feasible capital taxes that will finance enough of an asset base to eliminate distortions in the long run. Hence, he sets $\tau_t^k$ to its upper bound for $n$ periods, after which taxes take on an intermediate value for one period. Now, suppose that a government in period 1 were allowed to re-optimize. Since this government takes past decisions as given, its incentive in that period is to set a the highest possible tax on capital in an effort to increase public asset holdings. However, since the latter was already set at its maximum possible value, it ends up reproducing the commitment solution. Furthermore, the same logic applies to every government up to period $n$. Once period $n$ is reached, enough government assets have been accumulated that a date $n$ government is now in a position to implement the first best solution without violating the constraint. In effect, the problem at that date reduces to the one analyzed in this section. Thus, this argument pushes us to think further about situations where long-run distortional taxes arise both with and without commitment, even in the presence of both capital and government debt.

# 4 Optimal distortional taxes and the time inconsistency problem: an overlapping generations approach

In the environment studied by Chamley (1986), one of the fundamental reasons driving the time consistency of the optimal initial capital levy relates to households' ability to undo this burden in future periods by borrowing from the government. In essence, the presence of bonds lets the government frontload all taxes in the initial period in a lump sum fashion while still allowing households to meet their desired consumption path through sovereign lending. Using the proceeds from the lump sum tax, the government builds up enough of a public asset base at date 0 that it can credibly promise never having to set positive taxes in the future. And indeed, we showed above that when optimal policy is chosen sequentially by different governments, all with the objective of maximizing household utility, Ramsey tax rates arise.

In practice, however, the mechanism that disposes of distortional taxes breaks down simply because individuals have finite lives (among other reasons). Specifically, consider a world with overlapping generations. The current old, whom the government has an incentive to tax heavily because of their relatively inelastic capital position (i.e. their investment decisions are largely behind them and sunk), may simply not be alive in subsequent periods to borrow from the government in order to compensate for such a tax. Taking this fact into account, a planner who weighs all generations equally will not place the entire burden associated with the financing of government expenditures on the initial old generation. Therefore, even under commitment, a role for distortional taxes immediately emerges as the planner optimally apportions the cost of public spending across different generations.

More importantly, when individuals die in finite time, optimal taxes under commitment will generally be time inconsistent, thus restoring the government's ability to commit to future actions as a policy concern. In particular, because the cost of financing the present discounted value of government expenditures cannot be raised all at once, contrary to sections 2 and 3 above, the government can no longer credibly promise to set future taxes to zero. In fact, when optimal policy is chosen sequentially by different policymakers, each government in the sequence has an incentive to exploit the fact that for the old generation under their rule, past investment decisions cannot be undone and the capital stock is fixed. Therefore, one generally expects that a world with finitely lived individuals will produce time consistent optimal taxes on capital that are positive at all dates.

We can illustrate these ideas most simply by addressing the problem of optimal taxation in a two-period economy analogous to that in section 2, but with overlapping generations. For transparency, this section assumes Constant Relative Risk Aversion (CRRA) preferences and Cobb-Douglas technology. Time is indexed by $t = 0, 1$, and individuals live for two periods. The economic environment, therefore, is populated by three types of households: those who are already old at date 0, those who are born at date 0 and old and date 1, and those who are born at date 1. Let $c_{yt}$ and $c_{ot}$ denote the consumption of the young and old at time $t$ respectively. Individuals are endowed with one unit of time at birth, and use the first period of life to work and save for old age. In the second period of life, individuals use the return on their savings to finance their consumption expenditures and pass away.

Under these assumptions, consumption by the old generation at date 0 is simply given by

$$c_{o0} = \left[(1 - \tau_0^k)r_0 + 1 - \delta\right] a_0, \tag{17}$$

where $a_0 = b_0 + k_0$. From the standpoint of a Ramsey planner (i.e. one who can commit to a sequence of taxes), this generation's investment decisions are sunk, in the sense that $a_0$ is given, which creates an incentive to set $\tau_0^k > 0$. However, contrary to the representative agent construct of section 2, this generation will not be alive in the following period to make up for this tax through borrowing.

Taking policy and prices as given, the generation born at date 0 solves

$$\max_{c_{y0}, n_0, a_1, c_{o1}} \frac{c_{y0}^{1-\sigma}}{1-\sigma} - \chi \frac{n_0^{1-\nu}}{1-\nu} + \beta \frac{c_{o1}^{1-\sigma}}{1-\sigma}$$

subject to

$$\begin{aligned} c_{y0} &= (1 - \tau_0^n) w_0 n_0 - a_1, \text{ and} &(18)\\ c_{o1} &= \left[(1 - \tau_1^k) r_1 + 1 - \delta\right] a_1. &(19) \end{aligned}$$

Optimal labor and savings decisions for this generation can be summarized by

$$\chi n_0^{-\nu} = (1 - \tau_0^n) w_0 c_{y0}^{-\sigma}, \tag{20}$$

and

$$c_{y0}^{-\sigma} = \beta \left[(1 - \tau_1^k) r_1 + 1 - \delta\right] c_{o1}^{-\sigma}, \tag{21}$$

respectively.

Finally, the generation born at date 1 sets $a_2 = 0$ since the world ends on that date. Its budget constraint is given by

$$c_{y1} = (1 - \tau_1^n) w_1 n_1, \tag{22}$$

and its only relevant decision relates to optimal labor supply,

$$\chi n_1^{-\nu} = (1 - \tau_1^n) w_1 c_{y1}^{-\sigma}. \tag{23}$$

## 4.1 Optimal fiscal policy with commitment

Given a planner who weighs every generation equally, second-best Ramsey tax rates and corresponding allocations are found by solving

$$\max\ W = \underbrace{\frac{c_{o0}^{1-\sigma}}{1-\sigma}}_{\text{initial old generation}} + \underbrace{\frac{c_{y0}^{1-\sigma}}{1-\sigma} - \chi \frac{n_0^{1-\nu}}{1-\nu} + \beta \frac{c_{o1}^{1-\sigma}}{1-\sigma}}_{\text{initial young generation}} + \beta \underbrace{\left[ \frac{c_{y1}^{1-\sigma}}{1-\sigma} - \chi \frac{n_1^{1-\nu}}{1-\nu} \right]}_{\text{generation born at date 1}} \tag{P3}$$

subject to the sequence of government budget constraints,

$$g_0 + [(1 - \tau_0^k) r_0 + 1 - \delta] b_0 = \tau_0^n w_0 n_0 + \tau_0^k r_0 k_0 + b_1, \tag{24}$$

and

$$g_1 + [(1 - \tau_1^k) r_1 + 1 - \delta] b_1 = \tau_1^n w_1 n_1 + \tau_1^k r_1 k_1, \tag{25}$$

agents' budget constraints (17), (18), (19), and (22), as well as agents' optimal labor and savings decisions (20), (21), and (23).

Although closed form solutions for optimal taxes and corresponding allocations are not possible in this context, the numerical example depicted in Figure 2 is helpful in making two important points. First,.., $\tau_0^n$, $\tau_1^n \neq 0$. Talk about variations with respect to $k_0$. Second, ..., $\tau_1^k = 0$. **To be completed.**

## 4.2 Time consistent optimal fiscal policy

As in section 2, time consistent optimal tax rates can be found in two stages using backward induction. Thus, starting in the last period, the government solves

$$\max \frac{c_{o1}^{1-\sigma}}{1-\sigma} + \frac{c_{y1}^{1-\sigma}}{1-\sigma} - \chi \frac{n_1^{1-\nu}}{1-\nu}, \tag{26}$$

subject to agent's resource constraints (19) and (22), the equation describing the optimal labor-leisure decision of the generation born at date 1, (23), and the period 1 government budget constraint, (25). The set of first-order necessary conditions associated with this problem can be reduced to

$$\frac{\tau_1^n (1-\nu)}{c_{o1}^{\sigma}} + \left( \nu - \frac{\sigma w_1 n_1 (1 - \tau_1^n)}{c_{y1}} \right) \left( \frac{1}{c_{o1}^{\sigma}} - \frac{(1 - \tau_1^n)}{c_{y1}^{\sigma}} \right) = 0, \tag{27}$$

Given the states $\{k_1, b_1\}$, equations (19), (22), (23), (25), and (27), represent a set of five equations in five unknowns, $c_{o1}$, $c_{y1}$, $n_1$, $\tau_1^n$, and $\tau_1^k$.

At date 0, the government takes as given this set of equations, which implicitly determines $c_{o1}(k_1, b_1)$, $c_{y1}(k_1, b_1)$, and $n_1(k_1, b_1)$, and solves

$$\max \frac{c_{o0}^{1-\sigma}}{1-\sigma} + \frac{c_{y0}^{1-\sigma}}{1-\sigma} - \chi \frac{n_0^{1-\nu}}{1-\nu} + \beta V(k_1, b_1), \tag{28}$$

where $V(k_1, b_1) = \frac{c_{o1}(k_1,b_1)^{1-\sigma}}{1-\sigma} + \frac{c_{y1}(k_1,b_1)^{1-\sigma}}{1-\sigma} - \chi \frac{n_1(k_1,b_1)^{1-\nu}}{1-\nu}$, subject to agents' budget constraints (17) and (18), optimal behavior summarized by (20) and (21), as well as the period 0 government budget constraint (24). ....

**To be completed.**

## 5 Conclusions

**To be completed.**

## 6 Bibliography

To be completed.

# Appendix A

As stated in the text, we prove proposition 1 using backwards induction, starting with the last period.

## Optimal policy in the final period

A benevolent government in the last period is faced with states $\{k_1, b_1\}$, and maximizes household utility subject to the constraints (4) and (5), as well as the private sector decision rules (6), (7), (8), and (9), as they apply to date 1. To simplify notation, we shall use at times $u_c(t)$, $u_n(t)$, $F_k(t)$, etc... to denote time$-t$ values of the indicated functions, to be evaluated at their appropriate arguments. Thus, the Lagrangian corresponding to the date 1 problem is

$$\begin{aligned}
&\max\ u(c_1, 1-n_1) \\
&+\mu\left[u_c(1)(1-\tau_1^n)F_n(1) - u_l(1)\right] \\
&+\lambda\left[\left[F_k(1)(1-\tau_1^k)+1-\delta\right](k_1+b_1) + (1-\tau_1^n)F_n(1)n_1 - c_1\right] \\
&+\gamma\left[\tau_1^k F_k(1)(k_1+b_1) + \tau_1^n F_n(1)n_1 - g_1 - \left(F_k(1)+1-\delta\right)b_1\right].
\end{aligned} \tag{PA1}$$

The first order conditions associated with this problem are

$$c_1 \quad : \quad u_c(1) + \mu\left[u_{cc}(1)(1-\tau_1^n)F_n(1) - u_{lc}(1)\right] - \lambda = 0, \tag{29}$$

$$\tau_1^k \quad : \quad -\lambda + \gamma = 0, \tag{30}$$

$$\tau_1^n \quad : \quad -\mu u_c(1) - \lambda n_1 + \gamma n_1 = 0, \tag{31}$$

and

$$\begin{aligned}
n_1 \quad : \quad & -u_l(1) + \mu\left\{(1-\tau_1^n)\left[F_{nn}(1)u_c(1) - F_n(1)u_{cl}(1)\right] + u_{ll}(1)\right\} \\
& +\lambda\left\{F_{kn}(1)(1-\tau_1^k)(k_1+b_1) + (1-\tau_1^n)\left[F_{nn}(1)n_1 + F_n(1)\right]\right\} \\
& +\gamma\left\{\tau_1^k F_{kn}(1)(b_1+k_1) + \tau_1^n\left[F_{nn}(1)n_1 + F_n(1)\right] - F_{kn}(1)b_1\right\} \\
= \quad & 0
\end{aligned} \tag{32}$$

Given these conditions, we now solve for the decision rules, $\tau_1^n$, $\tau_1^k$, $n_1$, and $c_1$ as functions of the states, $k_1$ and $b_1$. It is easiest to first establish the following result in proposition 1.

*Result 1:* $\tau_1^n = 0$

From equations (30) and (31), we have that $\mu = 0$. This implies that

$$u_c(1) = \lambda \tag{33}$$

in (29). Furthermore, given that $\mu = 0$, that $\lambda = \gamma$, and that $F_{kn}(1)k_1 + F_{nn}(1)n_1 = 0$ under the assumption that $F$ is constant returns to scale, equation (32) reduces to

$$u_n(1) = u_c(1)F_n(1). \tag{34}$$

Since equation (6) describes households' optimal labor-leisure decision at all dates, it follows that

$$\tau_1^n = 0, \tag{35}$$

so that labor is not taxed in the final period.■

We now turn our attention to the remaining decision rules. Using the fact that $\tau_1^n = 0$, we obtain from the government budget constraint that

$$\tau_1^k = \frac{g_1 + (F_k(1) + 1 - \delta)b_1}{F_k(1)(k_1 + b_1)}, \tag{36}$$

which allows us to re-write the household resource constraint as

$$c_1 = \left[\left(1 - \frac{g_1 + (F_k(1) + 1 - \delta)b_1}{F_k(1)(k_1 + b_1)}\right) F_k(1) + 1 - \delta\right] (k_1 + b_1) + F_n(1)n_1. \tag{37}$$

This last expression yields, after some basic manipulations,

$$c_1 = F(1) - g_1 + (1 - \delta)k_1. \tag{38}$$

Therefore, we can write (34) as

$$u_n(F(k_1, n_1) - g_1 + (1 - \delta)k_1, 1 - n_1) = u_c\left(F(k_1, n_1) - g_1 + (1 - \delta)k_1, 1 - n_1\right) F_n(k_1, n_1)$$

which defines the solution for labor in the last period as a function of $k_1$, and the exogenous variable, $g_1$,

$$n_1 \equiv n_1(k_1). \tag{39}$$

Observe that $n_1$ is independent of the level of debt inherited in period 1. Moreover, we can then substitute (39) in equations (36) and (38) to obtain policy functions for the capital tax rate and consumption at date 1 respectively,

$$\tau_1^k \equiv \tau_1^k(k_1, b_1) \text{ and} \tag{40}$$

$$c_1 \equiv c_1(k_1), \tag{41}$$

where the level of consumption, $c_1$, in (38) is independent of $b_1$. Equations (35), (39), (40), and (41) are the optimal decision rules, as functions of the state, associated with the period 1 problem.

We now turn to the optimal policy problem from the standpoint of the initial period and address the remaining results in proposition 1.

## Optimal policy in period zero

Knowing how a benevolent government behaves in the last period, so that one can anticipate what tax rates emerge given the relevant states for that date, $\{k_1, b_1\}$, one can address the problem of optimal fiscal policy from the standpoint of date 0. The relevant Lagrangian for period 0 is

$$\begin{aligned}
&\max\ u(c_0, 1-n_0) + \beta\overbrace{u(c_1(k_1,), 1-n_1(k_1))}^{V(k_1)} \qquad \text{(PA2)}\\
&+\mu\left[u_c(0)(1-\tau_0^n)F_n(0) - u_l(0)\right]\\
&+\lambda\left\{u_c(0) - \beta\left[F_k(k_1, n_1(k_1))(1-\tau_1^k(k_1,b_1)) + 1 - \delta\right]u_c(c_1(k_1), n_1(k_1))\right\}\\
&+\gamma\left[\left(F_k(0)(1-\tau_0^k) + 1 - \delta\right)(k_0+b_0) + (1-\tau_0^n)F_n(0)n_0 - c_0 - k_1 - b_1\right]\\
&+\omega\left[\tau_0^k F_k(0)(k_0+b_0) + \tau_0^n F_n(0)n_0 - g_0 - \left(F_k(0)+1-\delta\right)b_0 + b_1\right],
\end{aligned}$$

where, for transparency, we have written solutions for date 1 variables explicitly in terms of the date 1 states as worked out above.

The first order conditions associated with (PA2) yield

$$c_0 \quad : \quad u_c(0) + \mu\left[u_{cc}(0)(1-\tau_0^n)F_n(0) - u_{lc}(0)\right] + \lambda u_{cc}(0) - \gamma = 0, \tag{42}$$

$$\tau_0^k \quad : \quad -\gamma + \omega = 0, \tag{43}$$

$$\tau_0^n \quad : \quad -\mu u_c(0) - \gamma n_0 + \omega n_0 = 0. \tag{44}$$

The optimal choice of bonds to be carried into date 1 yields

$$b_1 : \lambda\beta F_k(1)\tau_b^k(1)u_c(1) - \gamma + \omega = 0, \tag{45}$$

Observe that, since $\gamma = \omega$ in (43), (44) implies $\mu = 0$ and (45) implies $\lambda = 0$.

The optimality condition with respect to $k_1$ is simply

$$k_1 : \beta\left[u_c(1)c_k(1) - u_l(1)n_k(1)\right] - \gamma = 0. \tag{46}$$

The optimal allocation of labor is given by

$$\begin{aligned}
n_0 \quad : \quad & -u_l(0) + \qquad \text{(47)}\\
& +\gamma\left\{F_{kn}(0)(1-\tau_0^k)(k_0+b_0) + (1-\tau_0^n)\left[F_{nn}(0)n_0 + F_n(0)\right]\right\}\\
& +\omega\left\{\tau_0^k F_{kn}(0)(b_0+k_0) + \tau_0^n\left[F_{nn}(0)n_0 + F_n(0)\right] - F_{kn}(0)b_0\right\}\\
= \quad & 0.
\end{aligned}$$

We are now in a position to establish the following results regarding the optimal labor tax at date 0 and the optimal capital tax rate in the final period.

*Result 2:* $\tau_0^n = 0$ and $\tau_1^k \equiv \tau_1^k(k_1, b_1) = 0$.

Observe that (44) implies $\mu = 0$ since $\gamma = \omega$ in (43), and that equation (42) reduces to

$$u_c(0) = \gamma. \tag{48}$$

Hence, the optimal allocation of labor in (47) simplifies to

$$u_l(0) = u_c(0) F_n(0) \tag{49}$$

As before, since equation (6) holds in every period, it follows that

$$\tau_0^n = 0, \tag{50}$$

which confirms the first part of result 2. Moreover, note that so far, results 1 and 2 combine to give us $\tau_0^n = \tau_1^n = 0$. To prove the second part of result 2, take the derivative of (38) with respect to $k_1$ to obtain

$$c_k(1) = F_k(1) + F_n(1) n_k(1) + (1 - \delta).$$

Consequently, using (34), equation (46) simplifies to

$$u_c(0) = \beta \left[ u_c(1) F_k(1) + 1 - \delta \right]. \tag{51}$$

Therefore, by equation (7), it follows that

$$\tau_1^k = 0, \tag{52}$$

which establishes the second part of Result 2.■

As in the steady state associated with the Ramsey problem, time consistent tax rates on labor and capital are zero in the last period. It remains to find the policy functions for $\tau_0^k$, $n_0$, $c_0$, $k_1$ and $b_1$, given the initial levels of private capital and public debt, $k_0$ and $b_0$.

Given (52), equation (36) implies that

$$b_1 = \frac{-g_1}{F_k(1) + 1 - \delta}, \tag{53}$$

which is to say that government purchases at date 1 are financed solely by interest payments on loans made to the private sector in the initial period. Using the fact that $\tau_0^n = 0$ and the expression for $b_1$ above, we can re-write the government budget constraint at date 0 as

$$\tau_0^k = \frac{g_0 + (F_k(0) + 1 - \delta)b_0 + \frac{g_1}{F_k(1)+1-\delta}}{F_k(0)(k_0 + b_0)}. \tag{54}$$

Therefore, the size of the initial tax on capital is equal to that of the present discounted value of public expenditures less the return on any initial assets owned by the government. Note that *ceteris paribus*, $\tau_0^k$ can be negative if the government starts off with a large enough assets.

From the household's resource constraint, we have that

$$c_0 = F(0) - g_0 + (1 - \delta)k_1 - k_0. \tag{55}$$

Therefore, equation (51) can be re-written as

$$\begin{aligned} & u_c(F(k_0, n_0) - g_0 + (1 - \delta)k_0 - k_1, 1 - n_0) \qquad (56) \\ = \quad & \beta u_c(c_1(k_1), 1 - n_1(k_1)) \left[F_k(k_1, n_1(k_1)) + 1 - \delta\right] \end{aligned}$$

Observe also that equation (49) can be expressed as

$$\begin{aligned} & u_l(F(k_0, n_0) - g_0 + (1 - \delta)k_0 - k_1, 1 - n_0) \qquad (57) \\ = \quad & u_c(F(k_0, n_0) - g_0 + (1 - \delta)k_0 - k_1, 1 - n_0)F_n(k_0, n_0) \end{aligned}$$

Given the initial capital stock, $k_0$ (and exogenous government spending $g_0$), equations (56) and (57) make up a system of two equations in two unknowns, $n_0$ and $k_1$, which define the policy functions

$$n_0 \equiv n_0(k_0) \tag{58}$$

and

$$k_1 \equiv k_1(k_0). \tag{59}$$

Finally, we can now use (58) and (59) in equations (53), (54), and (55) to formally define the remaining policy functions $b_1(k_0)$, $\tau_0^k(k_0, b_0)$, and $c_0(k_0)$ respectively.■

## Appendix B

The Lagrangian corresponding to problem (P2) is

$$
\begin{aligned}
&\max_{c,n,\tau^k,\tau^n,k',b'} \; u(c,1-n)+\beta W(k',b') \\
&+\mu\left\{\left[(1-\tau^k)F_k+1-\delta\right](k+b)+(1-\tau^n)F_n n-k'-b'-c\right\} \\
&+\lambda\left\{u_l(c,1-n)-u_c(c,1-n)(1-\tau^n)F_n\right\} \\
&+\gamma\left(u_c(c,1-n)-\beta\left\{\left[(1-\psi(k',b'))F'_k+1-\delta\right]u_c(\widetilde{c}(k',b',H'),1-n')\right\}\right) \\
&+\omega\left\{\tau^k F_k(k+b)+\tau^n F_n n-g-(F_k+1-\delta)b+b'\right\},
\end{aligned}
\tag{PA3}
$$

where

$$\widetilde{c}(k',b',H')=\left(F'_k+1-\delta\right)k'+F'_n n'-H(k',b')-g',$$

denotes next period's consumption under the assumption that tomorrow's government abides by the Markov policy rules $\theta$ and $\psi$.

Since the one period deviation considered in (PA3) is such that households behave optimally thereafter under the Markov policy rules $\theta$ and $\psi$, observe from the dynamic program in (P2) that

$$W(k,b)=u(c(k,b),1-n(k,b))+\beta W(k',b')$$

where

$$c(k,b)=\left[(1-\psi(k,b))F_k+1-\delta\right](k+b)+(1-\theta(k,b)F_n n-\mathcal{H}(k,b)-\mathcal{B}(k,b)$$

and $n(k,b)$ solves

$$u_l(c(k,b),1-n)=u_c(c(k,b),1-n)F_n(1-\theta(k,b)).$$

The first order conditions associated with problem (PA3) are as follows (for notational convenience, when the context is clear, we do not explicitly write $u_c$, $F_k$, $F_n$, etc... as a function of their arguments):

$$
\begin{aligned}
\tau^k \quad &: \quad -\mu F_k(k+b)+\omega F_k(k+b)=0 \\
&\Rightarrow \quad \mu=\omega.
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
\tau^n \quad &: \quad -\mu F_n n+\lambda F_n u_c+\omega F_n n \\
&\Rightarrow \quad \lambda=0.
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
b' \quad : \quad & \beta W_b(k', b') - \mu \\
& -\gamma\beta \left\{ -\psi_b' F_k' u_c(\widetilde{c}', 1-n') + \left[(1-\psi')F_k' + 1 - \delta\right] \left[u_{cc}' \widetilde{c}_b' - u_{cl} n_b'\right] \right\} \\
& +\omega \\
= \quad & 0.
\end{aligned}
$$

Since $\mu = \omega$, this last expression reduces to

$$
W_b(k', b') = \gamma \left\{ -\psi_b' F_k' u_c(\widetilde{c}', 1-n') + \left[(1-\psi')F_k' + 1 - \delta\right] \left[u_{cc}' \widetilde{c}_b' - u_l n_b'\right] \right\}. \tag{62}
$$

$$
\begin{aligned}
c \quad : \quad & u_c - \mu + \lambda \left[u_{lc} - u_{cc}(1-\tau^n)F_n\right] + \gamma u_{cc} = 0 \\
\Rightarrow \quad & u_c + \gamma u_{cc} = \mu \text{ since } \lambda = 0.
\end{aligned} \tag{63}
$$

After some manipulations, the first-order conditions with respect to $n$ yields

$$
n : u_l = \mu F_n - u_{cl}\gamma. \tag{64}
$$

Finally, the optimality condition with respect to tomorrow's capital stock reads as

$$
\begin{aligned}
& \beta W_k'(k', b') - \mu - \\
& \gamma\beta \left\{ \left(-\psi_k' F_k' + (1-\psi') \left[F_{kk}' + F_{kn}' n_k'\right]\right) u_c' + \left[(1-\psi')F_k' + 1 - \delta\right] \left[u_{cc}' \widetilde{c}_k' - u_{cl} n_k'\right] \right\} \\
= \quad & 0.
\end{aligned} \tag{65}
$$

To arrive at the results stated in proposition 2, as well as to highlight some interesting properties of the problem above, we begin by focusing on equation (62). In particular, let us conjecture for now that $\mathcal{H}(k, b) = \mathcal{H}(k)$ and $W_b(k, b) = 0$.In other words, we conjecture that the value function associated with problem (PA3) is independent of the level of debt along the equilibrium path. Then, since the expression in brackets is strictly non-zero, we have that

$$
\gamma = 0. \tag{66}
$$

It immediately follows that

$$
u_c = \mu \tag{67}
$$

and that

$$u_l = u_c F_n \tag{68}$$

from equations (63) and (64) respectively. In particular, this last equation implies that $\tau^n = 0$ so that, in a Markov perfect equilibrium, $\theta(k, b) = 0\ \forall k$ and $b$. This establishes the first part of proposition 2.

From the first-order condition with respect to $k'$, we have that

$$u_c = \beta W'_k, \tag{69}$$

where

$$W'_k = u'_c \left[F'_k + F'_n n'_k + 1 - \delta - \mathcal{H}'_k\right] - u'_l n'_k + \beta(W''_k \mathcal{H}'_k + \underbrace{W''_b \mathcal{B}'_k}_{=0}).$$

Since $u'_l = u'_c F'_n$ and $W''_k = u'_c/\beta$ from equations (68) and (69) respectively, this last equation simplifies to

$$u_c = \beta \left[F'_k + 1 - \delta\right] u'_c,$$

from which it follows that $\psi(k', b') = 0$ when $k'$ and $b'$ are chosen optimally. Put another way, $\psi(\mathcal{H}(k, b), \mathcal{B}(k, b)) = 0$, which establishes the second part of proposition 2.

It now remains to verify that our conjectures $\mathcal{H}(k, b) = \mathcal{H}(k)$ and $W_b(k, b) = 0$ is actually correct in a Markov perfect equilibrium.

Evaluating the agents' first order condition for leisure in the political equilibrium, we see that labor is a function of capital holdings and savings (recall that $g$ is exogenously given): $n(k, k')$.

$$u_l(F(k, n) + (1 - \delta)k - k' - g, 1 - n) = u_c(F(k, n) + (1 - \delta)k - k' - g, 1 - n)F_n(k, n).$$

Replacing $n(k, k')$ and into the optimality condition for savings, we realize that $k'$ is indeed independent of bond holdings,

$$u_c(F(k, n(k, k')) + (1 - \delta)k - k' - g, 1 - n(k, k')) = \beta \left[F_k(k', n(k, k')) + 1 - \delta\right]$$
$$u_c(F(k', n(k', k'')) + (1 - \delta)k - k'' - g, 1 - n(k', k'')),$$

since $k' = \mathcal{H}(k)$ and $k'' = \mathcal{H}(\mathcal{H}(k))$ satisfy this equation.

The optimal labor choice is then, $n(k, \mathcal{H}(k)) \equiv \eta(k)$ and consumption equals $\mathfrak{c}(k) = F(k, \eta(k)) + (1 - \delta)k - \mathcal{H}(k) - g$. Finally, the value function becomes, $W(k, b) = u(\mathfrak{c}(k), 1 - \eta(k)) + \beta W(\mathcal{H}(k)) = W(k)$, which is independent of $b$.