Optimal Auction Design For Multiple Objects with Externalities *

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Abstract

In this paper we characterize the optimal allocation mechanism for N objects, (permits), to I potential buyers, (firms). Firms' payoffs depend on their costs, the costs of competitors and on the final allocation of the permits, allowing for externalities, substitutabilities and complementarities. Firms' cost parameter is private information and is independently distributed across firms. Due to fact that there are multiple objects to be allocated and buyers valuations are non-linear the problem fails to be "regular," even if the monotone hazard rate property is satisfied. Moreover the standard ironing technique does not apply. Our first insight is to develop a new method to solve for the optimal mechanism for the "general case." In this case the optimal mechanism may require randomization between allocations. Externalities in our model are type dependent. This has two consequences: first, even though the private information of each firm is one dimensional (its cost), an allocation's virtual valuation depends on the cost parameters of all firms. Second, the "critical"type of each buyer, (the type for which

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participation constraint binds) is not exogenously given but depends on the particular mechanism selected. Our model captures key features of many important multi-object allocation problems like the allocation of time slots for TV commercials, landing slots in airports, privatization and firm takeovers. Keywords: *Optimal Auctions, Multiple Objects, Externalities, Mechanism Design: JEL D44, C7, C72.*

1 Introduction

In this paper we characterize the optimal allocation mechanism for N objects (permits), to I potential buyers (firms). Firms' payoffs depend on their costs, the costs of competitors and on the final allocation of the permits, allowing for externalities, substitutabilities and complementarities. A firm cares not only whether it obtains a particular set of permits, but also cases about who obtained which licence. Firms' cost parameter is private information and is independently distributed across firms. Externalities are type dependent.

The model of this paper can be thought as follows: There is a revenue-maximizing seller [1](the government) trying to sell permits for operating in a certain market to some potential buyers. These permits represent a right to participate in the market. The profits of a given firm depend on three things: its own marginal cost, which is private information, the market structure and the marginal costs of the competitors that also participate in the market. After the permits have been allocated, the firms that got one or more of them face a perfectly anticipated demand and engage in some sort of oligopolistic competition. We can also allow for the possibility that these firms are already competing in different markets, so even if they do not get permits assigned, who gets the permits will affect their profits. The presence of such externalities allows the seller to extract extra payments from any given firm, just by threatening to setup a very damaging market structure in case it does not participate in the process

In a large variety of multi-object allocation problems the presence of externalities is of central role. Our model with small modifications can help address the following problems.

[1]The objective function can be modified to take into account the possibility that the government cares also for consumer surplus.

• **Firm Take-overs:** Externalities are of huge importance in firm take-overs: Recently (February 2004), Cingular bought AT&T wireless for $41 billion after a bidding war with Vodafone. Some perceive that the big winner of this sale will be Verizon even though it was not a participant in the auction (NY Times February 17, 2004 "Verizon Wireless May Benefit From Results of Auction").

• **Allocation of Airport Take-Off and Landing Slots.** Airport take-off and landing slots are a scarce resource yet not priced! There are important externalities since for instance if two airlines are fierce competitors in a big airport say United and American at O'Hare, then if United obtains critical landing slots in LAX, (Los Angeles International Airport), this may well affect its market position in O'Hare vis-a-vis American.

• **Auctioning of time slots for advertisements on TV, radio.** In reality airtime for advertisements is priced using conventional mechanisms, whereas if networks take-into account the presence of externalities and auction-off the time slots, we might end up with less (even zero) airtime of advertisement yet higher revenue. How much would a firm pay so that its fiercest competitor does not advertise in the intermission of Super-ball? One can imagine a network asking this question to Miller, Budweiser, Bud etc. Taking this to an extreme there may be a potential for a lot of revenue with actually no one airing a spot. In other words strongly opposed interests may permit the seller to extract payments just for doing nothing ![2]

• **Privatization - Mechanism Design with Endogenous Market Structure.** . Our model captures such scenarios and generalizes previous work by Dana and Spier (1994), who examine whether the government should sell a firm in one piece or cut it into two, (for a discussion on this, see Milgorm (1996)).[3] In the work of Dana and Spier (1994) the outcome of the mechanism depends heavily on the weight that the government assigns to revenue versus efficiency and on the type of competition that prevails in the market after privatization.

• **Selling licences for cellular networks, TV or radio broadcasting.**

• **Optimal Bundling:** Since we allow for complementarities and substitutability one can think

[2] This is similar to the common agency problem, as identified for example in Dixit (1997).

[3] Gale (1991) also considers a variation of this problem but because he imposes a very strong super-additivity condition to the profit function, he shows that an optimal mechanism always gives all the "permits" to at most one buyer, so the market structure is always the one of a monopoly.

of applications for cases of bundling of goods, (bundles like telecommunication services - internet - cable TV, computers-printers-software-digital cameras etc).

This paper is related to the optimal auction literature for multiple objects and to the literature of mechanism design with externalities. Maskin and Riley (1989) analyze the case of single dimensional private information and continuously divisible goods, Armstrong (2000) allows for multidimensional uncertainty but there are only two buyers and two types. Jehiel, Moldovanu and Stacchetti, (JMS) (1996) study optimal auction design in the presence of externalities in a *single* unit environment where externalities are type *independent*.[4] Because of the presence of externalities the seller can extract payment for the losers but the revenue maximizing allocation of the object is the same as in the case of the revenue maximizing auction without taking into account the presence of externalities. JMS (2001) consider again the design of optimal auctions of a single object in the presence of externalities. Here the externalities are type dependent: the type of each buyer is a vector of numbers that determines his/her utility as a function of who gets the object. The multi-dimensionality makes the solution of the general problem intractable: it is almost impossible to verify that the set of conditions that are implied by incentive compatibility are satisfied (the allocation rule has to be monotonic and conservative - or path independent).

Our innovation is to allow for multiple objects, general payoff functions that allow for complementarities and substitutabilities and type dependent externalities among buyers, but because private information is single-dimensional we can solve the problem. Due to fact that there are multiple objects to be allocated and buyers valuations are non-linear the problem fails to be "regular," even if the monotone hazard rate property is satisfied. Moreover the standard ironing technique does not apply. Our first insight is to develop a new method to solve for the optimal mechanism for the "general case." In this case the optimal mechanism may require randomization between allocations. Second, even though the private information of each firm is one dimensional (its cost), an allocation's virtual valuation depends on the cost parameters of all other firms. This captures nicely the existence of externalities among buyers: how much money the seller can extract from firm A depends on the technology of firm B, which captures together with other parameters how

[4] General models allowing for type dependent externalities like those of Jehiel-Moldovanu (2001), and Krishna and Perry (2001) are concerned with the design of efficient mechanisms.

strong of a competitor firm B is. As in JMS (1996) and JMS (2001) the critical type, (where the participation constraint binds), of the buyer is not exogenously given but depends on the range of the externalities. But unlike JMS (1996) and (2001) in our approach since we allow for more general payoff functions, the critical type also depends on the actual mechanism. This critical type of each agent determines how much money the seller can extract from the players. Hence the characterization of the optimum becomes intricate: given a mechanism there is a vector of critical types and a amount of payments that the seller can extract from the buyers: the mechanism depends not only on the virtual valuations, but also on which is the critical type. Moreover the vector of critical types is mechanism specific. A consequence of this interrelationship between the critical types and the mechanism is that the optimal allocation of the object in the presence of externalities is different from the one we would obtain with no externalities. In contrast, the presence of externalities in JMS (1996) affects only the payment that the seller can extract from the buyers and not the allocation of the object.

To summarize, the two main insights of our analysis are

1) With Non-Linear valuations the optimal mechanism may randomize between allocations: a new method to obtain the optimal mechanism.

2) With type-dependent externalities "punishments" depend on the allocation that the seller wants to implement: we still obtain revenue equivalence - but now revenue is not any more a linear function of the allocation.

2 The model

A seller owns N indivisible objects that are of 0 value to her and faces I risk-neutral buyers. Both N and I are finite natural numbers. The seller, (indexed by zero), can bundle these N objects in any way she sees fit. An allocation $z = (z_0, z_1, ..., z_I)$ is an assignment of objects to the buyers and to the seller. The set of possible allocations is given by $Z \subseteq [I \cup \{0\}]^N$ and hence Z is finite. Buyer i's valuation of allocation z is $\pi_i(z, c_i, c_{-i})$, which is indexed by $i's$ type c_i and by the types of all the other buyers c_{-i}. Buyer $i's$ type c_i is distributed on $C_i = [\underline{c_i}, \overline{c_i}]$, with $0 \leq \underline{c_i} \leq \overline{c_i} < \infty$ according to a distribution F_i that has has a strictly positive and continuous f_i. All buyers' types are independently distributed. We use

$$f_{-i}(c_{-i}) = f_1(c_1) \times f_2(c_2)...f_{i-1}(c_{i-1}) \times f_{i+1}(c_{i+1})...f_I(c_I)$$

and

$$f(c) = f_1(c_1)...f_I(c_I), \text{ where } c \in C = C_1 \times C_2 \times .. \times C_I.$$

We assume that[5]

π_i is decreasing and convex in c_i

$\pi_i(z, \cdot, c_{-i})$ is differentiable for all z and c_{-i}.

This specification makes clear that we are in the context of an auction with externalities, since each buyer cares not only for the objects that are assigned to him, but also for the allocation of the remaining ones. Notice also that we allow $\pi_i(z, c_i, c_{-i}) \neq 0$ even when the allocation z does not include any objects for i, so we can include the cases when the bidders are firms competing in a different markets, and whatever happens in the current sale one will affect their positioning and interaction relative to the other buyers, which will in turn affect their payoffs.

3 Characterization of Feasible Mechanisms

By the revelation principle we know that we can restrict attention to direct revelation mechanisms.

A *direct revelation mechanism* $M = (p, x)$ consists of an *assignment rule* $p : C \longrightarrow \Delta(Z)$ and a *payment rule* $C \longrightarrow \mathbb{R}^I$.

The assignment rule specifies the probability of each allocation for a given a vector of messages. We denote by $p^z(c)$ the probability that allocation z is implemented when the message tuple is c. The payment specifies a vector of expected payments given a vector of reports. For a fixed

[5] Gale's (1990) condition on profit function, would in our notation read as follows:

$$(\forall z \in \partial Z)(\forall z' \in Z/\partial Z)(\forall c \in [\underline{c_i}, \overline{c_i}]) \sum_{i=1}^{I} \pi_i(z, c_i, c_{-i}) \geq \sum_{i=1}^{I} \pi_i(z', c_i, c_{-i})$$

where $\partial Z = \{z \in Z | (\exists i \in \{1, ..., I\}) z_i = N\}$.

mechanism $M = (p, x)$, the ex-ante utility of a firm of type c_i when he participates and declares c_i' is:

$$U_i(c_i, c_i'; (p, x)) = \int_{C_{-i}} \sum_{z \in Z} (p^z(c_i', c_{-i})\pi_i(z, c_i, c_{-i}) - x_i(c_i', c_{-i})) f_{-i}(c_{-i}) dc_{-i}$$

The utility of a buyer if he decides not to participate in the mechanism is given by

$$\underline{U}_i(\rho_i, c_i) = \int_{C_{-i}} \sum_{z \in Z} \rho_i^z \pi(z, c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}$$

where $\rho_i \in \Delta(Z)$ is the (possibly random) allocation that the seller employs if i refuses to participate. The role of these "punishment allocations" will be discussed in the section that follows.

We say that a mechanism (p, x) is feasible iff

$$\begin{aligned} U_i(c_i, c_i; (p, x)) &\geq U_i(c_i, c_i'; (p, x)) \text{ for all } c_i, c_i' \in C_i \\ U_i(c_i, c_i; (p, x)) &\geq \underline{U}_i(\rho_i, c_i) \text{ for all } c_i \in C_i \\ \sum_{z \in Z} p^z(c) &\leq 1, p^z(c) \geq 0 \text{ for all } c \in C \end{aligned}$$

The first set of constraints are the incentive compatibility constraints, the second set of constraints are the voluntary participation constraints and the third set of constraints impose the requirements that probabilities sum up to one and are non-negative numbers. Then the problem of a revenue maximizing seller can be written as

$$\max_{(p,x) \text{ feasible}} \int_C \sum_{i=1}^{I} x_i(c) f(c) dc$$

Now we turn to characterize properties of feasible mechanisms. First let's define

$$P_i(c_i) \equiv \int_{C_{-i}} \sum_{z \in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i}.$$

This the expected "marginal cost" of allocation z. If π_i were increasing in c_i we could then call P_i the expected marginal value of allocation z. The analog of P_i in Myerson (1981) is $P_i(c_i) = \int\limits_{C_{-i}} p(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}$ since there is only one object and $\frac{\partial \pi_i(z,c_i,c_{-i})}{\partial c_i} = 1$.

Given an incentive compatible mechanism (p, x) $i's$ maximized payoff is given by

$$V_i(c_i) = \max_{c_i'} \int\limits_{C_{-i}} (\sum_{z \in Z} p^z(c_i', c_{-i}) \pi_i(z, c_i, c_{-i}) - x(c_i', c_{-i})) f_{-i}(c_{-i}) dc_{-i}.$$

We are now ready to investigate properties of feasible mechanisms.

Lemma 1. *A mechanism (p, x) is feasible iff*

$$P_i(c_i') \geq P_i(c_i) \qquad \textit{for all } c_i' > c_i \tag{1}$$

$$V_i(c_i) = V_i(\overline{c}_i) - \int\limits_{c_i}^{\overline{c_i}} P_i(s) ds \qquad \textit{for all } c_i \in C_i \tag{2}$$

$$V_i(c_i) \geq \underline{U}_i(\rho_i, c_i) \qquad \textit{for all } c_i \in C_i \tag{3}$$

$$p^z(c) \geq 0 \qquad \sum_{z \in Z} p^z(c) \leq 1 \tag{4}$$

Proof. By the convexity of $\pi_i(z, \cdot, c_{-i})$ we have V is a maximum of convex functions, so it is convex, and therefore differentiable a.e. It's also easy to check that the following are equivalent:

(a) (p, x) is incentive compatible

(b) $P_i(c_i) \in \partial V(c_i)$

(c) $U(c_i, c_i; (p, x)) = V(c_i)$

($\Longrightarrow$) Since the mechanism is incentive compatible, from the previous characterization we get that a feasible mechanism must satisfy (b). A result in Krishna and Maenner (1998) then implies (2). By the convexity of V, we know ∂V is monotone, so:

$$(P_i(c_i) - P_i(c_i'))(c_i - c_i') \geq 0$$

This immediately implies (1). Finally, individual rationality is the same as (3).

($\Longleftarrow$)Individual rationality is the same as (3). To prove incentive compatibility it's enough to show that $P_i(c_i) \in \partial V_i(c_i)$. By (1) and (2),

$$\begin{aligned} V(c_i') - V(c_i) &= \int\limits_{c_i}^{c_i'} P_i(s)ds \\ &\geq P_i(c_i)(c_i' - c_i) \end{aligned}$$

which shows $P_i(c_i) \in \partial V_i(c_i)$. $\square$

Lemma 2. *The expected payment of an agent of i can be written as*

$$\int_C x_i(c)f(c)dc = \int\limits_C \sum_{z\in Z} p^z(c_i, c_{-i})[\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i}]f(c)dc - V_i(\overline{c_i}) \tag{5}$$

Proof.

$$\int\limits_C x_i(c)f(c)dc = \int\limits_C [\sum_{z\in Z} p^z(c_i, c_{-i})\pi_i(z, c_i, c_{-i})]f(c)dc - \int\limits_{C_i} V(c_i)dc_i$$

But because of (2), and using changing the order of integration we get:

$$\begin{aligned} \int\limits_{C_i} V(c_i)dc_i &= \int\limits_{C_i} [V(\overline{c_i}) - \int\limits_{c_i}^{\overline{c_i}} P_i(s_i)ds_i]f_i(c_i)dc_i \\ &= V(\overline{c_i}) - \int\limits_{C_i} P_i(s_i)[\int\limits_{\underline{c_i}}^{s_i} f_i(c_i)dc_i]ds \\ &= V(\overline{c_i}) - \int\limits_{C_i} P_i(c_i)F_i(c_i)dc_i \\ &= V(\overline{c_i}) - \int\limits_{C_i}(\int\limits_{C_{-i}} \sum_{z\in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i})F_i(c_i)dc_i \\ &= V(\overline{c_i}) - \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} \frac{F_i(c_i)}{f_i(c_i)} f(c)dc \end{aligned}$$

>From the last expression the result follows. $\square$

4 Analysis of the Problem

In the environment under consideration, each buyer potentially cares about the ultimate allocation of objects even if no objects are assigned to him. The seller can take advantage of the presence of these external effects and extract higher payments by artificially creating unfavorable outside options. For example, consider the auction for licences of nth generation mobile services. Firms bidding in this auction may also operate in the market for the provision of internet services. The seller knows, and can take advantage of, the fact that if one firm does not participate in the auction for the licences, it will be for sure be left out of that market, and it will lose market share in the market of high speed internet connections. This idea appeared for the first time in a mechanism design problem in JMS (1996) and (2001). The mechanism designer has, is some loose sense, the power to impose outside options.

We denoted by $Z_i \subset Z$ denote the subset of allocations that the seller can employ to threaten buyer i [6]. Let ρ_i denote a probability distribution over elements of Z_i, then the seller can then threaten buyer i that he will implement ρ^i if he/she decides not to participate. Put it differently $\rho_i \in \Delta(Z_i)$ specifies the probability of each allocation when firm i decides not to participate in the auction. This is the threat allocation rule: because there are externalities the seller can threat i that in the event that i fails to participate, he will face a very unfavorable allocation. In turn ρ_i determines $\underline{U}_i$

$$\underline{U}_i(\rho_i, c_i) = \int\limits_{C_{-i}} \sum_{z \in Z} \rho_i^z \pi_i(z, c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i},$$

which as we will explain later determines $V_i(\bar{c}_i)$ in (5).

As a side note, we must say that we are making a very strong assumption about the commitment ability of the seller. His threat of choosing a particular allocation in case agent i does not participate is credible only under this assumption. To see how important this is, and a possible approach when

[6] We will assume that $Z_i \subset \{z \in Z | z_j \neq i \text{ for all } j \in \{1, ..., N\}\}$, so the seller cannot force the agent to get some objects

one removes this assumption, see subsection 5.5.

In general the payoff at the outside option, that is $\underline{U}_i(c_i;\rho_i)$, will depend on c_i. This interdependency will make the characterization of the optimal punishment complicated: these punishments $\{\rho_i\}_{i\in I}$ will depend on the particular assignment function p that the seller wants to implement. Apart from this complication the problem under consideration has complications that arise from the fact that there are multiple objects for sale and from the fact that valuations are non-linear. In order to address these complications one at a time we will deal with three cases separately.

In subsection 4.1 we consider the (simpler) cases where, for any threat rule ρ_i, the payoff of buyer i from non-participation does not depend on his type c_i, that is

$$\underline{U}_i(c_i;\rho_i) = \underline{U}_i(\rho_i) \text{ for all } c_i \in C_i.$$

From an economic point of view, we can consider these cases as the ones where the externality is created by the interaction in a *different* market, so it does not depend on the realization of the cost parameter for this particular one. Imagine, for example, the case of firms that already compete in the provision of internet services, and are bidding for permits in the telecom business.

In subsection 4.2 we consider the cases where the payoff of buyer i from non-participation, $\underline{U}_i(c_i;\rho_i)$, can indeed depend on his type c_i. A good example for this environment is the case of advertising. The externality suffered by a candidate because his competitor gets to air a spot depends on his strength.

4.1 The Optimal Mechanism with Type-Independent "Threats"

As we said before, we consider the cases where the payoff of buyer i from non-participation does not depend on his type c_i. More rigorously we have:

$\pi_i(z, c_i, c_{-i}) = \pi_i(z, c_{-i})$ for all $z \in Z_i$

The key consequence is that the optimal threat that the seller employs will be independent of the particular allocation rule p that he wants to implement. This will allow us to separate the optimization problem in two.

Step 1: Determination of the Optimal Punishments

In the case under consideration, where buyer i's payoff from a punishment allocation does not depend on i's type, we can determine $\{\rho_i\}_{i \in I}$ independently of $\{p, x\}$. Remembering that $Z_i \subset Z$ is the set of allocations that the seller can use to punish buyer i we have the following lemma.

Lemma 3. *The optimal punishments* $\{\rho_i\}_{i \in I}$ *are given by*

$$\begin{aligned} \rho_i^{z^i} &= 1 \textit{ for all } i \in I, \textit{ where} \\ z^i &\in \arg\min_{z \in Z_i} \int\limits_{C_{-i}} \pi_i(z, c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}. \end{aligned}$$

Proof. Given such a punishment allocation the payoff of buyer is given by

$$\underline{U}_i(\rho_i^{z^i}) = \int\limits_{C_{-i}} \pi(z^i, c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i},$$

which is by assumption independent of c_i. By the definition of z^i this is the worst possible outside option that the seller can induce for buyer i. □

Step 2: Determination of the Optimal Allocation Rule

Next we describe a simple program whose solution gives an optimal feasible mechanism. Before we do this we need to define the counterpart of virtual valuation for the allocation problem that we are considering here.

The Total Virtual Valuation of allocation z is given by

$$J_z(c) = \sum_{i=1}^{I} [\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i}]$$

Remarks:

1)In Myerson (1981) the concept of virtual valuation is buyer-specific. Letting v_i denote buyer i's valuation of the object, it is given by

$$J_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}.$$

2)In the current problem the concept of virtual valuation is *allocation* specific, and since an allocation may affect all the buyers we have to sum over all buyer's virtual valuations from allocation z.

3) The total virtual valuation of allocation z depends on the whole vector of types.

If in a mechanism $(\widehat{p}, \widehat{x}, \rho)$ the assignment function $\widehat{p}$ solves

$$\max_{p \text{ s.t.}(1),(??)} \int\limits_C \sum_{z \in Z} p^z(c) J_z(c) f(c) dc$$

, the payment function $\widehat{x}$ satisfies

$$\widehat{x}_i(c) = \sum_{z \in Z} \widehat{p}^z(c) \pi_i(z, c_i, c_{-i}) + \int\limits_{c_i}^{\overline{c_i}} \sum_{z \in Z} \frac{\partial \pi_i(z, s, c_{-i})}{\partial s} \widehat{p}^z(s, c_{-i}) ds - V_i(\bar{c}_i),$$

and

$$V_i(\bar{c}_i) = \underline{U}_i(\rho_i^{z^i}),$$

where

$$\rho_i^{z^i} = 1 \text{ iff } z^i \in \arg\min_{z \in Z_i} \int\limits_{C_{-i}} \pi_i(z, c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}$$

then the mechanism is optimal.

Proof. This result follows immediately from lemmas 2 and 3. □

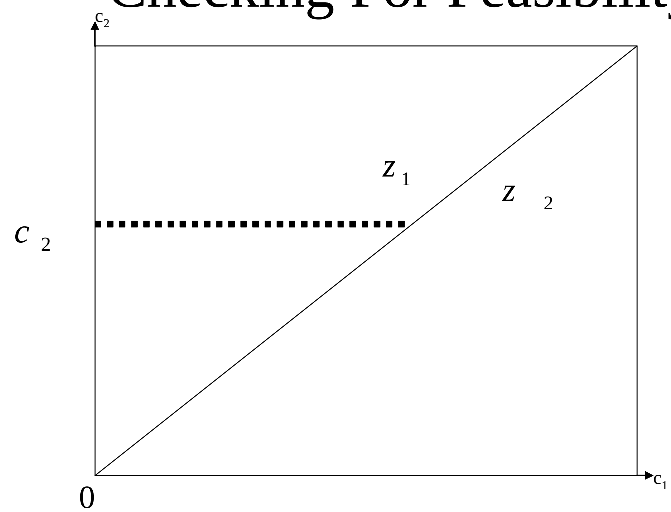
Proposition 4.1 describes a program whose solution gives us an optimal mechanism. The solution of this program is straightforward if the assignment function that solves the relaxed program

$$\max_{p \text{ s.t.}(??)} \int\limits_C \sum_{z \in Z} p^z(c) J_z(c) f(c) dc \tag{6}$$

also satisfies (1). This is so because the relaxed program can be solved by pointwise maximization. Following Myerson (1981) we will refer to this case as the *regular* case.

We now state three assumptions, each one of which guarantees precisely this.

Checking For Feasibility



[7] Let z_1,$z_2 \in Z$ be any two allocations. For a given cost realization (c_i, c_{-i}) if $z_1 \in arg \max_{z \in Z} J_z(c_i^-, c_{-i})$ and $z_2 \in arg \max_{z \in Z} J_z(c_i^+, c_{-i})$, then

$$\frac{\partial \pi_i(z_2, c_i, c_{-i})}{\partial c_i} \geq \frac{\partial \pi_i(z_1, c_i, c_{-i})}{\partial c_i}$$

A sufficient condition for Assumption 4.1 to hold is

$J_z(c_i, c_{-i})$ is decreasing in c_i and

$$\frac{\partial J_{z_1}(c_i, c_{-i})}{\partial c_i} \leq \frac{\partial J_{z_2}(c_i, c_{-i})}{\partial c_i} \Longrightarrow \frac{\partial \pi_i(z_1, c_i, c_{-i})}{\partial c_i} \leq \frac{\partial \pi_i(z_2, c_i, c_{-i})}{\partial c_i}$$

Even more stringent than this is

[8] Allocations $z_i \in Z$ are ranked: $\frac{\partial \pi_i(z_k,c_i,c_{-i})}{\partial c_i} \leq \frac{\partial \pi_i(z_l,c_i,c_{-i})}{\partial c_i}$ and $\frac{\partial J_{z_k}(c_i,c_{-i})}{\partial c_i} \leq \frac{\partial J_{z_l}(c_i,c_{-i})}{\partial c_i}$ for all $k < l$ and $c \in C$.

The Proposition that follows describes the solution to the seller's problem if pointwise maximization of the relaxed problem (6) leads to a feasible allocation.

Suppose that Assumption 4.1 is satisfied. Then the optimal allocation $\widehat{p}$ is given by:

$$\widehat{p}^{z^*}(c) = \begin{cases} 1 & \text{if } z^* \in arg \max_z J_z(c) \\ 0 & \text{otherwise} \end{cases}$$

Proof. The solution proposed corresponds to pointwise maximization, so the only possibility that is not optimal is that is not feasible. To check that feasibility is satisfied notice that

$$P_i(c_i) = \int_{C_{-i}} \sum_{z \in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i}$$

[7]In the case of no externalities, this condition is implied by $J_z(c_i)$ decreasing in c_i, which is the equivalent to the regularity condition in Myerson (1981).

[8]In their particular framework, this is the assumption made in Dana and Spier (1994).

and consider a fixed c_{-i}. In a region $[\underline{c}, \overline{c}]$ where $\overline{z} \in arg \max\limits_{z \in Z} J_z(c)\ p(c_i, c_{-i})$ does not change ($p^{\overline{z}} = 1$) and $P_i(c_i)$ is nondecreasing by the convexity of $\pi_i(z, \cdot, c_{-i})$. For a given c^* where $z_1 \in arg \max\limits_{z \in Z} J_z(c_i^-, c_{-i})$ and $z_2 \in arg \max\limits_{z \in Z} J_z c_i^+, c_{-i})$, $p^{z_1}(c_i^{*-}, c_{-i}) = 1$ and $p^{z_2}(c_i^{*+}, c_{-i}) = 1$, so $P_i(c_i)$ is nondecreasing because of Assumption 4.1. $\square$

In the problem considered in Myerson (1981), a sufficient condition for the problem to be regular is that the virtual valuations are increasing. For example, if the distribution F_i satisfies that $\frac{1-F_i(c_i)}{f_i(c_i)}$ is increasing (called the monotone hazard rate property,(MHR)) then the problem is regular. For the cases where the monotonicity of the virtual valuations fails, Myerson introduced an artificial program replacing virtual valuations with "ironed" ones (made monotonic in a "clever" way), and established that the solution of this artificial problem solves the original one.

Some remarks are in place

1) Myerson's ironing technique does not work here: even if virtual valuations are monotonic (decreasing, in our case), the assignment function obtained via pointwise optimization will in general fail to satisfy (1).
2) The assumptions that guarantee that a problem is regular are quite stringent, as the example that follows demonstrates.

Consider the case of the privatization of a public monopoly where the market structure can be decided by the seller. There are 2 potential buyers, A and B, and 4 possible allocations $Z = \{\text{monopoly A, monopoly B, duopoly, no sale}\} = \{z_A, z_B, z_{AB}, z_0\}$ The (constant) marginal cost of each firm is $c_i \sim U[0, 1]$. We will assume that there is a demand function $a - q$, where $a > 1$ (so that production is always positive). If we consider the case when the ex-post competition is Cournot, the valuations for buyers A and B are respectively given by:

$$\begin{array}{ll}
\pi_A(z_0, c_A, c_B) = 0 & \pi_B(z_0, c_A, c_B) = 0 \\
\pi_A(z_A, c_A, c_B) = \frac{(a-c_A)^2}{4b} & \pi_B(z_A, c_A, c_B) = 0 \\
\pi_A(z_B, c_A, c_B) = 0 & \pi_B(z_B, c_A, c_B) = \frac{(a-c_A)^2}{4b} \\
\pi_A(z_{AB}, c_A, c_B) = \frac{(a+c_B-2c_A)^2}{9b} & \pi_B(z_{AB}, c_A, c_B) = \frac{(a+c_A-2c_B)^2}{9b}
\end{array}$$

In this very simple case, Assumption 4.1 is satisfied iff $a \geq 8$.

Proof. : In fact, for Assumption 4.1 to hold, it must be the case that

$$\begin{aligned}
\frac{\partial J_{z_A}}{\partial c_A} &\leq \frac{\partial J_{z_{AB}}}{\partial c_A} \text{ or equivalently} && (7) \\
32c_B &\leq 6a + 25c_A && (8)
\end{aligned}$$

implies that

$$\begin{aligned}
\frac{\partial \pi_A(z_A, c_A, c_B)}{\partial c_A} &\leq \frac{\partial \pi_A(z_{AB}, c_A, c_B)}{\partial c_A} \text{ or equivalently} && (9) \\
8c_B - 7c_A &\leq a && (10)
\end{aligned}$$

Notice that if $a \geq \frac{7}{6}$, 8 is satisfied for all $c_A, c_B \in [0, 1]$, so 10 must also hold for all $c_A, c_B \in [0, 1]$. But this is true only for $a \geq 8$. This immediately implies that the condition is not satisfied if $a \in (\frac{7}{6}, 8)$.

If $a \in [1, \frac{7}{6})$, then there exist c_A, c_B such that $32c_B = 6a + 25c_A$. Then for that realization, 10 must hold, but we have:

$$\begin{aligned}
8c_B - 7c_A &= 32c_B - 25c_A - 3c_A \\
&= 6a - 3c_A \\
&= a + (5a - 3c_A)
\end{aligned}$$

So we need that $5a - 3c_A \leq 0$, but this implies $c_A \geq \frac{5}{3}a \geq \frac{5}{3}$, which is impossible.

Finally to see that if $a \geq 8$ the condition hold, we notice that in that case 10 is always satisfied. □

This example is in the same spirit as the ones in Section 3 in Dana and Spier (1994). There, they impose conditions that guarantee that $\frac{\partial J_{z_A}}{\partial c_A} \leq \frac{\partial J_{z_{AB}}}{\partial c_A}$ and $\frac{\partial \pi_A(z_A, c_A, c_B)}{\partial c_A} \leq \frac{\partial \pi_A(z_{AB}, c_A, c_B)}{\partial c_A}$ are always true, which is equivalent to our assumption 4.1[9].

What can be done if the regularity condition is not satisfied? Recall that the purpose of the regularity condition is to guarantee that pointwise optimization of the objective function will lead to a solution that satisfies the feasibility requirements. For the cases where this fails, Myerson has introduced the "ironing" technique, that essentially smooths out virtual valuations into monotone functions. This is done in a way that does not change the solution of the original problem. The possibility of using this technique relies on the fact that payoffs are linear in the allocation and that each agent cares only about one allocation (getting the object in that case).

When an agent cares about more than one allocation, and especially when one allocation affects more than one player (externalities), monotonicity conditions are not enough (as shown in example **??**). Also, when utilities are not linear in the type, we can have an allocation rule $\widehat{p} \notin \partial\Delta(Z)$. For a discussion about this, see section 6

4.2 The Optimal Mechanism with Type-Dependent "Threats"

Here we consider the seller's problem in the case the agents can be threatened with allocations that have type-dependent effects upon them. So in this subsection we will remove assumption 4.1.

In this case the seller's maximization problem can be written as:

$$\begin{array}{lll}
\max\limits_{p,x,\{\rho_i\}_{i\in I}} & \int\limits_C \sum\limits_{i=1}^{I} x_i(c) f(c) dc & \\
& U_i(c_i, c_i; (p,x)) \geq U_i(c_i, c_i^{'}; (p,x)) & \text{for all } c_i, c_i^{'} \in C_i \\
s.t. & U_i(c_i, c_i; (p,x)) \geq \int\limits_{C_{-i}} \sum\limits_{z\in Z_i} \rho_i^z \pi_i(z, c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i} & \text{for all } c_i \in C_i \\
& \sum\limits_{z\in Z} p^z(c) \leq 1, p^z(c) \geq 0 & \text{for all } c \in C
\end{array}$$

[9] In the illustrative cases they present in Section 3, they verify the assumption $\frac{\partial \pi_1^m}{\partial c_1} \leq \frac{\partial \pi_1^D(q_1(c_1), q_2(c_2))}{\partial c_1}$, whereas the assumption $\frac{\partial J_{10}}{\partial c_1} \leq \frac{\partial J_{11}}{\partial c_1}$ is imposed even in those illustrative cases.

Before we move on let us highlight in what respects this more general problem differs for the ones already examined in the literature. From the section that characterizes feasible mechanisms we obtain that

$$x_i(c) = \sum_{z \in Z} \left[(p^z(c_i', c_{-i})\pi_i(z, c_i, c_{-i}) - \int_{c_i}^{\bar{c}_i} \left[\sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi_i(z, s_i, c_{-i})}{\partial c_i} \right] ds \right] - V_i(\bar{c}_i) \tag{11}$$

In the standard problem without externalities, the worst that the seller can do to a buyer is not to assign him the object, hence the worse that the seller can do is to enforce a payoff of zero: the payment function is then given by (11), where $V_i(\bar{c}_i)$ is determined by the fact that the buyer always has the option not to participate which implies that

$$V_i(\bar{c}_i) = 0 \tag{12}$$

In the case of type-independent externalities, analyzed in the previous section, the only difference is that

$$V_i(\bar{c}_i) = \underline{U}_i(\rho_i^{z^i}) \tag{13}$$

This is essentially the same as the role of punishments as examined in JHM (1996).

Notice that the threat is independent of the mechanism, that is of the assignment function p, and the payment function x. Moreover, the "critical type" is always $\bar{c}_i$, so it's enough to check the participation constraint at $\bar{c}_i$. This won't be the case when the externalities are type-dependent.

The Determination of Punishments in the General Case

In the general case both the "optimal threat" and the "critical type" depend on the allocation rule that the seller wants to implement. Let's see how they are determined and how they depend on the allocation rule p.

A given allocation rule p determines up to a constant the expected payoff for each type of a buyer, which is given by the familiar expression

$$V_i(c_i) = V_i(\bar{c}_i; p) - \int_{c_i}^{\bar{c}_i} \int_{C_{-i}} \left[\sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi_i(z, s, c_{-i})}{\partial c_i} \right] f_{-i}(c_{-i}) dc_{-i} ds$$

At an optimal mechanism the constant $V_i(\bar{c}_i)$ is determined by the optimal threat that the seller can design. Let us call $\hat{V}(c_i)$ the payoff of type c_i of buyer i net of the constant that is

$$\hat{V}_i(c_i) = - \int_{c_i}^{\bar{c}_i} \int_{C_{-i}} \left[\sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi_i(z, s, c_{-i})}{\partial c_i} \right] f_{-i}(c_{-i}) dc_{-i} ds$$

which is, as we have shown, a decreasing and convex function of c_i. Now for each such expression there exists a "worst punishment" which is identified in two steps.

Step 1: Determination of the Critical Type $c_i^*(p, \rho_i)$

For each assignment function p (which determines $\hat{V}_i(c_i)$) and threat rule $\rho_i \in \Delta(Z_i)$, there exists a critical type $c_i^*(p, \rho_i)$. Let's define the expected payoff given a threat rule ρ_i for agent i if his type is c_i as

$$\overline{\pi}_i(c_i, \rho_i) = \int_{C_{-i}} \sum_{z \in Z_i} \rho_i^z \pi_i(z, c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}$$

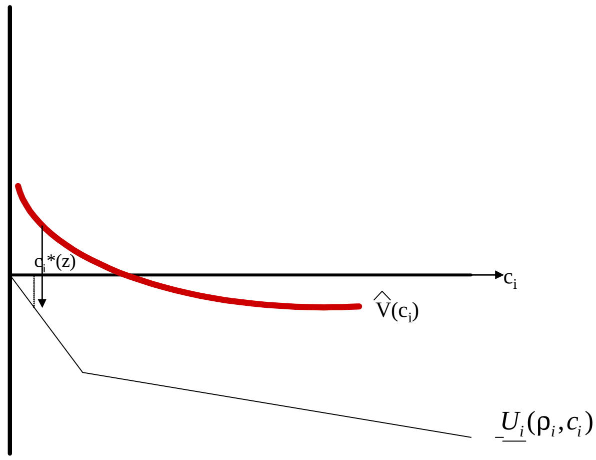
The seller is contemplating what would be the largest constant that he could reduce $i's$ payoff given a proposed allocation p and a threat rule ρ_i: this constant is going to be determined by the type where $\hat{V}_i$ would hit $\overline{\pi}_i$ first if we were to shift it down, we call this type $c_i^*(p, \rho_i)$. Formally $c_i^*(p, \rho_i)$ solves the following program:

$$c_i^*(p, \rho_i) \in \arg\min_{c_i} \left[\hat{V}_i(c_i) - \overline{\pi}_i(c_i, \rho_i) \right] \tag{14}$$

The constant by which the seller can reduce i's payoff is given by the difference

$$\hat{V}_i(c_i^*(p, \rho_i)) - \bar{\pi}_i(c_i^*(p, \rho_i), \rho_i).$$

The computation of c_i*(p,z)



Given the convexity of $V(\cdot)$ and $\overline{\pi}_i(\cdot, \rho_i)$, the Problem described in (14) can be written as:

$$c_i^*(p, \rho_i) \in \arg \min_{c_i \;\; s.t. \;\; \overline{\pi}_i'(c_i, \rho) \in \partial V(c_i)} \left[\hat{V}_i(c_i) - \overline{\pi}_i(c_i, \rho_i) \right].$$

This characterization, even if it looks more difficult, is extremely useful when the expected payoff functions $\overline{\pi}_i(\cdot, \rho_i)$ are linear, since the set of types where $\overline{\pi}_i(c_i, \rho_i) \in \partial V_i(c_i)$ is a singleton. Suppose for example that $\overline{\pi}_i(\cdot, \rho_i) = a_{\rho_i} c_i + b_{\rho_i}$. Then

$$c_i^*(p, \rho_i) = \begin{cases} \underline{c} & \text{if } P_i(\underline{c}) \geq a_{\rho_i} \\ \overline{c} & \text{if } P_i(\overline{c}) \leq a_{\rho_i} \\ P_i^{-1}(a_{\rho_i}) & \text{otherwise} \end{cases}$$

After finding the critical type for a particular threat rule ρ_i, we can compute the value of $V_i(\overline{c}_i)$, since from the characterization of incentive compatible mechanisms we have that $V_i(\bar{c}_i) = V_i(c_i^*(p, \rho_i)) + \int\limits_{c_i^*(p,\rho_i)}^{\bar{c}} P_i(s)ds$, so we can write

$$V_i(\bar{c}_i) = \overline{\pi}_i(c_i^*(p, \rho_i), \rho_i) + \int\limits_{c_i^*(p,\rho_i)}^{\bar{c}_i} P_i(s)ds$$

Step 2: Determination of the optimal punishments ρ_i

The next step for the seller is to find, for a given allocation rule p, the optimal threat $\rho_i^*(p)$, which satisfies

$$\rho_i^*(p) \in arg \min_{\rho_i \in \Delta(Z_i)} \overline{\pi}_i(c_i^*(p, \rho_i), \rho_i) + \int_{c_i^*(p,\rho_i)}^{\bar{c}} P_i(s)ds$$

As we remarked before, the fact that the optimal punishment depends on the allocation rule p that the seller has in mind is a big difference with the previous literature. The next example illustrates that.

Suppose that $Z_i = \{z_A, z_B\}$, the allocation rules p^1 and p^2 generate $\hat{V}_i(c_i; p_1) = 1 - 10c_i$ and $\hat{V}_i^2(c_i; p_2) = 0$ respectively in $C_i = [0, 1]$, and

$$\begin{aligned} \overline{\pi}_i(z_A, c_i) &= 1 - 10c_i \\ \overline{\pi}_i(z_B, c_i) &= 0 \end{aligned}$$

Then, denoting by ρ_A and ρ_B the respective degenerate measures, we get

$$\begin{aligned} V_i(1, \rho_A; p_1) &= -9 \\ V_i(1, \rho_B; p_1) &= 0 \end{aligned}$$

but

$$\begin{aligned} V_i(1, \rho_A; p_2) &= 1 \\ V_i(1, \rho_B; p_2) &= 0 \end{aligned}$$

so $\rho(p_1) = \rho_A$ and $\rho(p_1) = \rho_B$.

Notice that ρ_i^* does not necessarily lie in $\partial\Delta(Z_i)$, as shown by the next example:

Suppose that $Z_i = \{z_A, z_B\}$, the allocation rule p generates $\hat{V}_i(c_i) = 1 - c_i$ in $C_i = [0,1]$ and

$$\begin{aligned} \overline{\pi}_i(z_A, c_i) &= 1 - 10c_i \\ \overline{\pi}_i(z_B, c_i) &= 0 \end{aligned}$$

Then

$$\begin{aligned} \rho_i^{z_A} = 1 &\implies c_i^* = 0 \text{ and } V_i(1) = 0 \\ \rho_i^{z_B} = 1 &\implies c_i^* = 1 \text{ and } V_i(1) = 0 \end{aligned}$$

but

$$\rho_i^{z_A} = 0.1 \text{ and } \rho_i^{z_B} = 0.9 \implies c_i^* \in [0,1] \text{ and } V_i(1) = -\tfrac{9}{10}$$

Step 3: The Optimization Problem

From the previous steps, we see that the payments that can be extracted from the buyers can be written as a function only of the assignment rule p, since

$$V_i(\overline{c}_i) = \pi_i(c_i^*(p, \rho_i^*(p))) + \int_{c_i^*(p,\rho_i^*(p))}^{\overline{c}_i} P_i(s)ds.$$

Contrast this expression with (12) and (13) notice that in this case $V_i(\overline{c}_i)$ depends on p.

The previous results allow us to fully characterize the problem in terms of the assignment function p:

If in a mechanism $(\widehat{p}, \widehat{x}, \widehat{\rho})$ the assignment function $\widehat{p}$ solves:

$$\begin{aligned} \max_p \quad & \int_C \sum_{z\in Z} p^z(c) \sum_{i=1}^{I} \left[\pi_i(z, c_i, c_{-i}) + \tfrac{F_i(c_i)}{f_i(c_i)} \tfrac{\partial \pi_i(z,c_i,c_{-i})}{\partial c_i}\right] f(c)dc - \sum_{i=1}^{I} V_i(\overline{c}_i; p) \\ s.t. \quad & P_i \text{ increasing, } \sum_{z\in Z} p^z(c) \leq 1 \text{ and } p(c) \geq 0 \end{aligned}$$

where

$$V_i(\bar{c}_i;p) = \overline{\pi}_i(c_i^*(p,\rho_i(p)),\rho_i(p)) + \int\limits_{c_i^*(p,\rho_i(p))}^{\bar{c}_i} P_i(s)ds$$

and where in turn $c_i^*(p,\rho_i)$ satisfies

$$c_i^*(p,\rho_i) \in \arg\min_{c_i}\left[\hat{V}_i(c_i;p) - \overline{\pi}_i(c_i,\rho_i)\right]$$

and $\rho_i(p)$ satisfies

$$\rho_i(p) \in arg \min_{\rho_i \in \Delta(Z_i)} \pi_i(c_i^*(p,\rho_i),\rho_i) + \int_{c_i^*(p,\rho_i)}^{\bar{c}_i} P_i(s)ds$$

If also the payment function $\widehat{x}$ satisfies:

$$\widehat{x}_i(c) = \sum_{z\in Z}\widehat{p}^z(c)\pi_i(z,c_i,c_{-i}) + \int\limits_{c_i}^{\overline{c_i}}\sum_{z\in Z}\frac{\partial\pi_i(z,s,c_{-i})}{\partial s}\widehat{p}^z(s,c_{-i})f_i(s)ds - V_i(\bar{c}_i;\widehat{p})$$

and $\widehat{\rho}_i$ satisfies

$$\widehat{\rho}_i \in arg \min_{\rho_i \in \Delta(Z_i)} \pi_i(c_i^*(\widehat{p},\rho_i),\rho_i) + \int_{c_i^*(\widehat{p},\rho_i)}^{\bar{c}_i} P_i(s)ds$$

then the mechanism is optimal.
This program is not linear in p anymore, as we illustrate with an example in section 5, so the usual approach of pointwise maximization fails. Fortunately the problem has enough structure[10]to allow the use of variational methods without imposing additional restrictions on the mechanism (such as differentiability).

A simpler characterization can be found if the externalities are big, as we show in the next part.

The Optimal Mechanism in the Case of Large Externalities

[10]In particular enough differentiability on π_i will guarantee enough regularity on $c^*(p,\rho_i)$ as a function of the mechanism.

Suppose that the externalities are "big" in the following sense: for each i, there exists an allocation $z_i^* \in Z_i$ such that

$$\frac{\partial \overline{\pi}_i(z_i^*, c_i)}{\partial c_i} \le \frac{\partial \overline{\pi}_i(z, c_i,)}{\partial c_i} \text{ for all } z \in Z$$

and

$$\overline{\pi}_i(z_i^*, \underline{c}_i) \le \overline{\pi}_i(z, \underline{c}_i) \text{ for all } z \in Z$$

Suppose assumption 4.2 is satisfied. Then individual rationality for agent i has to be verified only at $c_i^* = \underline{c}_i$, and the problem of the seller can be rewritten as:

$$\begin{array}{ll} \max\limits_{p} & \int\limits_{C} \sum\limits_{z \in Z} p^z(c) \sum\limits_{i=1}^{I} \left[\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)-1}{f_i(c_i)} \frac{\partial \pi_i(z,c_i,c_{-i})}{\partial c_i} \right] f(c)dc \\ s.t. & P_i \text{ increasing, } \sum\limits_{z \in Z} p^z(c) \le 1 \text{ and } p(c) \ge 0 \end{array}$$

Proof. First we prove that under assumption 4.2, if individual rationality is satisfied for $c_i = \underline{c}_i$ then it is satisfied for all $c_i \in C_i$. It's clear that under assumption 4.2 the optimal punishment for agent i is given by the threat rule $\widehat{\rho}_i(z_i^*) = 1$. Since $P_i(c_i) \in \partial V(c_i)$ and

$$\begin{aligned} P_i(c_i) &= \int\limits_{C_{-i}} \sum_{z \in Z} p^z(c) \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i} \\ &\ge \int\limits_{C_{-i}} \sum_{z \in Z} p^z(c) \frac{\partial \pi_i(z_i^*, c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i} \\ &= \frac{\partial \overline{\pi}_i(z_i^*, c_i)}{\partial c_i} \end{aligned}$$

the fact that $V(\underline{c}_i) \ge \overline{\pi}(z_i^*, \underline{c}_i)$ implies that $V(c_i) \ge \overline{\pi}(z_i^*, c_i)$ for all $c_i \in C_i$.
The revenue of the seller is given by

$$\int\limits_{C} \sum_{z \in Z} p^z(c) \sum_{i=1}^{I} \left[\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} \right] f(c)dc - \sum_{i=1}^{I} V_i(\overline{c}_i)$$

Now, knowing that $c_i^* = \underline{c}$ we can write:

$$\begin{aligned} V_i(\bar{c}_i) &= \overline{\pi}_i(z^*, \underline{c}_i) + \int\limits_{C_i} P_i(s)ds \\ &= \overline{\pi}_i(z^*, \underline{c}_i) + \int\limits_{C_i} \left[\int\limits_{C_{-i}} \sum_{z \in Z} p^z(c) \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i} \right] dc_i \\ &= \overline{\pi}_i(z^*, \underline{c}_i) + \int\limits_{C} \sum_{z \in Z} p^z(c) \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} \frac{1}{f_i(c_i)} f(c) dc \end{aligned}$$

Now, using this in the previous expression we can rewrite the revenue of the seller as

$$\int\limits_{C} \sum_{z \in Z} p^z(c) \sum_{i=1}^{I} \left[\pi_i(z, c_i, c_{-i}) + \frac{F_i(c_i) - 1}{f_i(c_i)} \frac{\partial \pi_i(z, c_i, c_{-i})}{\partial c_i} \right] f(c) dc - \sum_{i \in I} \overline{\pi}_i(z_i^*, \underline{c}_i)$$

Since the last term does not depend on p the result follows. $\square$

Exactly as in subsection 4.1, the solution to this problem can be divided in two broad categories: the regular case (when pointwise maximization gives a feasible solution) and the non-regular one. The assumptions that guarantee regularity are analogous, only with slightly changed virtual valuations. As a simple corollary to the previous proposition, we can see that the allocation rule chosen by the seller is inefficient for a different reason than in the case of no externalities. Here, when compared to the ex-post efficient allocation rule, the seller allocates the objects "too much". Let's call $\widehat{p}$ the solution to the seller's problem and p^* the efficient allocation, that is

$$p^{*^{\bar{z}}}(c) = \begin{cases} 1 & \text{if } \bar{z} \in arg \max\limits_{z} \sum\limits_{i \in I} \pi_i(z, c_i, c_{-i}) \\ 0 & \text{otherwise} \end{cases}$$

Suppose that the allocation $z_0 = (0, ..., 0)$ satisfies that $\pi_i(z_0, c_i, c_{-i}) = 0$ for all $c \in C$. Then, compared to the ex-post efficient allocation rule, the solution to the seller's problem assigns the object too much, or more formally:

$$\{c \in C | \widehat{p}^{z_0}(c) = 1\} \subset \{c \in C | p^{*^{z_0}}(c) = 1\}$$

Proof. It follows immediately from the fact that $\frac{F_i(c_i)-1}{f_i(c_i)}\frac{\partial \pi_i(z,c_i,c_{-i})}{\partial c_i} \geq 0$

□

5 An Example

As an illustration of our previous analysis we present an example. Consider 2 firms fighting for a single slot to advertise their products. The value of actually airing a spot depends on the actual cost parameter c_i of the firm, which is private information. The cost is uniformly and independently distributed in $[0,1]$. We denote by $z=0$ the allocation when the object is not sold and $z=i$ the allocation when the object is given to agent i.

5.1 No externalities

Suppose that firms care only about getting the object. This case is one that can be just solved as in Myerson (1981). For example, suppose that profit functions for agent 1 are given by:

$$\begin{aligned} \pi_1(0,c_1,c_2) &= 0 \\ \pi_1(1,c_1,c_2) &= 1-c_1 \\ \pi_1(2,c_1,c_2) &= 0 \end{aligned}$$

and for agent 2 are given by:

$$\begin{aligned} \pi_2(0,c_1,c_2) &= 0 \\ \pi_2(1,c_1,c_2) &= 0 \\ \pi_2(2,c_1,c_2) &= 1-c_2 \end{aligned}$$

In this case the virtual valuations are

$$\begin{aligned}
J_1(c_1, c_2) &= \sum_{i=1}^{2}[\pi_i(1, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)}\frac{\partial \pi_i(1, c_i, c_{-i})}{\partial c_i}] \\
&= 1 - 2c_1 \\
J_2(c_1, c_2) &= 1 - 2c_2 \\
J_0(c_1, c_2) &= 0
\end{aligned}$$

The solution is then given by:

$$\begin{array}{ll}
p^1(c) = 1 & \text{if } c_1 \leq c_2 \text{ and } c_1 \leq \frac{1}{2} \\
p^2(c) = 1 & \text{if } c_1 \geq c_2 \text{ and } c_2 \leq \frac{1}{2} \\
0 & \text{otherwise}
\end{array}$$

This assignment function is illustrated in figure 2. The payments are given by

$$\begin{aligned}
x_1(c) &= \begin{cases} 1 - \max\{c_2, \frac{1}{2}\} & \text{if } c_1 \leq \min\{c_2, \frac{1}{2}\} \\ 0 & \text{otherwise} \end{cases} \\
x_2(c) &= \begin{cases} 1 - \max\{c_1, \frac{1}{2}\} & \text{if } c_2 \leq \min\{c_1, \frac{1}{2}\} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

It's also easy to compute the revenue for the seller:

$$\begin{aligned}
\int_0^1 \int_0^1 x_1(c_1, c_2) dc_2 dc_1 &= \int_0^{\frac{1}{2}} \int_{c_1}^{\frac{1}{2}} (1 - c_2) dc_2 dc_1 + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} \frac{1}{2} dc_2 dc_1 \\
&= \frac{5}{24}
\end{aligned}$$

Since the problem is symmetric we get a total revenue $R = \frac{5}{12}$

5.2 Type Independent Externalities

Now, let's suppose that firms also care about the competitor not getting the advertisement slot. But suppose that a firm's payoff if a competitor wins the auction is independent of that firms' own cost. For example, profit functions for agent 1 are given by:

$$\begin{aligned}\pi_1(0, c_1, c_2) &= 0 \\ \pi_1(1, c_1, c_2) &= 1 - c_1 \\ \pi_1(2, c_1, c_2) &= -\alpha\end{aligned}$$

and for agent 2 are given by:

$$\begin{aligned}\pi_2(0, c_1, c_2) &= 0 \\ \pi_2(1, c_1, c_2) &= -\alpha \\ \pi_2(2, c_1, c_2) &= 1 - c_2\end{aligned}$$

The virtual valuations are now

$$\begin{aligned}J_1(c_1, c_2) &= 1 - 2c_1 - \alpha \\ J_2(c_1, c_2) &= 1 - 2c_2 - \alpha \\ J_0(c_1, c_2) &= 0\end{aligned}$$

And the optimal allocation is exactly the same as in the previous case, that is:

$$\begin{array}{ll} p^1(c) = 1 & \text{if } c_1 \leq c_2 \text{ and } c_1 \leq \frac{1-\alpha}{2} \\ p^2(c) = 1 & \text{if } c_1 \geq c_2 \text{ and } c_2 \leq \frac{1-\alpha}{2} \\ 0 & \text{otherwise} \end{array}$$

Notice that now the seller keeps the object with a bigger probability, and he can also extract an extra payment of α from each bidder.

$$\begin{aligned}x_1(c) &= \begin{cases} 1 - \max\{c_2, \frac{1-\alpha}{2}\} + \alpha & \text{if } c_1 \leq \min\{c_2, \frac{1-\alpha}{2}\} \\ 0 & \text{otherwise} \end{cases} \\ x_2(c) &= \begin{cases} 1 - \max\{c_1, \frac{1-\alpha}{2}\} + \alpha & \text{if } c_2 \leq \min\{c_1, \frac{1-\alpha}{2}\} \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

The total revenue is then $R = \frac{(1-\alpha)^2}{4} + \frac{(1-\alpha)^3}{12} + \frac{(1+\alpha)^2(1-\alpha)}{2} + 2\alpha$

5.3 Type Dependent Externalities

Now, let's suppose that firms also care about the competitor not getting the advertisement slot. Even more, the cost of a competitor winning the auction is higher when their own cost realization is higher. For example, profit functions for agent 1 are given by:

$$\begin{aligned}
\pi_1(0, c_1, c_2) &= 0 \\
\pi_1(1, c_1, c_2) &= 1 - c_1 \\
\pi_1(2, c_1, c_2) &= -\alpha c_1
\end{aligned}$$

and for agent 2 are given by:

$$\begin{aligned}
\pi_2(0, c_1, c_2) &= 0 \\
\pi_2(1, c_1, c_2) &= -\alpha c_2 \\
\pi_2(2, c_1, c_2) &= 1 - c_2
\end{aligned}$$

With this we can write the virtual valuations associated to each allocation:

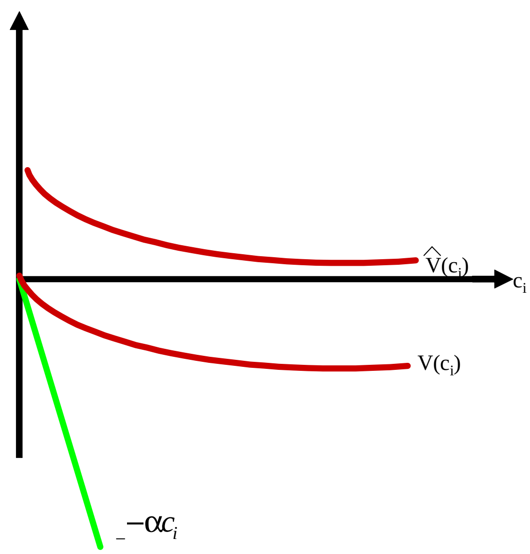
$$\begin{aligned}
J_1(c_1, c_2) &= \sum_{i=1}^{2} [\pi_i(1, c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i(1, c_i, c_{-i})}{\partial c_i}] \\
&= 1 - 2c_1 - 2\alpha c_2 \\
J_2(c_1, c_2) &= 1 - 2c_2 - 2\alpha c_1 \\
J_0(c_1, c_2) &= 0
\end{aligned}$$

The seller's problem can be written as:

$$\begin{aligned}
\max_p \quad & \int_{[0,1]} \int_{[0,1]} [p^1(c)[1 - 2c_1 - 2\alpha c_2] + p^2(c)[1 - 2c_2 - 2\alpha c_1]] dc_1 dc_2 - V_1(1) - V_2(1) \\
s.t. \quad & -\int [p^1(c) + \alpha p^2(c)] dc_2 \text{ is increasing} \\
& -\int [\alpha p^1(c) + p^2(c)] dc_1 \text{ is increasing} \\
& p_1(c) + p_2(c) \leq 1
\end{aligned}$$

We provide the solution only for the case that externalities are large.

The Critical Type: Large Externalities



5.4 The Solution for the Case that Externalities are Large: $\alpha > 1$

This is the easy case to handle. Since the "threat allocation" is linear ($\overline{\pi}_i(c_i) = -\alpha c_i$) we know that the critical type c_i^* is determined by

$$c_i = \begin{cases} 0 & \text{if } P(0) \geq -\alpha \\ 1 & \text{if } P(1) \leq \alpha \\ P^{-1}(\alpha) & \text{otherwise} \end{cases}$$

We can show now that **independently of the allocation rule selected**, the critical type for each agent is always $c_i^* = 0$. Notice that $P_1(c_1) = \int_0^1 [-p_1(c) - \alpha p_2(c)]$. Since $p_1(c) + p_2(c) = 1$ and $p_1(c), p_2(c) \geq 0$ we can conclude that $P_1(0) \geq -\alpha$, so $c_1^* = 0$. We analogously conclude that $c_2^* = 0$. Then

$$\begin{aligned} V_1(1) &= \overline{\pi}_i(c_i^*) + \int_{c_i^*}^1 P_1(c_1) dc_1 \\ &= \overline{\pi}_i(0) + \int_0^1 P_1(c_1) dc_1 \\ &= \int_0^1 \int_0^1 [-p_1(c) - \alpha p_2(c)] dc \end{aligned}$$

Analogously $V_2(1) = \int_0^1 \int_0^1 [-\alpha p_1(c) - p_2(c)] dc$

Then the problem of the seller becomes

$$\begin{aligned} \max_p \quad & \int_{[0,1]} \int_{[0,1]} [p^1(c)[1 - 2c_1 - 2\alpha c_2] + p^2(c)[1 - 2c_2 - 2\alpha c_1]] dc_1 dc_2 - V_1(1) - V_2(1) \\ s.t. \quad & \int [p^1(c)[-1] + p^2(c)[-\alpha]] dc_2 \text{ is nondecreasing} \\ & \int [p^1(c)[-\alpha] + p^2(c)[-1]] dc_1 \text{ is nondecreasing} \\ & p_1(c) + p_2(c) \leq 1 \end{aligned}$$

and can be rewritten as

$$\begin{array}{ll} \max\limits_{p} & \int\limits_{[0,1]}\int\limits_{[0,1]} [p^1(c)[2+\alpha-2c_1-2\alpha c_2]+p^2(c)[2+\alpha-2c_2-2\alpha c_1]]dc_1dc_2 \\ s.t. & \int[p^1(c)[-1]+p^2(c)[-\alpha]]dc_2 \text{ is nondecreasing} \\ & \int[p^1(c)[-\alpha]+p^2(c)[-1]]dc_1 \text{ is nondecreasing} \\ & p_1(c)+p_2(c)\leq 1 \end{array}$$

Pointwise maximization gives us

$$\begin{array}{ll} p^1(c)=1 & \text{if } 2+\alpha-2c_1-2\alpha c_2 \geq 2+\alpha-2c_2-2\alpha c_1 \text{ and } 2+\alpha-2c_1-2\alpha c_2\geq 0 \\ p^2(c)=1 & \text{if } 2+\alpha-2c_2-2\alpha c_1 \geq 2+\alpha-2c_1-2\alpha c_2 \text{ and } 2+\alpha-2c_2-2\alpha c_1\geq 0 \\ 0 & \text{otherwise} \end{array}$$

that can be rewritten as

$$\begin{array}{ll} p^1(c)=1 & \text{if } c_2\leq c_1 \text{ and } 2+\alpha-2c_1-2\alpha c_2\geq 0 \\ p^2(c)=1 & \text{if } c_1\leq c_2 \text{ and } 2+\alpha-2c_2-2\alpha c_1\geq 0 \\ 0 & \text{otherwise} \end{array}$$

Feasibility is satisfied since for a fixed $\bar{c}_2$ the function $c_1 \longrightarrow -p_1(c_1,\bar{c}_2)-\alpha p_2(c_1,\bar{c}_2)$ is nondecreasing. The same is true for a fixed $\bar{c}_1$ and the function $c_2 \longrightarrow -\alpha p_1(\bar{c}_1,c_2)-p_2(\bar{c}_1,c_2)$.

5.5 Sequentially Rational Punishments

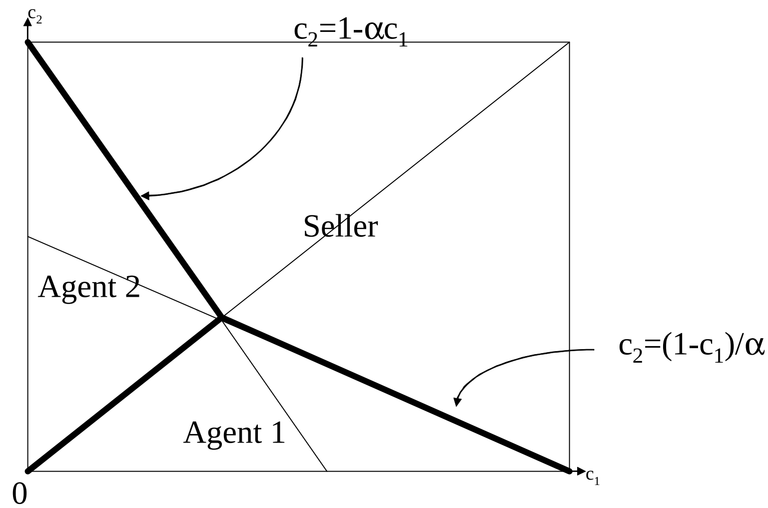
Let's consider again the case of section 4.1, but now let's suppose that the seller does not have commitment ability. Now he cannot get an extra payment of α based on the threat of giving the object to the other player, since it's not credible. Now player i knows that in case he does not participate, the seller will face a one bidder auction. In that case the virtual valuations would be

$$\begin{aligned} J_{-i}(c_{-i}) &= 1-2c_{-i} \\ J_0(c_{-i}) &= 0 \end{aligned}$$

so the optimal allocation would be

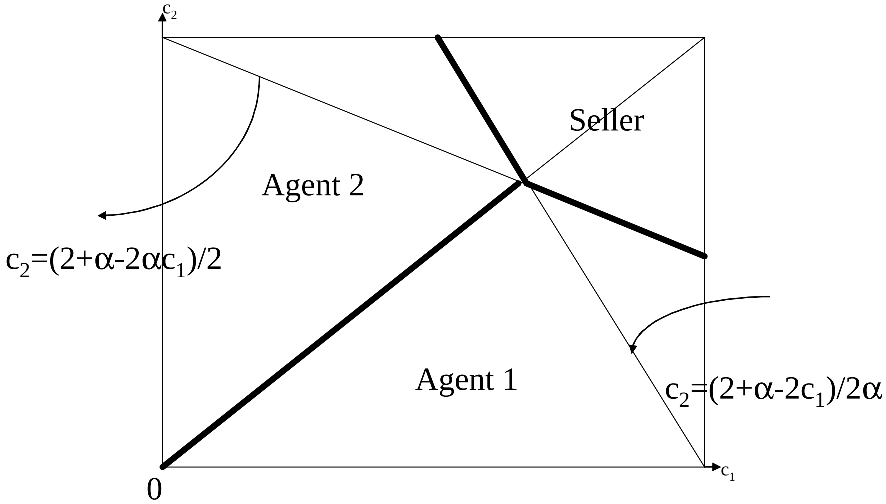
Example: Case of Large Externalities

α=2 Efficient Allocation



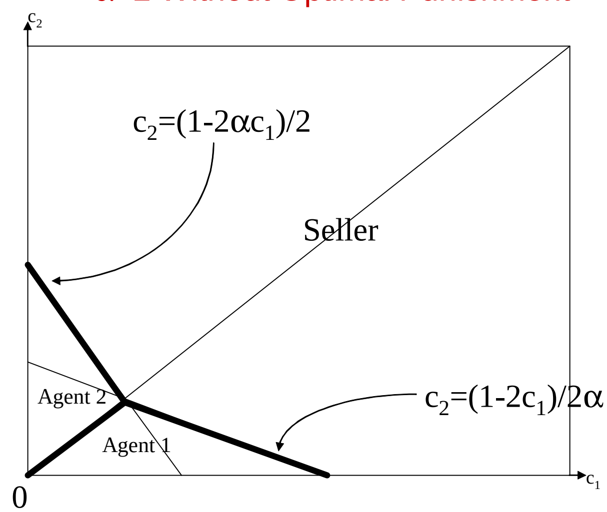
Example: Case of Large Externalities

α=2 With Optimal Punishment



Example: Case of Large Externalities

α=2 Without Optimal Punishment



$$p^{-i}(c_{-i}) = \begin{cases} 1 & \text{if } c_{-i} \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and the imposed externality on agent i would be (in expectation) only of $\frac{\alpha}{2}$. In that case, the extra payment that the seller would be able to extract would be reduced by half and the total revenue would be $R = \frac{(1-\alpha)^2}{4} + \frac{(1-\alpha)^3}{12} + \frac{(1+\alpha)^2(1-\alpha)}{2}$

6 New Ironing

In the cases where pointwise optimization leads to a solution that is not feasible, we have to take the feasibility constraints explicitly into account. Myerson (1981) presents a clever way of dealing with this difficulty. He proposes a rewriting of the objective function in terms of "ironed" virtual valuations that has the advantage that pointwise maximization of this artificial objective function leads to a feasible solution and moreover the solution of this artificial program solves the original one as well. Unfortunately this beautiful technique does not work here where there is more then one object, and payoffs are non-linear in the allocation.

In order to obtain the optimal allocation in this case without imposing regularity conditions we should do a different form of ironing. For simplicity let us first illustrate the technique in a special environment where there is a single buyer.

There is a single agent and $n+1$ allocations: $z_0, z_1, ..., z_n$.

Suppose also that the derivatives are ranked in the following way: $\frac{\partial \pi(z_0,c)}{\partial c} > \frac{\partial \pi(z_1,c)}{\partial c} > ....$

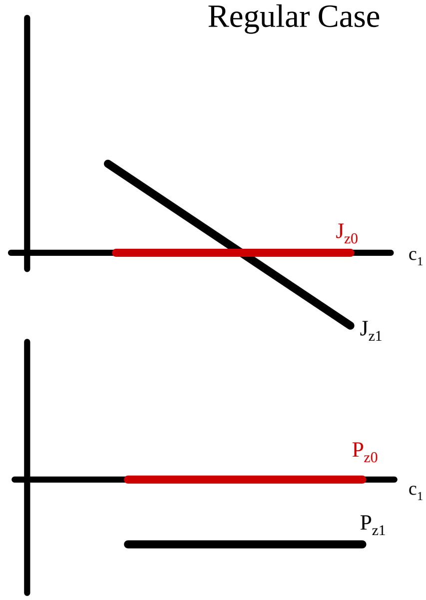
Now consider a point c^* where

$$\arg\max_z J_z(c^{*-}) = z_k \text{ and } \arg\max_z J_z(c^{*+}) = z_l.$$

with $k < l$. Furthermore for simplicity assume that there is only one such point.

Our first result states that the problem for the seller is simple: it's enough to find an optimal region $[\underline{x}, \overline{x}]$ where the feasibility constraint is satisfied with equality (that is $P_i(c)$ is constant). Outside of that region the solution maximizes the objective function pointwise.

One Buyer, Linear Valuations:
Regular Case



Lemma 4. *If* $\{\widehat{p}^{z_i}\}_{i\in\{0,\dots,n\}}$ *is a solution to the assignment problem, then there exist* $\underline{x}, \overline{x}$ *satisfying* $\underline{c} \le \underline{x} \le c^* \le \overline{x} \le \overline{c}$ *such that*

$$\begin{aligned}
\sum_{i=k}^{l} \widehat{p}^{z_i}(c)\frac{\partial \pi_i(z_i,c)}{\partial c} &= \sum_{i=k}^{l} \widehat{p}^{z_i}(\underline{x})\left.\frac{\partial \pi_i(z_i,c)}{\partial c}\right|_{c=\underline{x}} \text{ for all } c\in[\underline{x},\overline{x}] \\
\widehat{p}^{z_k}(\underline{x}) &= 1 \text{ if } \underline{x} > \underline{c} \\
\widehat{p}^{z_l}(\overline{x}) &= 1 \text{ if } \overline{x} < \overline{c}
\end{aligned}$$

Proof. Obvious $\square$

The problem can then be written as to minimize the loss in that region:

$$\begin{aligned}
&\min_{\substack{\underline{x},\bar{x} \\ m_i(c),\ i\in\{k,\dots,l\}}} \int_{\underline{x}}^{c^*} \left[J_{z_k}(c) - \sum_{k\le i\le l} m_i(c)J_{z_i}(c)\right] f(c)dc \\
&+ \int_{c^*}^{\bar{x}} \left[J_{z_l}(c) - \sum_{k\le i\le l} m_i(c)J_{z_i}(c)\right] f(c)dc
\end{aligned}$$

$$\begin{aligned}
s.t. \sum_{k\le i\le l} m_i(c)\frac{\partial \pi(z_i,c)}{\partial c} &= \sum_{k\le i\le l} m_i(c)\left.\frac{\partial \pi(z_i,c)}{\partial c}\right|_{\underline{x}} \\
m_k(\underline{x}) &= 1 \text{ if } \underline{x} > \underline{c} \\
m_l(\overline{x}) &= 1 \text{ if } \overline{x} < \overline{c} \\
\sum_{k\le i\le l} m_i(c) &= 1,\ m_i(c)\ge 0,
\end{aligned}$$

rewriting we can get:

$$\begin{aligned}
&\min_{\substack{\underline{x},\bar{x} \\ m_i(c),\ i\in\{k,\dots,l\}}} \int_{\underline{x}}^{c^*} \sum_{k+1\le i\le l} m_i(c)\left[J_{z_k}(c) - J_{z_i}(c)\right] f(c)dc \\
&+ \int_{c^*}^{\overline{x}} \sum_{k\le i\le l-1} m_i(c)\left[J_{z_l}(c) - J_{z_i}(c)\right] f(c)dc
\end{aligned}$$

$$
\begin{aligned}
s.t. \sum_{k\leq i\leq l} m_i(c)\frac{\partial \pi(z_i,c)}{\partial c} &= \sum_{k\leq i\leq l} m_i(c)\frac{\partial \pi(z_i,c)}{\partial c}\bigg|_{\underline{x}} \\
m_k(\underline{x}) &= 1 \text{ if } \underline{x} > \underline{c} \\
m_l(\overline{x}) &= 1 \text{ if } \overline{x} < \overline{c} \\
\sum_{k\leq i\leq l} m_i(c) &= 1,\ m_i(c) \geq 0
\end{aligned}
$$

These problem can be decomposed. For each $\underline{x}$ and $\{m_i(\underline{x})\}_{i=k}^{l}$, finding the optimal mixture at a point c is a simple linear program. Even more, the next result allows us to decompose the problem in two:

Lemma 5. *Consider the constrained problem*

$$
\min_{\substack{\underline{x},\overline{x} \\ m_i(c),\ i\in\{k,\ldots,l\}}} \int_{\underline{x}}^{c^*} \sum_{k+1\leq i\leq l} m_i(c)\left[J_{z_k}(c) - J_{z_i}(c)\right] f(c)dc
$$
$$
+ \int_{c^*}^{\overline{x}} \sum_{k\leq i\leq l-1} m_i(c)\left[J_{z_l}(c) - J_{z_i}(c)\right] f(c)dc
$$

$$
\begin{aligned}
s.t. \sum_{k\leq i\leq l} m_i(c)\frac{\partial \pi(z_i,c)}{\partial c} &= \frac{\partial \pi(z_k,c)}{\partial c} \\
m_l(\overline{x}) &= 1 \textit{ if } \overline{x} < \overline{c} \\
\sum_{k\leq i\leq l} m_i(c) &= 1,\ m_i(c) \geq 0
\end{aligned}
$$

If the solution to this problem has $\underline{x} > \underline{c}$, then the solution to the original problem also has $\underline{x} > \underline{c}$.

Proof. See appendix □

So we can solve first the problem constrained to a mixture with $m_k(\underline{x}) = 1$, if the solution is not a corner ($\underline{x} > \underline{c}$) then we have a solution. If not, we just need to maximize over the optimal mixture at $\underline{c}$.

To illustrate some properties of the solution, let's consider the case when $l = k+1$, so we are dealing only with two allocations.

In this case the problem becomes

$$\min_{\substack{\underline{x},\overline{x} \\ m_k(c)}} \int_{\underline{x}}^{c^*} (1-m_k(c))\left[J_{z_k}(c)-J_{z_l}(c)\right] f(c)dc$$
$$+\int_{c^*}^{\overline{x}} m_k(c)\left[J_{z_l}(c)-J_{z_k}(c)\right] f(c)dc$$

$$\begin{aligned} s.t.\ m_k(c)\frac{\partial \pi(z_k,c)}{\partial c}+(1-m_k(c))\frac{\partial \pi(z_l,c)}{\partial c} &= \left.\left[m_k(c)\frac{\partial \pi(z_k,c)}{\partial c}+(1-m_k(c))\frac{\partial \pi(z_l,c)}{\partial c}\right]\right|_{\underline{x}} \\ m_k(\underline{x}) &= 1 \text{ if } \underline{x}>\underline{c} \\ m_l(\overline{x}) &= 1 \text{ if } \overline{x}<\overline{c} \\ 0 &\leq m_k(c)\leq 1 \end{aligned}$$

From the first constraint we get

$$m_k(c)\frac{\partial \pi_k(c)}{\partial c}+(1-m_k(c))\frac{\partial \pi_l(c)}{\partial c}=\left.\frac{\partial \pi_k(c)}{\partial c}\right|_{\underline{x}}$$

and that gives us

$$m_k(c)=\frac{m_k(\underline{x})\left.\frac{\partial \pi_k(c)}{\partial c}\right|_{\underline{x}}+(1-m_k(\underline{x}))\left.\frac{\partial \pi_l(c)}{\partial c}\right|_{\underline{x}}-\frac{\partial \pi_l(c)}{\partial c}}{\frac{\partial \pi_k(c)}{\partial c}-\frac{\partial \pi_l(c)}{\partial c}}$$

The second constraint imposes a condition on $\overline{x}$ whenever $\overline{x}<\overline{c}$. Using the expression for $m_k(c)$ found above, this can be written as

$$m_k(\underline{x})\left.\frac{\partial \pi_k(c)}{\partial c}\right|_{\underline{x}}+(1-m_k(\underline{x}))\left.\frac{\partial \pi_l(c)}{\partial c}\right|_{\underline{x}}-\left.\frac{\partial \pi_l(c)}{\partial c}\right|_{\overline{x}}=0$$

Then $\overline{x}(\underline{x})$ is defined implicitly as the solution to the above equation as long as it is less or equal than $\overline{c}$, and as $\overline{c}$ otherwise. Notice that the convexity of π_k,π_l immediately implies that $\overline{x}(\underline{x})$ is nondecreasing.

Rewriting the problem (now only as a function of $\underline{x}$ and $m_k(\underline{x})$) we get:

$$\min_{\underline{x}} \int_{\underline{x}}^{c^*} \frac{\frac{\partial \pi_k(c)}{\partial c} - m_k(\underline{x}) \left.\frac{\partial \pi_k(c)}{\partial c}\right|_{\underline{x}} - (1-m_k(\underline{x})) \left.\frac{\partial \pi_l(c)}{\partial c}\right|_{\underline{x}} - \frac{\partial \pi_l(c)}{\partial c}}{\frac{\partial \pi_k(c)}{\partial c}} \left[J_{z_k}(c) - J_{z_l}(c)\right] f(c)dc$$

$$+ \int_{c^*}^{\overline{x}(\underline{x})} \frac{m_k(\underline{x}) \left.\frac{\partial \pi_k(c)}{\partial c}\right|_{\underline{x}} + (1-m_k(\underline{x})) \left.\frac{\partial \pi_l(c)}{\partial c}\right|_{\underline{x}} - \frac{\partial \pi_l(c)}{\partial c}}{\frac{\partial \pi_k(c)}{\partial c} - \frac{\partial \pi_l(c)}{\partial c}} \left[J_{z_l}(c) - J_{z_k}(c)\right] f(c)dc$$

Let's name the objective function $R(\underline{x}, m_k(\underline{x}))$. We get that

$$\begin{aligned}
\frac{dR(\underline{x})}{d\underline{x}} &= - \left.\left[\frac{\frac{\partial \pi_k(c)}{\partial c} - m_k(c)\frac{\partial \pi_k(c)}{\partial c} - (1-m_k(c))\frac{\partial \pi(c)}{\partial c}}{\frac{\partial \pi_k(c)}{\partial c} - \frac{\partial \pi_l(c)}{\partial c}} \right]\right|_{\underline{x}} \left[J_{z_k}(\underline{x}) - J_{z_l}(\underline{x})\right] f(\underline{x}) \\
&+ \left.\left[\frac{m_k(\underline{x}) \left.\frac{\partial \pi_k(c)}{\partial c}\right|_{\underline{x}} + (1-m_k(\underline{x})) \left.\frac{\partial \pi_l(c)}{\partial c}\right|_{\underline{x}} - \frac{\partial \pi_l}{\partial c}}{\frac{\partial \pi_k(c)}{\partial c} - \frac{\partial \pi_l(c)}{\partial c}} \right]\right|_{\overline{x}(\underline{x})} \left[J_{z_l}(\overline{x}(\underline{x})) - J_{z_k}(\overline{x}(\underline{x}))\right] \frac{d\overline{x}(\underline{x})}{d\underline{x}} f(\overline{x}(\underline{x})) \\
&+ \int_{\underline{x}}^{\overline{x}(\underline{x})} \frac{m_k(\underline{x}) \left.\frac{\partial^2 \pi_k(c)}{\partial c^2}\right|_{\underline{x}} + (1-m_k(\underline{x})) \left.\frac{\partial^2 \pi_l(c)}{\partial c^2}\right|_{\underline{x}}}{\frac{\partial \pi_k(c)}{\partial c} - \frac{\partial \pi_l(c)}{\partial c}} \left[J_{z_l}(c) - J_{z_k}(c)\right] f(c)dc
\end{aligned}$$

The second term exists only when $\overline{x}(\underline{x}) < \overline{c}$ and in that case it vanishes because of **??**. Then we get that

$$\begin{aligned}
\frac{dR(\underline{x})}{d\underline{x}} &= (1-m_k(\underline{x}))[J_{z_k}(\underline{x}) - J_{z_l}(\underline{x})]f(\underline{x}) \\
&+ \left[m_k(\underline{x}) \left.\frac{\partial^2 \pi_k(c)}{\partial c^2}\right|_{\underline{x}} + (1-m_k(\underline{x})) \left.\frac{\partial^2 \pi_l(c)}{\partial c^2}\right|_{\underline{x}} \right] \int_{\underline{x}}^{\overline{x}(\underline{x})} \frac{J_{z_l}(c) - J_{z_k}(c)}{\frac{\partial \pi_k(c)}{\partial c} - \frac{\partial \pi_l(c)}{\partial c}} f(c)dc
\end{aligned}$$

and that

$$\frac{dR(\underline{x})}{dm_k(\underline{x})} = \left.\left[\frac{\partial \pi_k(c)}{\partial c} - \frac{\partial \pi_l(c)}{\partial c} \right]\right|_{\underline{x}} \int_{\underline{x}}^{\overline{x}(\underline{x})} \frac{J_{z_l}(c) - J_{z_k}(c)}{\frac{\partial \pi_k(c)}{\partial c} - \frac{\partial \pi_l(c)}{\partial c}} f(c)dc$$

It's easy to see that $\int_{\underline{x}}^{\overline{x}(\underline{x})} \frac{J_{z_l}(c)-J_{z_k}(c)}{\frac{\partial \pi_k(c)}{\partial c} - \frac{\partial \pi_l(c)}{\partial c}} f(c)dc$ is nondecreasing in $\underline{x}$: the integrand is negative when $c < c^*$ and positive otherwise, and $\overline{x}(\underline{x})$ is nondecreasing in $\underline{x}$. This observation, plus the fact that

$\left[\frac{\partial\pi_k(c)}{\partial c} - \frac{\partial\pi_l(c)}{\partial c}\right]\Big|_{\underline{x}} > 0$ and $\int\limits_{c^*}^{\overline{x}(c)} \frac{J_{z_l}(c) - J_{z_k}(c)}{\frac{\partial\pi_k(c)}{\partial c} - \frac{\partial\pi_l(c)}{\partial c}} f(c)dc > 0$ allows us to characterize the solution

$$\underline{x} = \begin{cases} \underline{c} & \text{if } \int\limits_{\underline{c}}^{\overline{x}(\underline{c})} \frac{J_{z_l}(c) - J_{z_k}(c)}{\frac{\partial\pi_k(c)}{\partial c} - \frac{\partial\pi_l(c)}{\partial c}} f(c)dc \geq 0 \\ \left[\frac{dR(\underline{x})}{d\underline{x}}\right]^{-1}(0) & \text{if not} \end{cases}$$

and

$$m_k(\underline{x}) = \begin{cases} 0 & \text{if } \int\limits_{\underline{c}}^{\overline{x}(\underline{c})} \frac{J_{z_l}(c) - J_{z_k}(c)}{\frac{\partial\pi_k(c)}{\partial c} - \frac{\partial\pi_l(c)}{\partial c}} f(c)dc \geq 0 \\ 1 & \text{if not} \end{cases}$$

We illustrate this technique of obtaining the optimal allocation via a simple example.

Example: Suppose that there a single buyer whose cost parameter is private information and distributed on the interval $[\frac{1}{10}, \frac{2}{5}]$ according to $F(c) = Me^{c^2}$ and that there are two possible allocations, z_1 and z_2. For each cost realization the payoff arising from these two allocations is given by

$$\begin{aligned} \pi(z_1, c) &= Bc^2 + Ac + K_1 \\ \pi(z_2, c) &= Bc^2 + K_2, \end{aligned}$$

where $B > 0$, $A > 0$

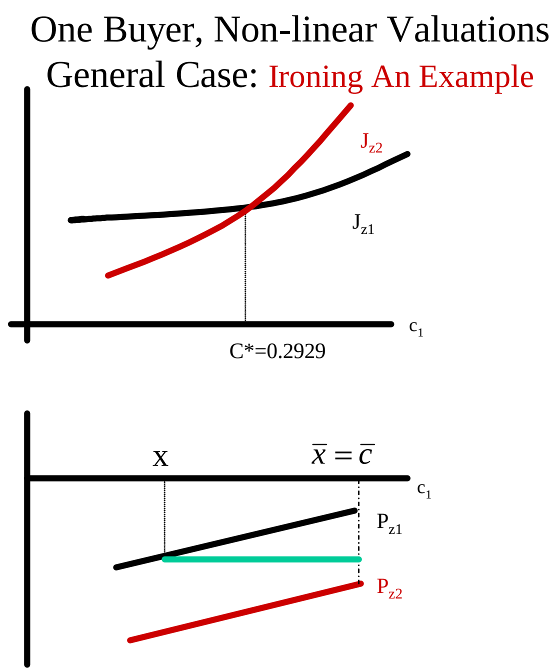
$$\begin{aligned} P_{z_2} &= \frac{\partial\pi(z_2, c)}{\partial c} = 2Bc \\ P_{z_1} &= \frac{\partial\pi(z_1, c)}{\partial c} = 2Bc + A \end{aligned}$$

For this example we have that

$$\frac{F(c)}{f(c)} = \frac{Me^{c^2}}{2Mce^{c^2}} = \frac{1}{2c}.$$

so we can write

$$\begin{aligned} J_{z_2}(c) &= Bc^2 + K_2 + \frac{1}{2c}2Bc \\ J_{z_1}(c) &= Bc^2 + Ac + K_1 + \frac{1}{2c}\left[2Bc + A\right] \end{aligned}$$



Consider parameters $A = 1, K_1 = -2$ and $K_2 = 0$. In that case there is a unique change of sign of the expression $J_{z_1}(c) - J_{z_2}(c)$ at

$$c^* = \frac{2 - \sqrt{2}}{2} = 0.2929.$$

For $c = 0.1$ we have that $J_{z_1}(c) - J_{z_2}(c) = 3.1$ and for $c = 0.4$ we have that $J_{z_1}(c) - J_{z_2}(c) = -0.35$. So at $c^* = 0.2929$ pointwise optimization would dictate that we should move from allocation 1 to allocation 2 but this is not feasible since for all c we have $P_{z_1} = \frac{\partial \pi(z_1,c)}{\partial c} = 2Bc + 1 \geq P_{z_2} = \frac{\partial \pi(z_2,c)}{\partial c} = 2Bc$.

The problem can be then written as:

$$
\begin{aligned}
\min_{\substack{\underline{x},\bar{x} \\ m_i(c),\ i\in\{0,1,\ldots,n\}}} & \int_{\underline{x}}^{c^*} m_2(c)\left[J_1(c)-J_2(c)\right]f(c)dc \\
& + \int_{c^*}^{\bar{x}} \left[1-m_2(c)\right]\left[J_2(c)-J_1(c)\right]f(c)dc
\end{aligned}
$$

$$
\begin{aligned}
s.t.\ (1-m_2(c))(2Bc+A)+m_2(c)2Bc &= 2B\underline{x}+A \\
m_2(\overline{x}) &= 1 \text{ if } \overline{x} < 0.4 \\
0 \leq m_2(c) \leq 1 &
\end{aligned}
$$

From the first constraint we get that

$$
\begin{aligned}
m_2(c) &= \frac{2B(c-\underline{x})}{A}, \text{ for } c \in [\underline{x}, \bar{x}) \text{ and} \\
1-m_2(c) &= \frac{A-2B(c-\underline{x})}{A}
\end{aligned}
$$

and from the second we get that for $\overline{x} < \overline{c}$ it must be the case that

$$
\begin{aligned}
m_2(\bar{x}) &= \frac{2B(\bar{x}-\underline{x})}{A} = 1 \\
\bar{x} &= \frac{A}{2B} + \underline{x}
\end{aligned}
$$

Then we conclude:

$$
\overline{x} = \begin{cases} \frac{A}{2B} + \underline{x} & \text{if } \frac{A}{2B} + \underline{x} \leq \overline{c} \\ \overline{c} & \text{if not} \end{cases}
$$

Now let's substitute $m_2(c)$ and $\overline{x}$ in the objective function and look for the optimal $\underline{x}$. We get:

$$
\min_{\underline{x}} \int_{\underline{x}}^{c^*} \frac{2B(c-\underline{x})}{A}\left[J_1(c)-J_2(c)\right]f(c)dc + \int_{c^*}^{\overline{x}(\underline{x})} \frac{A-2B(c-\underline{x})}{A}\left[J_2(c)-J_1(c)\right]f(c)dc
$$

Naming the objective function $R(\underline{x})$ and differentiating it, we obtain

$$
\begin{aligned}
\frac{\partial R(\underline{x})}{\partial \underline{x}} &= \frac{2B}{A}\int_{\underline{x}}^{\overline{x}(\underline{x})} \left[J_2(c)-J_1(c)\right]f(c)dc \\
&= 4BM \int_{\underline{x}}^{\overline{x}(\underline{x})} \left[2-c-\frac{1}{2c}\right] e^{c^2} dc
\end{aligned}
$$

Let's consider $B = 1$. In that case $\overline{x}(\underline{x}) = 0.4$. From previous work we know that $\frac{\partial R(\underline{x})}{\partial \underline{x}}$ is nondecreasing, and now we can find that $\frac{\partial R(\underline{x})}{\partial \underline{x}} = 0$ at $\underline{x} = 0.1928$. From the previous equations we can find the optimal mixture $m_2(c)$.

7 Concluding Remarks

In this paper we study the optimal allocation mechanism for N objects (permits), to I potential buyers (firms). Private information is single-dimensional and we can solve the problem even though our environment in all other respects is very general. Payoff functions allow for complementarities, substitutabilities and type dependent externalities among buyers. The presence of type-dependent externalities implies that even though the private information of each firm is one dimensional (its cost), virtual valuations depend on the cost parameters of all other firms. This captures nicely the existence of externalities among buyers: how much money the seller can extract from firm A depends on the technology of firm B, which captures together with other parameters how strong of a competitor firm B is. As in JHM (1996) and (2001) the type of the buyer that is indifferent between participating or not, is not exogenously given but depends on the range of the externalities. This critical type of each agent determines how much money the seller can extract from the players. Unlike JHM (1996) and (2001), in our model this type depends *also* on the actual mechanism that the mechanism designer employs. The reason for this is the presence of general type-dependent externalities. The characterization of the optimum then becomes intricate: given a mechanism there is a vector of critical types; the amount of payments that the seller can extract from the buyers depends on the vector of critical types therefore the mechanism depends not only on the virtual valuations, but also on which is the critical type. All these are novel but important insights which can have significant implications for the design of allocation mechanisms for multiple objects such as the design of mechanisms for the allocation of time-slots for advertisements, landing slots in airports and many others.

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