Can consumption spillovers be a source of equilibrium indeterminacy?*

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Abstract

In the framework of a one-sector exogenous growth model we show that consumption externalities are not a source of equilibrium indeterminacy when the labor supply is inelastic, whereas they are under endogenous labor supply. In particular, when the marginal rate of substitution between own consumption and the others' consumption is constant along the equilibrium path, the equilibrium may exhibit indeterminacy when the following two conditions are simultaneously satisfied: (i) both consumption and production externalities are present and (ii) the elasticity of the labor demand is larger than the elasticity of the Frisch labor supply. In contrast, when the marginal rate of substitution is not constant, the equilibrium may exhibit indeterminacy even if the previous two conditions are not met. In this case, the equilibrium can exhibit indeterminacy even if production externalities are absent. We also show that these results do not apply to an endogenous growth model, where indeterminacy may arise only when condition (ii) is met.

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1. Introduction

In this paper we study how the stability properties of the equilibrium path are modified by the introduction of average consumption in the utility function. This average consumption generates an externality resulting on either an increase or a reduction in the felicity that each individual obtains from his own consumption. This means that the individuals of our model could exhibit either jealousy or altruism. Moreover, consumption externalities may also increase or reduce the marginal utility of own consumption. Thus, our model encompasses the "keeping-up with the Joneses" feature considered by Galí (1994), where consumption spillovers raise the marginal value of own consumption.

The dynamic general equilibrium literature has introduced consumption externalities in order to study how they modify the price of financial assets (Abel (1990 and 1999) and Galí (1994)); the patterns of growth (Carroll (2000) and Carroll et al. (1997 and 2000)); and the properties of the business cycle (Lettau and Uhlig (2000)). This literature shows that these externalities are a potential source of inefficiency. On the one hand, Fisher and Hof (2000), Sieh et al. (2000) and Alonso-Carrera et al. (2004a and 2004b) show that consumption externalities may modify the intertemporal elasticity of substitution of consumption and, thus, be a source of intertemporal inefficiency. On the other hand, Ljungqvist and Uhlig (2000) show that these externalities also affect the consumption-leisure margin and, thus, generate an intratemporal inefficiency. These inefficiencies suggest that consumption externalities are a potential source of coordination failures that may result in the indeterminacy of the equilibrium path. In this paper we analyze when these externalities are a source of indeterminacy in three different one-sector growth models: an exogenous growth model with an exogenous labor supply, an exogenous growth model with an endogenous labor supply, and an endogenous growth model.

We first show that consumption externalities do not generate indeterminacy of the equilibrium path when the labor supply is exogenous. Thus, when externalities only give raise to intertemporal inefficiency, they are not a source of equilibrium indeterminacy. This negative result follows because the potential complementarities brought about by externalities would require a too large willingness to substitute consumption intertemporally in order to generate indeterminacy. Benhabib and Perli (1994) show that the required level of willingness to substitute consumption intertemporally is lower if the labor supply is endogenous, as agents can substitute consumption for leisure instead of consumption at different dates. Furthermore, we study how consumption externalities affect the uniqueness of the equilibrium path when the labor supply is endogenous and, hence, when consumption externalities also trigger intratemporal inefficiency.

The existing literature has studied the uniqueness of the equilibrium path of the standard one-sector growth model when the labor supply is endogenous. In particular, Benhabib and Farmer (1994) show that the equilibrium of a one-sector growth model with separable instantaneous utility and no consumption externalities may exhibit indeterminacy when the labor supply and the labor demand cross with the wrong slopes. If the labor supply is upward slopping, the condition for indeterminacy will require a sufficiently large degree of returns to labor that makes the labor demand upward slopping. Bennett and Farmer (2000) argue that the required degree of returns to labor is not plausible. These authors show that, if preferences are non-separable between consumption and leisure, then indeterminacy can arise when the labor demand and the labor supply cross with the normal slopes. In this case, the necessary condition for indeterminacy is that the elasticity of the labor demand is larger than the elasticity of the Frisch labor supply.[1] Thus, if the production function exhibits non-increasing returns to labor, indeterminacy requires that the Frisch labor supply has a negative elasticity, i.e., it must be downward slopping. However, the indeterminacy condition obtained by Bennett and Farmer (2000) implies that the utility function is not concave. In fact, we show that if there is no consumption externalities and the utility function is concave, the Frisch labor supply will be upward slopping and the equilibrium will not exhibit indeterminacy. We conclude in this paper that, if the utility function is concave and the production function does not exhibit increasing returns to labor, the equilibrium may only exhibit indeterminacy when consumption externalities are introduced.[2]

When consumption externalities are introduced in an exogenous growth model, the indeterminacy of the equilibrium path depends on the restricted homotheticity property of the utility function (RH property henceforth). We say that the utility function satisfies this property when the marginal rate of substitution (MRS) between consumption and consumption spillovers is constant along the equilibrium path. In this case, the equilibrium may exhibit indeterminacy when the following two conditions are simultaneously satisfied: (i) there are production externalities and (ii) the elasticity of the labor demand is larger than the elasticity of the Frisch labor supply (the Bennet and Farmer indeterminacy condition). In contrast, when the utility function does not satisfy the RH property, the equilibrium may exhibit indeterminacy even though these two conditions are not satisfied. Therefore, when the utility function does not satisfy the RH property, the equilibrium can exhibit indeterminacy when the only source of inefficiency is the presence of consumption externalities. The intuition behind this result is as follows. When the MRS is constant, consumption spillovers do not give raise to intertemporal inefficiency. In this case, the only source of intertemporal inefficiency is the presence of production externalities and, thus, these externalities are required for equilibrium indeterminacy, as happens in the model of Bennet and Farmer. In contrast, when the MRS is not constant, the equilibrium exhibits intertemporal inefficiencies even if there are no production externalities, which are then not any longer a requirement for equilibrium indeterminacy.

To summarize, we show that the equilibrium of a one-sector exogenous growth model with non-increasing returns to labor and a concave utility function may only exhibit indeterminacy when both intertemporal and intratemporal inefficiencies are present. Consumption externalities give raise to intratemporal inefficiency when they modify the consumption-leisure margin and they give raise to intertemporal inefficiency when the RH property is not satisfied. When these externalities are

[1] The Frisch labor supply is defined as the labor supply resulting from keeping the marginal utility of consumption constant.

[2] Weder (2004) considers a model where leisure instead of consumption exhibits externalities and show that indeterminacy becomes more likely in this setup.

introduced, we obtain the following two main results. First, when the utility function is concave, the equilibrium may only exhibit indeterminacy under consumption externalities. Second, when the utility function does not satisfy the RH property, the equilibrium may exhibit indeterminacy even if there are no production externalities. Therefore, we conclude that the only presence of consumption externalities may be a source of equilibrium indeterminacy in the one-sector exogenous growth model. This conclusion is in contrast with the negative result obtained by Guo (1999), who concluded that consumption externalities are not a source of equilibrium indeterminacy. However, Guo considers an instantaneous utility function that satisfies the RH property and, in this case, consumption externalities do not result in intertemporal inefficiencies. Weder (2000) also considers a model with consumption externalities and an utility function that satisfies the RH property. In his model productive externalities are thus needed to obtain indeterminacy of the dynamic equilibrium.

The results obtained in the exogenous growth model do not extend to the endogenous growth model. To show this, we consider an Ak growth model and we show that, even if the RH property is not satisfied, the equilibrium may only exhibit indeterminacy when the elasticity of the Frisch labor supply is smaller than the elasticity of the labor demand. Therefore, the indeterminacy condition obtained by Bennet and Farmer (2000) is also a necessary condition in the Ak growth model. We then conclude that consumption externalities do not trigger equilibrium indeterminacy in an endogenous growth model because the existence of balanced growth imposes conditions on the utility function that prevent indeterminacy from arising when the elasticity of the Frisch labor supply is larger than the elasticity of the labor demand.

The rest of the paper is organized as follows. Section 2 analyzes the uniqueness of the equilibrium path of an exogenous growth model when the labor supply is inelastic. Section 3 extends the analysis to a model with an endogenous labor supply and Section 4 considers an endogenous growth model. Finally, Section 5 concludes the paper.

2. Exogenous labor supply

The economy is populated by identical consumers facing an infinite horizon. Population remains constant. The consumers' utility function is $u(c,\overline{c})$, where c is the own consumption and $\overline{c}$ is the average consumption in the economy. The utility function is twice continuously differentiable and satisfies the following properties: $u_1(c,\overline{c}) > 0$, $u_{11}(c,\overline{c}) < 0$, $\lim_{c\longrightarrow\infty} u_1(c,\overline{c}) = 0$, and $\lim_{c\longrightarrow 0} u_1(c,\overline{c}) = \infty$ for all $\overline{c} > 0$.[3] The introduction of average consumption implies that consumption spillovers affect the consumers' utility. Moreover, when $u_{12}(c,\overline{c}) > 0$, average consumption increases the marginal utility of own consumption, whereas it decreases it when $u_{12}(c,\overline{c}) < 0$. The former case corresponds to the "keeping-up with the Joneses" formulation.

The production function in per capita terms is $f(k,\overline{k})$, where k is the private stock of capital per capita and $\overline{k}$ is the average stock of capital in the economy. This

[3] From now on, the subindex of a function referes to the position of the argument with respect to which the partial derivative is taken.

average stock of capital introduces thus a production externality. The production function satisfies the following properties: $f_1(k,\overline{k}) > 0$ and $f_{11}(k,\overline{k}) < 0$ for all $\overline{k}$; and $\lim\limits_{k->\infty} f_1(k,k) = A \geq 0$, $\lim\limits_{k->0} f_1(k,k) = \infty$, and $f_{11}(k,k)+f_{12}(k,k) < 0$. The last condition prevents capital from exhibiting increasing returns in equilibrium, which is a necessary condition for the existence of a stationary solution to the social planner optimization problem. Production is either consumed or invested in new capital. Thus, the resource constraint is

$$c + \dot{k} = f\left(k, \overline{k}\right). \tag{2.1}$$

The consumers maximize the discounted sum of utilities

$$\int_0^\infty e^{-\rho t} u\left(c, \overline{c}\right) dt,$$

subject to (2.1). To simplify the analysis, we assume that the discount rate ρ satisfies $\rho > A$, which implies that the equilibrium will not exhibit sustained growth. The solution to the consumers' maximization problem at a symmetric equilibrium (i.e., when $c = \overline{c}$ and $k = \overline{k}$) is characterized by the Euler equation

$$\frac{\dot{c}}{c} = \frac{f_1\left(k,k\right) - \rho}{\sigma\left(c\right)}, \tag{2.2}$$

where $\sigma\left(c\right) > 0$ is the inverse of the intertemporal elasticity of substitution at a symmetric equilibrium,

$$\sigma\left(c\right) = -\frac{c\left(u_{11}\left(c,c\right) + u_{12}\left(c,c\right)\right)}{u_1\left(c,c\right)},$$

and by the transversality condition

$$\lim_{t\to\infty} e^{-\rho t} k u_1\left(c,c\right) = 0. \tag{2.3}$$

From (2.1), we obtain the growth rate of capital per capita,

$$\frac{\dot{k}}{k} = \frac{f\left(k,k\right)}{k} - \frac{c}{k}. \tag{2.4}$$

Given an initial condition k_0, a competitive equilibrium is a path of consumption and capital per capita that solves the system formed by (2.2) and (2.4) and that satisfies the transversality condition (2.3). We proceed to characterize the equilibrium path. First, we show that the steady state (or stationary equilibrium) of this economy is unique.

Proposition 2.1. *There is a unique steady state in this economy and the corresponding values of the capital stock and of consumption are* $k^* = f_1^{-1}\left(\rho\right)$ *and* $c^* = f(k^*, k^*)$*, respectively.*

Proof. The steady state values of the variables are obtained from (2.2) and (2.4), and the uniqueness of the steady state follows from the monotonicity of the marginal product of capital. ■

Note that production externalities affect the steady state value of consumption and capital, whereas these values are not affected by consumption externalities. However, consumption externalities modify the path of the dynamic equilibrium by changing the intertemporal elasticity of substitution and, hence, they may generate intertemporal inefficiencies.[4] In fact, they give raise to the inefficiency of the equilibrium path when $\sigma(c)$ is different from the social planners' inverse of the intertemporal elasticity of substitution, which is defined as

$$\sigma^P(c) = -\frac{c\left(u_{11}(c,c) + u_{12}(c,c) + u_{21}(c,c) + u_{22}(c,c)\right)}{u_1(c,c) + u_2(c,c)},$$

since the instantaneous utility function viewed by the social planner is $\hat{u}(c) \equiv u(c,c)$.

The dynamic equilibrium may not be unique when it is inefficient. However, in this model, the dynamic equilibrium is unique when we introduce a constraint on the value of the intertemporal elasticity of substitution that is supported by the empirical evidence.

Proposition 2.2. *Assume that $\sigma(c) > 0$. Then, the steady state equilibrium is locally saddle-path stable and, thus, the dynamic equilibrium is locally unique.*

Proof. See the Appendix. ■

We have thus shown that the uniqueness of the equilibrium path is not affected by consumption externalities when they only modify the intertemporal elasticity of substitution. Thus, consumption externalities are not a source of equilibrium indeterminacy when they generate intertemporal inefficiencies only.[5] In the following section we show that consumption externalities are a source of equilibrium indeterminacy when leisure is introduced.

3. Endogenous labor supply

We now assume that each agent is endowed in each period with one unit of time that can be devoted to either labor supply l, or leisure $1-l$. The consumers' utility function $u(c,\overline{c},1-l)$ depends on consumption, on average consumption, and on leisure. The utility function is twice continuously differentiable and satisfies the following properties: $u_1(c,\overline{c},1-l) > 0$, $u_{11}(c,\overline{c},1-l) < 0$, $u_3(c,\overline{c},1-l) > 0$,

[4]See Alonso-Carrera, et. al. (2004a) for a discussion on the inefficiencies caused by consumption spillovers.

[5]Even though consumption externalities do not affect the unicity of the equilibrium path, they modify the path of the dynamic equilibrium by means of changing the speed of convergence. To see this, note that, if $u_{12} > (<)\, 0$, the introduction of the consumption externality increases (decreases) the intertemporal elasticity of substitution and, as follows from (2.2), it increases (decreases) the speed of convergence towards the steady state.

$u_{33}(c,\overline{c},1-l)<0,\qquad \lim_{c\to\infty} u_1(c,\overline{c},1-l)=0,\qquad \lim_{c\to 0} u_1(c,\overline{c},1-l)=\infty,$ $\lim_{l\to 0} u_3(c,\overline{c},1-l)=0,$ $\lim_{l\to 1} u_3(c,\overline{c},1-l)=\infty,$ $u_{13}(c,\overline{c},1-l)\neq 0,$ and

$$u_{11}(c,\overline{c},1-l)\,u_{33}(c,\overline{c},1-l)\geq u_{13}(c,\overline{c},1-l)\,u_{31}(c,\overline{c},1-l)\,. \tag{3.1}$$

for all $\overline{c}>0$. Condition (3.1) implies that the utility function is jointly concave with respect to consumption and leisure, which guarantees that the solution to the consumers' maximization problem is interior.

The production function $f\left(k,l,\overline{k}\right)$ satisfies the following properties: $f_1\left(k,l,\overline{k}\right)>0$, $f_2\left(k,l,\overline{k}\right)>0$, $f_{12}\left(k,l,\overline{k}\right)>0$, $f_{11}\left(k,l,\overline{k}\right)<0$, and $f_{22}\left(k,l,\overline{k}\right)<0$, for all $\overline{k}>0$; and $\lim_{k\to 0} f_1(k,l,k)=\infty$, $\lim_{k\to\infty} f_1(k,l,k)=0$, $f_{11}(k,l,k)+f_{13}(k,l,k)\leq 0$ and $f(k,0,k)=0$.

The consumers' maximization problem is

$$Max\int_0^{\infty} e^{-\rho t}u(c,\overline{c},1-l)\,dt,$$

subject to

$$f\left(k,l,\overline{k}\right)=c+\dot{k}.$$

Let us denote by λ the Lagrangian multiplier of this maximization problem. Then, the first order conditions evaluated at a symmetric equilibrium $\left(\text{i.e. when } k=\overline{k} \text{ and } c=\overline{c}\right)$ are

$$e^{-\rho t}u_1(c,c,1-l)=\lambda, \tag{3.2}$$

$$e^{-\rho t}u_3(c,c,1-l)=\lambda f_2(k,l,k)\,, \tag{3.3}$$

$$f_1(k,l,k)=-\frac{\dot{\lambda}}{\lambda}, \tag{3.4}$$

and the transversality condition is

$$\lim_{t\to\infty}\lambda k=0. \tag{3.5}$$

Combining (3.2) and (3.3), we obtain

$$\frac{u_3(c,c,1-l)}{u_1(c,c,1-l)}=f_2(k,l,k)\,, \tag{3.6}$$

which implicitly defines consumption as a function of capital and employment $c=c(k,l)$. By differentiating (3.6), we obtain

$$[\phi(k,l)+\sigma(k,l)]\left(\frac{\dot{c}}{c}\right)+[\varepsilon(k,l)-\delta(k,l)+1-\gamma(k,l)]\left(\frac{\dot{l}}{l}\right)=$$

$$(\alpha(k,l)+\beta(k,l))\left(\frac{\dot{k}}{k}\right), \tag{3.7}$$

where $\sigma(k,l)$ is the inverse of the intertemporal elasticity of substitution,

$$\sigma(k,l) = -\left(\frac{u_{11}+u_{12}}{u_1}\right)c(k,l),$$

and

$$\varepsilon(k,l) = \left(\frac{u_{13}}{u_1}\right)l,$$

$$\phi(k,l) = \left(\frac{u_{31}+u_{32}}{u_3}\right)c(k,l),$$

$$\delta(k,l) = \left(\frac{u_{33}}{u_3}\right)l,$$

$$\alpha(k,l) = \left(\frac{f_{21}}{f_2}\right)k,$$

$$\beta(k,l) = \left(\frac{f_{23}}{f_2}\right)k,$$

and

$$\gamma(k,l) = 1+\left(\frac{f_{22}}{f_2}\right)l.$$

We assume that $\sigma(k,l) > 0$ and that

$$[\varepsilon(k,l) - \delta(k,l)][\phi(k,l)+\sigma(k,l)] > 0. \tag{3.8}$$

The first inequality implies that the intertemporal elasticity of substitution is positive and the second one follows because both consumption and leisure are assumed to be normal goods.[6]

Combining (3.2) and (3.4), we obtain

$$f_1(k,l,k) - \rho = \sigma(k,l)\left(\frac{\dot{c}}{c}\right) + \varepsilon(k,l)\left(\frac{\dot{l}}{l}\right). \tag{3.9}$$

Moreover, using (3.9) and (3.7), we get

$$\frac{\dot{l}}{l} = \frac{\left(\frac{\phi(k,l)+\sigma(k,l)}{\sigma(k,l)}\right)[f_1(k,l,k)-\rho] - [\alpha(k,l)+\beta(k,l)]\left(\frac{\dot{k}}{k}\right)}{\gamma(k,l)-1-\zeta(k,l)}, \tag{3.10}$$

where

$$\zeta(k,l) = -\delta(k,l) - \left(\frac{\phi(k,l)}{\sigma(k,l)}\right)\varepsilon(k,l)$$

is the price elasticity of the Frisch labor supply. Recall that the Frisch labor supply is the labor supply obtained when the marginal utility of consumption is kept constant. Thus, to obtain that elasticity just note that (3.6) can be rewritten as

$$\frac{u_3\left(c(\overline{u}_1, 1-l), c(\overline{u}_1, 1-l), 1-l\right)}{\overline{u}_1} = f_2(k,l,k), \tag{3.11}$$

[6]The second inequality follows from applying the implicit function theorem in (3.6) and setting $\left.\frac{\partial c}{\partial(1-l)}\right|_{\frac{u_3}{u_1}} > 0$.

where $f_2(k,l,k)$ is the real wage, the upper bar in the marginal utility of consumption means that we keep it constant and the function $c(\overline{u}_1, 1-l)$ is obtained implicitly from

$$u_1(c,c,1-l) - \overline{u}_1 = 0. \tag{3.12}$$

Finally, from the resource constraint, we obtain

$$\dot{k} = f(k,l,k) - c(k,l), \tag{3.13}$$

where $c(k,l)$ is implicitly defined in (3.6).

Given an initial condition k_0, a competitive equilibrium is a path of employment and capital that solves the system of differential equations formed by (3.13) and (3.10) with $l \in (0,1)$ and that satisfies the transversality condition (3.5). Note that l is now the control variable, whereas k is the state variable.

Proposition 3.1. *The steady state values of capital k^* satisfy $k^* = k(l^*)$ and are such that*

$$f_1(k(l^*), l^*, k(l^*)) = \rho,$$

where the steady state values of employment l^* are a solution to the equation

$$Q(l) \equiv f(k(l), l, k(l)) - c(k(l), l) = 0. \tag{3.14}$$

Proof. The proof follows directly from (3.10) and (3.13). ■

It should be pointed out that equation (3.14) may have multiple solutions. To see this, consider the following instantaneous utility function:

$$u = \frac{\left(c\overline{c}^{\psi}\right)^{1-\nu}(1-l+\mu\overline{c}^{\omega})^{\theta(1-\nu)}}{1-\nu}, \tag{3.15}$$

which is jointly concave with respect to consumption and leisure when $\nu > \theta/(1+\theta)$ and $\theta > 0$. Consider also the following production function:

$$y = Ak^{\alpha}l^{\gamma}\overline{k}^{\beta}, \ \alpha \in (0,1), \ \gamma \in (0,1), \ \alpha+\beta < 1, \ \alpha+\gamma \leq 1. \tag{3.16}$$

Then, if we let $\mu = 0.5$, $\omega = 0.1$, $\theta = 0.8$, $\gamma = 0.6$, $\alpha = 0.4$, $\beta = 0$, $\rho = 0.045$ and $A = 0.05$, equation (3.14) has two roots: $l_1 = 0.29$ and $l_2 = 0.57$.

From now on, we assume that there is a unique steady. We denote the steady state values of the variables by a star and, hence, $c^* = c(k^*, l^*)$, $\sigma^* = \sigma(k^*, l^*)$, $\varepsilon^* = \varepsilon(k^*, l^*)$, $\phi^* = \phi(k^*, l^*)$, $\delta^* = \delta(k^*, l^*)$, $\zeta^* = \zeta(k^*, l^*)$, $\alpha^* = \alpha(k^*, l^*)$, $\beta^* = \beta(k^*, l^*)$ and $f^* = f(k^*, l^*, k^*)$ are the values of the corresponding variables at the steady state. The stability properties of the steady state are characterized in the following proposition:[7]

[7] This proposition can be extended to characterize the stability properties when there are multiple steady states.

Proposition 3.2. *Let*

$$N(l^*, k^*) = \left(\frac{\phi^* + \sigma^*}{\sigma^*}\right)\left(\frac{f_{12}^*}{f^*}k^* l^*\right) - (\alpha^* + \beta^*)\left(\frac{f_2^*}{f^*}l^* + \frac{\varepsilon^*}{\sigma^*}\right) + (\gamma^* - 1 - \zeta^*)\left(\frac{f_1^* + f_3^*}{f^*}\right)k^*.$$

Then,

(a) The steady state is unstable when either $\gamma - 1 > \zeta^*$, $\phi^* + \sigma^* > 0$ *and* $N(l^*, k^*) > 0$; *or* $\gamma - 1 < \zeta^*$, $\phi^* + \sigma^* < 0$ *and* $N(l^*, k^*) < 0$.

(b) The steady state is saddle path stable when either $\gamma - 1 > \zeta^*$ *and* $\phi^* + \sigma^* < 0$; *or* $\gamma - 1 < \zeta^*$ *and* $\phi^* + \sigma^* > 0$.

(c) The steady state is locally stable when either $\gamma - 1 > \zeta^*$, $\phi^* + \sigma^* > 0$ *and* $N(l^*, k^*) < 0$; *or* $\gamma - 1 < \zeta^*$, $\phi^* + \sigma^* < 0$ *and* $N(l^*, k^*) > 0$.

Proof. See the Appendix. ■

Obviously, the dynamic equilibrium exhibits local indeterminacy when the steady state is locally stable. From the results in Proposition 3.2, it follows that the equilibrium may exhibit indeterminacy in two different regions of the parameter space that are separated by the equation $\gamma - 1 = \zeta^*$. The left hand side of this equation is the elasticity of the labor demand and the right hand side is the elasticity of the Frisch labor supply at the steady state.

In the first region, indeterminacy arises when $\zeta^* < \gamma - 1 < 0$ and, hence, the Frisch labor supply has a negative slope. To see the implications of this negative slope, note that, since $u_{11} + u_{12} < 0$, the inequality $\zeta^* < 0$ implies that

$$u_{13}u_{32} - u_{12}u_{33} > u_{33}u_{11} - u_{13}u_{31} > 0,$$

where the last inequality follows from condition (3.1). Note that if consumption externalities are not present then $u_{32} = u_{12} = 0$ and the two inequalities cannot be simultaneously satisfied. Thus, when the utility function is concave and consumption spillovers are not introduced, the equilibrium will not exhibit indeterminacy in the first region. Bennet and Farmer (2000) show that the equilibrium exhibits indeterminacy in a model without consumption externalities when production externalities are sufficiently large and $\zeta^* < \gamma - 1$. However, as we have just proved, in their model indeterminacy requires that the utility function be non-concave.

In the second region indeterminacy arises when $\zeta^* > \gamma - 1$ and, hence, the Frisch labor supply may be upward slopping. In this case, indeterminacy arises when $\phi^* + \sigma^* < 0$. These two conditions can only be satisfied if consumption externalities are introduced. To see this, note that if there are no consumption externalities then (3.1) implies that $\varepsilon^*\phi^* + \sigma^*\delta^* \leq 0$. This inequality and (3.8) imply that $\phi^* + \sigma^* > 0$.[8]

We have thus shown that the dynamic equilibrium may exhibit indeterminacy only if consumption externalities are present. This result is obtained when two reasonable assumptions are imposed: (i) the utility function is concave and (ii) the production function does not exhibit increasing returns to labor. Note that this result is related with the indeterminacy results obtained in the literature.

[8]To prove this implication we proceed by contradiction. Therefore, we assume that $\phi^* + \sigma^* < 0$. Then, $-\sigma^*(\varepsilon^* - \delta^*) < \varepsilon^*\phi^* + \sigma^*\delta^* < 0$. Since $\sigma^* > 0$, it follows that $\varepsilon^* > \delta^*$. However, $\phi^* + \sigma^* < 0$ and $\varepsilon^* > \delta^*$ imply that (3.8) is not satisfied.

In particular, Benhabib and Farmer (1994) show that indeterminacy arises when there are increasing returns to labor and Bennet and Farmer (2000) show that indeterminacy arises when the utility function is not concave.

In what follows we analyze if the equilibrium may exhibit indeterminacy when production externalities are not present and, thus, the only source of inefficiency is the presence of consumption externalities. We first rewrite the indeterminacy regions in the following corollary:

Corollary 3.3. *Let us define*

$$\overline{\sigma} = \frac{\phi^* \varepsilon^*}{1 - \gamma - \delta^*},$$

and

$$\overline{\beta} = \frac{\left(\frac{\phi^* + \sigma^*}{\sigma^*}\right)\left(\frac{k^* l^* f_{12}^*}{f^*}\right) + \left(\gamma^* - 1 - \zeta^*\right)\left(\frac{f_1^* + f_3^*}{f^*}\right) k^*}{\frac{l^* f_2^*}{f^*} + \frac{\varepsilon^*}{\sigma^*}} - \alpha^*.$$

The equilibrium exhibits indeterminacy when one of the following sets of conditions holds:

(a) $\overline{\sigma} < \sigma^* < -\phi^*$ *and* $\left(\frac{l^* f_2^*}{f^*} + \frac{\varepsilon^*}{\sigma^*}\right)\left(\overline{\beta} - \beta^*\right) > 0.$

(b) $\overline{\sigma} > \sigma^* > -\phi^*$ *and* $\left(\frac{l^* f_2^*}{f^*} + \frac{\varepsilon^*}{\sigma^*}\right)\left(\overline{\beta} - \beta^*\right) < 0.$

Proof. The proof follows by using the conditions for indeterminacy given in Proposition 3.2 and by noticing that $\zeta^* = \gamma - 1$ when $\sigma^* = \overline{\sigma}$; and $N(l^*, k^*) = 0$ when $\beta^* = \overline{\beta}$. ■

Note that indeterminacy depends on the value of the intertemporal elasticity of substitution and on the value of the parameter β^* that measures the intensity of production externalities. On the one hand, if $\sigma^* > (<)\, \overline{\sigma}$ indeterminacy arises when the elasticity of the labor demand is larger (smaller) than the elasticity of the Frisch labor supply. Therefore, the Bennet and Farmer (2000) condition for indeterminacy implies that $\sigma^* < \overline{\sigma}$, i.e., the intertemporal elasticity of substitution must be sufficiently large. In contrast, we show that the equilibrium may exhibit indeterminacy even though the intertemporal elasticity of substitution is low, $\sigma^* > \overline{\sigma}$. On the other hand, the relationship between β^* and $\overline{\beta}$ implies that production externalities may be a requirement for equilibrium indeterminacy.

We next consider particular functional forms of the utility and production functions in order to show that the equilibrium may exhibit indeterminacy when there are no production externalities. We need first to define the concept of restricted homotheticity (RH). We say that the utility function satisfies this RH property if the marginal rate of substitution between consumption and average consumption is constant along the equilibrium path (i.e., when $c = \overline{c}$). This means that

$$\frac{u_1(c, c, 1-l)}{u_2(c, c, 1-l)} = \xi,$$

for some ξ constant. Consider now the utility function (3.15). We consider two extreme cases: (i) $\mu = 0$ and (ii) $\mu \neq 0$. In the first case, the utility function satisfies the RH property, whereas this property does not hold in the second case.[9]

Proposition 3.4. *Assume the production function (3.16) and the instantaneous utility function (3.15) with* $\mu = 0$, *which means that*

$$\overline{\sigma} = \frac{(1-\nu)^2 \gamma (1+\psi)}{1-(1-\nu)\gamma + \frac{\gamma}{\theta} - \gamma},$$

and

$$\overline{\beta} = \frac{\alpha\gamma}{\left(1+\frac{\gamma}{\theta}\right)\sigma^*} - \alpha,$$

where $\sigma^* = \nu - \psi(1-\nu)$. *Then, the equilibrium exhibits indeterminacy when* $\sigma^* < \overline{\sigma}$ *and* $\beta^* > \overline{\beta}$.

Proof. See the Appendix. ■

As shown in the proof of Proposition 3.4, if $\mu = 0$ indeterminacy may only arise when the following conditions are satisfied: (i) the elasticity of the Frisch labor supply is negative and lower than the elasticity of the labor demand; (ii) there are production externalities, that is, $\beta > 0$; (iii) $u_{12} > 0$, so that consumption externalities increase the marginal utility of own consumption and, hence, agents' exhibit the "keeping-up with the Joneses" feature; and (iv) the intertemporal elasticity of substitution is larger than 1, $\sigma < 1$.

Table 1 displays the stability properties of the equilibrium when different values of the parameters ψ and β are considered. Note that a positive value of β is required for equilibrium indeterminacy. Note also that the dynamic equilibrium exhibits saddle path stability and, hence, it is unique when ψ takes low values. This means that the equilibrium may only exhibit indeterminacy when both consumption and production externalities are introduced.

[Insert Table 1]

Proposition 3.5. *Assume the production function (3.16) and the instantaneous utility function (3.15) with* $\mu \neq 0$, $\beta = 0$, $\alpha = 1-\gamma$ *and* $\sigma^* > \overline{\sigma}$, *which means that*

$$\overline{\sigma} = \frac{(1-\nu)\gamma\left[(1-\nu)(1+\psi) + (\theta(1-\nu)-1)\pi\right]}{1-(1-\nu)\gamma + \frac{\gamma}{\theta} - \gamma},$$

and

$$\overline{\beta} = \frac{\alpha\gamma(1-\pi^*)}{\left(1+\frac{\gamma}{\theta}\right)\sigma^* + (1-\nu)\gamma\pi^*} - \alpha,$$

where

$$\sigma^* = \nu(1+\psi) - \psi - \theta(1-\nu)\pi^*,$$

[9] It can be shown that if $\mu = 0$ the steady state is unique and when $\mu \neq 0$ there may be several steady states. However, the results in Propositions 3.4 and 3.5 do not depend on the number of steady states.

$$\phi^* = 1 - \pi^* - \sigma^*,$$

and

$$\pi^* = \omega \left(1 - \frac{(1 - l^*)\,\gamma}{\theta l^*}\right).$$

Then, the equilibrium exhibits indeterminacy when $\sigma^* \in (\overline{\sigma}, v - 1)$ *and* $\overline{\beta} < 0$.

Proof. See the Appendix. ■

We have thus shown that if $\mu \neq 0$ then the equilibrium can exhibit indeterminacy when: (i) the elasticity of the Frisch labor supply is larger than the elasticity of the labor demand, $\sigma^* > \overline{\sigma}$; (ii) there are no production externalities, i.e., $\beta = 0$; and (iii) the production function exhibits constant returns to scale, i.e., $\gamma = 1 - \alpha$. Therefore, this example shows that if the utility function does not satisfy the RH property then indeterminacy arises in a model without production externalities, with constant returns to scale, and with a small value of the intertemporal elasticity of substitution.

Table 2 shows how the stability properties of the equilibrium depend on the parameters μ and ψ when $\beta = 0$ and $\gamma = 1 - \alpha$. Note that, when consumption externalities are absent ($\mu = \psi = 0$), the equilibrium exhibits saddle path stability, whereas the steady state may be either unstable or locally stable when there are consumption externalities. When it is locally stable, the equilibrium exhibits local indeterminacy. Table 2 provides also an example of indeterminacy when there are no production externalities and the intertemporal elasticity of substitution is low, $\sigma^* > \overline{\sigma}$ (i.e., the elasticity of the Frisch labor supply is larger than the elasticity of the labor demand).

[Insert Table 2]

The previous examples show that, if the utility function satisfies the RH property, then the equilibrium may only exhibit indeterminacy when production externalities are present. To see why, note that the equilibrium is characterized by (3.10) and (3.13). The latter equation is the resource constraint and the intratemporal inefficiencies associated to the decision between leisure and consumption are summarized by this equation.[10] Equation (3.10) is the Euler equation summarizing all the intertemporal inefficiencies, which may accrue from production externalities affecting the interest rate or from consumption externalities. By solving the planners' problem, it can be shown that consumption externalities give raise to inefficiency through the modification of both the intertemporal elasticity of substitution of consumption and the elasticity of the marginal utility of consumption with respect to leisure. The intertemporal elasticity of substitution in the planners' problem is

$$\sigma^P = -\left(\frac{u_{11} + u_{12} + u_{21} + u_{22}}{u_1 + u_2}\right) c,$$

[10] Consumption externalities cause intratemporal inefficiencies because they modify the leisure-labor margin in equation (3.6) and thus affect the relation between consumption and employment used to derive equation (3.13). Note that they cause an intratemporal inefficiency even if the utility function satisfies the RH property.

and the elasticity of the marginal utility of consumption in the planners' problem is

$$\varepsilon^P = \left(\frac{u_{13} + u_{23}}{u_1 + u_2}\right) l.$$

It is easy to show that if preferences satisfy the RH property then $\sigma^P = \sigma$ and $\varepsilon^P = \varepsilon$.[11] This means that consumption externalities do not result in intertemporal inefficiency and, hence, the intertemporal inefficiency is just due to production externalities, as in Bennet and Farmer (2000). Obviously, in this case indeterminacy may only arise when production externalities are introduced and it occurs in the region where $\gamma - 1 > \zeta^*$, which is the indeterminacy region found by Bennet and Farmer. Note that, because of consumption externalities, indeterminacy may arise under concavity of the utility function. However, when preferences do not satisfy the RH property, it holds that $\sigma^P \neq \sigma$ and $\varepsilon^P \neq \varepsilon$. In this case, consumption externalities generate intertemporal inefficiencies and the equilibrium may exhibit indeterminacy even if there are no production externalities and $\gamma - 1 < \zeta^*$ (see Table 2).

In Section 2 we showed that, if consumption externalities only generate intertemporal inefficiencies, then the dynamic equilibrium is unique. In this section we have allowed for endogenous labor supply so that consumption externalities modify the elasticity of substitution between consumption and leisure and, hence, they also generate an intratemporal inefficiency. We have shown that, if consumption externalities only generate intratemporal inefficiencies and production externalities are not present, then the dynamic equilibrium is also unique. We thus conclude that consumption externalities are a source of equilibrium indeterminacy when they give raise to both intratemporal and intertemporal inefficiencies. This obviously occurs when the utility function does not satisfy the RH property.

4. Endogenous Growth

We will next extend the analysis to an economy with endogenous growth. To this end, we assume that $\beta = 1-\alpha$ so that the production function (3.16) can be rewritten as

$$y = A\overline{k}^{1-\alpha} k^{\alpha} l^{\gamma}, \ \alpha \in (0,1), \ \gamma \in (0,1), \ \alpha + \gamma \leq 1. \tag{4.1}$$

Note that, along a symmetric equilibrium path (i.e., when $k = \overline{k}$), this production function can be rewritten as follows

$$y = Akl^{\gamma}.$$

The consumers' maximization problem is

$$Max \int_0^{\infty} e^{-\rho t} u(c, \overline{c}, 1-l)\, dt,$$

[11] In fact, it can be shown that, if there are no production externalities, then the determinant of the Jacobian matrix of the system associated with the planners problem is identical to the corresponding determinant in the competitive equilibrium with the only difference that σ and ε become σ^P and ε^P, respectively. Then, it is obvious that, if $\sigma^P = \sigma$, $\varepsilon^P = \varepsilon$ and $\beta = 0$, the determinant will be negative and the equilibrium will exhibit saddle path stability.

subject to

$$A\bar{k}^{1-\alpha}k^{\alpha}l^{\gamma} = c + \dot{k}.$$

The solution is characterized by the corresponding Euler and transversality equations together with the equation relating the marginal rate of substitution between consumption and leisure with the marginal product of labor,

$$\left(\frac{1}{c}\right)\frac{u_3}{u_1} = \gamma A l^{\gamma-1}x, \tag{4.2}$$

where $x = k/c$. This equation implicitly defines consumption as a function of employment and of the ratio x, $c = c(l,\ x)$. By differentiating this equation, we obtain

$$\left[\phi(x,l) + \sigma(x,l)\right]\left(\frac{\dot{c}}{c}\right) + \left[\varepsilon(x,l) - \delta(x,l) + 1 - \gamma\right]\left(\frac{\dot{l}}{l}\right) = \frac{\dot{k}}{k}, \tag{4.3}$$

where the growth rate of the stock of capital is obtained from the resource constraint as follows:

$$\frac{\dot{k}}{k} = Al^{\gamma} - \frac{c}{k} = Al^{\gamma} - \frac{1}{x}. \tag{4.4}$$

The Euler condition (3.9) can be rewritten as

$$\sigma(x,l)\left(\frac{\dot{c}}{c}\right) + \varepsilon(x,l)\left(\frac{\dot{l}}{l}\right) + \rho = \alpha A l^{\gamma}. \tag{4.5}$$

Combining (4.5) with (4.3), we obtain

$$\frac{\dot{l}}{l} = \frac{\left(\frac{\phi(x,l)+\sigma(x,l)}{\sigma(x,l)}\right)(\alpha A l^{\gamma} - \rho) - \left(Al^{\gamma} - \frac{1}{x}\right)}{\gamma - 1 - \zeta(x,l)}. \tag{4.6}$$

Next, use the definition of x to obtain

$$\frac{\dot{x}}{x} = \frac{\dot{k}}{k} - \frac{\dot{c}}{c},$$

and by using (4.5), (4.4) and (4.6), we get

$$\frac{\dot{x}}{x} = Al^{\gamma} - \frac{1}{x} - \left(\frac{\alpha A l^{\gamma} - \rho - \varepsilon(x,l)\left(\frac{\dot{l}}{l}\right)}{\sigma(x,l)}\right). \tag{4.7}$$

Given an initial condition k_0, a competitive equilibrium is a path of employment, consumption, and capital that solves the system of differential equations formed by (4.6) and (4.7) with $l \in (0,1)$ and that satisfies the transversality condition (3.5).[12]

A stationary equilibrium of this economy is an equilibrium path along which the employment l and the ratio x remain constant. The stationary equilibrium of a growing economy is dubbed Balanced Growth Path (BGP).

[12] For the dynamic properties of the model of this section, it should be pointed out that, even if the capital to consumption ratio x and the employment l are control-like variables, they are governed by equation (4.2) so that the initial values of capital and employment fully determine the value of x. Therefore, since there is just one degree of freedom, saddle path stability will occur when the Jacobian matrix of the linearized dynamic system have only one negative eigenvalue, whereas indeterminacy will appear when that matrix exhibit two negative eigenvalues.

Proposition 4.1. *The BGP values of employment l^* and of the ratio x^* are the solutions of the following system of equations:*

$$\phi\left(x^*, l^*\right) + \sigma\left(x^*, l^*\right) = 1, \tag{4.8}$$

$$A\left(l^*\right)^{\gamma} - \frac{1}{x^*} = \frac{\alpha A\left(l^*\right)^{\gamma} - \rho}{\sigma\left(x^*, l^*\right)}.$$

Consumption and capital grow at the following common and constant growth rate:

$$g^* = \frac{\alpha A\left(l^*\right)^{\gamma} - \rho}{\sigma\left(x^*, l^*\right)}.$$

Proof. The proof follows from (4.4), (4.6) and (4.7). ■

Note that equation (4.8) is just a condition needed for the existence of BGP. As occurs in the model of the previous section, it is possible to obtain multiple BGP's. For instance, under the utility function (3.15) and $\nu = 3$, $\psi = -1.43$, $\theta = 0.9$, $\gamma = 0.6$, $\alpha = 0.4$, $A = 1$ and $\rho = 0.24$, there are two BGP values of employment: $l_1 = 0.21$ and $l_2 = 0.66$. We next assume that there is a unique BGP and study the stability conditions of the BGP.

Proposition 4.2. *Assume that there is a unique BGP and let*

$$N\left(l^*, x^*\right) = \left(\frac{\alpha}{\sigma^*} - 1\right)\gamma A l^{\gamma} + \left(\gamma - 1 - \zeta^*\right)\left(\frac{1}{x^*} + \left(\frac{\partial \sigma^*}{\partial x}\right)\left(\frac{x^* g^*}{\sigma^*}\right)\right)$$
$$+ \left(\frac{\varepsilon^*}{\sigma^*}\right)\left(g^* x^*\left(\frac{\partial \sigma}{\partial x} + \frac{\partial \phi}{\partial x} - \frac{\frac{\partial \sigma}{\partial x}}{\sigma^*}\right) - \frac{1}{x^*}\right) - l^* g^*\left(\frac{\frac{\partial \sigma}{\partial l}}{\sigma^*} - \frac{\partial \sigma}{\partial l} - \frac{\partial \phi}{\partial l}\right).$$

Then,

(a) the BGP is saddle path stable if $\gamma - 1 < \zeta^*$.
(b) the BGP is unstable if $\gamma - 1 > \zeta^*$ and $N\left(l^*, x^*\right) > 0$.
(c) the BGP is locally stable if $\gamma - 1 > \zeta^*$ and $N\left(l^*, x^*\right) < 0$.

Proof. See the Appendix. ■

This proposition shows that in an endogenous growth model indeterminacy can only arise when the elasticity of the labor demand is larger than the elasticity of the Frisch labor supply. This necessary condition is also the condition found by Bennet and Farmer (2000) in a model without consumption externalities. We then conclude that consumption externalities are a source of equilibrium indeterminacy in the exogenous growth model, whereas they are not in the endogenous growth model. This occurs because in an endogenous growth model the utility function must satisfy condition (4.8) and, as shown in the proof of Proposition 4.2, this condition implies that the equilibrium may only exhibit indeterminacy when $\gamma - 1 > \zeta^*$.

We next show two examples of equilibrium indeterminacy by using the particular functional form of the utility function given in (3.15) and analyzing the two cases $\mu = 0$ and $\mu \neq 0$. In these two examples we assume that $\alpha < \sigma^*$, which guarantees the uniqueness of the BGP.

Proposition 4.3. *Consider the utility function (3.15) and assume that $\mu = 0$ and $\alpha < \sigma^*$. Then, there exists a unique BGP. The dynamic equilibrium is unique and does not exhibit transition when $\gamma - 1 < \zeta^*$, whereas it exhibits local indeterminacy when $\gamma - 1 > \zeta^*$.*

Proof. See the Appendix. ■

The previous proposition states that, when the production function exhibits constant returns to capital and the utility function satisfies the RH property, the equilibrium either exhibits indeterminacy or is unstable so that it does not exhibit transition. We next consider the case with $\mu \neq 0$. In this case, preferences do not satisfy the RH property.

Proposition 4.4. *Assume that $\mu \neq 0$, $\omega < 0$ and $\alpha < \sigma^*$. Then, the BGP is unique. Moreover, let*

$$N(l^*, x^*) = (\gamma - 1 - \zeta^*)\left(\frac{1}{x^*} + \frac{\theta(1-\nu)\,\omega g^*}{\sigma^*}\right) - \omega g^*\left(\frac{1-\gamma+\gamma l^*}{(1-l^*)}\right)\left(1 - \frac{\theta(1-\nu)}{\sigma^*}\right) + \left(\frac{\alpha}{\sigma^*} - 1\right)A\gamma l^{\gamma} + \left(\frac{\varepsilon^*}{\sigma^*}\right)\left(\omega g^*\left(1 - \frac{\theta(1-\nu)}{\sigma^*}\right) - \frac{1}{x^*}\right).$$

(a) If $\gamma - 1 > \zeta^*$, then the BGP is unstable when $N(l^*, x^*) > 0$ and it is locally stable when $N(l^*, x^*) < 0$.

(b) If $\gamma - 1 < \zeta^*$, then the BGP is saddle path stable.

Proof. See the Appendix. ■

As follows from Proposition 4.4, the equilibrium of an Ak growth model can exhibit transition when it is locally unique. Therefore, these two examples show that the transition of the equilibrium path depends on the way consumption externalities are introduced in the utility function. Table 3 provides examples of the different stability properties of the BGP equilibrium depending on the configuration of parameter values. We see from this table that indeterminacy may arise when $\gamma - 1 > \zeta^*$. We see that the BGP of an Ak growth model can be saddle path stable, which means that the dynamic equilibrium is unique and exhibits transition.

[Insert Table 3]

5. Concluding remarks

In this paper we have analyzed the uniqueness of the dynamic equilibrium of a one-sector growth model when we assume that average consumption affects individuals felicity and the average stock of capital affects the level of production, that is, both production and consumption externalities are present. We assume that the utility function of this model is concave and that the production function does not exhibit increasing returns to the private production factors. With these plausible assumptions, we show that the equilibrium is unique when either (i) the labor supply is exogenous or (ii) the labor supply is endogenous and average consumption does not affect the individuals felicity. The first condition implies that the introduction of intratemporal inefficiencies is a necessary condition for equilibrium indeterminacy. The second condition implies that the introduction of average consumption is also a necessary condition.

In the framework of an exogenous growth model, we also show that, if average consumption affects the utility function and the RH property holds, then the dynamic equilibrium may exhibit indeterminacy only when the following two conditions are satisfied: (i) production externalities are present and (ii) the elasticity of the labor demand is larger than the elasticity of the Frisch labor supply. In contrast, if the RH condition does not hold, the equilibrium can exhibit indeterminacy even if these two conditions are not met. Therefore, in this case, indeterminacy may arise when there are no production externalities and the elasticity of the labor demand is smaller than the elasticity of the Frisch labor supply.

We relate the previous results with the efficiency of the equilibrium path. On the one hand, when the labor supply is exogenous, consumption externalities may only modify the intertemporal elasticity of substitution of consumption and, hence, they only may generate intertemporal inefficiencies. On the other hand, when the labor supply is endogenous, consumption externalities modify the intratemporal elasticity of substitution between consumption and leisure and may affect the intertemporal elasticity of substitution of consumption. In fact, they modify the intertemporal of elasticity substitution of consumption when the RH property is not satisfied. In this case, consumption externalities give raise to both intertemporal and intratemporal inefficiencies and the equilibrium exhibits indeterminacy even if production externalities are not introduced. We thus conclude that consumption externalities are a source of equilibrium indeterminacy when they generate both intertemporal and intratemporal inefficiencies, which occurs when the utility function does not satisfy the RH property.

In the framework of an endogenous growth model, we show that the equilibrium may exhibit indeterminacy only when the elasticity of the labor demand is larger than the elasticity of the Frisch labor supply. Therefore, consumption externalities are not a source of equilibrium indeterminacy in the endogenous growth model, where the conditions on the utility function that are required for balanced growth prevent indeterminacy from arising when the elasticity of the labor demand is smaller than the elasticity of the Frisch labor supply.

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Appendix

Throughout this appendix the variables of the model are evaluated at a steady state.

Proof of Proposition 2.2 The Jacobian matrix associated with the system of differential equations (2.2) and (2.4) around the steady state is

$$J = \begin{pmatrix} \frac{\partial \dot{c}}{\partial c} & \frac{\partial \dot{c}}{\partial k} \\ \\ \frac{\partial \dot{k}}{\partial c} & \frac{\partial \dot{k}}{\partial k} \end{pmatrix} = \begin{pmatrix} 0 & \left(\frac{f_{11}(k,k)+f_{12}(k,k)}{\sigma(c)}\right) c \\ \\ -1 & f_1(k,k) + f_2(k,k) \end{pmatrix},$$

and its determinant is

$$Det = \left(\frac{f_{11}(k,k) + f_{12}(k,k)}{\sigma(c)}\right) c,$$

which is negative when $f_{11}(k,k) + f_{12}(k,k) < 0$ and $\sigma(c) > 0$. Thus, the eigenvalues of the Jacobian matrix have opposite sign, which in turn implies that the steady state is saddle path stable. ■

Proof of Proposition 3.2 The Jacobian matrix associated with the system of differential equations (3.10) and (3.13) around the steady state is

$$J = \begin{pmatrix} \frac{\partial \dot{l}}{\partial l} & \frac{\partial \dot{l}}{\partial k} \\ \\ \frac{\partial \dot{k}}{\partial l} & \frac{\partial \dot{k}}{\partial k} \end{pmatrix} =$$

$$\begin{pmatrix} l\left[\frac{\left(\frac{\phi+\sigma}{\sigma}\right)f_{12}-\left(\frac{\alpha+\beta}{k}\right)\left(\frac{\partial \dot{k}}{\partial l}\right)}{\gamma-1-\zeta}\right] & l\left[\frac{\left(\frac{\phi+\sigma}{\sigma}\right)(f_{11}+f_{13})-\left(\frac{\alpha+\beta}{k}\right)\left(\frac{\partial \dot{k}}{\partial k}\right)}{\gamma-1-\zeta}\right] \\ \\ f_2 - c_l & f_1 + f_3 - c_k \end{pmatrix},$$

where

$$c_l = \frac{\partial c(k,l)}{\partial l} = \left(\frac{\delta - \varepsilon + \gamma - 1}{\phi + \sigma}\right)\left(\frac{c}{l}\right),$$

and

$$c_k = \frac{\partial c(k,l)}{\partial k} = \left(\frac{\alpha + \beta}{\phi + \sigma}\right)\left(\frac{c}{k}\right).$$

The determinant of the Jacobian matrix is

$$\begin{aligned} Det &= \left(\frac{l}{\gamma - 1 - \zeta}\right)\left(\frac{\phi + \sigma}{\sigma}\right)\left(\frac{1}{f_{11} + f_{13}}\right)\left[(f_1 + f_3 - c_k)\left(\frac{f_{12}}{f_{11} + f_{13}}\right) - (f_2 - c_l)\right] \\ &= -\left(\frac{l}{\gamma - 1 - \zeta}\right)\left(\frac{\phi + \sigma}{\sigma}\right)\left(\frac{1}{f_{11} + f_{13}}\right)\left(\frac{\partial Q(l)}{\partial l}\right), \end{aligned}$$

where $Q(l)$ is defined in (3.14). Note that $Q(0) < 0$ and therefore $\frac{\partial Q(l)}{\partial l} > 0$ at the steady state. The trace of the Jacobian matrix is

$$Tr = \left(\frac{1}{\gamma - 1 - \zeta}\right)\left[\left(\frac{\phi + \sigma}{\sigma}\right) l f_{12} - l\left(\frac{\alpha + \beta}{k}\right)(f_2 - c_l) + (f_1 + f_3 - c_k)(\gamma - 1 - \zeta)\right].$$

By using the expressions for c_l and c_k, we obtain

$$Tr = \left(\frac{1}{\gamma - 1 - \zeta}\right)\left(\frac{f}{k}\right) N(l,k),$$

where $N(l,k)$ is defined in Proposition 3.2. The result follows immediately. ■

Proof of Proposition 3.4 Assume that $\mu = 0$ and use (3.15) and (3.16) to obtain in a steady state

$$\sigma = \nu(1 + \psi) - \psi,$$

$$\phi = (1 - \nu)(1 + \psi),$$

$$\varepsilon = \theta(1 - \nu)\left(\frac{l}{1 - l}\right) = \gamma(1 - \nu),$$

$$\delta = (\theta(1 - \nu) - 1)\left(\frac{l}{1 - l}\right) = \gamma(1 - \nu) - \frac{\gamma}{\theta},$$

and the steady state values are $l = \frac{\gamma}{\gamma + \theta}$, $k = \left(\frac{\alpha A l^{\gamma}}{\rho}\right)^{\frac{1}{1 - \alpha - \beta}}$ and $c = \frac{\rho}{\alpha} k$.

By using σ, ϕ, ε and δ, the values of $\overline{\sigma}$ and $\overline{\beta}$ in Proposition 3.4 are obtained. Note that $\phi + \sigma = 1 > 0$ implies that indeterminacy may only arise when $\sigma < \overline{\sigma}$ and $\left(\frac{f_2(k,l,k)}{f(k,l,k)} l + \frac{\varepsilon}{\sigma}\right)\left(\overline{\beta} - \beta\right) < 0$. The inequality $\sigma < \overline{\sigma}$ and the concavity condition imply that $\nu < 1$ and $\psi > 0$, which means that $\sigma < 1$ and $u_{12} > 0$. The inequalities $\nu < 1$ and $\psi > 0$ imply that $\frac{f_2(k,l,k)}{f(k,l,k)} l + \frac{\varepsilon}{\sigma} > 0$ and $\overline{\beta} > 0$. It then follows that indeterminacy may only arise when $\beta > 0$. ■

Proof of Proposition 3.5 Assume that $\mu \neq 0$. Then, by using (3.6), (3.14), (3.15) and (3.16) we obtain that

$$\frac{\omega \mu c^{\omega}}{1 - l + \mu c^{\omega}} = \omega\left(1 - \frac{(1 - l)\gamma}{\theta l}\right) = \pi,$$

and the elasticities are

$$\sigma = \nu(1 + \psi) - \psi - \theta(1 - \nu)\pi,$$

$$\phi = (1 - \nu)(1 + \psi) + (\theta(1 - \nu) - 1)\pi,$$

$$\varepsilon = (1 - \nu)\gamma,$$

$$\delta = (1 - \nu)\gamma - \frac{\gamma}{\theta},$$

where l is the steady state value of employment that solves

$$\mu \left[A \left(\frac{\rho}{\alpha} \right)^{-(\alpha+\beta)} l^{\gamma} \right]^{\frac{\omega}{1-\alpha-\beta}} = \frac{\theta l}{\gamma} - (1-l) .$$

By using σ, ϕ, ε and δ, the values of $\overline{\sigma}$, and $\overline{\beta}$ in the statement of this proposition are obtained. Note that indeterminacy may only arise when $\sigma + \phi < 0$ which implies that $\pi > 1$. Next, $\sigma > \overline{\sigma}$ implies that indeterminacy may only arise when $\overline{\beta} < 0$ and $\frac{f_2(k,l,k)}{f(k,l,k)} l + \frac{\varepsilon}{\sigma} < 0$. The later inequality implies in turn that $\sigma < v - 1$. ■

Proof of Proposition 4.2 The Jacobian matrix associated with the system of differential equations (4.6) and (4.7) around the steady state is

$$J = \begin{pmatrix} \frac{\partial \dot{l}}{\partial l} & \frac{\partial \dot{l}}{\partial x} \\ \frac{\partial \dot{x}}{\partial l} & \frac{\partial \dot{x}}{\partial x} \end{pmatrix} =$$

$$\begin{pmatrix} -l \left[\frac{g\left(\left(\frac{\partial \sigma(x,l)}{\partial l} \right) \left(\frac{1}{\sigma} \right) - \frac{\partial \sigma(x,l)}{\partial l} - \frac{\partial \phi(x,l)}{\partial l} \right) + \gamma A l^{\gamma-1} \left(1 - \frac{\alpha}{\sigma} \right)}{\gamma-1-\zeta} \right] & -l \left[\frac{g\left(\left(\frac{\partial \sigma(x,l)}{\partial x} \right) \left(\frac{1}{\sigma} \right) - \frac{\partial \sigma(x,l)}{\partial x} - \frac{\partial \phi(x,l)}{\partial x} \right) + \frac{1}{x^2}}{\gamma-1-\zeta} \right] \\ x \left(\left(1 - \frac{\alpha}{\sigma} \right) \gamma A l^{\gamma-1} + g \left(\frac{\partial \sigma(x,l)}{\partial l} \right) \left(\frac{1}{\sigma} \right) + \left(\frac{\partial \dot{l}}{\partial l} \right) \left(\frac{\varepsilon}{l\sigma} \right) \right) & x \left(\frac{1}{x^2} + \left(\frac{\partial \sigma(x,l)}{\partial x} \right) \left(\frac{g}{\sigma} \right) + \left(\frac{\partial \dot{l}}{\partial x} \right) \left(\frac{\varepsilon}{l\sigma} \right) \right) \end{pmatrix} .$$

The determinant of the Jacobian matrix is

$$Det = -Q'(l) \left(\frac{\partial M(x,l)}{\partial x} \right) \left[\frac{glx}{\gamma - 1 - \zeta} \right], \tag{A1}$$

where

$$Q(l) = A l^{\gamma} - \frac{1}{x(l)} - \left(\frac{\alpha A l^{\gamma} - \rho}{\sigma(x(l), l)} \right),$$

and $x(l)$ is implicitly defined by the equation $M(x,l) = 0$, where

$$M(x,l) = \sigma(x,l) + \phi(x,l) - 1.$$

Note that $Q(0) < 0$ and $Q(l) = 0$ at the unique BGP. This means that $Q'(l) > 0$ at the BGP. Next, by using the definitions of $\sigma(x,l)$ and $\phi(x,l)$ we can rewrite

$$M(x,l) = \left(\frac{\partial h(c(x,l), l)}{\partial c} \right) \left(\frac{c(x,l)}{h(c(x,l), l)} \right) - 1,$$

where $h(c(x,l), l) = \frac{u_3}{u_1}$ and $c(x,l)$ is implicitly defined by using (4.2) by the following equation:

$$h(c,l) = \gamma A l^{\gamma-1} x c.$$

From this equation we obtain that

$$\frac{\partial c(x,l)}{\partial x} = \frac{c}{xM}.$$

Next, we obtain

$$\frac{\partial M(x,l)}{\partial x} = \left(\frac{c}{xM} \right) \left(\frac{\partial M(x,l)}{\partial c} \right).$$

From equation (4.8), M converges to 0. This implies that $\left(\frac{\partial M(x,l)}{\partial c}\right)\left(\frac{1}{M}\right) < 0$ and thus $\frac{\partial M(x,l)}{\partial x} < 0$. Therefore, the determinant is positive when $\gamma - 1 > \zeta$ and negative otherwise. The trace is

$$Tr = \left(\frac{1}{\gamma - 1 - \zeta}\right) N\left(l, x\right), \tag{A2}$$

where $N\left(l, x\right)$ is defined in Proposition 4.2. The result then follows. ∎

Proof of Proposition 4.3 Assume that $\mu = 0$. Then, by using (4.2) it can be shown that

$$x = \left(\frac{\theta}{\gamma A}\right)\left(\frac{l^{1-\gamma}}{1-l}\right), \tag{A3}$$

and the steady state is characterized by $Q\left(l\right) = 0$, where

$$Q\left(l\right) = Al^{\gamma}\left[1 - \left(\frac{\gamma}{\theta}\right)\left(\frac{1-l}{l}\right)\right] - \frac{\alpha Al^{\gamma} - \rho}{\sigma}.$$

If $\sigma > \alpha$, then $Q'\left(l\right) > 0$ and $Q\left(l\right) = 0$ has at most one solution.[13] By using (A3) and the stationary values of σ, ϕ, ε and δ shown in the proof of Proposition 3.4, equation (4.6) can be rewritten as

$$\frac{\dot{l}}{l} = -\frac{Q\left(l\right)}{\gamma - 1 - \zeta}.$$

This equation drives the transition of the economy. Since $Q'\left(l\right) > 0$, we have that $\frac{\partial \dot{l}}{\partial l} < (>)\, 0$ if $\gamma - 1 > (<)\, \zeta$, where

$$\zeta = -\delta - \varepsilon\left(\frac{\phi}{\sigma}\right) = \left(\frac{l}{1-l}\right)\left(1 - \frac{\theta\left(1-\nu\right)}{\nu\left(1+\psi\right) - \psi}\right).$$

The result then follows. ∎

Proof of Proposition 4.4 Assume that $\mu \neq 0$. Then, by using (4.2), we obtain that

$$\frac{\omega\mu c^{\omega}}{1 - l + \mu c^{\omega}} = \omega\left(1 - \frac{\left(1-l\right)\gamma Axl^{\gamma}}{\theta l}\right) = \pi\left(x, l\right),$$

and the elasticities are

$$\sigma\left(x, l\right) = \nu\left(1+\psi\right) - \psi - \theta\left(1-\nu\right)\pi\left(x, l\right),$$

$$\phi\left(x, l\right) = \left(1-\nu\right)\left(1+\psi\right) + \left(\theta\left(1-\nu\right) - 1\right)\pi\left(x, l\right),$$

$$\varepsilon\left(x, l\right) = \left(1-\nu\right) A\gamma xl^{\gamma},$$

and

[13] The economy may have two different Balanced Growth Paths. However, there is a unique BGP when $\sigma > \alpha$, which is a plausible assumption. In order to see this, note that the share of capital income on national income $\alpha \approx 0.35$ and the plausible values of the inverse of the intertemporal elasticity of substitution are larger than 0.35. See Bennet and Farmer (2000) for a discussion on the values of these parameters.

$$\delta(x,l)=\left(\frac{\theta(1-\nu)-1}{\theta}\right)A\gamma x l^{\gamma}.$$

Along the steady state, $\sigma+\ \phi=1$ implies that $\pi=0$, which means that

$$x=\frac{\theta l^{1-\gamma}}{(1-l)\gamma A},$$

where l solves the following equation:

$$\left(1-\frac{\alpha}{\sigma}\right)Al^{\gamma}-\frac{(1-l)A\gamma}{\theta l^{1-\gamma}}+\frac{\rho}{\sigma}=0.$$

Note that there is a unique solution when $\sigma>\alpha$. Finally, the growth rate is

$$g=\frac{\alpha Al^{\gamma}-\rho}{\sigma}.$$

By using (A1) and (A2), we obtain that the determinant of the Jacobian matrix is

$$Det=-\omega\left(\frac{gA\gamma l^{\gamma-1}}{\gamma-1-\zeta}\right)\left[\left(1-\frac{\alpha}{\sigma}\right)l+\frac{1-\gamma+\gamma l}{\theta}\right],$$

where

$$\zeta=\left(\frac{l}{1-l}\right)\left(\frac{\nu(1+\psi)-\psi-\theta(1-\nu)}{(\nu(1+\psi)-\psi)}\right),$$

and the trace is

$$Tr=\left(\frac{1}{\gamma-1-\zeta}\right)N(l,x),$$

where $N(l,x)$ is defined in Proposition 4.4. The result then follows. ■

Table 1. Stability of the exogenous growth model when $\mu = 0$.[14]

$\gamma = 1 - \alpha$	$\beta = 0$	$\beta = 0.2$	$\beta = 0.3$
Saddle Path	$\psi < 0.58$	$\psi \leq 0.58$	$\psi \leq 0.58$
Indeterminacy	$\emptyset$	$\psi \in (0.58, 0.64)$	$\psi \in (0.58, 0.72)$
Unstable	$\psi \in [0.58, 1.22)$	$\psi \in [0.64, 1.22)$	$\psi \in [0.72, 1.22)$

Table 2. Stability of the exogenous growth model when $\mu \neq 0$.[15]

$\gamma = 1 - \alpha$, $\beta = 0$	$\mu = \omega = 0$	$\mu = 0.8$, $\omega = 1.7$
Unstable	$\emptyset$	$\psi > -5.4$
Saddle Path	$\psi > -1.25$	$\psi \in (-6.1, -5.8)$
Indeterminacy $\gamma - 1 < \zeta$	$\emptyset$	$\psi \in [-5.8, -5.4)$

Table 3. Stability of the endogenous growth model when $\mu \neq 0$.[16]

$\gamma = 1 - \alpha$	$\omega = 0$	$\omega = -1$
Unstable	$\psi < 0.64$	$\emptyset$
Saddle Path	$\emptyset$	$\psi < 0.64$
Indeterminacy $\gamma - 1 > \zeta^*$	$\psi \in [0.64, 1)$	$\psi \in [0.64, 1)$

[14] The parameters in the benchmark economy take the following values: $v = 0.55$, $\rho = 0.065$ so that the interest rate is 6.5%, $\alpha = 0.4$ is the share of capital income on national income, and $\theta = \gamma\left(\frac{1-l^*}{l^*}\right)$ with $l^* = 0.34$. Note that $\sigma^* > 0$ requires that $\psi < 1.22$.

[15] The parameters take the following values: $\rho = 0.045$, $A = 1.01$, $\theta = 4$, $\gamma = 0.6$, $\nu = 5$. The value of ω varies with μ so that $l^* = 0.34$. Note that $\sigma^* > 0$ requires that $\psi > -1.25$ when $\mu = \omega = 0$, whereas it requires that $\psi > -6.1$ when $\mu = 0.8$ and $\omega = 1.7$.

[16] The value of A is such that $l^* = 0.34$. The value of ρ is such that $g^* = 0.04$. The values of the other parameters are $\alpha = 0.4$, $\gamma = 1 - \alpha$, $v = 0.7$ and $\theta = 3$. Note that $\sigma^* > \alpha$ requires that $\psi < 1$.