Spurious Investment Specific Technological Change

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Abstract

We introduce a vintage capital model in which workers are matched with machines of increasing quality. Quality improvements of the machines are the sole source of technological change in this economy. However, the matching of workers with machines implies that there is no well defined capital aggregate in this economy. Hence, investment price indices are a spurious measure of price changes in capital goods. We show that the use of such spurious measures of investment price changes in conjunction with standard growth accounting methods can lead to the severe overstatement of the importance of quality improvements of capital goods in our model. That is it can lead to spurious measures of investment specific technological change.

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1 Introduction

Many recent empirical studies of technological change have used changes in the relative price of investment goods with respect to consumption goods as a measure of the degree of investment specific or embodied technological change. These studies include, among others, Greenwood, Hercowitz, and Krusell (1997,2000), Violante, Ohanian, Ríos-Rull, and Krusell (2000), Cummins and Violante (2002), Fisher (2002), and Altig, Christiano, Eichenbaum, and Linde (2005).

Greenwood, Hercowitz, and Krusell (1997) were the first to use the changes in a quality adjusted capital price index relative to the changes in the consumption price index as a measure of investment specific technological change in a general equilibrium framework. They use the capital price index to decompose productivity growth in to disembodied Total Factor Productivity (TFP) growth and growth induced by the decline of the quality adjusted relative price of capital goods, known as investment specific technological change.

Their analysis yields the result that, since the middle of the 1970's, the quality adjusted relative price decline of investment goods has accelerated, therefore increasing the contribution of investment specific technological change to U.S. output growth. This, in principle, is not inconsistent with the observation that quality improvements in computers and other IT capital goods have accelerated since the middle of the 1970's. There is, however, one catch.

The results in Greenwood, Hercowitz, and Krusell (1997) also yield that the rate of investment specific technological change measured using the investment price index that they use implies that TFP growth in the U.S. has been persistently negative between 1973 and 1990. The average annual decline in TFP for the period between 1973 and 1990 reported in their analysis is 0.9%.

In principle, it is not hard to come up with an explanation why TFP could temporarily decline. It is much harder, however, to come up with a story why TFP would decline persistently over a 17 year long period and why this decline would exactly coincide with the time that investment specific technological change accelerates. This begs the question whether the price index representing the relative price of capital might not be the appropriate measure of investment specific technological change and whether it might overstate the contribution of quality improvements of capital goods to economic growth.

In this paper we introduce a model in which it is the case that the application of such a price index would overstate the importance of quality improvements of capital goods for economic growth. Our model is a vintage capital model, in the spirit of Johansen (1959), Arrow (1962), and Jovanovic (1999,2004). In it, workers of different skill levels are matched with machines of different and increasing quality. The quality improvements of machines are the sole source of economic growth in our model. Each worker can only use one machine, such that the capital labor ratio is fixed. The assignment of workers across machines means that capital vintages and labor are intertwined to such a degree that there is no aggregate production function representation in terms of labor and an aggregate capital stock.

The non-existence of an aggregate capital stock is nothing new. Fisher (1969) already showed that, in case of embodied technological change, such a capital stock only exists if the vintage specific production functions are Cobb-Douglas. Because of the fixed capital labor ratio, in our model the vintage specific production functions are Leontieff instead. The problem is that the spurious application of a capital price index in the absence of an aggregate capital stock can lead to very deceiving conclusions.

We show that when the same growth accounting methods used by Greenwood, Hercowitz, and Krusell (1997) are applied in our model they would overestimate the contribution of capital deepening, i.e. quality improvements of capital goods, to output growth and would lead to a downward bias in estimated TFP growth.

The structure of this paper is as follows. In the next section we introduce our model economy. Because our argument does not hinge on transitional dynamics, we consider a model that is always on its balanced growth path. In Section 3 we derive the equilibrium balanced growth path of the economy and proof its relevant properties. In Section 4 we ask two questions. The first is what we would like a growth accounting analysis in this economy to yield. We derive an aggregate production function representation, show that there is no aggregate capital stock and that there is no TFP growth. The second question is what we would actually measure if we would apply the growth accounting methods applied by Greenwood, Hercowitz, and Krusell (1997) and would use a capital price index to identify quality improvements in capital goods. We show that there is nothing that assures us that we obtain the proper measure of TFP growth from this growth accounting exercise. In Section 5, we back this claim up with a numerical example in which we choose the parameters in our model to satisfy long-run properties of the U.S. economy. For these parameter values, the growth accounting exercise severely overstates quality improvements in capital goods and yields negative TFP growth, just like Greenwood, Hercowitz, and Krusell (1997) find for 1973-1990. In Section 6 we discuss the implications of our results for the analysis of productivity. We conclude in Section 7. Mathematical details are left for Appendix A.

2 Model

The model that we introduce is a model of endogenous embodied technological change. In our model a continuum of workers with heterogenous levels of human capital in each period choose a type of machine that they use to produce a homogenous final good. The machines are supplied by a set of firms that compete monopolistically. These firms each bid for a licensing fee to produce the type of machines that they supply. This licensing fee in its turn provides the resources for the R&D necessary to come up with a new machine. The final good is used as a consumption good and provides the workers with utility.

The main results of this paper are easiest explained along a balanced growth path. For this reason, we develop a model economy that is always on its balanced growth path. This allows us to make the simplifying assumptions of linear preferences and innovations of equal size at a constant frequency.

The following four subsections introduce the household, final goods, capital goods, and R&D sectors of our model economy respectively.

2.1 Households

A household in our economy consists of one infintely-lived worker. All households have linear preferences in the sense that a household, which, for reasons explained below, we index by h, that consumes $c_{t+s}(h)$ for $s = 0, 1, 2, \ldots$ gets the following level of utility

$$\sum_{s=0}^{\infty} \beta^s c_{t+s}(h) \text{ where } 0 < \beta < 1 \tag{1}$$

The household maximizes this objective subject to the intertemporal budget constraint that

$$a_{t+s+1}(h) = (1 + r_{t+s})\, a_{t+s}(h) + w_{t+s}(h) + \pi_{t+s} - c_{t+s}(h) \tag{2}$$

Here $a_{t+s}(h)$ denote the assetholdings of the household, in terms of the consumption good, at the beginning of period $t + s$, r_{t+s} is the real interest rate at time $t + s$, $w_{t+s}(h)$ is the labor income the household makes of running its own business, π_{t+s} are the dividend payments that the household receives over the shares it owns in capital goods producing and research and development firms[1].

The households in this economy will thus make choices to do two things. The first is that they will make choices that maximize the present discounted value of their proprietor's income. The second is that their intertemporal optimality condition implies that for consumption to be positive in each period, the real interest rate has to satisfy

$$r_t = \frac{1-\beta}{\beta} \equiv r \text{ for all } t \tag{3}$$

which is what we will assume throughout the rest of this paper.

2.2 Final goods producers

Each worker in our model economy supplies its labor to produce a homogenous final good. The labor is supplied to competitive firms in the final goods sector.

We will take a certain degree of heterogeneity among workers as given. The relevant dimension of heterogeneity across workers is their human capital levels. We denote the human capital level of a particular worker by h. There is a continuum of workers of measure one whose human capital levels are uniformly distributed over the unit-interval, such that $h \backsim unif\,(0, 1)$.

Each worker produces a homogenous final (consumption) good by combining the unit of labor supplied by a worker, of type h, with one machine.

Just like workers, machines are also heterogenous in this economy. There is a countable set of machines in each period. We denote a particular type,

[1] We will assume that the shares in these firms are equally distributed among the households, because of which they all get equal dividend payments. However, as Caselli and Ventura (2000) show, the aggregate behavior of our economy will not depend on the distribution of shares.

or vintage, of machine by τ[2]. Each vintage of machine embodies a different quality, where $A_{t-\tau} > 0$ denotes the number of efficiency units embodied in a machine of vintage age τ. Throughout, we will assume that there is no technological regress such that $A_t - A_{t-1} > 0$ for all t.

The combination of a worker of type h and a machine of vintage age τ yields $hA_{t-\tau}$ units of output[3].

In order to avoid having to consider intractable intertemporal optimization problems and having to make assumptions about possible second hand markets, we will assume that machines fully depreciate in one period. This assumption basically implies that the machines considered here are equivalent to intermediate goods in the sense of Aghion and Howitt (1992) and Romer (1990). The workers can not use these machines for nothing.

The workers can not use these machines for nothing. The price of a machine of quality $A_{t-\tau}$ at time t is $P_{t,\tau}$. This price is measured in units of the final good, which we will use as the numeraire good throughout.

Given this production technology, vintage profile of prices and the menu of available vintages of machines, in each period workers choose, from this menu, the type of machine that maximizes their proprietor's income level. This income level is the difference between the revenue generated by the sale of the final goods produced and the cost of the machine used to produce them.

That is, if $\mathbf{T}_t$ denotes the set of available technology vintage ages and $\mathbf{A}_t$ the set of associated productivity levels of the machines, then a worker with human capital level h will choose a technology from the technology choice set $\Upsilon_t(h)$, which is defined as

$$\Upsilon_t(h) = \left\{ \tau \in \mathbf{T}_t \middle| \tau \in \arg\max_{s \in \mathbf{T}_t} \{hA_{t-s} - P_{t,s}\} \right\}$$

Let $w_t(h)$ be the wage rate of a worker with human capital level h, then competition and free entry on the demand side of the labor market implies

[2]The notational convention that we will use in this paper follows Chari and Hopenhayn (1991) in the sense that τ represents 'vintage age'. That is, A_t represents the frontier technology level and $A_{t-\tau}$ is the frontier technology level of τ periods ago. For notational convenience, we will, every once in a while, switch between the notation of technology in its levels, i.e. A_t, and technology growth rates, i.e. $g_t = \frac{A_t - A_{t-1}}{A_{t-1}}$.

[3]This setup of the production function is similar to the preference setup used by Bresnahan (1981) to estimate marginal cost profiles and markups in the American Automobile Industry.

zero profits such that the wage rate of a worker with skill level h equals revenue minus capital expenditures. Mathematically, this implies

$$w_t(h) = hA_{t-\tau} - P_{t,\tau}, \text{ for all } \tau \in \Upsilon_t(h) \tag{4}$$

When we aggregate over workers of all human capital levels, we obtain the relevant demand sets. Let $\mathbf{P}_t$ be the vector of prices charged for the available machines, then the set of buyers of machines of vintage age τ, which we will denote by $D_t(\tau, \mathbf{P}_t, \mathbf{A}_t)$, is given by

$$D_t(\tau, \mathbf{P}_t, \mathbf{A}_t) = \left\{ h \in [0,1] \middle| \tau \in \arg\max_{s \in \mathbf{T}_t} (hA_{t-s} - P_{t,s}) \right\} \tag{5}$$

These sets determine the demand for each of the available vintages of machines.

2.3 Capital goods producers

Machine designs are assumed to be patented for M periods and each period there is one new machine design patented.

During the first M periods of a machine design's life, the particular machine is supplied by a monopolist firm. After the patent expires the machine design is public domain and there is perfect competition in the supply of these machines.

In order to show the generality of our results we will allow for one monopolist selling all M patented machines, M monopolistic competitors that each sell one particular vintage of machine, or any case in between.

Hence, each supplier may supply more than one patented machine design. We will denote the number of supplies of patented machines by $N \leq M$ and index them by n. The function $\iota_t(\tau)$ identifies the supplier of machines of vintage τ.

The technology used to produce machines is as follows. Units of the final (consumption) good are only input needed in machine production. We make this assumption to avoid having to deal with the selection of workers across the final goods and capital good producing sectors. Production of a continuum of mass $K_{t,\tau}$ of machines of type $A_{t-\tau}$ requires the use of

$$\frac{c_\tau}{2} A_{t-\tau} K_{t,\tau}^2 \tag{6}$$

units of the final good. The cost parameter $c_\tau > 0$ depends vintage age, in order to allow us to take into account potential learning by doing in the production of machines. For example, Irwin and Klenow (1994) show that learning by doing effects are important in the production of random access memory (RAM) chips.

The question that is left is how these machine producers end up choosing the prices of their machines. Suppose supplier n supplies a total of v_n vintages. Let $\tau_{n_1}, \tau_{n_2}, \tau_{n_3}, ...\tau_{n_{v_n}}$ be the vintages supplied by supplier n. Then the vector of prices chosen by supplier n can be denoted

$$\mathbf{P}_{t,n} = \left\{ P_{t,\tau_{n_1}}, \ldots, P_{t,\tau_{n_{v_n}}} \right\}. \tag{7}$$

Throughout this paper, we will focus on Pure Strategy Nash (PSN) equilibria. For the particular problem at hand here this implies that supplier n takes the prices set by the other supplies, which we will denote by the vector $\mathbf{P}'_{t,n}$, and the productivity levels the machines, i.e. the $A_{t-\tau}$ for $\tau \in \mathbf{T}_t$, as given.

Given these variables, producer n chooses the prices of his machines to maximize profits. This implies that $\mathbf{P}_{t,n}$ is an element of the best response set

$$BR_t\left(\tau; \mathbf{P}'_{t,n}, \mathbf{A}_t\right) = \left\{ \mathbf{P}_{t,n} \in \mathbb{R}_+^{v_n} \middle| \mathbf{P}_{t,n} \in \underset{\mathbf{P}\in\mathbb{R}_+^{v_n}}{\arg\max} \left\{ \sum_{i=1}^{v_n} \left(\mathbf{P}_i K_{t,\tau_{n_i}} - \frac{c_{\tau_{n_i}}}{2} A_{t-\tau_{n_i}} K^2_{t,\tau_{n_i}} \right) \right\} \right\} \tag{8}$$

Where $K_{t,\tau_{n_i}}$ equals the mass of workers that demand machines of vintage age τ_{n_i} at the prices set[4].

Because patents expire after M periods, these best response sets only apply to $\tau = 0, \ldots, M-1$. For machines that were designed M or more periods ago, perfect competition implies that price must equal average cost and that free entry drives both to zero. Hence, $P_{t,\tau} = 0$ for $\tau \geq M$.

The corresponding profits are

$$\pi_{t,n} = \sum_{i=1}^{v_n} \left(\mathbf{P}_i K_{t,\tau_{n_i}} - \frac{c_{\tau_{n_i}}}{2} A_{t-\tau_{n_i}} K^2_{t,\tau_{n_i}} \right) \text{ for all } \mathbf{P}_{t,n} \in BR_t\left(\tau; \mathbf{P}'_{t,n}, \mathbf{A}_t\right) \tag{9}$$

for $\tau = 0, \ldots, M-1$ and are zero for $\tau \geq M$.

[4]Formally, $X_{t,\tau_{n_i}}$ is the Lebesque measure of the demand set $D\left(\tau_{n_i}, \left(\mathbf{P}'_{t,n}, \mathbf{P}_{t,n}\right), \mathbf{A}_t\right)$.

2.4 Patent race and innovation

3 Equilibrium and balanced growth

In this section we derive the equilibrium outcome and prove the relevant properties of this economy along its balanced growth path. These are the properties that drive our spurious investment specific technological change

We derive the equilibrium in four steps. First of all, we solve for the machine demand decisions made by the workers in the final goods sector. Secondly, we obtain the optimal price setting strategies by the suppliers of the different vintages of machines in response to the demand functions derived in the first step. Thirdly, we derive the equilibrium in the R&D sector and the implied pattern of innovations financed by the profits made by the capital goods producers and derived in the second step. Finally, we combine the results of the first three steps to derive the balanced growth path our model economy. We only describe the main results and their intuition here. The details of the derivations are left for Appendix A.

3.1 Demand for machines

Because our setup in the final goods sector is similar to that of the car market in Bresnahan (1981), so are the resulting demand functions. They satisfy the following two main properties, independent of the set of technologies sold, i.e. $\mathbf{A}_t$, and the prices set for the patented designs, i.e. $\mathbf{P}_t$.

First of all, better workers end up using better machines. That is, there is endogenous assortative matching between workers and machines. Mathematically, this can be written as

$$\text{For } h' > h, \text{ if } h \in D_t(\tau, \mathbf{P}_t, \mathbf{A}_t) \text{ then } h' \notin D_t(\tau', \mathbf{P}_t, \mathbf{A}_t) \text{ forall } \tau' > \tau. \tag{10}$$

Assortative matching between machines and workers is a natural outcome when a technology exhibits capital-skill complementaries, like in the final goods sector in our model. Jovanovic (1999) is an example where this is the case as well.

This assortative matching result also implies that the demand sets are connected. That is, for vintages of machines for which there is positive demand, they are of the form

$$D_t(\tau, \mathbf{P}_t, \mathbf{A}_t) = \left(\underline{h}_{t,\tau}, \overline{h}_{t,\tau}\right] \text{ where } 0 \leq \underline{h}_{t,\tau} < \overline{h}_{t,\tau} \leq 1 \tag{11}$$

where the upper and lowerbounds of the set are determined by the prices and in the productivity levels of the vintages sold. It also follows from this assortative matching result that the set of workers that is indifferent between the choice of two machines is negligible. That is, the size of these demand sets, and thus the demand for each of the different vintages, is uniquely determined by the prices that are set and the productivity levels of the machines.

Secondly, perfect competition for the machines of vintage age M and older implies that machines of a design older than M, i.e. a design for which the patent has expired for more than one period, are not demanded anymore. Their demand set is the empty set in equilibrium. That is,

$$D_t\left(\tau, \mathbf{P}_t, \mathbf{A}_t\right) = \emptyset \text{ for } \tau > M \tag{12}$$

The derivation of this result is straightforward. The quadratic production technology for machines implies that perfect competition on the machines for which the patent has expired will drive their price to zero. Among the machines that are essentially free, the workers will always choose the best one, i.e. $\tau = M$, and will not use machines of an older vintage age.

3.2 Price schedule of machines

The properties of the demand sets proven above imply that the amount of machines of vintage age τ equals

$$K_{t,\tau} = \overline{h}_{t,\tau} - \underline{h}_{t,\tau} \tag{13}$$

This result can be used to derive the equilibrium price schedule of machines. Before doing so, we first formally define what we mean by the PSN price setting equilibrium in this market.

For a given set of available technologies, $\mathbf{A}_t$, a PSN equilibrium price schedule $\mathbf{P}_t^* = \left\{\mathbf{P}_{t,1}^*, \ldots, \mathbf{P}_{t,N}^*\right\}$ in this market satisfies two properties. First of all, for those vintages for which the patent has expired the price is zero. That is, $P_{t,\tau} = 0$ for all $\tau \geq M$. Secondly, each of the suppliers of patented vintage of machines chooses its prices as part of its best response set with respect to the prices set by the other producers. That is, let $\mathbf{P}_{t,n}^*$ be the prices set by supplier n for the machines it supplies and let $\mathbf{P}_{t,n}^{*\prime}$ the prices set by the other producers in the PSN equilibrium, then

$$\mathbf{P}_{t,n}^* \in BR_t\left(\tau; \mathbf{P}_{t,n}^{*\prime}, \mathbf{A}_t\right) \text{ for all } n = 1, \ldots, N \tag{14}$$

It turns out that, for all possible technology menus $\mathbf{A}_t$ and all possible permutations of suppliers over the M patented vintages, there exists a unique equilibrium price schedule. The equilibrium price schedule has several relevant properties that are independent of $\mathbf{A}_t$, of the way suppliers are distributed over the M newest machine designs, and of the cost coefficients $\{c_\tau\}_{\tau=0}^{\infty}$. The existence and uniqueness of the price schedule as well as the details underlying the properties are derived in A. Here we limit ourselves to the description of the properties that are relevant for the rest of our analysis.

The first property is that, in equilibrium, prices are set such that there is strictly positive demand for all M patented vintages. Mathematically, this boils down to that

$$D_t\left(\tau, \mathbf{P}_t, \mathbf{A}_t\right) \neq \emptyset \text{ for } \tau = 0, \ldots, M \tag{15}$$

in the PSN price setting equilibrium.

The second property is that in this equilibrium suppliers make strictly positive profits of the supply of each of the individual patented designs. That is,

$$P_{t,\tau} > c_\tau A_{t-\tau} K_{t,\tau} > 0 \text{ for all } \tau = 0, \ldots, M-1 \tag{16}$$

such that for each patented vintage, all of which are produced with a decreasing returns to scale technology, price exceeds average cost and thus profits are strictly positive.

The final two properties are most easily written in terms of prices per efficiency units. For this purpose, we define the price per efficiency unit of a machine of vintage age τ as $\widehat{P}_{t,\tau} \equiv P_{t,\tau}/A_{t-\tau}$.

In terms of the price schedule per efficiency unit, the third relevant property for what is to come is that prices per efficiency unit are increasing in the quality of the machines. Formally,

$$\widehat{P}_{t,\tau} \text{ is strictly decreasing in } \tau \tag{17}$$

That is, the older the vintage age of the machine, the lower the quality, and the lower the price per efficiency unit.

The final property of the price per efficiency unit schedule is that it only depends on the cost parameters, $\{c_\tau\}_{\tau=0}^{M-1}$, the patent length, M, and the productivity profile of the vintages, $\mathbf{A}_t = \{A_t, \ldots, A_{t-M}\}$. Moreover the price per efficiency unit schedule is homogenous of degree zero in the productivity levels of the vintages.

Formally, let $\widehat{\mathbf{P}}_t^*$ be the equilibrium schedule of prices per efficiency unit, then this last property implies

$$\widehat{\mathbf{P}}_t^* = \widehat{\mathbf{P}}\left(\mathbf{A}_t, \{c_\tau\}_{\tau=0}^{M-1}\right) \tag{18}$$

such that $\widehat{\mathbf{P}}_t^*$ is solely a function of the cost parameters for the vintages sold in the market, i.e. $\{c_\tau\}_{\tau=0}^{M-1}$, and the productivity profile, i.e. $\mathbf{A}_t$. Furthermore, the function $\widehat{\mathbf{P}}$ is homogenous of degree zero in $\mathbf{A}_t$. This property is relevant, because along the balanced growth path of our model the productivity profile will growt at a constant rate and thus the price per efficiency unit profile across vintages of machines will be constant.

3.3 Innovations and technological progress

3.4 Balanced growth path

4 Measurement

Now that we have derived the properties of prices in equilibrium it is time to consider what we would infer about investment specific (embodied) technological change in our model economy. That is, what would we measure if we would apply standard price index methods to calculate a capital price index in this economy?

Before we show what the application of price index methods yields in terms of investment specific technological change, we first illustrate what we would like to measure. We do so by deriving an aggregate production function for the final goods sector as well as for the individual workers. We show that in those two production functions capital and labor are not separable.

In the last subsection here, we show that the estimation of a spurious capital price index, i.e. a capital price index for a capital stock that does not exist, might lead to very misleading conclusions.

4.1 Aggregate production function representation

For the derivation of the aggregate production function for the final goods sector, we follow Fisher (1969). We consider the decision of a planner that is endowed with a continuum for workers of measure L_t that is uniformly distributed over the interval $\left[\underline{h}, \overline{h}\right]$ as well as with a sequence of capital stocks of different vintages $\{K_{t,\tau}\}_{\tau=0}^{M}$. Given these endowments of production factors, the planner chooses an allocation of labor over the capital stocks to maximize output.

Let $K_\tau(h) \geq 0$ be the amount of capital of vintage age τ that is assigned to workers of type h and,equivalently, let $L_h(\tau) \geq 0$ the amount of workers of human capital level h that is assigned to machines of vintage age τ.

The planner chooses these allocations to maximize output, which is given by the production function

$$Y_t = \sum_{\tau=0}^{M} A_{t-\tau} \int_{\underline{h}}^{\overline{h}} h \min\{K_\tau(h), L_h(h)\} dh \tag{19}$$

and subject to the resource constraints that the capital assigned does not exceed the capital available

$$\int_{\underline{h}}^{\overline{h}} K_\tau(h) dh \leq K_{t,\tau} \tag{20}$$

and that the amount of labor assigned does not exceed the amount of labor available

$$\sum_{\tau=0}^{M} L_h(\tau) \leq \frac{L_t}{\left[\overline{h} - \underline{h}\right]} \tag{21}$$

The solution to this optimization problem coincides with the decentralized equilibrium outcome in our model economy. It entails the assortative matching between workers and machines.

Denote the human capital level of the least skilled worker that is still assigned a machine as

$$\underline{h}^* = \overline{h} - \left(\overline{h} - \underline{h}\right) \min\left\{\frac{1}{L_t}\sum_{\tau=0}^{M} K_{t,\tau}, 1\right\} \tag{22}$$

and let the oldest vintage of machines assigned to workers be

$$\tau^* = \max_{\tau=0,\ldots,M} \left\{\sum_{s=0}^{\tau-1} K_{t,s} < L\right\} \tag{23}$$

These definitions allow us to write the optimal assignment as follows.

$$K_\tau(h) = L_h(\tau) = \begin{cases} L & \text{for } \tau \leq \tau^* \text{ and } h \in \left(h_{\tau-1}^*, h_\tau^*\right] \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

Where the boundaries of the matching sets are given by

$$h_\tau^* = \begin{cases} \overline{h} & \text{for } \tau = 0 \\ \max\left\{ \underline{h}, \overline{h} - \frac{(\overline{h}-\underline{h})}{L} \sum_{s=0}^{\tau-1} K_{t,s} \right\} & \text{otherwise} \end{cases} \tag{25}$$

The level of output that results from this assignment equals

$$\begin{aligned} Y\left(L, \{K_{t,\tau}\}_{\tau=0}^{M}\right) &= \frac{L}{\overline{h}-\underline{h}} \sum_{\tau=0}^{\tau^*} \int_{h_{\tau-1}^*}^{h_\tau^*} A_{t-\tau} h \\ &= \frac{1}{2} \frac{L}{\overline{h}-\underline{h}} \left[A_t \overline{h}^2 - \sum_{\tau=1}^{\tau^*} \left(A_{t-\tau-1} - A_{t-\tau}\right) h_\tau^{*2} - A_{t-\tau^*} \underline{h}^{*2} \right] \end{aligned} \tag{26}$$

This production function exhibits constant returns to scale. However, because of the assignment of capital over workers, capital and labor are not separable in this production function. On the contrary, the amounts of capital and labor interact in a complex manner through the assignment of machines to workers, which determines the h_τ^*'s.

Note that this result is both true at the aggregate level for the final goods sector, where $\overline{h} = 1$ and $\underline{h} = 0$ as well as for the individual worker where $\overline{h} = \underline{h} = h$.

Hence, there is no aggregate production function representation in terms of an aggregate capital aggregate $J\left(\{K_{t,\tau}\}_{\tau=0}^{\infty}\right)$ that is homogenous of degree one in the capital inputs $\{K_{t,\tau}\}_{\tau=0}^{\infty}$ and the aggregate labor input L. Therefore, the concept of a capital price index is ill-defined in this model. A capital price index does not exist, because there is no properly defined theoretical aggregate capital stock.

The non-existence of an aggregate capital stock in this model should not be such a big surprise. Fisher (1969) already showed that such a capital stock only exists when the underlying vintage production functions are Cobb-Douglas. However, the assumption of a fixed capital labor ratio in our model yields that the underlying vintage production function here is Leontieff rather than Cobb-Douglas.

4.2 What we would like to measure

So, what would we like to infer about technological change in the final goods sector of this economy? It is important to answer this question, because the answer to this question provides us with the theoretical benchmark.

The first observation about technological change is that there is no growth in total factor productivity in this model. This follows from the construction of the aggregate production function above. That is, if the final goods sector uses the same amounts of labor and the same number of machines for each particular vintage at two different points in time, then it would produce the same amount of output at both points in time. There is no technological progress in this model that shifts the productivity of all factors of production in the same way, where each vintage of machine is considered a seperate production factor because there is no aggregate capital stock, and thus TFP growth is zero.

All productivity growth in this model is embodied in the new machines that come available over time. Without the adoption of the new machines productivity levels in the final goods sector would not be increasing over time.

Hence, what we would like to get out of an accounting exercise that distinguishes between total factor productivity and embodied technological change is that TFP growth is zero in the final goods sector and that all growth is due to the quality improvements of machines.

Would our current methods of measuring investment specific technological change (and of growth accounting) yield this result in the model economy here? What would happen if we would apply growth accounting techniques in our model economy to assess the contributions of total factor productivity growth and of investment specific technological change?

Using growth accounting for the final goods sector involves dividing the growth of output in this sector into its three contributing factors. The first is the growth of the labor input. The second is capital deepening, i.e. the growth of capital inputs as measured by a "quality adjusted" capital stock. We will elaborate on how such a "quality adjusted" capital stock is measured below. The final part is TFP growth, i.e. the Solow residual, it is simply the part of output growth that is not attributed to growth of the capital and labor inputs.

In practice, this boils down to applying a log-linear approximation to obtain the decomposition

$$(\ln Y_t - \ln Y_{t-1}) = (\ln Z_t - \ln Z_{t-1}) + s_{L,t} (\ln L_t - \ln L_{t-1}) + (1 - s_{L,t}) (\ln K_t - \ln K_{t-1}) \tag{27}$$

where Z_t represents the measured level of TFP, $s_{L,t}$ is the share of labor in the final goods sector, and K_t is the measured quality adjusted capital aggregate.

As derived above, on the balanced growth path, output of the final goods sector grows at a constant rate g, the labor share in the final goods sector is constant, i.e. $s_{L,t} = s_L$, and the labor inputs are constant and equal one, i.e. $L_t = 1$ for all t. This implies that, along the balanced growth path in our model economy, this decomposition simplifies to

$$g = (\ln Z_t - \ln Z_{t-1}) + (1 - s_L)(\ln K_t - \ln K_{t-1}) \tag{28}$$

Thus, on the balanced growth path our growth accounting exercise will attribute output growth either to TFP growth, i.e. t7o the growth of Z_t, or to capital deepening, i.e. the growth of K_t. The growth rate of TFP is the residual, after the subtraction of the capital deepening contribution from g.

Hence, to understand what we would infer about TFP and embodied technological change in our model economy, we have to consider how the capital aggregate K_t would be constructed in our model economy.

Since there is full depreciation of machines in every period, the capital aggregate K_t in our model economy would equal the capital expenditures in period t deflated by a capital price index. Since firms in the final goods sector make zero profits in equilibrium, capital expenditures equal total revenue minus the wage bill. That is, capital expenditures equal $(1 - s_L) Y_t$. Consequently, the capital aggregate K_t is constructed as

$$K_t = (1 - s_L) \frac{Y_t}{P_{K,t}} \tag{29}$$

where $P_{K,t}$ is the capital price index which represents the relative price of the capital goods in terms of the consumption good.

Substitution of the above capital aggregate into the growth accounting equation yields that TFP will be measured as a weighted average of output growth and the capital price declines. That is,

$$(\ln Z_t - \ln Z_{t-1}) = s_L g + (1 - s_L)(\ln P_{K,t} - \ln P_{K,t-1}) \tag{30}$$

Hence, what is crucial for the growth accounting results in our model is the capital price index $P_{K,t}$ used for it.

Since we already argued that all growth in the final goods sector of this economy is due to quality increases in capital and that there is no TFP growth, i.e.

$$(\ln Z_t - \ln Z_{t-1}) = 0 \tag{31}$$

in the sector, we would like our capital price index to satisfy that

$$(\ln P_{K,t} - \ln P_{K,t-1}) = -\frac{s_L g}{(1-s_L)} \tag{32}$$

However, there is nothing in our model that assures us that this is the actual percentage change in the relative price of capital, $P_{K,t}$, measured using common price index methods.

4.3 What would capital price indices measure?

There are, in principle, many different ways to construct such a price index $P_{K,t}$, each of which essentially employs a different price index formula. Furthermore, since in every period some machines exit the market while others enter, one also has to decide on how to deal with the inclusion of new goods. The aim of this paper is not to be an exposition on the many price index methods. Instead, it is meant to illustrate a conceptual problem with the application of them in the simple model economy introduced. Therefore, we will limit our analysis to one of the most common price index formulas. Furthermore, we will consider only two ways of dealing with the inclusion of new goods. The qualitative results derived from the resulting price indices also hold for the application of other common price index methods. That is, we will emphasize the conceptual issues with constructing a capital price index in this model and these issues are robust to what type of capital price index is constructed.

The price index formula we use is the Laspeyres formula. It is a useful benchmark, because as Frisch (1936) and Konüs (1939) already showed, it yields an upperbound on inflation in the standard case in which there are no new capital goods and there exists a proper capital aggregate.

The first way we deal with new goods to ignore them and simply apply the price index formulas to models of machines that are sold in the two periods between which we calculate capital price inflation. This yields the matched model indices used in, for example, Aizcorbe et al. (2000) and that are commonly applied to capital price indices by the Bureau of Labor Statistics.

The Laspeyres matched model index that aims to measure capital price inflation between $t-1$ and t in our model would yield

$$\pi_t^{(M)} = \frac{\sum_{\tau=1}^{M} P_{t,\tau} K_{t-1,\tau-1}}{\sum_{\tau=0}^{M-1} P_{t-1,\tau} K_{t-1,\tau}} - 1 \tag{33}$$

It measures the percentage change in the cost from $t-1$ to t of buying the period $t-1$ machines that are available in period t.

For this matched model Layspeyres index we find that, on the balanced growth path of our economy, it yields a constant percentage decline in the relative price of capital goods relative to consumption goods. That is,

$$\pi_t^{(M)} = \pi^{(M)} < 0 \quad \text{for all } t \tag{34}$$

The magnitude of the measured price declines depends on cross-vintage profile of the price declines

$$\frac{P_{t,\tau} - P_{t-1,\tau-1}}{P_{t-1,\tau-1}} = \frac{\widehat{P}_{t,\tau} - \widehat{P}_{t-1,\tau-1}}{\widehat{P}_{t-1,\tau-1}} \tag{35}$$

which in its turn depends on the length of the patent M, the cost parameters $\{c_\tau\}_{\tau=0}^{M-1}$ and the growth rate g.

The second way we deal with new goods is to include them by using a hedonic regression model to impute the price of the models that enter and exit for the periods that their prices are not observed. This would result in a hedonic price index.

The Laspeyres hedonic price index that aims to measure capital price inflation between $t-1$ and t in our model would yield

$$\pi_t^{(H)} = \frac{\sum_{\tau=1}^{M} P_{t,\tau} K_{t-1,\tau-1} + P'_{t,M+1} K_{t-1,M}}{\sum_{\tau=0}^{M} P_{t-1,\tau} K_{t-1,\tau}} - 1 \tag{36}$$

where $P'_{t,M+1}$ is the imputed price of the machines of vintage age $M+1$ at time t that is imputed using a hedonic regression. In general $P'_{t,M+1}$depends on the specific hedonic regression applied. However, on the balanced growth path the price of the vintage of age M already equals zero, so any reasonable imputation method would infer that all worse vintages should also have a price equal to zero. Consequently, if $P'_{t,M+1} = 0$, then

$$\pi_t^{(H)} = (1 - s_{t-1,M})\, \pi_t^{(M)} \tag{37}$$

where $s_{t-1,M}$ is the share of the vintage of age M at time $t-1$ and $\pi_t^{(L,M)}$ is the inflation rate measured using the Laspeyres matched model index defined above. Because $s_{t-1,M} > 0$ and $\pi_t^{(M)} < 0$ are both constant over time on the balanced growth path, we obtain that

$$\pi_t^{(H)} = \pi^{(H)} < 0 \quad \text{for all } t \tag{38}$$

Thus, just like the matched model index, the hedonic Laspeyres capital price index implies that a constant rate of decline in the relative price of capital compared to the consumption good along the balanced growth path.

Hence, both price indices that we consider here would find a constant rate of decline in the relative price of investment goods, consistent with the observation that drives the results in Greenwood, Hercowitz, and Krusell (1997). These measured declines, i.e. $\pi^{(M)}$ and $\pi^{(H)}$, are in no way related to the relative price decline that we would like to measure. That is, there is nothing that assures us that either

$$\pi^{(M)} = -\frac{s_L g}{(1-s_L)} \text{ or } \pi^{(H)} = -\frac{s_L g}{(1-s_L)} \tag{39}$$

In order to see why, it is useful to consider what mechanisms underly the price declines measured by the capital price indices.

5 Numerical example

6 Implications

Greenwood, Hercowitz, and Krusell (1997) assume a Cobb-Douglas technology in their analysis and thus the a capital aggregate in their model is well-defined. From an empirical point of view the question is thus whether this

7 Conclusion

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A Proofs

Proof of equations (10) and (11): To see why (10) is true, consider $h' > h$ and $\tau' > \tau$, then $h \in D(\tau, \mathbf{P}_t, \mathbf{A}_t)$ implies that

$$\forall\ s \in \mathbf{T}_t :\ A_{t-\tau}h - P_{t,\tau} \geq A_{t-s}h - P_{t,s}$$

or, equivalently, in terms of marginal benefits and costs

$$\forall\ s \in \mathbf{T}_t :\ (A_{t-\tau} - A_{t-s})\, h \geq P_{t,\tau} - P_{t,s}$$

Consequently, because for all $\tau' > \tau$ strictly positive technological progress implies $A_{t-\tau'} > A_{t-\tau}$, the marginal benefits from updating for the worker of type h' exceed those of the worker of type h. That is,

$$\forall \tau' > \tau :\ (A_{t-\tau} - A_{t-\tau'})\, h' > (A_{t-\tau} - A_{t-\tau'})\, h \geq P_{t,\tau} - P_{t,\tau'}$$

This implies that it must thus be true that $h' \notin D_t(\tau', \mathbf{P}_t, \mathbf{A}_t)$ for all $\tau' > \tau$.

The result of equation (10) implies that the demand sets are connected for the following reason. Suppose there would be a demand set that was not connected, then there exist $h'' > h' > h$ such that $h'' \in D_t(\tau, \mathbf{P}_t, \mathbf{A}_t)$, $h' \in D_t(\tau', \mathbf{P}_t, \mathbf{A}_t)$, and $h \in D_t(\tau, \mathbf{P}_t, \mathbf{A}_t)$ where $\tau \neq \tau'$. However, if $\tau > \tau'$, then the choices of h'' and h' do not satisfy assortative matching. On the other hand, if $\tau' > \tau$, then the choices of h' and h do not satisfy assortative matching. Hence, the demand sets need to be connected.

If the demand sets are connected and subsets of the unit interval, then they have to be of the form given in equation (11).

The proof that the set of all workers that is indifferent between two machines is negligible is a bit more involved. Let $\mathcal{H}_t$ denote the set of all human capital levels for which the workers are indifferent between two vintages of machines at time t. Since the human capital levels are uniformly distributed on the unit interval, it suffices to prove that $\mathcal{H}_t$ contains a finite number of elements. Since we have already derived that workers will only use technologies $\{0, \ldots, M\}$ there are only a finite number of combinations between which workers can be indifferent.

We will show that, if a worker of type h is indifferent between two intermediate goods, then no other worker will be. That is, define the set

$$\mathcal{H}_t^*(\tau, \tau') = \{h \in [0,1] \mid h \in D_t(\tau) \wedge h \in D_t(\tau')\}$$

such that

$$\mathcal{H}_t = \bigcup_{\tau=0}^{M-1} \bigcup_{\tau'=\tau+1}^{M} \mathcal{H}_t^*(\tau, \tau')$$

and, denoting the Lebesque measure as $\mu(.)$, we obtain

$$\mu(\mathcal{H}_t) \leq \frac{1}{\bar{h}} \sum_{\tau=0}^{M-1} \sum_{\tau'=\tau+1}^{M} \mu(\mathcal{H}_t^*(\tau, \tau'))$$

We will simply show that $\forall \tau' > \tau : \mu\left(\mathcal{H}_t^*\left(\tau, \tau'\right)\right) = 0$. Let $h \in [0,1]$ be such that $h \in D_t(\tau)$ as well as $h \in D_t(\tau')$ for $\tau' > \tau$. In that case

$$A_{t-\tau}h - P_{t,\tau} = A_{t-\tau'}h - P_{t,\tau'}$$

or equivalently

$$\left(A_{t-\tau} - A_{t-\tau'}\right) h = P_{t,\tau} - P_{t,\tau'}$$

This, however implies that for all $h' > h > h''$

$$\left(A_{t-\tau} - A_{t-\tau'}\right) h' > P_{t,\tau} - P_{t,\tau'} > \left(A_{t-\tau} - A_{t-\tau'}\right) h''$$

such that the workers of type $h' > h$ will prefer τ over τ', while workers of type $h'' < h$ will do the opposite. Hence, $\mathcal{H}_t^*\left(\tau, \tau'\right) = \{h\}$ and is of measure zero.

Proof of equations (15) through (18): We will prove these equations in three steps. In the first step, we prove equation (15) and show that, irrespective of $\mathbf{A}_t$, M, and $\{c_\tau\}_{\tau=0}^{M}$, the suppliers will set their prices such that there is demand for each of the vintages. In the second step, we derive the first order conditions that, given that it is interior, determine the optimal price schedule and show that the suppliers make strictly positive profits of the supply of each of the patented vintages. That is, we prove equation (16). In the final step, we prove the properties of the price schedule per efficiency unit that are formalized in equations (17) and (18).

Strictly positive demand for all M newest vintages: In order to prove equation (15), it turns out to be useful to introduce the function that relates a vintage back to its supplier. We denote this function by $\iota(\tau)$. It is equal to the index number of the supplier that supplies machines of vintage τ.

Furthermore, to keep track of which vintages are supplied by the same supplier and which are not, we define the indicator function

$$I\left[a = b\right] = \left\{ \begin{array}{l} 1 \text{ if } a = b \\ 0 \text{ if } a \neq b \end{array} \right\} \tag{40}$$

so that $I\left[\iota(\tau) = \iota(\tau')\right]$ is equal to one if vintages τ and τ' are supplied by the same supplier and zero otherwise.

For this proof we will consider the supplier of vintage τ and consider the effect of its price setting on the profits made from the supply of vintage τ, as well as that of vintage ages $\tau-1$ and $\tau+1$. Here we assume, without loss of generality that these adjacent vintages have prices set such that $K_{t,\tau-1}, K_{t,\tau+1} > 0$ in case vintage τ would not be supplied. We will distinguish the cases $\tau = 0$, for which $K_{t,\tau-1}$ is irrelevant, and $\tau = M-1$, for which we know that there are no profits made of vintage $\tau+1$.

For this vintage τ, we will show that there exists a price $P_{t,\tau} > 0$ such that the supplier makes strictly positive profits of the supply of vintage τ as well as that this price increases the sum of the profits over all three vintages $(\tau-1, \tau, \tau+1)$, or any two of these vintages that include τ. That is, independent of the prices of the adjacent vintages for which there is demand, the supplier of vintage τ can increase its profits, no matter whether it only owns the patent for vintage τ or any of the patents for the adjacent vintages. The assortative matching result implies that looking at three adjacent vintages is enough for this argument, because the price set for vintage τ at the margin only affects the demand for the adjacent vintages.

Let us first determine the reservation price level, above which vintage τ will not be demanded at all. This price level is determined by the type of worker, that, without the availability of vintage τ, is indifferent between vintage $\tau-1$ and $\tau+1$. We denote the human capital level of this worker by $\widetilde{h}$. It has to satisfy

$$A_{t-\tau+1}\widetilde{h} - P_{t,\tau-1} = A_{t-\tau-1}\widetilde{h} - P_{t,\tau+1} \tag{41}$$

such that

$$\widetilde{h} = \begin{cases} 1 & \text{for } \tau = 0 \\ \frac{P_{t,\tau-1}-P_{t,\tau+1}}{A_{t-\tau+1}-A_{t-\tau-1}} & \text{for } \tau > 0 \end{cases} \tag{42}$$

Hence, demand for vintage τ implies that its price level much be such that

$$A_{t-\tau}\widetilde{h} - P_{t,\tau} \geq A_{t-\tau+1}\widetilde{h} - P_{t,\tau-1} = A_{t-\tau-1}\widetilde{h} - P_{t,\tau+1} \tag{43}$$

In terms of the price per efficiency unit, this implies that

$$\hat{P}_{t,\tau} \leq \begin{cases} \frac{A_t - A_{t-1}}{A_t} + \frac{A_{t-1}}{A_t}\widehat{P}_{t,1} & \text{for } \tau = 0 \\ \left[\begin{array}{c} \frac{A_{t-\tau+1}}{A_{t-\tau}} \left(\frac{A_{t-\tau}-A_{t-\tau-1}}{A_{t-\tau+1}-A_{t-\tau-1}} \right) \widehat{P}_{t,\tau-1} \\ +\frac{A_{t-\tau-1}}{A_{t-\tau}} \left(\frac{A_{t-\tau+1}-A_{t-\tau}}{A_{t-\tau+1}-A_{t-\tau-1}} \right) \widehat{P}_{t,\tau+1} \end{array} \right] & \text{for } \tau > 0 \end{cases} \tag{44}$$

$$\equiv \widetilde{P}_{t,\tau} \tag{45}$$

Hence, $\widetilde{P}_{t,\tau}$ is the maximum price per efficieny unit at which the supplier of vintage τ faces positive demand.

The supplier of vintage τ has three options. First all, it can choose to make vintage τ available for free, in which case $P_{t,\tau} = 0$ and the firm would make non-positive profits. Secondly, it could choose $P_{t,\tau} \geq A_{t-\tau}\widetilde{P}_{t,\tau}$ at which it faces no demand and profits are zero. Finally, it can choose a price $P_{t,\tau} \geq A_{t-\tau}\left(\widetilde{P}_{t,\tau} - \varepsilon\right)$ where $0 < \varepsilon < \widetilde{P}_{t,\tau}$.

The firm will choose the third option, whenever that option increases the profits its makes over all the vintages its supplies. In the following we will show that, independent of the prices $P_{t,\tau-1}$ and $P_{t,\tau+1}$, there exists an $\varepsilon > 0$ for which this is the case.

We will consider the profits that the the supplier of vintage τ makes when it chooses a price equal to

$$\hat{P}_{t,\tau} = \widetilde{P}_{t,\tau} - \varepsilon \text{ for } \varepsilon > 0 \tag{46}$$

For a small enough $\varepsilon > 0$ when $K_{t,\tau-1}, K_{t,\tau+1} > 0$ the choice of $\hat{P}_{t,\tau}$will not affect the demand of vintages other than those of vintage ages $\tau-1$, τ and $\tau+1$. Hence, for small $\varepsilon > 0$, which turns out to be the relevant case in this proof, what matters for the supplier of vintage τ and what determines the price it chooses is whether it also supplies vintage $\tau-1$, and/or $\tau+1$, or neither of them.

At the price $\hat{P}_{t,\tau} = \widetilde{P}_{t,\tau} - \varepsilon$ the demand for vintage τ can be shown to equal

$$K_{t,\tau} = \left(I\left[\tau \neq 0\right] \frac{A_{t-\tau}}{A_{t-\tau+1} - A_{t-\tau}} + \frac{A_{t-\tau}}{A_{t-\tau} - A_{t-\tau-1}} \right) \varepsilon \tag{47}$$

The new profits over the three adjacent vintages for the supplier of vintage τ is given by

$$\begin{aligned}
& I\left[\iota\left(\tau-1\right) = \iota\left(\tau\right)\right] A_{t-\tau+1} \left(\widehat{P}_{t,\tau-1} - \frac{c_{\tau-1}}{2} \left(K_{t,\tau-1} - \frac{A_{t-\tau}}{A_{t-\tau+1} - A_{t-\tau}} \varepsilon \right) \right) \times \\
& \left(K_{t,\tau-1} - \frac{A_{t-\tau}}{A_{t-\tau+1} - A_{t-\tau}} \varepsilon \right) + \\
& A_{t-\tau} \left(\widetilde{P}_{t,\tau} - \left[1 + \frac{c_\tau}{2} \left(I\left[\tau \neq 0\right] \frac{A_{t-\tau}}{A_{t-\tau+1} - A_{t-\tau}} + \frac{A_{t-\tau}}{A_{t-\tau} - A_{t-\tau-1}} \right) \right] \varepsilon \right) \times \\
& \left(I\left[\tau \neq 0\right] \frac{A_{t-\tau}}{A_{t-\tau+1} - A_{t-\tau}} + \frac{A_{t-\tau}}{A_{t-\tau} - A_{t-\tau-1}} \right) \varepsilon + \\
& I\left[\iota\left(\tau+1\right) = \iota\left(\tau\right)\right] A_{t-\tau-1} \left(\widehat{P}_{t,\tau+1} - \frac{c_{\tau+1}}{2} \left(K_{t,\tau+1} - \frac{A_{t-\tau}}{A_{t-\tau} - A_{t-\tau-1}} \varepsilon \right) \right) \times \\
& \left(K_{t,\tau+1} - \frac{A_{t-\tau}}{A_{t-\tau} - A_{t-\tau-1}} \varepsilon \right)
\end{aligned} \tag{48}$$

Which simplifies to

$$\begin{aligned}
& I\left[\iota\left(\tau-1\right) = \iota\left(\tau\right)\right] A_{t-\tau+1} \left(\widehat{P}_{t,\tau-1} - \frac{c_{\tau-1}}{2} K_{t,\tau-1} \right) K_{t,\tau-1} + \\
& I\left[\iota\left(\tau+1\right) = \iota\left(\tau\right)\right] A_{t-\tau-1} \left(\widehat{P}_{t,\tau+1} - \frac{c_{\tau+1}}{2} K_{t,\tau+1} \right) K_{t,\tau+1} + \\
& a\varepsilon - b\varepsilon^2
\end{aligned} \tag{49}$$

where $a > 0$ and $b > 0$. In particular, they equal

$$a = A_t I\left[\tau = 0\right] + \left(1 - I\left[\iota\left(\tau\right) = \iota\left(\tau-1\right)\right]\right) \frac{A_{t-\tau+1} A_{t-\tau}}{A_{t-\tau+1} - A_{t-\tau}} \widehat{P}_{t,\tau-1} \tag{50}$$

$$+ \left(1 - I\left[\iota\left(\tau+1\right) = \iota\left(\tau\right)\right]\right) \frac{A_{t-\tau} A_{t-\tau-1}}{A_{t-\tau} - A_{t-\tau-1}} \widehat{P}_{t,\tau+1} \tag{51}$$

$$+ I\left[\iota\left(\tau-1\right) = \iota\left(\tau\right)\right] \frac{A_{t-\tau+1} A_{t-\tau}}{A_{t-\tau+1} - A_{t-\tau}} c_{\tau-1} K_{t,\tau-1}$$

$$+ I\left[\iota\left(\tau+1\right) = \iota\left(\tau\right)\right] \frac{A_{t-\tau-1} A_{t-\tau}}{A_{t-\tau} - A_{t-\tau-1}} c_{\tau+1} K_{t,\tau+1} \tag{52}$$

and

$$
\begin{aligned}
b \quad = \quad & \frac{c_{\tau-1}}{2} I\left[\iota\left(\tau-1\right)=\iota\left(\tau\right)\right] A_{t-\tau+1}\left(\frac{A_{t-\tau}}{A_{t-\tau+1}-A_{t-\tau}}\right)^{2} && (53)\\
& +\left[1+\frac{c_{\tau}}{2}\left(I\left[\tau\neq 0\right]\frac{A_{t-\tau}}{A_{t-\tau+1}-A_{t-\tau}}+\frac{A_{t-\tau}}{A_{t-\tau}-A_{t-\tau-1}}\right)\right]\times \\
& \left(I\left[\tau\neq 0\right]\frac{A_{t-\tau}}{A_{t-\tau+1}-A_{t-\tau}}+\frac{A_{t-\tau}}{A_{t-\tau}-A_{t-\tau-1}}\right)+ && (54)\\
& +I\left[\iota\left(\tau+1\right)=\iota\left(\tau\right)\right] A_{t-\tau-1}\frac{c_{\tau+1}}{2}\left(\frac{A_{t-\tau}}{A_{t-\tau}-A_{t-\tau-1}}\right)^{2}
\end{aligned}
$$

Note that the first two terms of equation (49) equal the profits that the supplier of vintage τ would have made of the two adjacent vintages, if it would have owned any of them. The term $a\varepsilon - b\varepsilon^2$ represents the additional profits earned due to the supply of vintage τ at price $\widetilde{P}_{t,\tau} - \varepsilon$. Hence, the supplier of vintage τ would always set a price that generates strictly positive demand for that vintage if there exists an $\varepsilon > 0$ for which this additional profit is strictly positive. Since there always is an $\varepsilon > 0$ for which $a\varepsilon - b\varepsilon^2 > 0$, it always the case that the supplier of vintage τ will supply that vintage at a price that generates strictly positive demand.

Strictly positive profits: This follows as a corollary from the proof above. The supplier of the vintage τ can always choose its price to strictly increase its profits relative to zero.

$\widehat{P}_{t,\tau}$ **is strictly decreasing in** τ**:** This follows from an induction argument. We have proven above that in the equilibrium there must be strictly positive demand for each of the vintages of age $\tau = 0, \ldots, M-1$, i.e. $K_{t,\tau} > 0$ in equilibrium. In terms of the prices per efficiency unit, the demand sets are

$$
K_{t,\tau} = \begin{cases} \left[1 - \frac{A_t}{A_t - A_{t-1}}\widehat{P}_{t,0} + \frac{A_{t-1}}{A_t - A_{t-1}}\widehat{P}_{t,1}\right] & \text{for } \tau = 0 \\ \left[\frac{A_{t-\tau+1}}{A_{t-\tau+1}-A_{t-\tau}}\left(\widehat{P}_{t,\tau-1} - \widehat{P}_{t,\tau}\right) - \frac{A_{t-\tau-1}}{A_{t-\tau}-A_{t-\tau-1}}\left(\widehat{P}_{t,\tau} - \widehat{P}_{t,\tau+1}\right)\right] & \text{for } \tau = 1, \ldots, M-1 \end{cases} \tag{55}
$$

This implies that for $\tau = 1, \ldots, M-1$

$$
\frac{A_{t-\tau+1}}{A_{t-\tau+1}-A_{t-\tau}}\left(\widehat{P}_{t,\tau-1} - \widehat{P}_{t,\tau}\right) > \frac{A_{t-\tau-1}}{A_{t-\tau}-A_{t-\tau-1}}\left(\widehat{P}_{t,\tau} - \widehat{P}_{t,\tau+1}\right) \tag{56}
$$

Hence, if the price per efficiency unit for vintage age τ is larger than that for $\tau + 1$, then it must be the case that the price per efficiency unit for vintage age $\tau - 1$ is higher than that of vintage τ. The only thing we need to proof our claim is a initial result and then we can apply an induction argument.

We do know that in equilibrium the supplier of vintage age $M-1$ will choose a price that is strictly positive, such that $\widehat{P}_{t,M-1} > 0$. Furthermore, we know that perfect competition in the supply of vintage M will drive its price to zero, such that $\widehat{P}_{t,M} = 0$. Hence, we know that $\left(\widehat{P}_{t,M-1} - \widehat{P}_{t,M}\right) > 0$. Applying our induction argument thus yields that this implies that $\left(\widehat{P}_{t,\tau} - \widehat{P}_{t,\tau+1}\right) > 0$ for $\tau = 0, \ldots, M-1$. Hence $\widehat{P}_{t,\tau}$ is strictly decreasing in τ.

$\widehat{\mathbf{P}}_t^* = \widehat{\mathbf{P}}\left(\mathbf{A}_t, \{c_\tau\}_{\tau=0}^M\right)$ **where** $\widehat{\mathbf{P}}(.)$ **is homogenous of degree zero in** A_t**:** Supplier i sets the prices of the vintages of machines its supplies to maximize the profits

$$\pi_{t,i} = \sum_{\tau=0}^{M-1} I\left[\iota(\tau) = i\right] A_{t-\tau}\left(\widehat{P}_{t,\tau} - \frac{c_\tau}{2}K_{t,\tau}\right)K_{t,\tau} \tag{57}$$

The necessary and sufficient conditions for profit maximization in equilibrium imply that this supplier will set the price of each vintage τ which it supplies, i.e. $\iota(\tau) = i$, to satisfy the condition

$$0 = K_{t-\tau} + \sum_{s=0}^{M-1} I\left[\iota(\tau) = i\right]\left(\widehat{P}_{t,\tau} - c_\tau K_{t,\tau}\right)\frac{\partial K_{t,\tau}}{\partial \widehat{P}_{t,\tau}} \tag{58}$$

However, note that these optimality conditions are homogenous of degree zero in $\mathbf{A}_t = \{A_t, \ldots, A_{t-M}\}$. This is because the demand functions that determine $K_{t,\tau}$ are homogenouse of degree zero in $\mathbf{A}_t = \{A_t, \ldots, A_{t-M}\}$ and so are the marginal demand functions $\partial K_{t,\tau}/\partial \widehat{P}_{t,\tau}$. Furthermore, besides the productivity levels in $\mathbf{A}_t$ the only other parameters that show up in these equilibrium conditions are the cost parameters $\{c_\tau\}_{\tau=0}^{M-1}$. Thus the equilibrium price per efficiency unit profile is only a function of the productivity levels and the cost parameters and it is homogenous of degree zero in the productivity levels.

Furthermore, the system of equilibrium conditions, implied by the optimality conditions above, is linear in the prices per efficiency unit and it turns out to be straightforward to show that it has one unique solution. That is, the PSN equilibrium exists and it is unique.

Proof of equation (34) : The following is the proof of equation (34). The matched model Laspeyres index yields a capital price inflation estimate of

$$\pi_t^{(M)} = \frac{\sum_{\tau=1}^M P_{t,\tau}K_{t-1,\tau-1}}{\sum_{\tau=0}^{M-1} P_{t-1,\tau}K_{t-1,\tau}} - 1 = \frac{\sum_{\tau=0}^{M-1} P_{t,\tau+1}K_{t-1,\tau}}{\sum_{\tau=0}^{M-1} P_{t-1,\tau}K_{t-1,\tau}} - 1 \tag{59}$$

$$= \sum_{\tau=0}^{M-1} s_{t-1,\tau}^* \widehat{\pi}_{t,\tau} \tag{60}$$

where the shares $s_{t,\tau}^*$ are given by

$$s_{t-1,\tau}^* = \frac{P_{t-1,\tau}K_{t-1,\tau}}{\sum_{s=0}^{M-1} P_{t-1,s}K_{t-1,s}} = \frac{A_{t-1-\tau}\widehat{P}_{t-1,\tau}K_{t-1,\tau}}{\sum_{s=0}^{M-1} A_{t-1-s}\widehat{P}_{t-1,s}K_{t-1,s}} \tag{61}$$

and represent the expenditure share in period $t-1$ of vintage age τ in the expenditures on machines that are also available at time t. The inflation rates $\widehat{\pi}_{t,\tau}$ equal

$$\widehat{\pi}_{t,\tau} = \left(\widehat{P}_{t,\tau+1} - \widehat{P}_{t-1,\tau}\right)\Big/\widehat{P}_{t-1,\tau} \tag{62}$$

On the balanced growth path both $s_{t-1,\tau}^*$ and $\widehat{\pi}_{t,\tau}$ will be constant over time. Furthermore, because the price per efficiency unit is declining in the vintage age, $\widehat{\pi}_{t,\tau} < 0$ for all τ. And thus $\pi_t^{(M)}$ is constant over time and negative.