

# TWO-FUND SEPARATION IN DYNAMIC GENERAL EQUILIBRIUM

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ABSTRACT. The purpose of this note is to examine the two-fund separation paradigm in the context of an infinite-horizon general equilibrium model with dynamically complete markets and heterogeneous consumers with time and state separable utility functions. With the exception of the dynamic structure, we maintain the assumptions of the classical static models that exhibit two-fund separation with a riskless security. In addition to a security with state-independent payoffs agents can trade a collection of assets with dividends following a time-homogeneous Markov process. We make no further assumptions about the distribution of asset dividends, returns, or prices. Agents have equi-cautious HARA utility functions. If the riskless security in the economy is a consol then agents' portfolios exhibit two-fund separation. But if agents can trade only a one-period bond, this result no longer holds. Examples show this effect to be quantitatively significant. The underlying intuition is that general equilibrium restrictions lead to interest rate fluctuations that destroy the optimality of two-fund separation in economies with a one-period bond and result in different equilibrium portfolios.

Keywords: Portfolio separation, dynamically complete markets, consol, one-period bond, interest rate fluctuation.

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## 1. INTRODUCTION

The two-fund separation theorem – among the most remarkable results of classical finance theory – states that investors who must allocate their wealth between a number of risky assets and a riskless security should all hold the same mutual fund of risky assets. An investor’s risk aversion only affects the proportions of wealth that (s)he invests in the risky mutual fund and the riskless security. But the allocation of wealth across the different risky assets does not depend on the investor’s preferences.

Canner et al. (1997) point out that popular financial planning advice violates the separation theorem and call this observation the “asset allocation puzzle.” They document recommendations from different investment advisors who all encourage conservative investors to hold a higher ratio of bonds to stocks than aggressive investors. Bossaerts et al. (2003) state that the separation result cannot be reconciled with casual empirical observations and conclude that *“most tests of asset pricing models address only the pricing predictions – perhaps because the portfolio choice predictions are obviously wrong.”*

These critiques assume that classical two-fund separation, a result from static models such as the Capital Asset Pricing Model, is applicable to the dynamic nature of modern financial markets. Attempting to verify the two-fund separation theorem in actual financial markets assumes the existence of a riskless asset. In dynamic markets (inflation-indexed) bonds with maturities matching all possible investment horizons would allow investors access to such an asset. But bonds with very long maturities and in the limit a consol – a bond yielding safe coupon payments ad infinitum – do not exist. Instead, investors are told to hold cash as a “safe” asset. However, because an investor must continually reinvest cash in the future at unknown and fluctuating interest rates, cash is safe only in the short term, as emphasized by Campbell and Viceira (2002). Therefore, it is unclear whether one should expect investors’ observed portfolios to satisfy the static two-fund separation property.

In this note we prove that in a dynamic model of asset trading with a consol the two-fund separation theorem holds. But if only risky assets and a one-period bond (cash) can be traded on financial markets then two-fund separation typically fails. To keep the analysis as close as possible to the classical static presentations (Cass and Stiglitz (1970)), we use an infinite-horizon general equilibrium model with dynamically complete markets and heterogeneous consumers with time and state separable utility functions. With the exception of the dynamic structure we maintain the assumptions of the static models. All agents have HARA utilities with linear absolute risk tolerances having identical slopes. We assume that there is a security with state-independent payoffs (consol or cash) and that all asset dividends follow a time-homogeneous Markov process, but do not make any further assumptions about the distribution of asset dividends, returns, or prices. Efficient equilibria in this model have time-homogeneous consumption and asset price processes. Portfolios are constant over time. All endogenous variables lie in a finite-dimensional space, so we can apply transversality theory on Euclidean spaces (see Magill and Quinzii (1996b)) to derive generic results.

The underlying intuition for the very different portfolio properties is that general equilibrium restrictions create interest rate fluctuation. When the safe asset is a consol these

fluctuations do not affect portfolios since the agents have a trade-once-and-hold-forever strategy. But if the agents can only trade a one-period bond, they have to reestablish the constant portfolio each period. In that case, interest rate fluctuation destroys the optimality of two-fund separation and leads to different equilibrium portfolios.

In light of our results, it should come as no surprise that observed investors' portfolios<sup>1</sup> do not satisfy two-fund separation. The aforementioned critiques of Canner et al. (1997) and Bossaerts et al. (2003) are based on the implicit – but incorrect – assumption that investors have access to a truly riskless asset.

The classical papers on two-fund separation are Cass and Stiglitz (1970) and Ross (1978). Both papers address two-fund separation of agents' portfolio demand. Cass and Stiglitz (1970) provide conditions on agents' preferences that ensure two-fund separation. Ross (1978) presents conditions on asset return distributions under which two-fund separation holds. Russell (1980) presents a unified approach of Cass and Stiglitz and Ross. Ingersoll (1987) provides a detailed overview of various separation results and highlights the distinction between restrictions on utility functions and restrictions on asset return distributions. Gollier (2001) states the separation result of Cass and Stiglitz in the context of a static equilibrium model. The first discussion of the two-fund separation idea is Tobin (1958) who analyzes portfolio demand in a mean-variance setting. Two-fund separation has been examined in great detail in the CAPM, see for example Black (1972). We can't possibly do justice to the huge literature on portfolio separation and mutual fund theorems in the CAPM and just refer to textbook overviews such as Ingersoll (1987) or Huang and Litzenberger (1988).

Section 2 presents the basic model for our analysis. In Section 3 we show some preliminary results. Section 4 develops the two-fund separation theory for our dynamic model, proving the generalization of the classical static result when the safe asset is a consol and showing that two-fund separation fails generically when there is only a one-period bond. In Section 5 we analyze a variation of the basic model with few assets. Section 6 concludes the analysis and the Appendix contains all technical proofs.

## 2. THE ASSET MARKET ECONOMY

We examine a standard Lucas asset pricing model (Lucas (1978)) with heterogeneous agents and dynamically complete asset markets. Time is indexed by  $t \in \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$ . A time-homogeneous recurrent Markov process of exogenous states  $(y_t)_{t \in \mathbb{N}_0}$  takes values in a discrete set  $\mathcal{Y} = \{1, 2, \dots, S\}$ . The Markov transition matrix is denoted by  $\Pi$ . A date-event  $\sigma_t$  is the history of shocks up to time  $t$ , i.e.  $\sigma_t = (y_0, y_1, \dots, y_t)$ . Let  $\Sigma_t$  denote the possible histories  $\sigma_t$  up to time  $t$  and let  $\Sigma = \cup_t \Sigma_t$  denote all possible histories of the exogenous

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<sup>1</sup>The asset allocation puzzle of Canner et al. (1997) has received a lot of attention in the finance literature. Among others, Brennan and Xia (2000) claim to solve the puzzle. In a continuous-time portfolio selection model they examine the behavior of a single investor with a constant relative risk-aversion utility and a finite horizon who can invest in a single stock, cash, and bonds of different maturities. The interest rate process is given by an exogenously specified Markov process. They show that the ratio of bonds to stock in the optimal portfolio is increasing in the risk aversion coefficient. But they neither examine an equilibrium model nor do they address the classical HARA set-up of Cass and Stiglitz (1970).

states. We denote the predecessor of a date-event  $\sigma \in \Sigma$  by  $\sigma^*$ . The starting node  $\sigma_0 = y_0$  has a predecessor  $\sigma_0^* = \sigma_{-1}$ .

There is a finite number of types  $\mathcal{H} = \{1, 2, \dots, H\}$  of infinitely-lived agents. There is a single perishable consumption good, which is produced by firms. The agents have no individual endowment of the consumption good. The firms distribute their output each period to its owners through dividends. Investors trade shares of the firms and other securities in order to transfer wealth across time and states. There are  $J = S$  assets traded on financial markets. An asset is characterized by its state-dependent dividends. We denote asset  $j$ 's dividend or payoff by  $d^j : \mathcal{Y} \rightarrow \mathbb{R}_+$ ,  $j = 1, \dots, S$ , which solely depends on the current state  $y \in \mathcal{Y}$ . Each security is either an infinitely-lived (long-lived) asset or a single-period asset. There are  $J^l \geq 1$  long-lived assets in the economy. The remaining  $S - J^l$  securities are short-lived assets that are issued in each period. A short-lived asset  $j$  issued in period  $t$  pays  $d^j(y)$  in period  $t + 1$  if state  $y$  occurs and then expires. For ease of exposition we collect the infinitely lived assets in a set  $L \equiv \{1, \dots, J^l\}$  and the one-period assets in a set  $O \equiv \{J^l + 1, \dots, S\}$ .

Agent  $h$ 's portfolio at date-event  $\sigma \in \Sigma$  is  $\theta^h(\sigma) \equiv (\theta^{hL}(\sigma), \theta^{hO}(\sigma)) = (\theta^{h1}(\sigma), \dots, \theta^{hS}(\sigma)) \in \mathbb{R}^S$ . His initial endowment asset  $j$  prior to time 0 is denoted by  $\theta_{-1}^{hj}$ ,  $j \in L$ . Each agent has zero initial endowment of the short-lived assets and so these assets are in zero net supply. The infinitely lived assets which represent firm dividends are in unit net supply. Other financial assets, such as a consol, are in zero net supply. We write  $\theta_{-1}^L \equiv (\theta_{-1}^{hL})_{h \in \mathcal{H}}$ . The aggregate endowment of the economy in state  $y$  is  $e(y) = \sum_{j \in L} d^j(y)$ . Agent  $h$ 's initial endowment of dividends before time 0 is given by  $\omega^h(y) = \sum_{j \in L} \theta_{-1}^{hj} d^j(y) > 0$ . In order to avoid unnecessary complications we assume that all agents have nonnegative initial holdings of each asset and a positive initial holding of at least one asset.

Let  $q(\sigma) \equiv (q^1(\sigma), \dots, q^S(\sigma))$  be the ex-dividend prices of all assets at date-event  $\sigma$ . At each date-event  $\sigma = (\sigma^* y)$  agent  $h$  faces a budget constraint,

$$c^h(\sigma) = \sum_{j \in L} \theta^{hj}(\sigma^*) (q^j(\sigma) + d^j(y)) + \sum_{j \in O} \theta^{hj}(\sigma^*) d^j(y) - \sum_{j=1}^S \theta^{hj}(\sigma) q^j(\sigma).$$

Each agent  $h$  has a time-separable utility function

$$U_h(c) = E \left\{ \sum_{t=0}^{\infty} \beta^t u_h(c_t) \right\},$$

where  $c = (c_0, c_1, c_2, \dots)$  is a consumption process. All agents have the same discount factor  $\beta \in (0, 1)$ . We assume that the Bernoulli functions  $u_h : X \rightarrow \mathbb{R}$  are strictly monotone, twice differentiable, and strictly concave on some interval  $X \subset \mathbb{R}$ . Below we discuss conditions that ensure equilibrium consumption at every date-event to always lie in the interior of an appropriately chosen consumption set  $X$ .

Let the matrix

$$d = \begin{pmatrix} d^1(1) & \cdots & d^S(1) \\ \vdots & \ddots & \vdots \\ d^1(S) & \cdots & d^S(S) \end{pmatrix}$$

represent security dividends or payoffs. The vector of utility functions is  $U = (U_1, \dots, U_H)$ . We denote the primitives of the economy by the expression  $\mathcal{E} = (d, X, \beta, U; \theta_{-1}^L, \Pi)$ .

We define a standard notion of a financial market equilibrium.

**Definition 1.** *A financial market equilibrium for an economy  $\mathcal{E}$  is a process of portfolio holdings  $\{(\bar{\theta}^1(\sigma), \dots, \bar{\theta}^H(\sigma))\}$  and asset prices  $\{(\bar{q}^1(\sigma), \dots, \bar{q}^J(\sigma))\}$  for all  $\sigma \in \Sigma$  satisfying the following conditions:*

$$(1) \sum_{h=1}^H \bar{\theta}^h(\sigma) = \sum_{h=1}^H \theta_{-1}^h \quad \text{for all } \sigma \in \Sigma.$$

(2) For each agent  $h \in \mathcal{H}$ ,

$$(\bar{\theta}^h(\sigma))_{\sigma \in \Sigma} \in \arg \max_{\theta} U_h(c) \quad \text{s.t.}$$

$$c^h(\sigma) = \sum_{j \in L} \bar{\theta}^{hj}(\sigma^*) (\bar{q}^j(\sigma) + d^j(y)) + \sum_{j \in O} \bar{\theta}^{hj}(\sigma^*) d^j(y) - \sum_{j=1}^S \bar{\theta}^{hj}(\sigma) \bar{q}^j(\sigma)$$

$$\sup_{\sigma \in \Sigma} |\bar{\theta}^h(\sigma) \bar{q}(\sigma)| < \infty$$

### 3. EQUILIBRIUM IN DYNAMICALLY COMPLETE MARKETS

We use the Negishi approach (Negishi (1960)) of Judd et al. (2003) to characterize efficient equilibria in our model. Efficient equilibria exhibit time-homogeneous consumption processes and asset prices, that is, consumption allocations and asset prices in date-event  $\sigma = (\sigma^*y)$  only depend on the last shock  $y$ . We take advantage of this recursivity in our notation and express the dependence of variables on just the exogenous shock through a subscript. For example,  $c_y^h$  will denote the consumption of agent  $h$  in state  $y$ . We define  $p_y = u'_1(c_y^1)$  to be the price of consumption in state  $y$  and  $p = (p_y)_{y \in \mathcal{Y}} \in \mathbb{R}_{++}^S$  to be the vector of prices. We denote the  $S \times S$  identity matrix by  $I_S$ , Negishi weights by  $\lambda^h$ ,  $h = 2, \dots, H$ , and use  $\otimes$  to denote element-wise multiplication of vectors.

If the economy starts in the state  $y_0 \in \mathcal{Y}$  at period  $t = 0$ , then the Negishi weights and consumption vectors must satisfy the following equations.

$$(1) \quad u'_1(c_y^1) - \lambda^h u'_h(c_y^h) = 0, \quad h = 2, \dots, H, \quad y \in \mathcal{Y},$$

$$(2) \quad \left( [I_S - \beta \Pi]^{-1} (p \otimes (c^h - \omega^h)) \right)_{y_0} = 0, \quad h = 2, \dots, H,$$

$$(3) \quad \sum_{h=1}^H c_y^h - \sum_{h=1}^H \omega_y^h = 0, \quad y \in \mathcal{Y}.$$

The system of equations (1, 2, 3) has  $HS + (H-1)$  unknowns,  $HS$  unknown state-contingent, agent-specific consumption levels  $c_y^h$ , and  $H-1$  Negishi weights  $\lambda^h$ . Once we know the consumption vectors we obtain closed-form expressions for asset prices. The prices of a long-lived asset  $j$  are given by

$$(4) \quad q^j \otimes p = [I_S - \beta \Pi]^{-1} \beta \Pi (p \otimes d^j).$$

The price of a short-lived asset  $j$  in state  $y$  is

$$(5) \quad q_y^j = \frac{\beta E \left\{ p \otimes d_{y+}^j \mid y \right\}}{p_y} = \frac{\beta \Pi_{y \cdot} (p \otimes d^j)}{p_y},$$

where  $\Pi_y$  denotes row  $y$  of the matrix  $\Pi$ .

Define the matrix  $D = (d^1, \dots, d^{J^l}, d^{J^l+1} - q^{J^l+1}, \dots, d^S - q^S)$ . Judd et al. (2003) show under conditions ensuring that the matrix  $D$  has full rank  $S$  and if all transition probabilities are strictly positive that, after one initial round of trading at time 0, all agents hold a state-independent portfolio vector  $\Theta^h \equiv \theta_y^h$  for all  $y \in \mathcal{Y}$  which is given by

$$(6) \quad \Theta^h = D^{-1}c^h, \quad \forall h \in \mathcal{H}.$$

In summary, a financial market equilibrium is a solution to equations (1)–(6).

Judd et al. (2003) impose an Inada condition ( $\lim_{x \rightarrow 0} u'_h(x) = \infty$ ) to ensure that the solutions to equations (1)–(3) yield positive consumption allocations. We cannot make that assumption here since some of the classical utility functions that yield two-fund separation (e.g., quadratic utility) do not satisfy such an Inada condition. Instead we allow for the possibility of negative consumption. Those of our utility functions that do not satisfy an Inada condition have the property  $\lim_{x \rightarrow -\infty} u'_h(x) = \infty$ . Therefore, equations (1)–(3) have a solution that is bounded below and thus an interior point of a consumption set (interval)  $X$  that allows for sufficiently negative consumption. In addition, we need to ensure that consumption remains non-satiated since we want to avoid free disposal of income. We do not state (tedious) assumptions on fundamentals and refer to Magill and Quinzii (2000, Proposition 3) who show for quadratic utilities how to restrict parameters to ensure positive and non-satiated consumption. In summary, for an appropriately chosen consumption set  $X$  equations (1)–(3) are necessary and sufficient for a consumption allocation of an efficient financial market equilibrium. (And ideally we think of specifications of the model that result in strictly positive consumption allocations.)

For our analysis of agents' portfolios we adopt the following two assumptions from Judd et al. (2003).

**[A1]** All elements of the transition matrix  $\Pi$  are positive,

$$\Pi \in \{A \in \mathbb{R}^{S \times S} : A_{ys} > 0 \quad \forall y, s \in \mathcal{Y}, \quad \sum_{s=1}^S A_{ys} = 1 \quad \forall y \in \mathcal{Y}\}.$$

**[A2]**  $\text{Rank}[d] = S$ .

For our application of the parametric transversality theorem (see Appendix A.1) using Assumption [A1] it is useful to define an open set that is diffeomorphic to the set of admissible transition matrices,

$$\Delta_{++}^{S \times (S-1)} \equiv \{A_{ys}, y \in \mathcal{Y}, s \in \{1, \dots, S-1\} : A_{ys} > 0, \quad \sum_{s=1}^{S-1} A_{ys} < 1 \quad \forall y \in \mathcal{Y}\}.$$

We identify transition matrices with elements in  $\Delta_{++}^{S \times (S-1)}$ . Remark 1 below explains why it is sensible to have genericity statements with respect to transition probabilities. We want to examine two-fund separation for the classical families of utility functions and so cannot allow for the popular perturbations of utility functions as, for example, in Cass and Citanna (1998) and Citanna et al. (2004).

The following assumption is not crucial but simplifies our genericity arguments.

[A3] All agents have a positive initial position of the first long-lived asset.

We define the open set  $\Delta_{++}^{H-1} \equiv \{x \in \mathbb{R}_{++}^{H-1} : \sum_{i=1}^{H-1} x_i < 1\}$ . The assumption requires  $(\theta_{-1}^{h1})_{h \geq 2} \in \Delta_{++}^{H-1}$ .

**3.1. Some Equilibrium Properties.** Economies without aggregate risk are well known to have equilibria of special structure. For completeness we summarize the equilibrium properties of such economies.

**Proposition 1** (Equilibrium Without Aggregate Risk). *Suppose the aggregate endowment is constant,  $e_y = \hat{e}$  for all  $y \in \mathcal{Y}$ .*

- (1) *Consumption allocations and asset prices are the same in every efficient financial market equilibrium. Allocations are state-independent. Consumption allocations, asset prices, and portfolios are independent of agents' utility functions.*
- (2) *If [A1] and [A2] hold, then the equilibrium is unique. Each agent holds constant shares of all long-lived assets (in unit net supply) and does not trade short-lived assets.*

Proposition 1 completely characterizes efficient financial market equilibria in economies without aggregate uncertainty. Portfolios satisfy what one could call a “one-fund” property. Such a simple equilibrium makes any further analysis of two-fund separation superfluous. Our main results in this paper are for economies with a “riskless” asset. For such economies the full-rank assumption [A2] immediately implies that the social endowment in the economy is not constant. That is, there exist  $y_1, y_2 \in \mathcal{Y}$  such that  $e(y_1) \neq e(y_2)$ .

Judd et al. (2003) prove existence of efficient financial market equilibria for generic dividends of the short-lived assets. We cannot use this existence result here since the analysis of two-fund separation requires particular dividend structures. Therefore we prove an alternative existence result that suits our analysis.

**Proposition 2** (Equilibrium with Aggregate Risk). *Consider an economy  $\mathcal{E}$  satisfying assumption [A2].*

- (1) *If all  $S$  assets are long-lived, then the economy  $\mathcal{E}$  has an efficient financial market equilibrium.*
- (2) *Suppose also [A1] and [A3] hold. If there are  $S-1$  long-lived assets and a one-period bond, then  $\mathcal{E}$  has an efficient equilibrium for generic subsets  $\mathcal{T} \subset \Delta_{++}^{H-1}$  of initial holdings of the first asset and  $\mathcal{P} \subset \Delta_{++}^{S \times (S-1)}$  of transition matrices.*

We prove genericity with respect to transition probabilities as they are a natural choice for the exogenous parameters in the genericity proofs of our analysis, see Remark 1. We show the following lemma also in Appendix A.2.

**Lemma 1.** *In an efficient financial market equilibrium of the economy  $\mathcal{E}$  the price of a one-period bond,  $q^b$ , has the following characteristics.*

- (1) *The price  $q^b$  is constant if and only if the aggregate endowment is constant. In that case the bond price equals the discount factor,  $q_y^b = \beta$  for all  $y \in \mathcal{Y}$ .*

- (2) Suppose  $S \geq 3$  and  $[A1],[A2]$  and  $[A3]$  hold. For generic subsets  $\mathcal{T} \subset \Delta_{++}^{H-1}$  of initial holdings of the first asset and  $\mathcal{P} \subset \Delta_{++}^{S \times (S-1)}$  of transition matrices the price of a one-period bond is not a linear function of the aggregate endowment. That is, there do not exist numbers  $a, f \in \mathbb{R}$  such that  $q_y^b = a \cdot e_y + f$  for all  $y \in \mathcal{Y}$ .

At first it may be surprising that Part 2 of the lemma only holds for a generic set of transition probabilities. We explain why this condition is needed in Remark 1 below.

**3.2. Linear Sharing Rules.** Linear sharing rules for consumption are the foundation of two-fund separation on financial markets. Using standard terminology we say that equilibrium consumption adheres to a linear sharing rule if it satisfies

$$c_y^h = m^h e_y + b^h \quad \forall h \in \mathcal{H}, y \in \mathcal{Y},$$

for real numbers  $m^h, b^h$  for all agents  $h \in \mathcal{H}$ . Obviously, in equilibrium it holds that  $\sum_{h=1}^H m^h = 1$  and  $\sum_{h=1}^H b^h = 0$ . Our results in this paper show that we have to carefully distinguish between linear sharing rules with nonzero intercepts and those for which  $b^h = 0$  for all  $h \in \mathcal{H}$ .

Recall that the absolute risk tolerance of agent  $h$ 's utility function  $u_h : X \rightarrow \mathbb{R}$  is defined as  $T_h(c) = -\frac{u_h'(c)}{u_h''(c)}$ . Of particular interest for linear sharing rules are utility functions with linear absolute risk tolerance, that is,  $T_h(c) = a^h + g^h c$ , for real numbers  $g^h$  and  $a^h$ . These utility functions comprise the well-known family of HARA (hyberbolic absolute risk aversion) utilities (see Gollier (2001), Hens and Pilgrim (2002)). If all agents have HARA utilities and all their linear absolute risk tolerances have identical slopes, that is,  $g^h \equiv g$  for all  $h \in \mathcal{H}$  for some slope  $g$ , then the agents are said to have equ-cautious HARA utilities.

Utility functions exhibiting linear absolute risk tolerance with constant but nonzero slope for all agents have the form

$$[\text{EC}] \quad u_h(c) = \begin{cases} K \left( A^h + \frac{c}{\gamma} \right)^{1-\gamma} & \text{for } \gamma \neq 0, 1, \quad c \in \{c \in \mathbb{R} | A^h + \frac{c}{\gamma} > 0\} \\ \ln(A^h + c) & \text{for } \gamma = 1, \quad c \in \{c \in \mathbb{R} | A^h + c > 0\} \end{cases}$$

with  $K = \text{sign}(\frac{1-\gamma}{\gamma})$  to ensure that  $u$  is strictly increasing and strictly concave (on some appropriate consumption set). The absolute risk tolerance for these utility functions is  $T_h(c) = A^h + \frac{c}{\gamma}$ . If  $\gamma > 0$  and  $A^h = 0$  for all  $h \in \mathcal{H}$  then we have the special case of utility functions with identical constant relative risk aversion (CRRA). If  $\gamma = -1$  then all agents have quadratic utility functions.

The limit case for utility functions of the type [EC] as  $\gamma \rightarrow \infty$  are utility functions with constant absolute risk aversion (CARA). We write

$$[\text{CARA}] \quad u_h(c) = -\frac{1}{a^h} e^{-a^h c}$$

with constant absolute risk tolerance of  $T_h(c) = \frac{1}{a^h} \equiv \tau^h$ .

We need the following lemma for our analysis. It follows from the classical results on Pareto-efficient sharing rules by Wilson (1968) and Amershi and Stoekenius (1983). (See Gollier (2001) for a textbook treatment of a static equilibrium problem.)



**Lemma 2.** *If all agents have equi-cautious HARA utilities, then the consumption allocation of each agent in an efficient equilibrium satisfies a linear sharing rule.*

We calculate the sharing rules directly by solving the Negishi equations (1) for given weights  $\lambda^h$  for all  $h \in \mathcal{H}$ . For utility functions of type [EC] equations (1) become

$$(7) \quad \left(A^1 + \frac{c^1}{\gamma}\right)^{-\gamma} - \lambda^h \left(A^h + \frac{c^h}{\gamma}\right)^{-\gamma} = 0, \quad h \in \mathcal{H}, \quad y \in \mathcal{Y},$$

where we include the trivial equation for agent 1 with weight  $\lambda^1 = 1$  to simplify the subsequent expressions. Some algebra leads to the following linear sharing rule,

$$c_y^h = e_y \cdot \left( \frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \right) + \gamma \left( -A^h + \frac{(\lambda^h)^{\frac{1}{\gamma}}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \sum_{i \in \mathcal{H}} A^i \right).$$

Note that for the special case of CRRA utility functions,  $A^h = 0$  for all  $h \in \mathcal{H}$ , the sharing rule has zero intercept. For CARA utility functions the linear sharing rules are as follows.

$$c_y^h = e_y \cdot \frac{\tau^h}{\sum_{i \in \mathcal{H}} \tau^i} + \left( \tau^h \ln(\lambda^h) - \frac{\tau^h}{\sum_{i \in \mathcal{H}} \tau^i} \sum_{i \in \mathcal{H}} \tau^i \ln(\lambda^i) \right).$$

**Remark 1.** *Now the necessity of genericity with respect to transition probabilities in Lemma 1 is apparent. If transition probabilities are i.i.d. and all agents have HARA utility with  $\gamma = 1$  (but possibly  $A^h \neq 0$ ), then the linear sharing rule leads to the bond price being a linear function of the endowment for any set of dividends and initial portfolios. If  $c_y^h = m^h e_y + b^h$  for all  $y \in \mathcal{Y}$  then*

$$q_y = \beta \left( \sum_{s \in \mathcal{Y}} \Pi_{ys} \frac{1}{m^h e_s + b^h} \right) (m^h e_y + b^h).$$

Two-fund separation in models with a one-period bond (see Section 4.2) depends crucially on whether the intercept of the sharing rules is zero. We prove the following lemma in Appendix A.2.

**Lemma 3.** *Suppose all agents have equi-cautious HARA utility functions of the type [CARA] or the type [EC] with  $\sum_{h \in \mathcal{H}} A^h \neq 0$  and [A3] holds. Then, for a generic set  $\mathcal{T} \subset \Delta_{++}^{H-1}$  of initial holdings of the first asset, each agent's sharing rule is linear with nonzero intercept, that is,  $b^h \neq 0$  for all  $h \in \mathcal{H}$ .*

#### 4. TWO-FUND SEPARATION: CONSOL VS. ONE-PERIOD BOND

We define the concept of two-fund monetary separation (see Cass and Stiglitz (1970)) in the context of our general equilibrium model. Agents' portfolios satisfy two-fund monetary separation if each agent has the same share of every risky asset in the economy.

**Definition 2.** *Consider an economy  $\mathcal{E}$  with an asset that has a riskless payoff vector,  $d_y^S = 1$  for all  $y \in \mathcal{Y}$ . We say that agent  $h$ 's portfolios exhibits two-fund monetary separation if  $\Theta^{hj} = \Theta^{hk}$  for all  $j, k \neq S$ .*

**4.1. Infinitely Lived Securities.** In this subsection we assume that there are no short-lived assets, that is,  $J^l = S$ . Then equation (6) immediately yields that the consumption vector of every agent  $h$  is a linear combination of the asset dividends,

$$(8) \quad c^h = (d^1, \dots, d^S)\Theta^h.$$

Intuitively, the state-dependent security prices do not affect consumption since the agents do not trade the assets. Under the assumptions that all assets are infinitely lived and that there is a safe asset we recover the classical two-fund monetary separation result for static demands of Cass and Stiglitz (1970) in our dynamic equilibrium context.

**Theorem 1** (Two-Fund Separation Theorem). *Suppose the economy  $\mathcal{E}$  satisfies Assumption [A1] has  $J \leq S$  infinitely lived assets with linearly independent payoff vectors. The first  $J-1$  assets are in unit net supply and asset  $J$  is a consol in zero net supply. If the agents have equi-cautious HARA utilities then their portfolios exhibit two-fund monetary separation.*

**Proof:** Lemma 2 implies that sharing rules are linear and  $c_y^h = m^h e_y + b^h \forall h \in \mathcal{H}, y \in \mathcal{Y}$ . Under the assumptions of the theorem equation (8) has the unique solution  $\Theta^{hJ} = b^h$  and  $\Theta^{hj} = m^h \forall j = 1, \dots, J-1$ .  $\square$

Under the assumptions of Theorem 1 markets are dynamically complete with fewer assets than states and so portfolios exhibit two-fund separation for  $J < S$ .

Kang (2003) observes that the results of Judd et al. (2003) can be generalized to economies with time-varying positive transition probabilities. In addition, he notices that the results also hold for finite-horizon economies with only long-lived assets. We can use these observations to extend the result of Theorem 1 to economies with a finite time horizon and time-varying (positive) transition matrices. Either change to our model would affect the Negishi weights  $\lambda^h, h \in \mathcal{H}$ , and sharing rules  $m^h, b^h, h \in \mathcal{H}$ , but two-fund monetary separation would continue to hold.

**4.2. A One-period Riskless Bond.** Now we assume that the riskless asset is not a consol but instead a one-period bond. In addition the economy has  $S-1$  infinitely lived assets in unit net supply. In such an economy two-fund monetary separation generically fails even when sharing rules are linear with nonzero intercepts.

**Theorem 2.** *Consider an economy  $\mathcal{E}$  that satisfies the following conditions.*

- (i) *There are  $J = S \geq 3$  assets.*
- (ii) *There are  $S-1$  infinitely lived securities in unit net supply. The last asset is a one-period riskless bond.*
- (iii) *Assumptions [A1] – [A3] hold.*
- (iv) *All agents have equi-cautious HARA utility functions of the type [CARA] or the type [EC] with  $\sum_{h \in \mathcal{H}} A^h \neq 0$ .*

*Then there are generic subsets  $\mathcal{T} \subset \Delta_{++}^{H-1}$  of initial portfolios of the first asset and  $\mathcal{P} \subset \Delta_{++}^{S \times (S-1)}$  of transition matrices such that each agent's equilibrium portfolio does not exhibit two-fund monetary separation.*

**Proof:** All agents' consumption allocations follow a linear sharing rule. Now suppose that equilibrium portfolios exhibit two-fund monetary separation, so agent  $h$  holds a portion  $\vartheta^h \equiv \Theta^{hj}$  of all infinitely lived assets  $j = 1, \dots, S-1$ , and  $\Theta^{hS}$  of the one-period bond. Then equation (6) implies that the portfolio shares must satisfy

$$(9) \quad m^h \cdot e + b^h 1_S = \vartheta^h \cdot e + \Theta^{hS}(1_S - q^S) \quad \text{for all } h \in \mathcal{H},$$

where  $q^S$  denotes the bond price and  $1_S$  the vector of all ones. If  $b^h = 0$  for all  $h \in \mathcal{H}$  then  $\Theta^{hS} = 0$  and  $m^h = \vartheta^h$  is a solution to this equation. Thus, two-fund monetary separation holds. But Lemma 3 states that under conditions (iii) and (iv) it holds that  $b^h \neq 0$  for all  $h \in \mathcal{H}$  for a generic set of initial portfolio holdings.

Now suppose  $b^h \neq 0$  for all  $h$ . Then any solution to equation (9) must have  $\Theta^{hS} \neq 0$ . Thus we can rewrite the equation as

$$q^S = \frac{\vartheta^h - m^h}{\Theta^{hS}} \cdot e + \frac{\Theta^{hS} - b^h}{\Theta^{hS}} \cdot 1_S.$$

But now the price of the one-period bond is a linear function of the aggregate endowment. Lemma 1, Part 2, states that for a generic set of initial portfolios and transition matrices this cannot happen. Hence, equation (9) does not have a solution generically. The intersection of generic sets is generic. The statement of the theorem now follows.  $\square$

Equation (9) in the proof of the theorem is very instructive in providing intuition for the lack of two-fund monetary separation when the bond is short-lived. Recall that for an economy with a consol the corresponding equation would be as follows.

$$(10) \quad m^h \cdot e + b^h 1_S = \vartheta^h \cdot e + \Theta^{hS} 1_S \quad \text{for all } h \in \mathcal{H}.$$

So, the only difference that the short-lived bond induces in the portfolio equation is that the bond position  $\Theta^{hS}$  is multiplied by the coupon payment minus the price instead of just being multiplied by the coupon payment. The economic reason for this difference is that the agent does not trade the consol after time 0 but must reestablish the position in the short-lived bond in every period. This change has no impact on the portfolio weights if agents' sharing rules have zero intercept and so the riskless security is not traded. But if sharing rules have nonzero intercept, then the bond price affects the portfolio weights. It still does not destroy two-fund monetary separation if it is a linear function of the social endowment. But if that relationship does not hold, then the fluctuations of the bond price lead to a change of the portfolio weights that implement equilibrium consumption.

In summary, fluctuations in the equilibrium interest rates of the short-term bond lead to the breakdown of two-fund monetary separation. These fluctuations affect an agent holding a nonzero bond position in equilibrium because he must rebuild that position in every period. On the contrary, in an economy with a consol, the agent establishes a position in the consol at time 0 once and forever. Fluctuations in the price of the consol therefore do not affect the agent just like he is unaffected by stock price fluctuations. This fact allows him to hold a portfolio exhibiting two-fund monetary separation.

The fact that interest rate variability has significant economic consequences in a dynamic equilibrium model has also been noted by Magill and Quinzii (2000). They examine an

infinite-horizon CAPM economy with stochastic endowments and observe that with fewer assets than states an Arrow-Debreu allocation can only be achieved if a constant consumption stream can be spanned by the payoff matrix. But such a spanning condition may not hold if the interest rate fluctuates in equilibrium. As a consequence markets will be incomplete.

**4.3. A Numerical Example.** The purpose of this numerical example is to show that the equilibrium interest rate fluctuations in an economy with a one-period bond have a quantitatively nontrivial impact on agents' portfolios.

Consider an economy with  $H = 3$  agents who have CARA utility functions with coefficients of absolute risk-aversion of 1, 3, and 5, respectively. The agents' discount factor is  $\beta = 0.95$ . There are three stocks with the following dividend vectors.

$$\begin{aligned} d^1 &= (1.1, 1.1, 0.9, 0.9), \\ d^2 &= (1.2, 1.1, 0.9, 0.8), \\ d^3 &= (0.9, 1.1, 1.1, 0.9). \end{aligned}$$

All elements of the Markov transition matrix are 0.25. The economy starts in state  $y_0 = 1$ . The agents' initial holdings of the three stocks are  $\theta_{-1}^{hj} = \frac{1}{3}$  for  $h = 1, 2, 3$ ,  $j = 1, 2, 3$ .

The state-contingent stock prices are

$$\begin{aligned} q^1 &= (21.64129, 23.09972, 17.79559, 14.63329), \\ q^2 &= (21.41381, 22.85691, 17.60854, 14.47947), \\ q^3 &= (21.84659, 23.31885, 17.96441, 14.77210). \end{aligned}$$

Suppose that the fourth asset in this economy is a consol. The price vector  $q^c$  of the consol is then

$$q^c = (22.00141, 23.48411, 18.09172, 14.87679).$$

The economy satisfies the conditions of Theorem 1 and so agents' portfolios (written as  $(\Theta^{h1}, \Theta^{h2}, \Theta^{h3})$ ) exhibit two-fund monetary separation.

$$\Theta^1 = \left(\frac{15}{23}, \frac{15}{23}, \frac{15}{23}, -0.94401\right), \quad \Theta^2 = \left(\frac{5}{23}, \frac{5}{23}, \frac{5}{23}, 0.34328\right), \quad \Theta^3 = \left(\frac{3}{23}, \frac{3}{23}, \frac{3}{23}, 0.60073\right).$$

Suppose the fourth asset is a one-period bond instead of a consol. The bond price  $q^b$  is

$$q^b = (1.10007, 1.17421, 0.90459, 0.74384).$$

The agents' portfolios do not exhibit two-fund monetary separation.

$$\begin{aligned} \Theta^1 &= (0.21024, 0.44302, 0.36858, -0.48296) \\ \Theta^2 &= (0.37809, 0.29345, 0.32052, 0.17562) \\ \Theta^3 &= (0.41166, 0.26353, 0.31091, 0.30734) \end{aligned}$$

The change in the bond maturity strongly affects portfolios. The least risk-averse agent holds considerably less of the three stocks while the two more risk-averse agents hold considerably more of the two stocks than in the economy with a consol.

## 5. DYNAMICALLY COMPLETE MARKETS WITH FEW ASSETS

Up to this point the analysis has been based on the assumption that the transition matrix  $\Pi$  had no zero entries. In the presence of  $J = S$  independent assets this assumption is needed to rule out any possibility of trade in equilibrium after the initial period. Without this assumption there would exist a continuum of portfolio allocations in financial markets supporting an efficient financial market equilibrium. From an economic point of view this assumption means that at any time  $t + 1$  any dividend state can occur, no matter what previous state in period  $t$  occurred. However, it may economically be more intuitive to believe that dividends move more “smoothly” and do not experience jumps of arbitrary size. In addition, it may also be more reasonable to assume that markets contain fewer assets than the total number of states but are still complete due to dynamic trading, as in Kreps (1982). For our model this assumption means that all rows of the transition matrix  $\Pi$  contain zeros. Any given exogenous state  $y \in \mathcal{Y}$  can only be succeeded by states from a strict subset of  $\mathcal{Y}$ . In turn, a smaller number of assets will suffice to implement a consumption plan. In this section we formalize this idea.

**5.1. Economies with  $J < S$  Assets.** We change two features of our basic model from Section 2. First, the number  $J$  of assets is now smaller than the number  $S$  of exogenous states,  $J < S$ . Second, the Markov chain of exogenous states remains recurrent but now every row of the transition matrix  $\Pi$  has only exactly  $J$  positive entries. The remaining  $S - J$  elements in each row are zero. We define the set of successors of a state  $y \in \mathcal{Y}$  as

$$S(y) = \{z \in \mathcal{Y} | \Pi_{yz} > 0\} = Y - \{z \in \mathcal{Y} | \Pi_{yz} = 0\}.$$

Our assumption on the transition matrix ensures that the cardinality of all successors sets is identical to  $J$ , so  $|S(y)| = J$  for all  $y \in \mathcal{Y}$ . We also refer to the set of possible predecessors of a state  $y \in \mathcal{Y}$  and denote it by  $P(y) = \{x \in \mathcal{Y} | \Pi_{xy} > 0\}$ . For a given set of successor sets  $S(y), y \in \mathcal{Y}$ , we denote the set of permissible transition matrices by

$$\{A \in \mathbb{R}^{S \times S} : A_{ys} > 0 \forall s \in S(y), y \in \mathcal{Y}, A_{yz} = 0 \forall z \notin S(y), y \in \mathcal{Y}, \sum_{s \in S(y)} A_{ys} = 1 \forall y \in \mathcal{Y}\}.$$

We identify such transition matrices again with elements of a set that is diffeomorphic to an open subset of  $\mathbb{R}_{++}^{S \times (J-1)}$ . Let  $S^-(y)$  denote the successor set  $S(y)$  without its largest element and define

$$\Delta_{++}^{S \times (J-1)} = \{A_{ys}, y \in \mathcal{Y}, s \in S^-(y) : A_{ys} > 0, \sum_{s \in S^-(y)} A_{ys} < 1 \forall y \in \mathcal{Y}\}.$$

We denote economies with such restricted transition matrices by  $\mathcal{E}_f$ .

We can easily adapt our computational approach for the calculation of efficient equilibria to the new model. The system of equations (1, 2, 3) for the computation of the state-contingent equilibrium consumptions does not change. Similarly, the equations (4, 5) for the asset prices still apply, too. The only change occurs in the agents' budget constraints which are used to determine the portfolio choices. Equations (6) no longer apply but must

be replaced by the following set of equations,

$$(11) \quad c_x^h = \sum_{j \in L} \theta_y^{hj} (q_x^j + d_x^j) + \sum_{j \in O} \theta_y^{hj} d_x^j - \theta_x^h q_x \quad \forall x \in S(y), y \in \mathcal{Y}, h \in \mathcal{H}.$$

For this set of  $S \times J$  equations with the  $S \times J$  unknowns  $\theta_y^{hj}, y \in \mathcal{Y}, j = 1, \dots, J$ , to have a unique solution the  $J \times J$ -payoff matrix for the  $J$  states in the set  $S(y)$  must have full rank for all  $y \in \mathcal{Y}$ . The proof of Proposition 2 can be modified to show a generic existence result under the assumption that the dividend submatrix  $d_{S(y)}$  has full rank  $J$  for all  $y \in \mathcal{Y}$ . We state the modified assumptions for this model.

[A1'] Each row of  $\Pi$  has exactly  $J$  positive elements.

[A2']  $\text{Rank}[d_{S(y)}] = J$  for  $y \in \mathcal{Y}$ .

5.2. **Consol.** We can easily adapt the proof of Theorem 1 in order to establish the corresponding result for the revised model.

**Theorem 3** (Two-Fund Separation Theorem). *Suppose the economy  $\mathcal{E}_f$  satisfies Assumptions [A1'] and has  $K \leq J$  infinitely lived assets with linearly independent payoff vectors. The first  $K - 1$  assets are in unit net supply and asset  $K$  is a consol in zero net supply. If the agents have equi-cautious HARA utilities then there is no trade after the initial period and the agents' portfolios exhibit two-fund monetary separation.*

**Proof:** The statement of the proposition follows from Lemma 2 and equations (11). Sharing rules are linear,  $c_y^h = m^h e_y + b^h \forall h \in \mathcal{H}, y \in \mathcal{Y}$ . Equations (11) simplify to  $c_x^h = \sum_{j=1}^K \theta_y^{hj} (q_x^j + d_x^j) - \sum_{j=1}^K \theta_x^{hj} q_x^j \forall x \in S(y)$ , since all assets are infinitely lived. Portfolio shares of  $\theta_y^{hj} = m^h$  for all  $j = 1, \dots, K - 1, y \in \mathcal{Y}$  and  $\theta_y^{hK} = b^h$  for  $y \in \mathcal{Y}$  are the unique solution to the budget equations.  $\square$

5.3. **One-period Bond.** The next theorem states that in economies with fewer assets than states and a one-period riskless bond there is trade on security markets.

**Theorem 4.** *Consider an economy  $\mathcal{E}_f$  that satisfies the following conditions.*

- (i) *There are  $J \geq 2$  assets.*
- (ii) *There are  $J - 1$  infinitely lived securities in unit net supply. The last asset is a one-period riskless bond.*
- (iii) *Assumptions [A1'], [A2'], and [A3] hold.*
- (iv) *All agents have equi-cautious HARA utility functions of the type [CARA] or the type [EC] with  $\sum_{h \in \mathcal{H}} A^h \neq 0$ .*

*Then there are generic subsets  $\mathcal{T} \subset \Delta_{++}^{H-1}$  of initial portfolios of the first asset and  $\mathcal{P}_f \subset \Delta_{++}^{S \times (J-1)}$  of transition matrices such that the agents trade on security markets.*

Without proof we state the failure of two-fund separation when the bond is short-lived.

**Corollary 1.** *Under the assumptions of Theorem 4 portfolios generically do not exhibit two-fund separation if there are  $J - 1 \geq 2$  infinitely lived securities in unit net supply.*

An intuitive explanation for the occurrence of trade is as follows. Suppose there is no trade in this economy. Then equations (11) simplify to the following equations.

$$(12) \quad c^h - \sum_{j \in L} \Theta^{hj} d^j - \Theta^{hJ} (1_S - q^J) = 0 \quad \forall h \in \mathcal{H}.$$

If sharing rules have zero intercepts then the bond is not needed for the implementation of the equilibrium consumption allocation and so  $\Theta^{hJ} = 0$  for all  $h \in \mathcal{H}$ . There is no trade in the economy and the sharing rules determine the agents' portfolio positions. But if sharing rules have nonzero intercepts then any solution of (12) must have  $\Theta^{hJ} \neq 0$  and the consumption vectors  $c^h$  must lie in the span of the  $J$  vectors  $d^1, \dots, d^{J-1}$  and  $1_S - q^J$ . Contrary to the model with  $S = J$  and strictly positive transition matrices this property is nongeneric when  $S > J$ . So equations (12) typically do not have a solution and there must be trade. The proof of the theorem (see Appendix A.2) formalizes this intuition. In order to keep the proof simple we continue to restrict ourselves to the case of equi-cautious HARA utility functions although the trade result holds for much broader classes of utility functions. (The proof of the corollary is along the lines of our other genericity proofs but the details are very tedious and thus are omitted.)

**5.4. Numerical Example.** We modify the example from Section 4.3 to illustrate Theorem 4. We eliminate the third stock from the economy and alter the Markov transition matrix but do not change any other parameters. The revised transition matrix is

$$\Pi = \begin{bmatrix} 0.5 & 0.25 & 0.25 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0.25 & 0 & 0.5 & 0.25 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Every state has only three possible successors,  $J = 3$  and  $S = 4$ . The stock prices are

$$\begin{aligned} q^1 &= (22.85754, 21.47990, 16.58465, 15.56036), \\ q^2 &= (22.88138, 21.41006, 16.53213, 15.46785). \end{aligned}$$

Suppose the third asset in this economy is a consol. The price vector of the consol is

$$q^c = (23.13575, 21.76886, 16.92729, 15.86207).$$

The economy satisfies the conditions of Theorem 3 and agents' portfolios (written as  $(\Theta^{h1}, \Theta^{h2}, \Theta^{h3})$ ) exhibit two-fund monetary separation,

$$\Theta^1 = \left(\frac{15}{23}, \frac{15}{23}, -0.63461\right), \quad \Theta^2 = \left(\frac{5}{23}, \frac{5}{23}, 0.23077\right), \quad \Theta^3 = \left(\frac{3}{23}, \frac{3}{23}, 0.40384\right).$$

If markets are completed with a one-period bond then the bond prices are,

$$q^b = (1.05757, 1.05209, 0.89992, 0.84189).$$

In order to implement the consumption allocation the agents need to engage in some trade whenever the state of the economy changes. The agents' portfolios never exhibit two-fund

monetary separation and involve considerable amounts of trade.

$y$	$\theta_y^{11}$	$\theta_y^{12}$	$\theta_y^{13}$	$\theta_y^{21}$	$\theta_y^{22}$	$\theta_y^{23}$	$\theta_y^{31}$	$\theta_y^{32}$	$\theta_y^{33}$
1	0.23356	0.46714	-0.83205	0.36961	0.28468	0.30256	0.39682	0.24818	0.52949
2	0.18441	0.51023	-0.69226	0.38749	0.26901	0.25173	0.42810	0.22076	0.44053
3	0.52491	0.18352	-0.98206	0.26367	0.38781	0.35711	0.21142	0.42867	0.62494
4	0.45437	0.24558	-0.83052	0.28932	0.36524	0.30201	0.25631	0.38918	0.52851

## 6. CONCLUSION

The celebrated two-fund separation theorem holds in a dynamic general equilibrium model of asset trading only if agents can trade an infinitely-lived bond. If agents have access to a one-period bond only, two-fund separation typically fails.

On modern financial markets, investors can trade a multitude of financial assets including many finite-maturity bonds. But they do not have access to a truly safe asset. Therefore, the critiques of Canner et al. (1997) and Bossaerts et al. (2003), which are based on the implicit assumption that investors have access to such an asset, are not justified. We cannot expect two-fund separation to hold when the key assumption is not satisfied.

Our results naturally lead us to question whether modern financial markets may enable investors to synthesize a consol through a variety of other assets, thereby leading to two-fund separation. Judd et al. (2004) study this question by examining families of bonds with variable but finite maturity structures. They argue that for some nongeneric transition matrices and dividend structures a finite number of bonds can span a consol. In such situations agents hold the same fund of risky stocks. But their computational exercises show that this result does not hold in general. Thus the insights of this note appear relevant beyond our standard general equilibrium model with its the simple asset structures.

## APPENDIX

**A.1. Parametric Systems of Equations.** We state the theorem on a parametric system of equations that we use in the genericity proofs below.

**Theorem 5. (Parametric Systems of Equations)** *Let  $\Omega \subset \mathbb{R}^k$ ,  $X \subset \mathbb{R}^n$  be open sets and let  $h : \Omega \times X \rightarrow \mathbb{R}^m$  be a smooth function. If  $n < m$  and for all  $(\bar{\omega}, \bar{x}) \in \Omega \times X$  such that  $h(\bar{\omega}, \bar{x}) = 0$  it holds that  $\text{rank} [D_{\omega, x} h(\bar{\omega}, \bar{x})] = m$ , then there exists a set  $\Omega^* \subset \Omega$  with  $\Omega - \Omega^*$  a set of Lebesgue measure zero, such that  $\{x \in X : h(\omega, x) = 0\} = \emptyset$  for all  $\omega \in \Omega^*$ .*

For a detailed discussion of this theorem see Magill and Quinzii (1996a, Paragraph 11; 1996b). This theorem is a specialized version of the parametric transversality theorem, see Guillemin and Pollack (1974, Chapter 2, Paragraph 3) and Mas-Colell (1985, Chapter 8). Billingsley (1986, Section 12) provides a detailed exposition on the  $k$ -dimensional Lebesgue measure in Euclidean space. For an exposition on sets of measure zero see Guillemin and Pollack (1974, Chapter 1, Paragraph 7). A set is said to have full measure if its complement is a set of Lebesgue measure zero. An open set of full Lebesgue measure is called generic.



**A.2. Proofs.** This section contains all proofs that are omitted in the main body of the paper.

**Proof of Proposition 1:** Market-clearing and collinearity of marginal utilities imply that in an efficient equilibrium all agents have state-independent consumption allocations. Define  $\hat{c}^h \equiv \bar{c}_y^h$  for all  $y \in \mathcal{Y}$ . Equations (2) imply that the agents' consumption allocations are

$$\hat{c}^h = \frac{([I_S - \beta\Pi]^{-1})_{y_0} \cdot \omega^h}{\sum_{s \in \mathcal{Y}} ([I_S - \beta\Pi]^{-1})_{y_0 s}}$$

for all  $h \in \mathcal{H}$ . The resulting asset prices are for infinitely lived assets,  $q^j = [I_S - \beta\Pi]^{-1} \beta\Pi d^j$ ,  $j \in L$ , and for one-period assets,  $q^j = \beta\Pi d^j$ ,  $j \in O$ . If the matrix  $d$  has full column rank then the solution to equations (6) is unique and gives the agents' holdings of infinitely-lived assets  $j \in L$  in unit net supply,

$$\Theta^{hj} = \frac{\hat{c}^h}{\hat{e}} = \frac{1}{\hat{e}} \frac{([I_S - \beta\Pi]^{-1})_{y_0} \cdot \omega^h}{\sum_{s \in \mathcal{Y}} ([I_S - \beta\Pi]^{-1})_{y_0 s}}.$$

(If the matrix  $d$  does not have full column rank then this solution is only one in a continuum of optimal portfolios.) The agents do not trade any of the other assets. Note that all expressions in this proof are independent of agents' utility functions.  $\square$

**Proof of Proposition 2:** The existence result of Mas-Colell and Zame (1991) implies that there exist equilibrium state-contingent consumption values  $c_y^h$ ,  $h \in \mathcal{H}$ ,  $y \in \mathcal{Y}$ , which solve the system of equations (1, 2, 3). The critical remaining issue for the existence of an efficient financial market equilibrium is now whether the matrix  $D$  has full rank. In that case equations (6) yield the agents' equilibrium portfolios.

If all assets are long-lived then  $D = d$  and so  $D$  has full rank. We now show that for economies with a one-period riskless bond the matrix  $D$  generically has full rank. If  $D$  does not have full rank then the following set of equations must have a solution.

$$(13) \quad u'_1(c_y^1) - \lambda^h u'_h(c_y^h) = 0, \quad h = 2, \dots, H, \quad y \in \mathcal{Y},$$

$$(14) \quad \left( [I_S - \beta\Pi]^{-1} (p \otimes (c^h - \sum_{j \in L} \theta_{-1}^{hj} d^j)) \right)_{y_0} = 0, \quad h = 2, \dots, H,$$

$$(15) \quad \sum_{h=1}^H c_y^h - \sum_{h=1}^H \omega_y^h = 0, \quad y \in \mathcal{Y},$$

$$(16) \quad q_y^S p_y - \beta\Pi_y \cdot p = 0, \quad y \in \mathcal{Y},$$

$$(17) \quad \sum_{j \in L} d^j(y) a_j + (1 - q^S(y)) = 0, \quad y \in \mathcal{Y}.$$

We denote the system of equations (13)–(17) by  $F((c^h)_{h \in \mathcal{H}}, (\lambda^h)_{h \geq 2}, q^S, a; (\theta_{-1}^h)_{h \geq 2}, \Pi_{\cdot 1}) = 0$ . The expression  $F_{(i)} = 0$  denotes equations (i). We now show that this system has no solutions for generic sets of individual asset holdings and transition probabilities.

The system (13)–(17) has  $HS + (H - 1) + S + (S - 1)$  endogenous unknowns  $c^h$ ,  $h \in \mathcal{H}$ ,  $\lambda^h$ ,  $h = 2, \dots, H$ ,  $q^S$ , and  $a_j$ ,  $j = 1, \dots, S - 1$ , in  $(H - 1)S + (H - 1) + S + S + S$

equations. In addition, the function  $F$  depends on the  $(H - 1) + S$  exogenous parameters  $(\theta_{-1}^{h1})_{h \geq 2} \in \Delta_{++}^{H-1}$  and  $\Pi_{\cdot 1}$  where  $\Pi^{-S} \in \Delta_{++}^{S \times (S-1)}$  denotes the first  $S - 1$  columns of  $\Pi$ . Assumption [A2] allows us to assume without loss of generality that  $e_1 \neq e_S$ .

We now prove that the Jacobian of  $F$  taken with respect to  $c^h, q^S, \theta_{-1}^{h1}$  and  $\Pi_{\cdot 1}$  has full row rank  $(H - 1)S + (H - 1) + S + S + S$ . Denote by  $\Lambda_S(x) \in \mathbb{R}^{S \times S}$  the diagonal matrix whose diagonal elements are the elements of the vector  $x \in \mathbb{R}^S$ . We denote the derivative of the budget constraints (14) with respect to the agent's initial holding in the first infinitely lived asset,  $-([I_S - \beta\Pi]^{-1}(p \otimes d^1))_{y_0}$ , by  $\eta^1$ . Note that  $\eta^1 < 0$ . In order to keep the display tractable, we show the Jacobian of  $F$  for the special case of  $H = 3$ .

	$c^1$	$c^2$	$c^3$	$q^S$	$\theta_{-1}^{21}$	$\theta_{-1}^{31}$	$\Pi_{\cdot 1}$	
$F_{(13)_{h=2}}$	$\Lambda_S(u_1''(c^1))$	$\Lambda_S(-\lambda^2 u_2''(c^2))$	0	0	0	0	0	$S$
$F_{(13)_{h=3}}$	$\Lambda_S(u_1''(c^1))$	0	$\Lambda_S(-\lambda^3 u_3''(c^3))$	0	0	0	0	$S$
$F_{(14)_{h=2}}$			0	0	$\eta^1$	0		1
$F_{(14)_{h=3}}$		0		0	0	$\eta^1$		1
$F_{(15)}$	$I_S$	$I_S$	$I_S$	0	0	0	0	$S$
$F_{(16)}$		0	0		0	0	$-\beta\Lambda((p_1 - p_S) \cdot 1_S)$	$S$
$F_{(17)}$	0	0	0	$-I_S$	0	0	0	$S$
	$S$	$S$	$S$	$S$	1	1	$S$	

The variables above the matrix indicate the variables with respect to which derivatives have been taken in the column underneath. The numbers to the right and below the matrix indicate the number of rows and columns, respectively. The terms to the left indicate the equations. Missing entries are not needed for the proof.

Now we perform column operations to obtain zero matrices in the first set of columns of the Jacobian. The sets of columns for  $c^h$ ,  $h \in \mathcal{H}$ , of the Jacobian then appear as follows.

	$c^1$	$c^2$	$c^3$	
$F_{(13)_{h=2}}$	0	$\Lambda_S(-\lambda^2 u_2''(c^2))$	0	$S$
$F_{(13)_{h=3}}$	0	0	$\Lambda_S(-\lambda^3 u_3''(c^3))$	$S$
$F_{(14)_{h=2}}$			0	1
$F_{(14)_{h=3}}$		0		1
$F_{(15)}$	$I_S + \Lambda\left(\frac{u_1''(c^1)}{\lambda^2 u_2''(c^2)}\right) + \Lambda\left(\frac{u_1''(c^1)}{\lambda^3 u_3''(c^3)}\right)$	$I_S$	$I_S$	$S$
$F_{(16)}$	0	0	0	$S$

The transformed matrix has submatrices of the following ranks.

	$c^1$	$c^2$	$c^3$	$q^S$	$\theta_{-1}^{21}$	$\theta_{-1}^{31}$	$\Pi_{\cdot 1}$	
$F_{(13)_{h=2}}$	0	$S$	0	0	0	0	0	$S$
$F_{(13)_{h=3}}$	0	0	$S$	0	0	0	0	$S$
$F_{(14)_{h=2}}$			0	0	1	0		1
$F_{(14)_{h=3}}$		0		0	0	1		1
$F_{(15)}$	$S$	$S$	$S$	0	0	0	0	$S$
$F_{(16)}$		0	0		0	0	$S$	$S$
$F_{(17)}$		0	0	$S$	0	0	0	$S$
	$S$	$S$	$S$	$S$	1	1	$S$	

The term  $D_{\Pi_1} F_{(16)} = -\beta \Lambda((p_1 - p_S) \cdot 1_S)$  has rank  $S$  since  $p_1 \neq p_S$  due to  $e_1 \neq e_S$ .

This matrix has full row rank  $(H-1)S + (H-1) + 3S$  which exceeds the number of endogenous variables by 1. The function  $F$  is defined on open sets with  $c^h \in \text{int}(X)$  for all  $h \in \mathcal{H}$ ,  $\lambda^h \in \mathbb{R}_{++}^S$  for  $h \geq 2$ ,  $a \in \mathbb{R}^{S-1}$ ,  $q^S \in \mathbb{R}_{++}^S$ ,  $(\theta_{-1}^{h1})_{h \geq 2} \in \Delta_{++}^{H-1}$ , and  $\Pi^{-S} \in \Delta_{++}^{S \times (S-1)}$ . Hence,  $F$  satisfies the hypotheses of the theorem on parametric systems of equations, Theorem 5. We conclude that there exist subsets  $\mathcal{T} \subset \Delta_{++}^{h-1}$  and  $\mathcal{P} \subset \Delta_{++}^{S \times (S-1)}$  of full Lebesgue measure such that the solution set of the system (13)–(17) is empty. The sets  $\mathcal{T}$  and  $\mathcal{P}$  are open. The solutions to (13)–(17) change smoothly with the exogenous parameters. A small variation in initial portfolios and probabilities cannot lead to a solvable system if there was no solution for the original parameters.

We conclude that the matrix  $D$  has full rank  $S$  and so an equilibrium exists for initial holdings  $(\theta_{-1}^{h1})_{h \geq 2} \in \mathcal{T}$  of the first asset and transition matrices such that  $\Pi^{-S} \in \mathcal{P}$ .  $\square$

**Proof of Lemma 1, Part 1:** The price of the one-period bond is  $q_y^b = \beta \frac{\Pi_y \cdot u'_1(c)}{u'_1(c_y)}$  where  $u'_1(c)$  denotes the column vector of utilities  $u'_1(c_y)$ ,  $y \in \mathcal{Y}$ . If the social endowment  $e$  is not constant, every agent must have nonconstant consumption. Choose  $y_1 \in \arg \min\{c_y^1 | y \in \mathcal{Y}\}$  such that  $\Pi_{y_1 s} > 0$  for some  $s \notin \arg \min\{c_y^1 | y \in \mathcal{Y}\}$ . Similarly, choose  $y_2 \in \arg \max\{c_y^1 | y \in \mathcal{Y}\}$  such that  $\Pi_{y_2 s} > 0$  for some  $s \notin \arg \max\{c_y^1 | y \in \mathcal{Y}\}$ . Obviously,  $y_1 \neq y_2$ . Then  $\Pi_{y_1} \cdot u'_1(c) < u'_1(c_{y_1})$  and  $\Pi_{y_2} \cdot u'_1(c) > u'_1(c_{y_2})$  and so  $q_{y_2}^b > \beta > q_{y_1}^b$ . If there is no aggregate risk in the economy, then the bond price equation immediately yields  $q^b = \beta$ .  $\square$

**Part 2:** The price of the one-period bond in state  $y \in \mathcal{Y}$  satisfies  $q_y p_y = \beta \Pi_y \cdot p$ . If in equilibrium the price is a linear function of the social endowment  $e$ , then the following set of equations must have a solution.

$$(18) \quad u'_1(c_y^1) - \lambda^h u'_h(c_y^h) = 0, \quad h = 2, \dots, H, \quad y \in \mathcal{Y},$$

$$(19) \quad \left( [I_S - \beta \Pi]^{-1} (p \otimes (c^h - \sum_{j \in L} \theta_{-1}^{hj} d^j)) \right)_{y_0} = 0, \quad h = 2, \dots, H,$$

$$(20) \quad \sum_{h=1}^H c_y^h - \sum_{h=1}^H \omega_y^h = 0, \quad y \in \mathcal{Y},$$

$$(21) \quad (a e_y + f) p_y - \beta \Pi_y \cdot p = 0, \quad y \in \mathcal{Y}.$$

We denote the system of equations (18)–(21) by  $F((c^h)_{h \in \mathcal{H}}, (\lambda^h)_{h \geq 2}, a, f; (\theta_{-1}^{h1})_{h \geq 2}, \Pi_1) = 0$ . The expression  $F_{(i)} = 0$  denotes equations (i). The system has  $HS + (H-1) + 2$  endogenous unknowns  $c^h$ ,  $\lambda^h$ ,  $h = 2, \dots, H$ ,  $a$ , and  $f$  and  $(H-1)S + (H-1) + S + S$  equations. In addition,  $F$  depends on the  $(H-1) + S$  exogenous parameters  $\theta_{-1}^{h1}$ ,  $h = 2, \dots, H$ , and  $\Pi_1$ . The aggregate endowment is not constant and so we can assume w.l.o.g. that  $p_1 \neq p_S$ .

The Jacobian of  $F$  taken with respect to  $c^h$ ,  $\theta_{-1}^{h1}$  and  $\Pi_1$  is identical to the respective columns of the corresponding matrix in the proof of Proposition 2. After performing the same column operations as in that proof we obtain a transformed matrix with submatrices

of the following ranks.

	$c^1$	$c^2$	$c^3$	$\theta_{-1}^{21}$	$\theta_{-1}^{31}$	$\Pi_{-1}$	
$F_{(18)_{h=2}}$	0	$S$	0	0	0	0	$S$
$F_{(18)_{h=3}}$	0	0	$S$	0	0	0	$S$
$F_{(19)_{h=2}}$			0	1	0		1
$F_{(19)_{h=3}}$		0		0	1		1
$F_{(20)}$	$S$	$S$	$S$	0	0	0	$S$
$F_{(21)}$		0	0	0	0	$S$	$S$
	$S$	$S$	$S$	1	1	$S$	

This matrix has full row rank  $(H-1)S + (H-1) + S + S$  which exceeds the number of endogenous variables by  $S-2 \geq 1$ . The function  $F$  is defined on open sets with  $c^h \in \text{int}(X)$  for all  $h \in \mathcal{H}$ ,  $\lambda^h \in \mathbb{R}_{++}^S$  for  $h \geq 2$ ,  $a, f \in \mathbb{R}$ ,  $(\theta_{-1}^{h1})_{h \geq 2} \in \Delta_{++}^{H-1}$  and  $\Pi^{-S} \in \Delta_{++}^{S \times (S-1)}$ . Hence,  $F$  satisfies the hypotheses of Theorem 5 and the proof proceeds as that of Proposition 2.  $\square$

**Proof of Lemma 3:** We first consider an economy where all agents have equi-cautious HARA utility functions of the type [EC]. Then if  $b^h = 0$  for some agent  $h$  in equilibrium the following equations must hold.

$$(22) \quad u'_1(c_y^1) - \lambda^h u'_h(c_y^h) = 0, \quad h = 2, \dots, H, \quad y \in \mathcal{Y},$$

$$(23) \quad \left( [I_S - \beta \Pi]^{-1} (p \otimes (c^h - \omega^h)) \right)_{y_0} = 0, \quad h = 2, \dots, H,$$

$$(24) \quad \sum_{h=1}^H c_y^h - \sum_{h=1}^H \omega_y^h = 0, \quad y \in \mathcal{Y},$$

$$(25) \quad \left( -A^{\bar{h}} + \frac{(\lambda^{\bar{h}})^{\frac{1}{\gamma}}}{\sum_{i \in \mathcal{H}} (\lambda^i)^{\frac{1}{\gamma}}} \sum_{i \in \mathcal{H}} A^i \right) = 0, \quad \text{for one } \bar{h} \in \mathcal{H}.$$

We denote the system of equations (22)–(25) by  $F((c^h)_{h \in \mathcal{H}}, (\lambda^h)_{h \geq 2}; (\theta_{-1}^h)_{h \geq 2}) = 0$ . The system has  $HS + (H-1)$  endogenous unknowns  $c^h$ ,  $h \in \mathcal{H}$  and  $\lambda^h$ ,  $h = 2, \dots, H$ , and  $(H-1)S + (H-1) + S + 1$  equations. In addition, the function  $F$  depends on the  $(H-1)$  exogenous parameters  $\theta_{-1}^{h1}$ ,  $h = 2, \dots, H$ . We show that the Jacobian of  $F$  taken with respect to  $c^h$ ,  $\lambda^h$ , and  $\theta_{-1}^{h1}$  has full row rank  $(H-1)S + (H-1) + S + 1$ .

For  $\bar{h} \geq 2$  denote the derivative in equation (25) with respect to  $\lambda^{\bar{h}}$  by

$$\eta_{\bar{h}}^2 = \frac{\frac{1}{\gamma} (\lambda^{\bar{h}})^{\frac{1}{\gamma}-1} \cdot \left( \sum_{i=1}^H (\lambda^i)^{\frac{1}{\gamma}} - (\lambda^{\bar{h}})^{\frac{1}{\gamma}} \right)}{\left( \sum_{i=1}^H (\lambda^i)^{\frac{1}{\gamma}} \right)^2} \left( \sum_{i=1}^H A^i \right).$$

For  $\bar{h} = 1$  we cannot take the derivative in (22) with respect to  $\lambda^1$ , since it does not appear (it is normalized to one). Instead we differentiate with respect to  $\lambda^2$  and obtain

$$\eta_1^2 = -\frac{\frac{1}{\gamma} (\lambda^2)^{\frac{1}{\gamma}-1}}{\left( \sum_{i=1}^H (\lambda^i)^{\frac{1}{\gamma}} \right)^2} \left( \sum_{i=1}^H A^i \right).$$

Note that under the condition from the lemma,  $\sum_{i \in \mathcal{H}} A^i \neq 0$ , it holds that  $\eta_h^2 \neq 0$ . For the special case of  $H = 3$  the Jacobian  $D_{c^h, \theta_{-1}^{h1}, \lambda^{\bar{h}}} F$  appears as follows.

	$c^1$	$c^2$	$c^3$	$\theta_{-1}^{21}$	$\theta_{-1}^{31}$	$\lambda^{\bar{h}}$	
$F_{(22)_{h=2}}$	$\Lambda_S(u_1''(c^1))$	$\Lambda_S(-\lambda^2 u_2''(c^2))$	0	0	0		$S$
$F_{(22)_{h=3}}$	$\Lambda_S(u_1''(c^1))$	0	$\Lambda_S(-\lambda^3 u_3''(c^3))$	0	0		$S$
$F_{(23)_{h=2}}$			0	$\eta^1$	0	0	1
$F_{(23)_{h=3}}$		0		0	$\eta^1$	0	1
$F_{(24)}$	$I_S$	$I_S$	$I_S$	0	0	0	$S$
$F_{(25)}$	0	0	0	0	0	$\eta_h^2$	1
	$S$	$S$	$S$	1	1	$S$	

After the same column operations as in the proof of Proposition 2 we obtain the following ranks for the various submatrices of the transformed matrix.

	$c^1$	$c^2$	$c^3$	$\theta_{-1}^{21}$	$\theta_{-1}^{31}$	$\lambda^{\bar{h}}$	
$F_{(18)_{h=2}}$	0	$S$	0	0	0		$S$
$F_{(18)_{h=3}}$	0	0	$S$	0	0		$S$
$F_{(19)_{h=2}}$			0	1	0	0	1
$F_{(19)_{h=3}}$		0		0	1	0	1
$F_{(20)}$	$S$	$S$	$S$	0	0	0	$S$
$F_{(21)}$	0	0	0	0	0	1	1
	$S$	$S$	$S$	1	1	$S$	

This matrix has full row rank  $HS + (H - 1) + 1$  which exceeds the number of endogenous variables by 1. The function  $F$  is defined on open sets with  $c^h \in \text{int}(X)$  for all  $h \in \mathcal{H}$ ,  $\lambda^h \in \mathbb{R}_{++}^S$  for  $h \geq 2$ , and  $(\theta_{-1}^{h1})_{h \geq 2} \in \Delta_{++}^{H-1}$ . Hence,  $F$  satisfies the hypotheses of Theorem 5 and the proof proceeds as that of Proposition 2. We can perform the proof for each agent and then take the intersection of generic sets which in turn yields a generic set for which no agent has a linear sharing rule with zero intercept.

In the proof for CARA utilities we must replace equation (25) with the following equation.

$$(26) \quad \tau^{\bar{h}} \ln(\lambda^{\bar{h}}) - \frac{\tau^{\bar{h}}}{\sum_{i \in \mathcal{H}} \tau^i} \sum_{i \in \mathcal{H}} \tau^i \ln(\lambda^i) = 0$$

for some agent  $\bar{h} \in \mathcal{H}$ . The proof is now identical to the one for [EC] type utilities.  $\square$

**Proof of Theorem 4:** Suppose there is no trade in this economy. Then equations (11) simplify to the following equations,

$$(27) \quad c^h - \sum_{j \in L} \Theta^{hj} d^j - \Theta^{hJ} (1_S - q^J) = 0 \quad \forall h \in \mathcal{H}.$$

If there is no trade then the following set of equations must have a solution.

$$(28) \quad u'_1(c_y^1) - \lambda^h u'_h(c_y^h) = 0, \quad h = 2, \dots, H, \quad y \in \mathcal{Y},$$

$$(29) \quad \left( [I_S - \beta \Pi]^{-1} (p \otimes (c^h - \sum_{j \in L} \theta_{-1}^{hj} d^j)) \right)_{y_0} = 0, \quad h = 2, \dots, H,$$

$$(30) \quad \sum_{h=1}^H c_y^h - \sum_{h=1}^H \omega_y^h = 0, \quad y \in \mathcal{Y},$$

$$(31) \quad q_y^J p_y - \beta \Pi_y \cdot p = 0, \quad y \in \mathcal{Y}$$

$$(32) \quad c^1 - \sum_{j \in L} \Theta^{1j} d^j - \Theta^{1J} (1_S - q^J) = 0.$$

We denote the system of equations (28)–(32) by  $F((c^h)_{h \in \mathcal{H}}, (\lambda^h)_{h \geq 2}, q^J, \Theta^1; (\theta_{-1}^h)_{h \geq 2}, \Pi^{-S}) = 0$ . The system has  $HS + (H-1) + S + J$  endogenous unknowns  $c^h$ ,  $h \in \mathcal{H}$ ,  $\lambda^h$ ,  $h = 2, \dots, H$ ,  $q^J$ , and  $\Theta^{1j}$ ,  $j = 1, \dots, J$ , in  $(H-1)S + (H-1) + S + S + S$  equations. In addition,  $F$  depends on the  $(H-1) + S(J-1)$  parameters  $\theta_{-1}^h$ ,  $h = 2, \dots, H$ , and  $\Pi_{ys}$  for  $y \in \mathcal{Y}$ ,  $s \in S^-(Y)$ . Define  $y^{min} \equiv \min S(y)$  and  $y^{max} \equiv \max S(y)$ . The social endowment is not constant across all the states in  $S(y)$ , so we can assume w.l.o.g. that  $p_{y^{min}} \neq p_{y^{max}}$ . We show that the Jacobian of  $F$  taken with respect to  $c^h$ ,  $q^J$ ,  $\theta_{-1}^h$  and  $\Pi_{y^{min}}$  (the first nonzero element in every row of the matrix  $\Pi$ ) has full row rank  $(H-1)S + (H-1) + S + S + S$ . We use  $\Lambda_S$  and  $\eta^1$  as in the proof of Proposition 2. Define  $\eta^2 = -\beta \Lambda((p_{y^{min}} - p_{y^{max}}) \cdot 1_S)$  and note that  $\eta^2$  has full rank  $S$ .

	$c^1$	$c^2$	$c^3$	$q^J$	$\theta_{-1}^{21}$	$\theta_{-1}^{31}$	$\Pi_{y^{min}}$	
$F_{(28)_{h=2}}$	$\Lambda_S(u''_1(c^1))$	$\Lambda_S(-\lambda^2 u''_2(c^2))$	0	0	0	0	0	$S$
$F_{(28)_{h=3}}$	$\Lambda_S(u''_1(c^1))$	0	$\Lambda_S(-\lambda^3 u''_3(c^3))$	0	0	0	0	$S$
$F_{(29)_{h=2}}$			0	0	$\eta^1$	0		1
$F_{(29)_{h=3}}$		0		0	0	$\eta^1$		1
$F_{(30)}$	$I_S$	$I_S$	$I_S$	0	0	0	0	$S$
$F_{(31)}$		0	0		0	0	$\eta^2$	$S$
$F_{(32)}$	$I_S$	0	0	$\Lambda_S(\Theta^{1J})$	0	0	0	$S$
	$S$	$S$	$S$	$S$	1	1	$S$	

After the same column operations as in the proof of Proposition 2 we obtain a transformed matrix having submatrices of the following ranks.

	$c^1$	$c^2$	$c^3$	$q^J$	$\theta_{-1}^{21}$	$\theta_{-1}^{31}$	$\Pi_{y^{min}}$	
$F_{(28)_{h=2}}$	0	$S$	0	0	0	0	0	$S$
$F_{(28)_{h=3}}$	0	0	$S$	0	0	0	0	$S$
$F_{(29)_{h=2}}$			0	0	1	0		1
$F_{(29)_{h=3}}$		0		0	0	1		1
$F_{(30)}$	$S$	$S$	$S$	0	0	0	0	$S$
$F_{(31)}$		0	0		0	0	$S$	$S$
$F_{(32)}$	$S$	0	0	$S$	0	0	0	$S$
	$S$	$S$	$S$	$S$	1	1	$S$	

The assumptions of Lemma 3 are satisfied and so there exists a generic set  $\mathcal{T}$  of initial holdings of the first asset such that all agents' sharing rules have nonzero intercepts. Therefore any solution to equations (32) must satisfy  $\Theta^{1J} \neq 0$ . Thus, the matrix  $\Lambda(\Theta^{1J})$  has rank  $S$ .

The Jacobian has full row rank  $(H - 1)S + (H - 1) + 3S$  which exceeds the number of endogenous variables by  $S - J > 0$ . The function  $F$  is defined on open sets with  $c^h \in \text{int}(X)$  for all  $h \in \mathcal{H}$ ,  $\lambda^h \in \mathbb{R}_{++}^S$  for  $h \geq 2$ ,  $\Theta^1 \in \mathbb{R}^J$ ,  $q^J \in \mathbb{R}_{++}^S$ ,  $(\theta_{-1}^{h1})_{h \geq 2} \in \Delta_{++}^{H-1}$  and  $\Pi^{-S^-(y)} \in \Delta_{++}^{S \times (J-1)}$ . Hence,  $F$  satisfies the hypotheses of Theorem 5 and the proof proceeds as the previous genericity proofs.  $\square$

## REFERENCES

- [1] Amershi, Amin H., and Jan. H. W. Stoeckenius, 1983, The Theory of Syndicates and Linear Sharing Rules, *Econometrica* 51, 1407–1416.
- [2] Black, Fischer, 1972, Capital Market Equilibrium with Restricted Borrowing, *Journal of Business* 45, 444–455.
- [3] Billingsley, Patrick, 1986. *Probability and Measure* (John Wiley & Sons, New York).
- [4] Bossaerts, Peter, Charles Plott, and William R. Zame, 2003, Prices and Portfolio Choices In Financial Markets: Theory and Experiment, working paper, CalTech and UCLA.
- [5] Brennan, Michael J., and Yihong Xia, 2000, Stochastic Interest Rates and Bond-Stock Mix, *European Finance Review* 4, 197 – 210.
- [6] Campbell, John Y., and Luis M. Viceira, 2002. *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors* (Oxford University Press, New York).
- [7] Canner, Niko, Gregory N. Mankiw, and David N. Weil, 1997, An Asset Allocation Puzzle, *American Economic Review* 87, 181–191.
- [8] Cass, David, and Alessandro Citanna, 1998, Pareto Improving Financial Innovation in Incomplete Markets, *Economic Theory* 11, 467–494.
- [9] Cass, David, and Joseph E. Stiglitz, 1970, The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds, *Journal of Economic Theory* 2, 122–160.
- [10] Citanna, Alessandro, Herakles M. Polemarchakis, and M. Tirelli, 2004, The Taxation of Trades in Assets, working paper.
- [11] Gollier, Christian, 2001. *The Economics of Risk and Time* (The MIT Press, Cambridge, MA).
- [12] Guillemin, Victor, and Alan Pollack, 1974. *Differential Topology* (Prentice-Hall, Englewood Cliffs, NJ).
- [13] Hens, Thorsten, and Beate Pilgrim, 2002. *General Equilibrium Foundations of Finance* (Kluwer Academic Publishers, Boston, MA).
- [14] Huang, Chi-fu, and Robert H. Litzenberger, 1988. *Foundations of Financial Economics* (North-Holland, New York, NY).
- [15] Ingersoll, Jonathan E. Jr., 1987. *Theory of Financial Decision Making* (Rowman & Littlefield, Savage, Maryland).
- [16] Judd, Kenneth L., Felix Kubler, and Karl Schmedders, 2003, Asset Trading Volume with Dynamically Complete Markets and Heterogeneous Agents, *The Journal of Finance* 58, 2203–2217.
- [17] Judd, Kenneth L., Felix Kubler, and Karl Schmedders, 2004, Stock Portfolios and Finite-Maturity Bonds, working paper.
- [18] Kang, Jong-Gu, 2003, A Multi-Period Complete Market Economy, Ph.D dissertation (Chapter 4), University of Essex, UK.
- [19] Kreps, David, 1982, Multi-Period Securities and the Efficient Allocation of Risk: A Comment on the Black-Scholes Option Pricing Model, in J. McCall, ed.: *The Economics of Information and Uncertainty* (The University of Chicago Press, Chicago, IL).
- [20] Lucas, Robert E. Jr., 1978, Asset Prices in an Exchange Economy, *Econometrica* 46, 1429–1445.

- [21] Magill, Michael, and Martine Quinzii, 1996a. *Theory of Incomplete Markets*, Vol 1 (The MIT Press, Cambridge, MA).
- [22] Magill, Michael, and Martine Quinzii, 1996b. *Theory of Incomplete Markets*, Chapter 8, unpublished chapter of Vol 2.
- [23] Magill, Michael, and Martine Quinzii, 2000, Infinite-Horizon CAPM Equilibrium, *Economic Theory* 15, 103–138.
- [24] Mas-Colell, Andreu, 1985. *The Theory of General Economic Equilibrium: A Differentiable Approach* (Cambridge University Press, Cambridge).
- [25] Mas-Colell, Andreu, and William Zame, 1991, Equilibrium Theory in Infinite Dimensional Spaces, in W. Hildenbrand and H. Sonnenschein, eds. *Handbook of Mathematical Economics* Vol IV (Cambridge University Press, Cambridge).
- [26] Negishi, Takashi, 1960, Welfare Economics and The Existence of an Equilibrium for a Competitive Economy, *Metroeconomica* 7, 92–97.
- [27] Ross, Stephen A., 1978, Mutual Fund Separation in Financial Theory – The Separating Distributions, *Journal of Economic Theory* 17, 254–286.
- [28] Russell, Thomas, 1980, Portfolio Separation: The Analytic Case, *Economics Letters* 6, 59–66.
- [29] Tobin, J., 1958, Liquidity Preferences as Behavior Towards Risk, *Review of Economic Studies* 25, 65–86.
- [30] Wilson, Robert, 1968, The Theory of Syndicates, *Econometrica* 36, 119–132.