GROUP FORMATION AND VOTER PARTICIPATION

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ABSTRACT. We present a mobilization model of two party winner-take-all elections with endogenous voter group formation: agents decide whether to be followers or become leaders and try to bring people to vote for their preferred party-candidate. The model gives a closed form solution and uniquely determines the number of leaders in equilibrium. Expected turnout and winning margin in the election are predicted as a function of the equilibrium number of leaders, their ability to mobilize voters and the importance of the election.

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JEL Classification: D72

"Si nos habitudes naissent de nos propres sentiments dans le retraite, elles naissent de l'opinion d'autrui dans la societe. Quand on ne vit pas en soi, mais dans les autres, ce sont leurs jugements qui reglent tout"
Jean-Jaques Rousseau, Lettre a M.d'Alembert (1758)

"Most people are other people. Their thoughts are someone else's opinions, their lives a mimicry, their passions a quotation"
Oscar Wilde, De Profundis (1905)

1. Introduction

Economists tend to view social phenomena from two different perspectives. When dealing with situations that are rife in strategic interactions, such as markets with a few participants, or bargaining among a small group of people, we favor game theoretic models. When dealing with situations where individuals' influence in the environment is negligible, such as large, anonymous markets, we favor competitive models. While this division serves us generally well, there are social phenomena which are not easily assigned to either perspective. Consider, for instance, elections with very many potential voters. A "fully strategic" treatment of voters' behavior in large elections is possible, but its implications are disappointing. If voters are motivated only by the effect of their actions in the result of the election, and there is but

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a slight cost involved in the act of voting, game-theoretic models predict a dismally low turnout, as long as voters are somewhat uncertain about the preferences of others.[1] This is clearly at odds with mass participation in elections. A "fully competitive" treatment of voters' behavior, emphasizing the private gains from the act of voting, is also possible. From this perspective, voters vote motivated by the desire to express preferences or allegiance to certain group, much as cheering crowds in a sport event.[2] But the competitive perspective seems to miss what the strategic perspective overemphasizes. Some individuals, such as political leaders, party activists, journalists, or other opinion makers, devote time and effort to influence the result of the election by mobilizing others to vote. These individuals do act strategically – the time and effort they devote to the election depend to some extent on the time and effort they expect others to devote. Moreover, there is no clear line between voters and opinion makers. Under different circumstances, an individual may decide to watch the election as a passive spectator or to invest time and other resources to influence the result.

Of course, it is true that by nature of their professional activity some individuals – government functionaries, editorialists, prominent economists, actors and other entertainers – may have an advantage in influencing others to vote. But it is also true that not everyone in each of these categories becomes an opinion maker in every large election. For some, perhaps many, voters there is a decision to be made in an election regarding whether to try to influence others or not. This margin of decision has been neglected in the economic literature on elections, be it "strategic" or "competitive." Nonetheless, we believe analyzing this margin of decision is crucial to understand voter participation in large elections. In this paper, we propose a model where voters decide whether to become opinion leaders, and extract implications in terms of the distribution of voter turnout and the distribution of the winning margin in large elections.

In our model, opinion leaders arise *endogenously* out of a large electorate of citizens with different political preferences. First, voters decide first whether to be followers or leaders that try to influence the outcome of the election by spending effort to bring followers to vote. Being a leader is costly as it is voting and is driven by a cost-benefit calculation. Second, followers are randomly assigned to their leaders and must decide whether to vote independently or vote in compliance with

[1] See e.g. Palfrey and Rosenthal [16, 17], who build on earlier work by Ledyard [10]. Another treatment of the subject is provided by Myerson [**?**].

[2] See e.g. Brennan and Buchanan [5] and for a very nice recent treatment, Schuessler [20].

their assigned leader in exchange for a reward. In the unique equivalence class of equilibria, only a fixed number of voters for each party become leaders: the influence of each leader, that is, the leader's ability to sway the election one way or the other is random but statistically declines with the number of leaders. More leaders imply a lower influence on followers of each single leader. As in mobilization models, electoral turnout remains high because most voters are willing to vote in agreement with their leader and their group.

Our model gives a closed form solution that allows us to derive the distribution of the electoral participation rate and the winning margin in two-party plurality elections as functions of the importance of the issue at stake in the election for voters, the cost of voting, and the cost of becoming a leader. Comparative statics on these parameters of the model is aligned with standard stylized facts. For instance, expected turnout increases with the importance of the elections and the distribution of the winning margin generated by the model remains non-degenerate for the arbitrarily large electorate assumed here.[3]

Citizen-candidate models a la Osborne and Slivinsky [15] or Besley and Coate [2] have in common with this work the idea of endogenizing political activism. Namely, out of a population of citizens in equilibrium some citizens decide to become politically active candidates/leaders. The goals and the type of political activism are radically different though. In the present paper leaders arise not because they want to be elected to office themselves but because they can affect the chance that their preferred party wins the election (pivotality). In other words, this paper tries to address the problem of the paradox of not voting with a group-based mobilization model. The issue of turnout is not addressed and is not the objective of citizen-candidate models which rather try to endogenize party platforms and characterize party formation. By contrast in the present paper party platforms are given.

Our work is also related to the social interactions literature pioneered, *inter alia*, by Glaeser, Sacerdote and Scheinkman [9]; some different approaches are presented by Becker and Murphy [4] and Durlauf and Young [7]. We borrow from the model of Glaeser *et al.* the arrangement of agents in a circle and the idea that some agents imitate their group behavior while others act independently, depending on their voting cost. We deviate in that the group formation, that is the number of agents that act as leaders is derived endogenously in the model, as well

[3]This stands in contrast with strategic models of elections. For instance, Palfrey and Rosenthal [16] obtain a high turnout rate in a large election without uncertainty and with no net benefits of voting, but at the cost of predicting nearly a tie.

as the decisions of followers to vote with their leader or independently. Lastly, voters are concerned not only by the decisions of their neighbors (group) as in local interaction models, but also by the decision of the majority (local and global interactions).

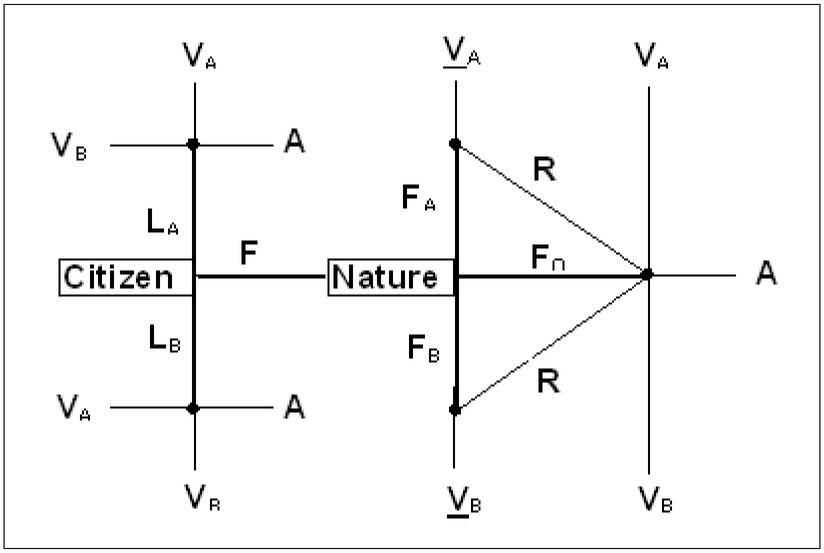
Six sections follow this introduction. The first illustrates the structure of the multi-stage game, the second gives the particular circular specification of how followers are assigned to leaders. The third section finds the unique class of equilibria after computing explicitly the pivotality of leaders. The fourth gives important comparative statics results, while the fifth shows how all the results are robust to various heterogeneities of potential voters. The last section summarizes and concludes.

2. The Model

We consider a large election with two alternatives, A and B, (e.g. two election candidates or two issues in a referendum). There are is a continuum of citizens (potential voters) of measure one of which a strictly positive measure are "A-partisans," and "B-partisans", the rest are non-partisans O. The only difference in the preferences of these three types is that partisans enjoy a gross gain of $G > 0$ if their preferred party wins the election and zero otherwise, non-partisans have a gain of zero whatever party wins.

A simple way to describe this game is to break it down into three stages, in which citizens have to make sequential decisions. In the first stage every citizen chooses whether to become an active supporter of a party or not. We refer to an active supporter as an opinion *leader* (L), a political entrepreneur committed to his party, and to an uncommitted voter as a *follower*.(F). Namely, each player of any type initially has one out of three choices to make (L_A, L_B, F), the first two choices involve a cost $C > 0$ of becoming a leader, the last involves no cost. In the second stage once all the leaders are chosen, any follower may (or may not) randomly fall under the influence of a leader of one of the parties. Hence, ex-post there are three types of followers (F_A, F_B, F_O). All followers that fall under the range of influence of some leader of A or B, namely F_A and F_B, are offered by their leader a compensation of $(v + \varepsilon)$ in exchange for committing to vote for the party of that leader (A or B). These "influentiable" followers have the option of accepting this compensation, or rejecting it. In case of acceptance they will have to vote for that party ($\underline{V}_A$ or $\underline{V}_B$). In case of rejection (R) they receive no compensation but are uncommitted and free to abstain or vote for who they want, just like the independent followers

F_O. In the third stage *every* citizen (leader or uncommitted follower) must choose whether to vote for one party or abstain (V_A, V_B, A) the first two choices involve a cost of voting $v > 0$, abstaining involves no cost. This last third voting stage can also thought as simultaneous to the previous second stage, the results do not change. The picture below tries to illustrate this game.



The first stage leader-follower decision and then the choice of nature, that randomly assigns followers to leaders, are depicted with thicker lines. The second and third (voting stages) take place in correspondence to the dots in the picture. Recall that the only ex-ante difference among citizen is their political view or type (A, B, O). In principle any citizen can end up in any of the final nodes of this game. For sake of clarity we introduce only after we present the main results of this paper other important initial differences among citizens such as voting costs, gains from the outcome of the election and cost of becoming leader. All these additional heterogeneities do not generically change the results of this paper.

We specify the move of nature after we solve for the last stage of the game. The only property we need for now is that a leader can influence some strictly positive measure of voters with probability one.

2.1. **Last Stages.**
Case of No Leaders. Note first that only A partisan citizens may decide to be leaders of party A, likewise for B partisans. No other type of citizen would want to incur the cost C of becoming a leader if he cannot get a gain from doing so by affecting the election in his preferred direction. If no citizen chooses to be a leader (which may happen if the

cost C is high relative to the gain G from having the preferred party win) then all citizens end up in the node F_0 and then play a winner-take all election voting game with a continuum of potential voters. For any positive voting cost, no equilibrium of this subgame can yield a positive turnout, because that would imply a positive measure of voters, each with a zero chance of being pivotal. Zero turnout and the paradox of not voting is a natural outcome of this subgame and for this reason this subgame will not be reached in equilibrium as it will become clear later.

Case of Some Leaders. If at least one citizen decides to become leader of one party (say party A) then he exercises some influence over a random positive measure of followers, namely a subset of the followers that belong to the node F_A. At this point the followers F_A have to decide whether to vote for the preferred party A of their assigned leader in exchange for a compensation $(v+\varepsilon)$, or reject the compensation and be free to vote for who they want or abstain.

Proposition 1. *In equilibrium all followers of type F_A (F_B) commit to vote $\underline{V}_A$ ($\underline{V}_B$) and have a net gain of $\varepsilon > 0$, all followers of type F_O abstain A and obtain a net gain of zero.*

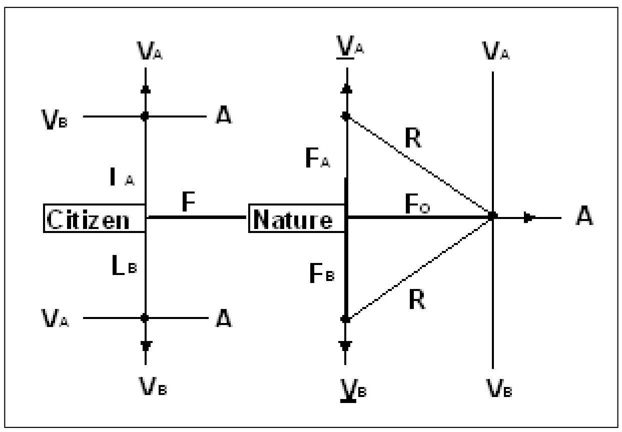
Proof. The above strategy profile an equilibrium because nobody wants to deviate from that profile: WLOG given that profile a single F_A follower would never be pivotal and by choosing R he would give away a net gain of ε to gain zero if he abstains or $-v$ if he votes. Likewise, any F_O follower abstains because he cannot be pivotal. Furthermore there are no other equilibria. It is not an equilibrium that any positive number of followers may F_A rejects R and votes for A. A positive measure of F_A followers (all the non partizan O-type and B-type followers, but also *most* of the A-types) always accept the offer of their A-leader and make voting for pivotal reasons pointless for any individual voter of any type. □

In summary in equilibrium all followers F_A and F_B vote for whatever leader gives them some compensation *regardless* of their preferences. In any equilibrium with some leaders, all followers know that their probability of changing the outcome of the election is zero regardless of whom they vote for.

It is easy to see that this outcome is similar with heterogeneity of voting costs. In this case only F_A followers with voting costs below the leader's compensation $(v+\varepsilon)$ (assuming this compensation is high enough so that there is a positive measure of such followers) commit to vote $\underline{V}_A$, the rest reject and then abstain together with all the F_O

followers. We elaborate further on this issue in the section on heterogeneity.

Now that we are done with the voting decision of the followers, we turn to what leaders vote for. Leaders know by there mere presence that they cannot affect the election with their vote, so they either abstain or vote for their preferred party if we assume that they compensate themselves with $(v+\varepsilon)$ for doing so. Either assumption does not change the turnout, because there will only be a discrete number of leaders in any equilibrium. Assuming they do vote without loss of generality, the last stage voting decisions of all citizens are summarized by the arrows in the picture.



We solved for the last voting stages and we are left with the initial leader-follower decision that all citizens face initially regardless of their party preferences. Solving backwards to find the Subgame Perfect Nash Equilibrium (SPNE) of this game, we rewrite the payoffs resulting from the subsequent voting stages. The payoffs of F_A and F_B followers become

$G\text{Pw}_A+\varepsilon$	A-partisans
$G\text{Pw}_B+\varepsilon$	B-partisans
ε	O-partisans

where Pw_A is the probability that party A wins. The payoffs of the F_O followers are the same as above without the ε. If ε is small the payoffs of *all* followers are

$G\text{Pw}_A$	A-partisans
$G\text{Pw}_B$	B-partisans
0	O-partisans

All citizens take into account these reservation values when deciding whether to become leaders or not. Given that leaders affect a positive measure of followers then they consider their chances of changing the outcome of the election when deciding whether to pay the cost C to become a leader. If there are (A, B) leaders already, a citizen becomes an additional leader of party A if and only if

$G\left(\mathrm{Pw}_A(A+1,B) - \mathrm{Pw}_A(A,B)\right) > C$	A-partisans
$G\left(\mathrm{Pw}_B(A,B+1) - \mathrm{Pw}_B(A,B)\right) > C$	B-partisans
$0 > C$	O-partisans

Assuming (later deriving) that $\mathrm{Pw}_A(A,B)$ is non-decreasing in A, only A partisans may consider to become leaders L_A, likewise only B types may decide to become leaders L_B. We rewrite the initial leader-follower trade-off the A types are facing as

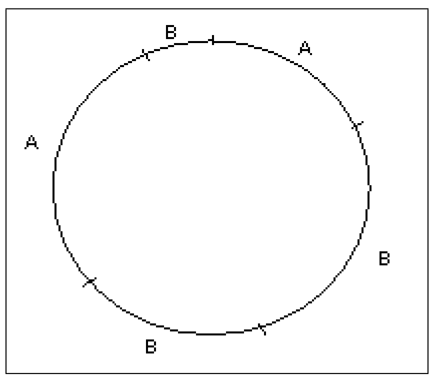
$$\mathrm{Pv}_A(A+1,B) > \frac{C}{G}$$

where $\mathrm{Pv}_A(A+1,B)$ is pivotality of the additional leader A, namely by how much an A leader can increase the likelihood that the party for whom he is a leader wins the election. As we show in the extensions of the model, introducing heterogeneity on the cost and/or benefit of becoming leader $\frac{G}{C}$, does not change the results either.

3. Influence of Leaders

Being a leader is costly (with fixed cost C). The number of followers that a given leader gets is random and it depends negatively on how many other citizens become leaders. This is realistic in the sense that political entrepreneurs do not know how many people they can bring to vote but they do know that their personal influence over followers decreases the more entrepreneurs there are that compete with them to bring citizens to vote. Namely, the number of leaders is a sufficient statistic for the distribution of followers.

We capture this random dependence assuming that all leaders are dropped uniformly on a circle of measure one, which represents the population. Each leader brings to vote for his party an interval of agents (to his right say), until his interval of influence is interrupted. This cluster of influence can be interrupted by another leader or may just die out exogenously. How likely it is to die out exogenously is a measure of the strength (or rather weakness) of the influence of a leader in the absence of other leaders. Ex post followers can fall in the sphere of influence of a leader of party A if the nearest leader to their left is of party A and the influence has not died out, in which case in the SPNE they will vote for party A regardless of their preferences. Likewise they will vote for B regardless of their preference if the nearest leader to their left is of party B and the influence has not died out. Finally they abstain if the influence of the nearest leader to their left has died out. If no citizens decide to be leaders, citizens have no external influence or reward from voting and, as a consequence, all citizen play a regular voting game which gives the low turnout outcome described in the paradox of not voting. The alternative with more votes wins the election, ties are zero probability events. An equivalence class of strategy profiles of this one shot simultaneous game with a continuum of players can be summarized by (A, B) that is, the *number* of leaders for party A and for party B, not who they are in particular. We only know that in equilibrium they are part of the A-partisans and B-partisans respectively and that they are a finite number or zero measure relative to all voters in equilibrium.



We assume that the population (e.g. political views) of the voters belonging to any of the clusters in the picture above is not different from the overall population. Followers are not selecting who is their leader in this model, rather some leader is assigned to them (or emerges among them randomly). This leader may or may not have the same political views of the majority of the voters in his cluster.

3.1. **Distribution without Abstainers.** We want to find how a given number of leaders (A, B) (considered as a sufficient statistic) maps into the distribution of votes. As a preliminary step for sake of exposition, we assume that *all* followers fall under the influence of some leader. That is, there are no F_O followers or exogenously fading influences. The influence of a leader can be interrupted only by the presence of another leader. To find this distribution we first need some statistical results.

Theorem 2. *The joint distribution of the spacings*

$$(x_1 = y_1, .., x_k = y_k - y_{k-1}, ..., x_{n+1} = 1 - y_n)$$

of the uniform order statistic $0 \leq y_1 < y_2 < ... < y_n \leq 1$*, i.e., the distribution* $g(x_1, .., x_k, ..x_{n+1})$ *is invariant under the permutation of its components.*

Proof. 2 See Reiss. p.40. □

This implies, in particular, that

Corollary 3. *All marginal distributions of* $(x_1, .., x_k, ..x_{n+1})$ *of equal dimension are equal.*

Assume that there are no exogenous interruptions of the spheres of influence of leaders. There are $A + B$ leaders in total with $A, B > 0$.

Proposition 4. *The distribution of the number of votes a for party A has the following pdf*

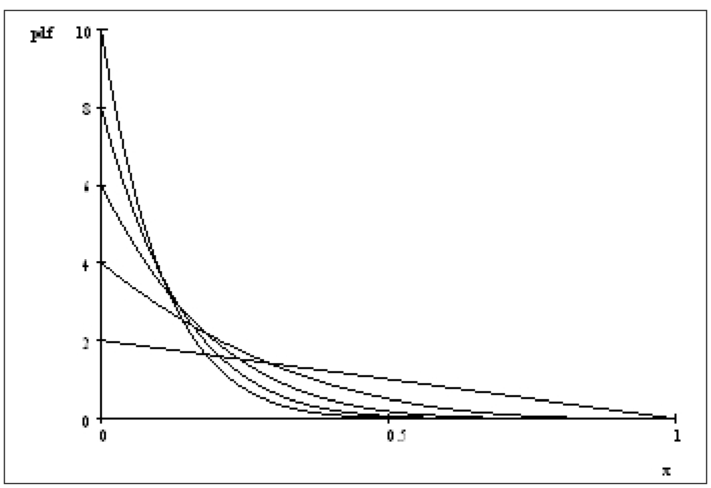
$$h_{(A,B)}(a) = \frac{(A+B-1)!}{(A-1)!\,(B-1)!} a^{A-1} (1-a)^{(B-1)} \qquad 0 < a < 1$$

Proof. See Appendix. □

The unconditional pdf of the size of a single cluster x, among other n clusters, i.e. the influence region of one leader among other additional n leaders is

$$n(1-x)^{n-1} \qquad 0 \leq x \leq 1$$

As you can see the influence of any single leader shrinks as n increases and his influence is crowded out by more other leaders



Distribution of Cluster Lenght

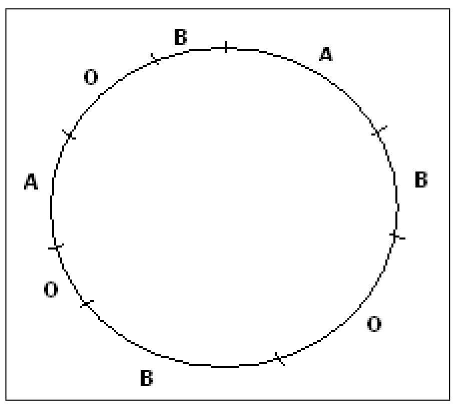
The intersection with the y-axis of any of the densities happens to also represent the number of additional leaders n that correspond to that density.

The turnout for party A is a random variable that has expectation

$$\mathrm{E}_{AB}\left(a\right)=\frac{A}{A+B}$$

which is intuitive: the party with more leaders is expected to get more votes.

3.2. **Distribution with Abstainers.** Assume that the influence of leaders can fade exogenously regardless of the presence of other leaders. This generates F_O followers that then become abstainers in the SPNE. Abstention in this model is lack of leadership and is obtained in the following way. An exogenous number O of interruptions of the spheres of influence fall uniformly on the circle. The smaller this number O the stronger is the social interaction and the stronger is the effect of leadership on potential voters



. Now (A,B,O) becomes the sufficient statistic for the distribution of the turnout (votes) for each party (a,b). It is a bivariate distribution defined over the unit simplex.

Proposition 5. *The bivariate distribution of the number of votes (a,b) for both parties given that there are A,B,O leaders is:*

$$\begin{aligned} h_{(A,B,O)}\left(a,b\right) &= \frac{(A+B+O-1)!}{(A-1)!\,(B-1)!\,(O-1)!}a^{(A-1)}b^{(B-1)}\left(1-a-b\right)^{(O-1)} \\ 0 &\leq a+b\leq 1 \end{aligned}$$

for $A,B,O\geq 1$

Proof. See Appendix. □

If e.g. $B=0$ we have a degenerate univariate density. If e.g. $A=B=O=1$ the bivarate distribution is uniform on the simplex and the marginal distributions are linear e.g.

$$h_a = \int_0^{1-a} 2db = 2(1-a)$$

The probability that party A wins the election is

$$\mathrm{Pw}_A = \Pr(a > b) = \int_0^{\frac{1}{2}} \int_0^{a} hdbda + \int_{\frac{1}{2}}^{1} \int_0^{1-a} hdbda$$

The results should hold in any circumstance in which abstainers are introduced by lack of leadership through blind or indiscriminate interruptions of the influence of any given leader.

3.3. **Probability of Winning.** We need to show that the probability that party A wins the election $\mathrm{Pw}_A(A,B,O) \equiv \Pr(a > b)$:

$$\begin{aligned} \mathrm{Pw}_A(A,B,O) &= \int_0^{\frac{1}{2}} \int_0^{a} h(A,B,O)\, dbda + \int_{\frac{1}{2}}^{1} \int_0^{1-a} h(A,B,O)\, dbda \\ \text{with} \quad h_{(A,B,O)}(a,b) &= \frac{(A+B+O-1)!}{(A-1)!\,(B-1)!\,(O-1)!} a^{A-1} b^{B-1} (1-a-b)^{O-1} \end{aligned}$$

is independent of O.

Proposition 6. *When $O \geq 1$ the probability of winning is independent of O and equal to*

$$Pw_A(A,B,O) = 1 - \sum_{k=1}^{B} \left(\frac{1}{2}\right)^{A+B-k} \frac{(A+B-k-1)!}{(A-1)!\,(B-k)!}$$

Proof. See Appendix. □

Since the probability of winning is independent of O for O grater or equal to one, we are left to prove that the same hods also if $O = 0$.

Proposition 7.

$$Pw_A(A,B,1) = Pw_A(A,B,0)$$

Proof. See Appendix. □

Having established that the probability of winning does not depend on O we can preform the calculation with the simpler univariate distribution with $O = 0$ derived earlier in the no abstainer case, which gives a more tractable expression than what we originally obtained for $\mathrm{Pw}(A,B,O)$.

4. Pivotality and Equilibrium

In the following we refer to Pv_A as the probability of any A leader of being pivotal, the same calculations hold reversed for the probability of being pivotal of B. We omit the variable $O = 0$ since we have shown it does not affect the probability of winning and therefore the pivotality. WLOG referring to party A, the probability of being pivotal for the A-th potential leader is the difference between the probability of winning with him and winning without him keeping everything else constant, that is

$$\mathrm{Pv}_A(A, B) \equiv \mathrm{Pw}_A(A, B) - \mathrm{Pw}_A(A - 1, B)$$

Proposition 8.

$$Pv_A(A, B) = \frac{1}{2^{A+B-1}} \frac{(A + B - 2)!}{(A - 1)!\,(B - 1)!}$$

Proof. See Appendix.[4] □

The expression for the pivotality when $A = B$ is useful later when we look for the equilibrium

$$\mathrm{Pv}_A(A, A) = \frac{1}{2^{2A-1}} \frac{(2A - 2)!}{(A - 1)!\,(A - 1)!} \qquad \text{for} \quad A \geq 1$$

For instance for $A = B = 1$, the probability of being pivotal is the change a leader can make from losing the election for sure to losing it with a 50-50 chance

$$\mathrm{Pv}_A(A, A) = \frac{1}{2}$$

The following monotonicity result is insightful and will be useful later.

Proposition 9. *For every $k = 1, \ldots n$*

$$Pv_A(B - k, B) < \ldots < (Pv_A(B - 1, B) = Pv_A(B, B)) > \ldots > Pv_A(B + k, B)$$

Proof.

$$\begin{aligned}
\frac{\mathrm{Pv}_A(A, B)}{\mathrm{Pv}_A(A - 1, B)} &= \frac{A + B - 2}{2A - 2} \lesseqgtr 1 \quad \Longleftrightarrow \quad A \lesseqgtr B \\
\frac{\mathrm{Pv}_A(A, B)}{\mathrm{Pv}_A(A - 1, B)} &= \frac{A + B - 2}{2A - 2} = 1 \quad \Longleftrightarrow \quad A = B
\end{aligned}$$

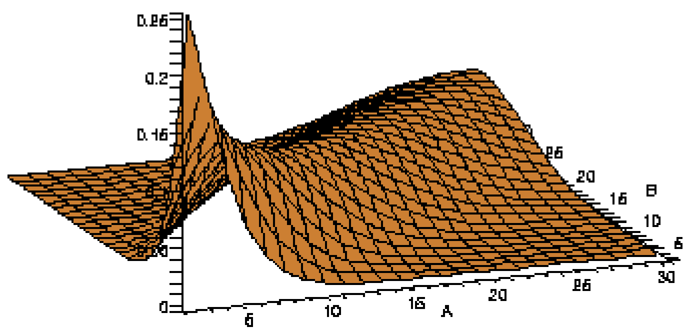
□

[4] I am very grateful to Aaron Robertson for illustrating me the Wilf-Zeilberger method to solve hypergeometric identities.

So the probability of changing the outcome of the election is higher when the difference between the number of leaders of the two parties is smaller. The other important monotonicity result is that $\mathrm{Pv}_A\,(A, A)$ decreases in A, because

$$\frac{\mathrm{Pv}_A(A+1, A+1)}{\mathrm{Pv}_A(A, A)} = 1 - \frac{1}{2A} < 1$$

The pivotality is highest when there are the same number of leaders for both parties, but this value decreases the more leaders there are. The 3-D plot of the pivotality function (done by using the gamma function rather than the factorials to be able to plot over $\mathcal{R}^2$) shows how the pivotality is highest in the diagonal ($A = B$) and decreases as we move along the diagonal.



$(A, B) \in (2, 30) \times (2, 30)$

The pivotality function also has the following symmetry property.

Proposition 10.

$$Pv_A\,(A, B) = Pv_B\,(A, B) \tag{1}$$

Proof. By definition we must have

$$\mathrm{Pv}_A\,(A, B) = \mathrm{Pv}_B\,(B, A)$$

and from proposition (8) we see that

$$\mathrm{Pv}_A\,(A, B) = \mathrm{Pv}_A\,(B, A)$$

□

which means that for any (A, B), how much an A-leader increases the chance that party A wins $(\mathrm{Pv}_A(A, B))$ equals how much a B-leader decreases that chance $(\mathrm{Pv}_B(A, B))$. We are now ready to find the equilibrium of the game.

4.1. **Equilibrium.** If the cost of becoming leader is constant then we have that an agent becomes a leader of party A if

$$G\mathrm{Pv}_A(A, B) - C > 0$$

We have the following equilibrium result.

Proposition 11. *There exist a unique class of equilibria. They are of the form* $(\overline{A}, \overline{A})$

Proof. See Appendix. □

Intuitively given the shape of the pivotality function asymmetric equilibria are not possible. In any configuration (A, B) with $A \neq B$ whenever the partisans of the winning party have no incentive deviate increasing or decreasing the number of their leaders, then necessarily some partisan of the losing party has incentives to become an additional leader.

Note that asymptotically (using Stirling's formula)

$$\mathrm{Pv}_A(A, A) \simeq \frac{1}{\sqrt{4\pi(A-1)}}$$

which is a slow decrease of the pivotality along the diagonal $B = A$, so the equilibrium number of leaders is not necessarily very small.

In any equilibrium, the number of leaders increases with the importance of the election and decreases with the cost of being a leader. This drives the following comparative statics results.

5. Comparative Statics

5.1. **Turnout.** The expected turnout in any equilibrium (A, A) with level of abstention O is

$$\begin{aligned}
&= E(T = a + b) \\
&= \int_0^1 \int_0^{1-a} (a+b)\, h(A, A, O)\, dbda \\
&= \frac{(2A+O-1)!}{(A-1)!\,(A-1)!\,(O-1)!} \int_0^1 \int_0^{1-a} (a+b)\, a^{(A-1)} b^{(A-1)} (1-a-b)^{(O-1)}\, dbda \\
&= \frac{(2A+O-1)!}{(A-1)!\,(A-1)!\,(O-1)!} \int_0^1 \int_0^{1-a} \left(\begin{array}{c} a^A b^{(A-1)} (1-a-b)^{(O-1)} \\ +a^{(A-1)} b^A (1-a-b)^{(O-1)} \end{array} \right) dbda
\end{aligned}$$

From the normalization (see appendix) note that

$$\int_0^1 \int_0^{1-a} a^{A-1} b^{B-1} (1-a-b)^{O-1} \, dbda = \frac{1}{\left(\frac{(A+B+O-1)!}{(A-1)!(B-1)!(O-1)!}\right)}$$

Hence

$$\begin{aligned} \mathrm{E}(T) &= \frac{(2A+O-1)!}{(A-1)!\,(A-1)!\,(O-1)!} \frac{2}{\left(\frac{(2A+1+O-1)!}{(A-1)!(A)!(O-1)!}\right)} \\ \mathrm{E}(T) &= \frac{2A}{2A+O} = \frac{1}{1+\frac{O}{2A}} \end{aligned}$$

Expected turnout increases with the equilibrium number of leaders and decreases with the level of abstention. It is 50% when the number of leaders of both parties equals the number of exogenous interruptions of leaders' influence.

5.2. **Closeness.** The expected closeness of the election, i.e. the expected winning margin of any party, is in any equilibrium (A, A) equal to

$$\begin{aligned} CL(A,O) &= \mathrm{E}(|a-b|) \\ &= 2\mathrm{E}(a-b|a>b) = 2\int_0^{\frac{1}{2}} \int_a^{1-a} (a-b)\, h(A,A,O)\, dbda \end{aligned}$$

$$\begin{aligned} &= CL(A,O) \\ &= \frac{2(2A+O-1)!}{(A-1)!\,(A-1)!\,(O-1)!} \int_0^{\frac{1}{2}} \int_a^{1-a} (a-b)\, a^{A-1} b^{A-1} (1-a-b)^{O-1} \, dbda \\ &= \frac{2A}{2A+O} \left(\mathrm{Pw}(A+1,A) - \mathrm{Pw}(A,A+1)\right) \\ &= \frac{2A}{2A+O} \left(2\mathrm{Pv}(A+1,A+1)\right) \\ &= \frac{1}{2A+O} \left(\frac{2}{2^{2A}} \frac{(2A)!}{A!\,(A-1)!}\right) \end{aligned}$$

with more leaders, that is a more important election, we have the following result.

Proposition 12. *More leaders make a closer election if and only if there are enough of them*

$$2A > O$$

Proof.

$$CL(A+1,O) = \frac{1}{2A+2+O}\left(\frac{2}{2^{2A+2}}\frac{(2A+2)!}{(A+1)!\,(A)!}\right)$$

$$\begin{aligned}\frac{CL(A+1,O)}{CL(A,O)} &= \frac{2A+O}{2A+2+O}\frac{2A+1}{2A} = \frac{1+\frac{1}{2A}}{1+\frac{2}{2A+O}} < 1\\ \iff &\quad 2A > O\end{aligned}$$

□

We have that a more important election is likely to have a smaller winning margin if there are sufficient leaders relative to abstainers that is if the turnout is more than 50%. If on the one hand, the number of voters is smaller than the number of abstainers, then extra leaders are more likely to bring abstainers to vote rather than to steal voters from other leaders. If on the other hand, in expectation there are more voters than abstainers, then new leaders tend more to steal voters from each other rather than bringing abstainers to vote. In the latter case, the number of votes for A or B, i.e. the random size of the sum of the cluster sizes of party A or B tends to stabilize more (lower variance). This decreases the difference between the aggregate voter shares, i.e. increases the closeness of the election.

6. Extensions: Heterogeneity

This model assumed ex-ante identical agents except for their party preference. It can be extended to allow for different voting costs v and different gains from winning G (or equivalently different costs of being leader C), under some regularity assumptions.

6.1. **Different Voting Costs.**

Assume that voting costs are heterogeneous and distributed according some continuous pdf

$$d(v) \quad \text{with } v \in [\underline{v}, \overline{v}]$$

with $\underline{v} > 0$. In this case all F_O followers still abstain, and for a given compensation c, F_A and F_B followers vote when their voting cost is lower than the compensation promised by leaders and abstain otherwise

$$\begin{array}{cl} \underline{V}_A \text{ or } \underline{V}_B & \text{if } v < c \\ A & \text{if } v > c \end{array}$$

As a result the pivotality calculation of leaders is the same and so is their equilibrium number. The fact that the results do not change is clear if you think that followers are never pivotal when there is at least one leader affecting the turnout. Only the turnout number for the homogeneous case needs to be scaled down in the heterogeneous case by the factor $D(c)$ (the CDF evaluated at c), that is, the fraction of people that vote in every leader's cluster.

Note that the citizens with low voting cost that become F_A or F_B followers may obtain a positive net benefit from voting for a leader. This raises their expected reservation value from being followers and biases the first stage leader-follower decision towards being followers. Citizens with low voting costs are more likely to be followers than leaders (unless we assume as we did that leaders reward themselves the same way too for voting). In either case these different reservation values do not change the results. They can be seen as a special case of the heterogeneity of gains from leadership that we illustrate in the next section.

6.2. **Different Benefits.** Assume that citizens may have different benefits from the outcome of the election (or different costs of leadership) distributed according to come continuous pdf

$$z(G) \quad \text{with } G \in \left[\underline{G}, \overline{G}\right]$$

Recall that the decision to become a leader depends on the pivotality cost-benefit calculation

$$\mathrm{Pv}(A, A) > \frac{C}{G} \equiv g$$

with $g \in \left[\underline{g}, \overline{g}\right]$. As long as the above is true for some value of g there will be additional citizens that become leaders. This process stops when the LHS, which does not depend on the parameter g, crosses the threshold $\underline{g}$, more precisely when

$$\mathrm{Pv}_A(A, A) > \underline{g} > \mathrm{Pv}_A(A+1, A)$$

The integer A that satisfies the above condition, defines the unique class of equilibria in this heterogeneous case. If the above inequality is not true, we can always find some citizens willing to become leader. So there is no difference with the homogeneous case and all the results and comparative statics follow through. There may be some inefficiency though since the chosen leaders not necessarily are the citizens with lower costs and higher benefits. This is due to the discrete nature of the leaders, all and only the citizens with

$$g \in \left[\underline{g}, \mathrm{Pv}_A(A, A)\right]$$

may become leaders in this heterogeneous case.

6.3. **Different Effectiveness of Leaders.** We assumed so far that the region of influence or cluster of a leader was random but the effectiveness of the leaders of each party was the same. Namely for any given region of influence we assumed the leader could attract all the agents in that region to vote for his party. More generally we can assume that leaders of party/issue A (B) can attract only a fraction $\alpha \in [0,1]$ ($\beta \in [0,1]$) of the potential voters of his region of influence. These setup is general and allows for proportions α and β to depend on the original preferences of the voters. For instance, an A leader could attract all A partisans only within his cluster or all A partisans and all non-partisans within his cluster or all agents within his cluster like we assumed in the benchmark model, and so forth. For $\alpha = \beta \in [0,1]$ the equilibrium number of leaders of the model is the one we previously obtained: only the turnout should be scaled down accordingly if $\alpha = \beta < 1$. In general if leaders of different parties have different effectiveness $\alpha \neq \beta$, then the equilibrium is no longer necessarily symmetric and the equilibrium number of leaders for each party need not be the same. The model extends as follows.

Proposition 13. *The pivotality of leaders is*

$$Pv_A = \gamma \left((1-\gamma)^{A-1} \gamma^{B-1} \frac{(A+B-2)!}{(A-1)!\,(B-1)!} \right)$$

$$Pv_B = (1-\gamma) \left((1-\gamma)^{A-1} \gamma^{B-1} \frac{(A+B-2)!}{(A-1)!\,(B-1)!} \right)$$

$$\gamma = \frac{\alpha}{\alpha+\beta}$$

Proof. See Appendix. □

The constant ratio of the pivotalities

$$\delta = \frac{\mathrm{Pv}_B\,(A,B)}{\mathrm{Pv}_A\,(A,B)} = \frac{1-\gamma}{\gamma}$$

is a measure of the advantage of B relative to A. Namely, if $\delta > 1$ then party B has a relative electoral advantage. Then

$$\begin{aligned}
\frac{\mathrm{Pv}_A\,(A,B)}{\mathrm{Pv}_A\,(A-1,B)} &= (1-\gamma)\left(1+\frac{B-1}{A-1}\right) \\
\frac{\mathrm{Pv}_B\,(A,B)}{\mathrm{Pv}_B\,(A,B-1)} &= \gamma\left(1+\frac{A-1}{B-1}\right)
\end{aligned}$$

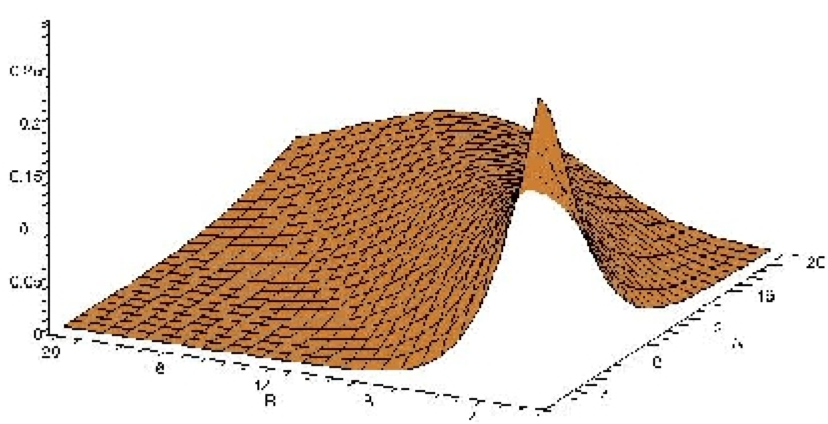
So

$$\begin{array}{rclcl} \mathrm{Pv}_A\left(A,B\right) & \gtreqless & \mathrm{Pv}_A\left(A-1,B\right) & \iff & \left(A-1\right)\lesseqgtr\delta\left(B-1\right) \\ \mathrm{Pv}_B\left(A,B\right) & \gtreqless & \mathrm{Pv}_B\left(A,B-1\right) & \iff & \left(A-1\right)\gtreqless\delta\left(B-1\right) \end{array}$$

That is the pivotality of A increases when $A < \delta\left(B-1\right)+1$ and similarly for B. Both pivotalities peak around the value where the number of leaders for each party are in the relation

$$\frac{A-1}{B-1}=\delta$$

Note that where both pivotalities peak the party with the electoral disadvantage has more leaders then the party with the advantage, that is pivotalities peak where the election is more likely to be a close election. This can be checked numerically by looking at the probabilities of winning, making sure they are the closest to 1/2 when $\frac{A-1}{B-1}=\delta$. In equilibrium the party with more effective leaders (e.g. party A when $\gamma > \frac{1}{2}$) will have less leaders so that the probability of winning stays close to 50%. and the pivotality peaks. The following picture shows the pivotality when party B has an advantage.



$\gamma = \frac{2}{5}, \quad (A,B)\in(2,20)\times(2,20)$

To have a pure strategy interior equilibrium the two conditions

$$\begin{aligned} \mathrm{Pv}_A(A,B) &> \frac{C}{G} > \mathrm{Pv}_A(A+1,B) \\ \mathrm{Pv}_B(A,B) &> \frac{C}{G} > \mathrm{Pv}_B(A,B+1) \end{aligned}$$

must be satisfied simultaneously, which implies that pivotalities must peak: $\frac{A-1}{B-1} = \delta$. The equilibrium of course depends on the cost benefit ratio and on its heterogeneity, if any.

7. Conclusions

There is no canonical rational choice model of voting in elections with costs to vote. But, as Feddersen point out in his recent survey article [8], while a canonical model does not yet exist, the literature appears to be converging toward a "group-based" model of turnout, in which members participate in elections because they are directly coordinated and rewarded by leaders. This paper is a contribution to this literature in two ways. First, it treats all agents as ex ante identical (except their political inclination) and has leaders self-select endogenously out of this homogeneous population, shedding some light on how these groups of voters can be formed out of the voter population in the first place. How and why people join or identify with their groups is (according to Feddersen) still a major concern that these group-based model have not addressed in a satisfactory way. The second contribution is more technical. This model gives a nice closed form solution, which is desirable because it allows to obtain immediate and intuitive comparative statics results. Moreover, the solution of the problem gives always a *pure strategy SPNE* that pins down uniquely the number of leaders for each party, the expected turnout, and the closeness of the election. Existence of equilibria (let alone uniqueness) is a central problem in this literature, compounded with the fact that the mixed strategy often used present conceptual problems of interpretation in group-based models. This model solves all these technical aspects. Of course, further research is needed to understand better the inner mechanisms of voter group formation. We consider this paper one step in that direction.

8. Appendix

8.1. Proofs.

Proof. 4

WLOG pick one of the $(n+1)$ leaders that are uniformly distributed over the unit circle. Just to start counting from there call that leader 0 and call that point 0. From 0 to 1 the remaining n leaders are distributed uniformly. The size of cluster of the leader at zero (which has the same distribution as the cluster of any other leader) is equal to the coordinate of the lowest of the remaining leaders, that is, it is distributed as the first order statistic of (n) iid uniform draws on the unit line

$$n\left(1-a\right)^{n-1} \qquad 0 \leq a \leq 1$$

Similarly the cumulative cluster size of k adjacent leaders (WLOG the first k leaders including 0) is equal to the coordinate of leader k or the k-th order statistic.

$$\frac{n!}{(k-1)!\,(n-k)!}a^{k-1}\left(1-a\right)^{n-k} \qquad 0 \leq a \leq 1$$

The cumulative cluster size of k non-adjacent leaders is distributed in the same way as above because the underlying distribution is uniform (see Corollary 3). If the total number of leaders is $(n+1) = A+B$ then , the cumulative cluster size of $k = A$ of them is distributed as in the statement of the theorem. $\square$

Proof. 5

The joint pdf of two order statistics of order n for a uniform underlying distribution is

$$\begin{aligned} f\left(a_i, a_j\right) &= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\left(a_i\right)^{i-1}\left(a_j-a_i\right)^{j-i-1}\left(1-a_j\right)^{n-j} \\ 0 &\leq a_i < a_j \leq 1 \end{aligned}$$

In this case, reordering the clusters (see Corollary 3) so that there are first A leaders then B leaders, the cluster sizes are

$$\begin{aligned} a &= a_i, \quad b = a_j - a_i \\ A &= i, \quad A+B = j, \quad A+B+O = n+1 \end{aligned}$$

Hence we have

$$h_{(A,B,O)}\left(a,b\right) = \frac{(A+B+O-1)!}{(A-1)!\,(B-1)!\,(O-1)!}a^{(A-1)}b^{(B-1)}\left(1-a-b\right)^{(O-1)}$$

$\square$

Proof. 6

Calculate the inner integrals first obtaining

$$\int_0^a h_{(A,B,O)} dbda = \left(\begin{array}{c} \frac{(A+B+O-1)!}{(A-1)!(O+B-1)!} a^{A-1} (1-a)^{O+B-1} - \\ -\frac{(A+B+O-1)!}{(A-1)!} \left(\sum_{k=1}^{B} \frac{a^{A-1} a^{B-k} (1-2a)^{O-1+k}}{(B-k)!(O-1+k)!} \right) \end{array} \right)$$

$$\int_0^{1-a} h_{(A,B,O)} dbda = \frac{(A+B+O-1)!}{(A-1)!\,(O+B-1)!} a^{A-1} (1-a)^{O+B-1}$$

Hence

$$\begin{aligned} \mathrm{Pw}_A(A,B) &= 1 - \frac{(A+B+O-1)!}{(A-1)!} \sum_{k=1}^{B} \left(\int_0^{\frac{1}{2}} \frac{a^{A+B-k-1} (1-2a)^{O-1+k}}{(B-k)!\,(O-1+k)!} da \right) \\ &= 1 - \sum_{k=1}^{B} \left(\frac{1}{2} \right)^{A+B-k} \frac{(A+B-k-1)!}{(A-1)!\,(B-k)!} \end{aligned}$$

Because integrating by parts iteratively we obtain

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} \frac{a^{A+B-k-1} (1-2a)^{O-1+k}}{(O-1+k)!} da \\ &= \left(\frac{1}{2} \right) (A+B-k-1) \int_0^{\frac{1}{2}} \frac{a^{A+B-k-2} (1-2a)^{O+k}}{(O+k)!} da \\ &= \left(\frac{1}{2} \right)^{A+B-k-1} (A+B-k-1)! \int_0^{\frac{1}{2}} \frac{(1-2a)^{O+A+B-2}}{(O+A+B-2)!} da \\ &= \frac{1}{(A+B+O-1)!} \left(\frac{1}{2} \right)^{A+B-k} (A+B-k-1)! \end{aligned}$$

That is

$$\int_0^{\frac{1}{2}} \left(\frac{a^{A+B-k-1} (1-2a)^{O-1+k}}{(B-k)!\,(O-1+k)!} \right) da = \frac{\left(\frac{1}{2}\right)^{A+B-k}}{(A+B+O-1)!} \frac{(A+B-k-1)!}{(B-k)!}$$

□

Proof. 7

When $O = 0$, the distribution is a univariate and the probability of winning is the probability that $a > 0.5$

$$\begin{aligned} \mathrm{Pw}_A(A,B,0) &= \frac{(A+B-1)!}{(A-1)!\,(B-1)!} \int_{\frac{1}{2}}^{1} a^{A-1} (1-a)^{B-1} da \\ \mathrm{Pw}_A(A,B,1) &= \frac{(A+B)!}{(A-1)!\,(B-1)!} \int_0^{\frac{1}{2}} \int_b^{1-b} a^{A-1} b^{B-1} dadb \end{aligned}$$

Need to show that:

$$\begin{aligned}(A+B)\int_0^{\frac{1}{2}}\int_b^{1-b} a^{A-1}b^{B-1}dadb &= \int_{\frac{1}{2}}^1 a^{A-1}(1-a)^{B-1}da \\ &= \int_0^{\frac{1}{2}} a^{B-1}(1-a)^{A-1}da\end{aligned}$$

The LHS is

$$(A+B)\int_0^{\frac{1}{2}}\int_b^{1-b} a^{A-1}b^{B-1}dadb = \frac{A+B}{A}\int_0^{\frac{1}{2}} b^{B-1}\left((1-b)^A - b^A\right)db$$

Hence subtracting the RHS

$$\begin{aligned}&\frac{A+B}{A}\int_0^{\frac{1}{2}} b^{B-1}\left((1-b)^A - b^A\right)db - \int_0^{\frac{1}{2}} a^{B-1}(1-a)^{A-1}da \\ =\ & \frac{1}{A}\int_0^{\frac{1}{2}}\left((A+B)\left(a^{B-1}(1-a)^A - a^{A+B-1}\right) - Aa^{B-1}(1-a)^{A-1}\right)da \\ =\ & \frac{1}{A}\left(\int_0^{\frac{1}{2}}\left((A+B)\left(a^{B-1}(1-a)^A\right) - Aa^{B-1}(1-a)^{A-1}\right)da - \left(\frac{1}{2}\right)^{A+B}\right) \\ =\ & \frac{1}{A}\left(\int_0^{\frac{1}{2}}\left(Ba^{B-1}(1-a)^A - Aa^B(1-a)^{A-1}\right)da - \left(\frac{1}{2}\right)^{A+B}\right) \\ =\ & 0\end{aligned}$$

Where in the last step we integrating by parts the second term obtaining

$$\int_0^{\frac{1}{2}} Aa^B(1-a)^{A-1}da = \left(\frac{1}{2}\right)^{A+B} + \int_0^{\frac{1}{2}} Ba^{B-1}(1-a)^A da$$

□

Proof. 8

By definition

$$\mathrm{Pv}_A(A,B) \equiv \mathrm{Pw}_A(A,B) - \mathrm{Pw}_A(A-1,B)$$

and from proposition (6) we have

$$\mathrm{Pw}_A(A,B) = 1 - \sum_{k=1}^{B}\left(\frac{1}{2}\right)^{A+B-k}\frac{(A+B-k-1)!}{(A-1)!\,(B-k)!}$$

Hence

$$\mathrm{Pv}_A(A,B) = \sum_{k=1}^{B}\left(\frac{1}{2}\right)^{A+B-k}\frac{(A+B-k-2)!}{(A-1)!\,(B-k)!}(A-B+k-1)$$

So we are left to prove the following identity

$$\sum_{k=1}^{B}\left(\frac{1}{2}\right)^{A+B-k}\frac{(A+B-k-2)!}{(A-1)!\,(B-k)!}(A-B+k-1)=\frac{1}{2^{A+B-1}}\frac{(A+B-2)!}{(A-1)!\,(B-1)!}$$

The identity is equivalent to

$$\begin{aligned} S(A,B,k) &= (2)^{k-1}\frac{(A+B-k-2)!}{(A+B-2)!}\frac{(B-1)!}{(B-k)!}(A-B+k-1) \\ \sum_{k=1}^{B}S(A,B,k) &= 1 \end{aligned}$$

Define $G(A,B,k)$ as

$$\begin{aligned} R(A,B,k) &= -\frac{A+B-k-1}{A-B+k-1} \\ G(A,B,k) &= R(A,B,k)S(A,B,k) \end{aligned}$$

Then

$$G(A,B,k)=-(2)^{k-1}\frac{(A+B-k-1)!}{(A+B-2)!}\frac{(B-1)!}{(B-k)!}$$

With simple algebra you can check that

$$\begin{aligned} S(A,B,k) &= G(A,B,k+1)-G(A,B,k) \quad &\text{for } k=1,...,B-1 \\ S(A,B,B) &= -G(A,B,B) \quad &\text{for } k=B \end{aligned}$$

Hence

$$\sum_{k=1}^{B}S(A,B,k)=-G(A,B,1)=1$$

□

Proof. 11 Any interior equilibrium $(A.B)$ must satisfy simultaneously

$$\begin{aligned} \mathrm{Pv}_A(A,B) &> \frac{C}{G}>\mathrm{Pv}_A(A+1,B) \\ \mathrm{Pv}_B(A,B) &> \frac{C}{G}>\mathrm{Pv}_B(A,B+1) \end{aligned}$$

which implies that the pivotality functions must be decreasing. By proposition (9) this implies both $A \geq B$ and $B \geq A$. Hence we must look for interior equilibria of the form (A,A). From proposition (8) it is always the case that

$$\mathrm{Pv}_A(A,A)>\mathrm{Pv}_A(A+1,A+1)=\mathrm{Pv}_A(A+1,A)$$

So for a given fixed cost we can find some $\overline{A}$

$$\mathrm{Pv}_A(\overline{A},\overline{A})>\frac{C}{G}>\mathrm{Pv}(\overline{A}+1,\overline{A}+1)=\mathrm{Pv}_A(\overline{A}+1,\overline{A})$$

Given that there are $(\overline{A}, \overline{A})$ leaders nobody else wants to be a leader of party A. And no leader of party A wants to be a follower. The same calculations and reasoning holds for the pivotality of B.

We are left to check if there are corner equilibria, i.e. equilibria of the form $(0, B)$, or equivalently $(A, 0)$. For this to be the case we need the following conditions to be satisfied

$$\begin{aligned}\mathrm{Pv}_B(0, B) &> \frac{C}{G} > \mathrm{Pv}_B(0, B+1)\\ \mathrm{Pv}_A(1, B) &< \frac{C}{G}\end{aligned}$$

From the property (1) the above conditions are equivalent to

$$\begin{aligned}\mathrm{Pv}_A(0, B) &> \frac{C}{G} > \mathrm{Pv}_A(0, B+1)\\ \mathrm{Pv}_A(1, B) &< \frac{C}{G}\end{aligned}$$

From the monotonicity result (9) we have

$$\mathrm{Pv}_A(1, B) > \mathrm{Pv}_A(0, B)$$

so the property can never be satisfied.

Note that no leaders is the unique equilibrium when

$$\frac{C}{G} \geq \frac{1}{2}$$

because in that case the probability of being pivotal is one half, assuming the election is decided by a fair coin toss in the case of no leaders.

Hence $(\overline{A}, \overline{A})$ is the unique class of Nash Equilibria. □

Proof. 13

The proof is divided into four parts or lemmas. In the first two lemmas we show the result

$$\begin{aligned}\mathrm{Pv}_A(A, B) &= \gamma \left((1-\gamma)^{A-1} \gamma^{B-1} \frac{(A+B-2)!}{(A-1)!\,(B-1)!}\right)\\ \mathrm{Pv}_B(A, B) &= (1-\gamma) \left((1-\gamma)^{A-1} \gamma^{B-1} \frac{(A+B-2)!}{(A-1)!\,(B-1)!}\right)\end{aligned}$$

for $O = 0$. In the last two lemmas we show that the result holds for any $O = 0, 1, 2, ...$ by showing that the probability of winning of any party Pw does not depend on O. □

Lemma 14. *The Case $O = 0$. The probability of winning of party A when $O = 0$ is*

$$Pw_A(A,B,0) = \frac{(A+B-1)!}{(A-1)!\,(B-1)!}\int_{\frac{\beta}{\alpha+\beta}}^{1} a^{A-1}(1-a)^{B-1}\,da$$

Integrating by parts we obtain

$$\begin{aligned}
&= \frac{(A+B-1)!}{(A-1)!\,(B-1)!}\int_{\frac{\beta}{\alpha+\beta}}^{1} a^{(A-1)}(1-a)^{(B-1)}\,da \\
&= \frac{(A+B-1)!}{(A)!\,(B-2)!}\int_{\frac{\beta}{\alpha+\beta}}^{1} a^{A}(1-a)^{B-2}\,da + \frac{(A+B-1)!}{(A)!\,(B-1)!}\left(\frac{\beta}{\alpha+\beta}\right)^{A}\left(\frac{\alpha}{\alpha+\beta}\right)^{B-1}
\end{aligned}$$

and integrating by parts iteratively we obtain the sum

$$Pw_A(A,B) = \sum_{k=1}^{A}\frac{(A+B-1)!}{(A-k)!\,(B+k-1)!}(1-\gamma)^{A-k}\gamma^{B+k-1}$$

where

$$\gamma = \frac{\alpha}{\alpha+\beta}$$

Lemma 15. *We need to show that for $O = 0$ we have*

$$\begin{aligned}
Pv_B(A,B) &= Pw_B(A,B) - Pw_B(A,B-1) \\
&= (1-\gamma)\left((1-\gamma)^{A-1}\gamma^{B-1}\frac{(A+B-2)!}{(A-1)!\,(B-1)!}\right)
\end{aligned}$$

The probability of winning is

$$\begin{aligned}
Pw_A(A,B) &= 1 - \sum_{k=1}^{B} H(B,k) = 1 - Pw_B(A,B) \\
H(B,k) &= \frac{(A+B-1)!}{(A+k-1)!\,(B-k)!}(1-\gamma)^{A+k-1}\gamma^{B-k}
\end{aligned}$$

Hence

$$\begin{aligned}
&= H(B,k) - H(B+1,k) \\
&= -\frac{(A+B)!}{(A+k-1)!\,(B+1-k)!}(1-\gamma)^{A+k-1}\gamma^{B+1-k} \\
&\quad +\frac{(A+B-1)!}{(A+k-1)!\,(B-k)!}(1-\gamma)^{A+k-1}\gamma^{B-k} \\
&= \frac{(A+B-1)!}{(A+k-1)!\,(B-k)!}(1-\gamma)^{A+k}\gamma^{B-k} \\
&\quad -\frac{(A+B-1)!}{(A+k-2)!\,(B-k+1)!}(1-\gamma)^{A+k-1}\gamma^{B-k+1} \\
&= D(B,k+1) - D(B,k)
\end{aligned}$$

where

$$D(B,k) = \frac{(A+B-1)!}{(A+k-2)!\,(B-k+1)!}(1-\gamma)^{A+k-1}\gamma^{B-k+1}$$

Note that summing the terms we have a telescopic sum on the RHS

$$\begin{aligned}
\sum_{k=1}^{B} H(B,k) - \sum_{k=1}^{B} H(B+1,k) &= \sum_{k=1}^{B}\left(D(B,k+1) - D(B,k)\right) \\
Pw_B(A,B) - \left(Pw_B(A,B+1) - H(B+1,B+1)\right) &= D(B,B+1) - D(B,1)
\end{aligned}$$

That is

$$\begin{aligned}
H(B+1,B+1) - Pv_B(A,B+1) &= D(B,B+1) - D(B,1) \\
(1-\gamma)^{A+B} - Pv_B(A,B+1) &= (1-\gamma)^{A+B} - \frac{(A+B-1)!}{(A-1)!\,(B)!}(1-\gamma)^{A}\gamma^{B}
\end{aligned}$$

Hence

$$Pv_B(A,B+1) = (1-\gamma)^{A}\gamma^{B}\frac{(A+B-1)!}{(A-1)!B!}$$

To obtain $Pv_A(A,B)$ (or in general Pw_A from Pw_B) given the symmetry of the problem, you can switch A with B and γ with $(1-\gamma)$ (α with β).

Lemma 16. *We show first that*

$$Pw_A(A,B,O=1) = Pw_A(A,B,O=0)$$

By straightforward calculation we have

$$
\begin{aligned}
&= Pw_A(A,B,1) = 1 - Pw_B(A,B,1) \\
&= \frac{(A+B)!}{(A-1)!\,(B-1)!}\left(\int_0^{\frac{\beta}{\alpha+\beta}} a^{(A-1)}\int_0^{\frac{\alpha}{\beta}a} b^{(B-1)}dbda + \int_{\frac{\beta}{\alpha+\beta}}^1 a^{(A-1)}\int_0^{1-a} b^{(B-1)}dbda\right) \\
&= \frac{(A+B)!}{(A-1)!\,(B)!}\left(\int_0^{\frac{\beta}{\alpha+\beta}} a^{(A-1)}\left(\frac{\alpha}{\beta}a\right)^B da + \int_{\frac{\beta}{\alpha+\beta}}^1 a^{(A-1)}(1-a)^B da\right) \\
&= \frac{(A+B)!}{(A-1)!\,(B)!}\left(\frac{\alpha}{\beta}\right)^B\int_0^{\frac{\beta}{\alpha+\beta}} a^{(A+B-1)}da + Pw_A(A,B+1,0) \\
&= \frac{(A+B-1)!}{(A-1)!\,(B)!}\left(\frac{\alpha}{\beta}\right)^B\left(\frac{\beta}{\alpha+\beta}\right)^{A+B} + Pw_A(A,B+1,0) \\
&= \frac{(A+B-1)!}{(A-1)!\,(B)!}\gamma^B(1-\gamma)^A + (1 - Pw_B(A,B+1,0))
\end{aligned}
$$

Thus, it must be the case that

$$Pw_B(A,B+1,0) - Pw_B(A,B,1) = \frac{(A+B-1)!}{(A-1)!\,(B)!}\gamma^B(1-\gamma)^A$$

But since by lemma (2) we have

$$\frac{(A+B-1)!}{(A-1)!\,(B)!}\gamma^B(1-\gamma)^A = Pw_B(A,B+1,0) - Pw_B(A,B,0)$$

Then

$$Pw_B(A,B,1) = Pw_B(A,B,0)$$

Lemma 17. *It is left to show that for $O = 1, 2, ...$*

$$Pw_A(A,B,O) = Pw_A(A,B,O+1)$$

Integrating by parts we have $Pw_A(A,B,O) =$

$$
\begin{aligned}
&\frac{(A+B+O-1)!}{(A-1)!\,(B-1)!\,(O-1)!}\int_0^{\frac{\alpha}{\alpha+\beta}} b^{B-1}\int_{\frac{\beta}{\alpha}b}^{1-b} a^{A-1}(1-a-b)^{O-1}\,dadb \\
&= \left(\begin{array}{c} \frac{(A+B+O-1)!}{A!(B-1)!(O-2)!}\int_0^{\frac{\alpha}{\alpha+\beta}} b^{B-1}\int_{\frac{\beta}{\alpha}b}^{1-b} a^A(1-a-b)^{O-2}\,dadb - \\ \frac{(A+B+O-1)!}{A!(B-1)!(O-1)!}\left(\frac{\beta}{\alpha}\right)^A\int_0^{\frac{\alpha}{\alpha+\beta}} b^{A+B-1}\left(1-\frac{\alpha+\beta}{\alpha}b\right)^{O-1}db \end{array}\right)
\end{aligned}
$$

Now the first term is

$$
\begin{aligned}
&\frac{(A+B+O-1)!}{A!\,(B-1)!\,(O-2)!}\int_0^{\frac{\alpha}{\alpha+\beta}} b^{B-1}\int_{\frac{\beta}{\alpha}b}^{1-b} a^A(1-a-b)^{O-2}\,dadb \\
&= Pw_A(A+1,B,O-1)
\end{aligned}
$$

The second term after iterated integration by parts is

$$\begin{aligned}
& \frac{(A+B+O-1)!}{A!\,(B-1)!\,(O-1)!}\left(\frac{\beta}{\alpha}\right)^{A}\int_{0}^{\frac{\alpha}{\alpha+\beta}} b^{A+B-1}\left(1-\frac{\alpha+\beta}{\alpha}b\right)^{O-1} db \\
= & \frac{(A+B+O-1)!}{A!\,(B-1)!\,(O-2)!\,(A+B)}\left(\frac{\beta}{\alpha}\right)^{A}\left(\frac{\alpha+\beta}{\alpha}\right)\int_{0}^{\frac{\alpha}{\alpha+\beta}} b^{A+B}\left(1-\frac{\alpha+\beta}{\alpha}b\right)^{O-2} db \\
= & \frac{(A+B+O-1)!}{A!\,(B-1)!\,(A+B)\ldots(A+B+O-2)}\left(\frac{\beta}{\alpha}\right)^{A}\left(\frac{\alpha+\beta}{\alpha}\right)^{O-1}\int_{0}^{\frac{\alpha}{\alpha+\beta}} b^{A+B+O-2} db \\
= & \frac{(A+B-1)!}{A!\,(B-1)!}(1-\gamma)^{A}\gamma^{B} \\
= & Pv_{A}(A+1,B,1) = Pw_{A}(A+1,B,1) - Pw_{A}(A,B,1)
\end{aligned}$$

Thus for $O = 1, 2, \ldots$

$Pw_{A}(A,B,O) = Pw_{A}(A+1,B,O-1) - (Pw_{A}(A+1,B,1) - Pw_{A}(A,B,1))$

Now proceed by induction. For $O = 2$*, we have for all* A *and* B

$$Pw_{A}(A,B,2) = Pw_{A}(A,B,1)$$

Hence substituting this relation in the previous one we obtain

$Pw_{A}(A,B,O) = Pw_{A}(A+1,B,O-1) - (Pw_{A}(A+1,B,2) - Pw_{A}(A,B,2))$

So for $O = 3$*, we have*

$$Pw_{A}(A,B,3) = Pw_{A}(A,B,2)$$

and so forth for all values of O*.*

8.2. **Normalization.**

Inner integral. Integrally on the b variable:

$$\begin{aligned}
& \int_{0}^{1-a} b^{B-1}(1-a-b)^{O-1} db \\
= & -\int_{0}^{1-a}\left(-(B-1)\,b^{B-2}\frac{(1-a-b)^{O}}{O}\right) db \\
= & \int_{0}^{1-a}(B-1)(B-2)\,b^{B-3}\frac{(1-a-b)^{O+1}}{O(O+1)} db \\
= & \int_{0}^{1-a}(B-1)!\frac{(1-a-b)^{O+B-2}}{(O+B-2)\ldots(O+1)\,O} db \\
= & \frac{(B-1)!}{\frac{(O+B-2)!}{(O-1)!}}\int_{0}^{1-a}(1-a-b)^{O+B-2} db
\end{aligned}$$

Resulting in the Inner integral

$$\int_0^{1-a} b^{B-1}\,(1-a-b)^{O-1}\,db = \frac{(B-1)!\,(O-1)!}{(O+B-1)!}\,(1-a)^{O+B-1}$$

Outside Integral. Finally the integral over a :

$$\int_0^1 a^{A-1}\,(1-a)^{O+B-1}\,da = \frac{(A-1)!}{\frac{(O+B+A-2)!}{(O+B-1)!}}\int_0^1 (1-a)^{O+B+A-2}\,da$$

Resulting in:

$$\int_0^1 a^{A-1}\,(1-a)^{O+B-1}\,da = \frac{(A-1)!\,(O+B-1)!}{(O+B+A-1)!}$$

Hence, the integral without the normalization coefficient yields the inverse of the normalization coefficient

$$\frac{(B-1)!\,(O-1)!}{(O+B-1)!}\frac{(A-1)!\,(O+B-1)!}{(O+B+A-1)!} = \frac{1}{\left(\frac{(A+B+O-1)!}{(A-1)!(B-1)!(O-1)!}\right)}$$

So the integral is normalized to one.

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