Default Risk, the Real Exchange Rate and Income Fluctuations in Emerging Economies

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Abstract

Volatile and countercyclical country interest rates and dollar-denominated debt are common features of emerging economies. This paper develops a two-sector small open economy model to study endogenous default risk and its interaction with the real exchange rate and income. Default probabilities and country risk premia depend on incentives for repayment. Default occurs in equilibrium because asset markets are incomplete. The model predicts that default incentives and default premia are higher in recessions, as observed in the data. The reason is that in a recession, a risk averse borrower finds it more costly to repay non-contingent debt and is more likely to default. In a quantitative exercise, the model matches several features of the Argentinian economy and can account for the dynamics observed during the recent default episode. Prior to default, the economy experiences higher interest rate premia, capital outflows, real exchange rate depreciation, and collapses in consumption. An important feature of the model is that economies with relatively small tradable sectors have higher incentives to default on dollar denominated debt and thus have larger default probabilities.

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1 Introduction

Emerging markets tend to have volatile business cycles and experience economic crises more frequently than developed economies. Recent evidence suggests that this may be related to cyclical changes in the access to international credit. In particular, emerging market economies face volatile and highly counter-cyclical interest rates, usually attributed to counter-cyclical default risk.[1] In addition, interest rates and default risk are also systematically correlated with real exchange rate devaluations: devaluations increase the probability of default and default increases the probability of devaluations. The exposure of the economy to such devaluations is magnified by high levels of liability dollarization - that is, "dollar" denominated debt, leveraged to a large extent on domestic income and assets.[2] Figure (1) illustrates these correlations by plotting aggregate consumption, real exchange rates, interest rates and the current account for Argentina.[3] In December 2001, Argentina defaulted on its international debt and fell into a deep economic crisis. During the crisis, consumption and real exchange rates collapsed, interest rates increased, and the current account experienced sharp reversals.[4] This evidence indicates that a priority for theoretical work in emerging markets macroeconomics must be understanding markets for international credit and in particular the joint analysis of default risk, interest rates and real exchange rates.

This paper develops a stochastic general equilibrium model with endogenous default risk. The model allows the study of the relation between interest rates, real exchange rates and output, shedding light on potential mechanisms generating the co-movements described above. The terms of international loans are endogenous to domestic fundamentals and depend on incentives to default. The paper extends the approach developed by Eaton and Gersovitz (1981) in their the seminal study on international lending, by analyzing how endogenous default probabilities and fluctuations in output are related. In addition, the model considers a default penalty that is limited to temporary financial autarky, and introduces a nontradable sector to analyze dynamics of interest rates and real exchange rates. The paper characterizes the equilibrium country interest rate schedule and its relation with output, and provides conditions under which default can be an equilibrium outcome. In a quantitative exercise the model is applied to analyze the default experience of Argentina. The results show that the

[1] Neumeyer and Perri (2001) and Uribe and Yuen (2003) document the counter-cyclicality of country interest rates for Argentina, Brazil, Ecuador, Mexico, Peru, Philippines, and South Africa.

[2] Reinhart (2002), Calvo and Reinhart (2000), Hausmann et. al (2001) document the high levels of liability dollarization in emerging markets foreign debt and its effect on crises and devaluations.

[3] Consumption and real exchange rate data are log, current account data are reported as percentage of output. Consumption and current account data are filtered with a linear trend.

[4] Mendoza (2002) documents similar dynamics for all the emerging market crises in the 1990's.

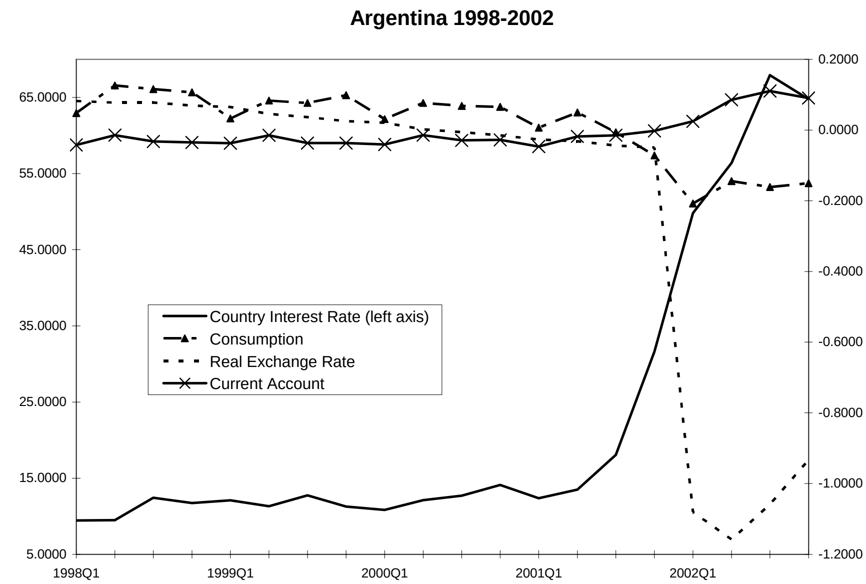

Figure 1:

model can account for the dynamics of consumption, interest rates, current account and the real exchange rate.

The model in this paper accounts for the empirical regularities in emerging markets as an equilibrium outcome of the interaction between risk-neutral competitive creditors and a risk averse borrower that has the option to default. The borrower is a benevolent government of a small open economy with two sectors, tradables and nontradables. The government in this economy can buy or sell tradable denominated bonds with foreign creditors. The model starts from the premise that default probabilities are endogenous to the economy's incentives to default and they affect the equilibrium interest rate the economy faces. Default is an equilibrium outcome of the model because the asset structure is incomplete since it includes only one period discount bonds that pay a non-contingent face value. Asset incompleteness is necessary in this framework to study time-varying default premia due to equilibrium default. With non-contingent assets, risk neutral competitive lenders are willing to provide debt contracts that in some states will result in default by charging a higher premium on these contracts. In addition to more closely reflecting the actual terms of international contracts where foreign debt is largely dollar denominated and contracted at non-contingent interest rates, this market structure has the potential to deliver counter-cyclical default risk, since repayment of non-contingent, nonnegotiable loans in low output, low consumption times is more costly than repayment in boom times. The existence of nontradable goods is important because even though foreign debt cannot be used to smooth consumption of nontradable goods, nontradable output fluctuations affect repayment incentives and equilibrium interest rates through changes in the real exchange rate.

In the first part of the paper, a simpler version of the model with *i.i.d.* shocks is considered in order to characterize analytically the equilibrium properties of credit markets and in particular the conditions under which default is an equilibrium outcome. It is shown that default is more likely the lower the output of tradables. This result contrasts with standard participation constraint models that have a complete set of assets, which have the feature that default incentives are higher in good times. The key intuition for why asset market incompleteness can reverse this outcome is that after a long series of low endowment shocks, an economy with incomplete markets can experience capital outflows in bad times. Risk averse agents with large debt holdings that can experience capital outflows have more incentives to default in times of low shocks. In essence, in times of low output, the asset available to the economy does not help agents smooth consumption, thus it is not very valued and default can be preferable than repayment. A similar intuition holds for the relationship between the level of nontradable output and default, although here it depends on how nontradable shocks affect the marginal utility of tradable consumption. We find that default is more tempting for low nontradable shocks when a low nontradable shock increases the marginal utility of tradable consumption. When this condition holds, capital outflows are more costly for lower nontradable shocks and thus default more likely.

In the quantitative part of the paper, the model is calibrated to Argentina to study its recent default episode. The model can match the observed business cycle correlations in Argentina. Specifically, it can account for the negative correlations between output and consumption with interest rates. It also matches the positive correlation between real exchange rate devaluations and interest rates, and the positive correlation between the current account and interest rates. The model matches the data in that before a default occurs, the economy faces high interest rates and features sharp reductions in capital inflows, sharp reversals from large current account deficits into much smaller deficits or even surpluses, and collapses in consumption and real exchange rates. The relative sizes of the tradable and nontradable sectors are also very important. We find that relatively large nontradable sectors reduce the economy's commitment for debt repayment and further limit the financial integration of economies. A model with only a tradable sector would face much lower equilibrium interest rates and default would be a much less likely outcome.

The paper is related to several studies from different strands of the macroeconomics and finance literatures. Some papers have looked at the relation between interest rates and business cycles. Neumeyer and Perri (2001) model the effect that exogenous interest rate fluctuations have on business cycles and find that interest rate shocks can account for 50% of the volatility of output. Uribe and Yue (2003) construct an empirical VAR to uncover the relationship between country interest rates and output, and then estimate with a theoretical model this relationship. They find that country spreads explain 12% of movements in output, and that output explains 12% in the movements of country interest rates. These papers, however, do not model endogenous country spreads responding to default probabilities.

The debt contractual arrangement in this paper is related to the optimal contract arrangements in the presence of commitment problems such as the analysis by Kehoe and Levine (1993) and Kocherlakota (1996). These studies, however, assume that a complete set of contingent assets are available and search for allocations that are efficient subject to a lack of enforceability. Alvarez and Jermann (2000) show that in this framework each state-contingent asset is associated with a state-contingent borrowing limit. This limit is such that in each state the borrowing country prefers to repay loans rather than default. While it is useful to be able to characterize allocations under the efficiency benchmark, this market structure may not be useful as a framework for understanding actual emerging markets. First, default never arises in equilibrium, so default risk premia are never observed. Second, default incentives in this class of models are typically higher in periods of high output, which is when efficiency dictates loan repayment. These features put these models at odds with the empirical evidence regarding default risk in emerging markets where bond yields are counter-cyclical and where debt prices reflect largely the risk of default. This paper delivers the correct empirical prediction because it assumes an incomplete set of assets where default occurs with a positive probability. In this regard the paper is specially related to the analysis on unsecured consumer credit with the risk of default by Chaterrjee, Corbae, Nakajima and Rios Rull (2002) where they model equilibrium default in an incomplete markets setting. Aguiar and Gopinath (2004) introduce shocks to the growth rate to a model similar to one developed here and find that these shocks can help account for the positive correlation between current account and interest rates at the cost of generating acyclical interest rates.

Other authors have developed models to study liability dollarization and have concentrated on how the level and volatility of output and relative price of nontradables affect the ability to pay dollar debt. Calvo (1998) shows how collapses in the price of nontradables due to constraints on tradable denominated debt can lead to crises by limiting the ability for debt repayment. Mendoza (2002) examines a stochastic environment where households face a liquidity requirement borrowing constraint (in which debt cannot be larger than an exogenous fraction of current income) to study emerging markets crises. He shows how crises can be the outcome of the equilibrium dynamics of an economy with imperfect credit markets. In these models though, the borrowing constraints do not guarantee that debtors have incentives to repay debts. In addition, they do not model the endogeneity of interest rates and the relation between interest rates and liability dollarization.

The focus in this paper is on understanding how the level and volatility of tradable and nontradable output affect incentives to default and thus equilibrium country interest rates in an environment of liability dollarization. Results match the empirical facts in that default incentives are higher when the economy has large debt positions, is in recession, and has relatively small tradable sectors.

The paper is organized as follows: section 2 presents the theoretical model, section 3 characterizes the equilibrium, section 4 assesses the quantitative implications of the model in explaining the data, and section 5 concludes.

2 The Model Economy

Consider small open economy with two sectors, a tradable sector and a nontradable sector. A set of one period discount bonds is available to the government of the small open economy at contingent prices. Debt contracts are not enforceable as the government can choose to default in its debt contracts if it finds it optimal. If the government defaults in its debts, it is assumed to be temporarily excluded from international financial markets and that a portion of the aggregate endowment is lost during the autarky periods. The economy trades discount bonds with risk neutral, competitive foreign creditors.

Households are identical and have preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^T, c_t^N) \tag{1}$$

where $0 < \beta < 1$ is the discount factor, c^T and c^N are consumption of the tradable and nontradable goods, and $u(\cdot)$ is strictly concave and increasing, and twice continuously differentiable. Households receive stochastic endowment streams of tradable $y^T$and nontradable y^N goods. The exogenous state vector of sector-endowment shocks is defined as $y \equiv (y^T, y^N) \in Y$. Shocks are assumed to have a compact support and to be a Markov process drawn from a probability space $(y, B(y))$ with a transition function $f(y'|y)$. In addition households receive a transfer of tradable goods from the government in a lump sum fashion. Households trade in the spot market tradable and nontradable goods with p^N being the relative price of non-tradable goods.

The government is benevolent in that its objective is to maximize the utility of households. The government has access to international financial markets, where it can buy one period discount bonds B' denominated in terms of tradables at price q. The government also decides whether to repay or default on its debt. The bond price function $q(B', y)$ depends on the size of the bond B' and on the aggregate shock y because default probabilities depend on both. A purchase of a discount bond with a positive value for B' means that the government has entered into a contract where it saves $q(B', y)\ B'$ units of period t tradable goods to receive $B' \geq 0$ units of tradable goods the next period. A purchase of a discount bond with negative face value for B' means that households have entered into a contract where they receive $q(B', y)\ B'$ units of period t tradable goods, and promise to deliver, conditional on not declaring default, B' units of tradable goods the next period. The government rebates back to the households all the proceedings from its international credit operations in a lump sum fashion.

When the government chooses to repay its debts, the tradable sector market clearing condition is the following:

$$c^T = y^T + B - q(B', y)B'$$

The market clearing condition for the nontradable sector, requires that nontradable consumption equals nontradable output at all times.

$$c^N = y^N$$

Given that the government is benevolent, it effectively uses international borrowing to smooth consumption. But the idiosyncratic income uncertainty induced by y^T and y^N cannot be insured away with the set of bonds available to the economy, which pay a time and state invariant amount of tradable goods only. That is, asset markets in this model are incomplete not only because of the endogenous default risk, but also because of the set of assets available.

If the government defaults, it is assumed that current debts are erased and that it is not able to save or borrow. The government will remain in financial autarky for a stochastic number of periods. There is an exogenous constant probability θ that the government can reenter the market. This is a simple way to model that governments that default on their international debt lose access to international financial markets only temporarily. In addition, in the period when the government defaults, endowments fall a proportion $(1-\lambda)$, where $\lambda \leq 1$. The assumption that default may reduce output can be rationalized by the common view that after default there is a disruption in the countries' ability to engage in international trade, and this reduces the value of output (Cole and Kehoe 2000, Conklin 1998, Obstfeld and Rogoff 1989).

When the government chooses to default market clearing conditions require that consumption equals output :

$$c^T = \lambda y^T$$

$$c^N = \lambda y^N$$

Foreign creditors have access to an international credit market in which they can borrow or lend as much as needed at a constant international interest rate $r > 0$. Creditors have perfect information regarding the economy's endowment processes and can observe each period the endowment levels.[5] Creditors are assumed to be risk neutral and behave competitively. Creditors engage in Bertrand competition, offering contracts to the government that gives them expected zero profits.

They choose loans B' to maximize expected profits ϕ, taking as given prices:

$$\phi = -qB' + \frac{(1-\delta)}{1+r}B' \tag{2}$$

where δ is the probability of default.

For positive levels of foreign asset holdings, $B' \geq 0$, the probability of default is zero and thus the price of a discounted bond will be equal to the opportunity cost for investors. For negative asset holdings, $B' < 0$, the equilibrium price accounts for the risk of default that creditors are facing, that is, the price of a discount bond will be equal to risk adjusted opportunity cost.[6] This requires that bond prices satisfy:

$$q = \frac{(1-\delta)}{1+r} \tag{3}$$

The probability of default δ is endogenous to the model and depends on the government

[5] We assume for simplicity that each period the government enters into a debt contract with only one creditor. This can be generalized by assuming that creditors can observe previously written contracts and contracts that are written first have to be honored first.

[6] Risk adjustment in this framework is not due to compensation for risk aversion as lenders are risk neutral. It reflects the risk neutral compensation for a lower expected payoff.

incentives to repay debt. Since $0 \leq \delta \leq 1$, the zero profit requirement implies that bond prices q lie in the closed interval $[0, (1+r)^{-1}]$. The country gross interest rate can be interpreted as the inverse of the discount bond price, $1 + r^c = \frac{1}{q}$.

The timing of decisions within each period is as follows The government starts with initial assets B, observes the endowment shock y, and decides whether to repay its debt obligations or default. If the government decides to repay, then taking as given the schedule bond price $q(B', y)$ the government choose B' subject to the resource constraints. Then creditors taking as given q choose B'. Finally consumption of tradable and nontradable goods c^T, c^N takes place.

3 Recursive Equilibrium

We define a recursive equilibrium in which the government does not have commitment and in which the government, foreign creditors and households act sequentially. Given aggregate states $s = (B, y)$, the policy functions for the government B', the price function for bonds q and the policy functions for the consumers c^T, c^N together with the relative price p^N determine the equilibrium.

Households' problem is static. They observe their endowment shocks and taking as given the aggregate states, government transfers and the nontradable price they choose optimal consumption . Their first order condition equates the marginal rate of substitution between nontradable and tradable consumption to the relative price.

$$\frac{u_{c^N}(c^T, c^N)}{u_{c^T}(c^T, c^N)} = p^N \tag{4}$$

Given that purchasing power parity (PPP) is assumed to hold for the tradable sector, the real exchange rate for this small economy is the domestic consumption price index p^c which is a function of the nontradable price, p^N, and the tradable price, which is normalized to 1.

Foreign creditors in the model are risk neutral and behave competitively. They lend the amount of bonds wanted by the government as long as the price satisfies

$$q(B', y) = \frac{(1 - \delta(B', y))}{1 + r}$$

Foreign creditors in the model are pretty passive, they lend bonds as long as the gross return on the bonds equals $1 + r$.

The government observes the aggregate endowment shocks y, and given initial foreign assets B, choose whether to repay or to default. If the government chooses to repay its debt obligations and remain in the contract, then it chooses the new level of foreign assets B'. The government understands that the price of new borrowing $q(B', y)$ depends on the states y and on its choice of B'. The government also understands that its choices will affect the households choices c^T, c^N and p^N and internalizes the domestic market clearing conditions. The government objective is to maximize the lifetime utility of households.

Define $v^o(B, y)$ as the value function for the government who has the option to default, and that start the current period with assets B and endowment shocks y. The government decides on whether to default or repay its debts, to maximize the welfare of households. Note that only when the government has debt (i.e. negative assets) could the default option be optimal.

Given the option to default, $v^o(B, y)$ satisfies:

$$v^o(B, y) = \max_{\{c,d\}} \left\{ v^c(B, y), v^d(y) \right\} \tag{5}$$

where $v^c(B, y)$ is the value associated with not defaulting and staying in the contract and $v^d(y)$ is the value associated with default.

When the government defaults, the economy will be in temporary financial autarky; θ is the probability that it will regain access to international credit markets. The value of default is given by the following:

$$v^d(y) = u(\lambda y^T, \lambda y^N) + \beta \int_{y'} \left[\theta v^o(0, y') + (1 - \theta) v^d(y') \right] f(y'|y) dy \tag{6}$$

If the government defaults endowments fall, and consumption equal output.

When the government chooses to remain in the credit relation, the value conditional on not defaulting is the following

$$v^c(B, y) = \max_{(B')} \left\{ u(y^T - q(B', y)B' + B, y^N) + \beta \int_{y'} v^o(B', y') f(y'|y) dy \right\} \tag{7}$$

The government decides on optimal policies B' to maximize utility. The decision to remain in the credit contract and not default is a period by period decision. That is, the expected value from next period forward incorporates the fact that the government could choose to default in the future. The government also faces a lower bound on debt, $B' \geq -Z$, which prevents the government from running ponzi schemes but is otherwise not binding in equilibrium.

The government default policy can be characterized by default sets and repayment sets.

Let $A(B)$ be the set of $y's$ for which repayment is optimal when assets are B such that[7]*:*

$$A(B) = \left\{ y \in Y : v^c(B, y) > v^d(y) \right\}$$

and $D(B) = \widetilde{A}(B)$ be the set of $y's$ for which default is optimal for a level of assets B, such that:

$$D(B) = \left\{ y \in Y : v^c(B, y) \leq v^d(y) \right\} \tag{8}$$

The value function can then be represented more precisely by the following dynamic problem where the government decides on optimal borrowing taking into account that its choice on assets may imply defaulting in some states.

$v^c(B, y) =$

$$\max_{(B')} \left\{ u(y^T - q(B', y)B' + B, y^N) + \beta \left[\int_{A(B')} v^c(B', y') f(y'|y) dy + \int_{D(B')} v^d(y') f(y'|y) dy \right] \right\}$$

The above centralized government borrowing problem can be decentralized in multiple ways, with the simplest being lump sum taxation as presented here. In a separate appendix, it is shown that the above problem can also be decentralized as in Kehoe and Perri (2004) by letting households borrow directly from foreign creditors. The government in this case makes the economy wide default decision, and levies a savings tax that gives households the right incentives for holding the optimal level of debt. For simplicity in the exposition we have assumed here that the government has access to lump sum taxes.

Having developed the problem for each of the agents in the economy, the equilibrium is defined. Let $s = \{B, y\}$ be the the aggregate states for the economy.

Definition 1. *The recursive equilibrium for this economy is defined as a set of policy functions for (i) consumption $c^T(s)$ and $c^N(s)$, nontradable price $p^N(s)$ (ii) government's asset holdings $B'(s)$, repayment sets $A(B)$ and default sets $D(B)$, and (iii) the price for bonds $q(s, B')$, such that:*

1. *Taking as given the government policies, policy functions for households $c^T(s)$, $c^N(s)$, and the relative price $p^N(s)$ satisfy the households optimization problem and domestic market clearing conditions hold.*

2. *Taking as given the bond price function $q(B', y)$, the government's policy functions $B'(s)$, repayment sets $A(B)$ and default sets $D(B)$, satisfy the government optimization*

[7] We assume that if the government is indifferent between repayment and default, default is chosen.

problem.

3. *Bonds prices $q(B', y)$ are such that they reflect the government default probabilities and they are consistent with creditor's expected zero profits such that the loan market clears.*

The equilibrium bond price function $q(B', y)$ has to be consistent with government's optimization and with expected zero profits for foreign creditors. That is, q correctly assesses the probability of default of the government.[8]

Default probabilities $\delta(B', y)$ and default sets $D(B')$ are then related in the following way:

$$\delta(B', y) = \int_{D(B')} f(y'|y)dy' \tag{9}$$

When default sets are empty, $D(B') = \emptyset$, equilibrium default probabilities $\delta(B', y)$ are equal to 0. That is, the economy with assets B' never chooses to default for all realization of the endowment shocks. When $D(B') = Y$, default probabilities $\delta(B', y)$ are equal to 1. More generally, default sets are shrinking in assets, as the following proposition shows:

Proposition 1. (Default sets are shrinking in assets*) For all $B^1 \leq B^2$, if default is optimal for B^2, in some states y, then default will be optimal for B^1 for the same states y, that is $D(B^2) \subseteq D(B^1)$.*

Proof. See Appendix.

This result is similar to Eaton and Gersovitz (1981) and Chatterjee, et al. (2002). Intuitively, the result follows from the property that the value of staying in the contract is increasing in B, and that the value of default is independent of B. As assets decrease, the value of the contract monotonically decreases, while the value of default is constant. Thus, if default is preferred for some level of assets B, for a given state y, the value of the contract is less than the value of default. As assets decrease, the value of the contract will be even lower than before, and so default will continued to be preferred.

Since stochastic shocks are assumed to have a bounded support and the value of the contract is monotonically decreasing as assets fall, there exists a level of B that is low enough, such that for all endowment shocks default is optimal and default sets are equal to the entire endowment set. On the other hand, given that default can only be preferable when assets have a negative value (i.e. when the government is holding debts), there exists a level of

[8] Chatterjee et al (2001) prove the existence of an equilibrium price schedule in a similar environment for their work on consumer default risk. We conjecture that the existence proof for this model follows that of Theorem 4 in their paper.

assets $B \leq 0$, such that default sets are empty. These two properties of default sets can be summarized as follows.

Definition 2. *Denote $\underline{B}$ the upper bound of assets for which the default set constitutes the entire set and let $\overline{B}$ be the lower bound of assets for which default sets are empty, where $\underline{B} \leq \overline{B} \leq 0$ due to Proposition 1.*

$$\underline{B} = \sup \{B : \ D(B) = Y\}$$

$$\overline{B} = \inf \{B : \ D(B) = \emptyset\}$$

For asset holdings greater than $\overline{B}$, default is never optimal for all y and equilibrium bond prices are equal to $(1+r)^{-1}$ because default probabilities are zero.[9] For asset holdings $B \leq \underline{B}$ default is always optimal and equilibrium prices for these bonds are zero because default probabilities are one. Given that default sets are shrinking in the level of assets, condition (9) implies that equilibrium default probabilities are decreasing in B', and the equilibrium price function $q(B', y)$ is an increasing function of B'. Lower levels of assets will be associated with larger default probabilities, and thus discount prices for those bonds will be lower to compensate risk neutral investors for a lower expected payoff. That is, larger loans are generally more expensive. Equilibrium bond prices are also contingent on the endowment shocks, because the probability distribution from which shocks are drawn the next period depends on today's shocks. Since the risk of default varies with the level of debt and depends on the stochastic structure of the endowment shocks, competitive risk neutral pricing requires that equilibrium bond prices depend on both B' and y.

3.1 Case of *i.i.d.* Tradable Endowment Shocks

This section characterizes the equilibrium bond price function and the default decision for the case of a constant nontradable endowment and i.i.d. tradable endowment shocks. When endowment shocks are i.i.d., equilibrium bond prices are independent of the endowment realization and are only a function of the level of assets $q(B')$ because today's shock gives no information on the likelihood of tomorrow's shock. We will assume, without loss of generality, that $\lambda = 1$, no output loss in autarky, and $\theta = 0$, financial autarky is permanent after default. The results can be generalized for other parameters of λ and θ.

[9] Zhang (1997) introduced this debt limit as the no default debt limit in his work on participation contraints under incomplete markets.

Proposition 2. *Default incentives are stronger the lower the tradable endowment. For all* $y_1^T < y_2^T$ *, if* $y_2^T \in D(B)$*, then* $y_1^T \in D(B)$.

Proof. See Appendix.

This result comes from the property that utility is increasing and concave in tradable consumption and by noting that default can be optimal only if under no default the economy experiences net capital outflows ($B - q(B')B' < 0$). In fact, when for some state default is optimal, there are no contracts available to the government such that the economy can experience capital inflows given that level of debt for all states. Given that utility is increasing and concave in consumption, and that the economy is not able to borrow more when it has the low endowment, repayment is more costly in this low endowment state and thus default more likely.

Intuitively, the asset available to the economy is not a very useful insurance instrument for a highly indebted economy, because in times of a low endowment it cannot raise enough resources to smooth consumption. Thus, the asset the economy is giving up is not very valuable and default may be preferable in times of low endowments.

Endowment shocks have generally two opposing effects on default incentives. When output is high, the value of default is relatively high, increasing default incentives. But at the same time, when output is high, the value of repayment is relatively higher too, decreasing default incentives. With an incomplete set of assets, and *i.i.d.* shocks, the latter effect dominates and thus default is more likely the lower the tradable endowment.

This result contrasts with the participation constraint models that have a complete set of contingent assets. These models have the feature that default incentives are higher in times of good endowment shocks. In fact, for small open economy models with participation constraints and a complete set of contingent assets, default incentives are always higher in the good states because only autarky is affected by the current endowment, as the value of staying in the contract is constant in the long run and independent of the economy's specific endowment.

Proposition 3. *If default sets are non-empty, then they are closed intervals where only the upper bound depends on the level of assets:*

$$D(B) = [\underline{y}, y^*(B)] \qquad \text{for } B \leq \overline{B} \tag{10}$$

where $y^*(B)$ *is a continuous, non-increasing function of* B*, such that:*

$$y^*(B) = \left\{ \begin{array}{lr} y^*(B) : v^d(y^*(B)) = v^c(B, y^*(B)) & \text{for } \underline{B} \leq B \leq \overline{B} \\ \overline{y} & \text{for } B < \underline{B} \end{array} \right\} \tag{11}$$

Proof. See Appendix.

Proposition 3 proves that for $B \in [\underline{B}, \overline{B}]$, there will be a unique y^* where the contract value and continuation value cross. Default sets can then be characterized solely by a closed interval where only the upper bound is a function of the level of assets. For a given a level of assets $B \in [\underline{B}, \overline{B}]$, default will be optimal for endowment levels less than or equal to $y^*(B)$, and repayment will be optimal for endowment levels greater than $y^*(B)$. The function $y^*(B)$ is the default boundary that divides the y, B space into the repayment and default regions.[10]

The first order condition of the government with respect to asset holdings can be presented more sharply in this case by the following condition:

$$\frac{\partial[q(B')B']}{\partial B'} u_{c^T} = \beta \int_{A(B')} v^c_{B'}(B', y') f(y')\, dy' \tag{12}$$

Equation (12) equates the marginal utility of consumption today to the expected marginal value of wealth tomorrow for the states where repayment is optimal. The marginal cost from borrowing a loan of size B' in the current period is the expected marginal disutility of consumption from repaying that loan the following period. As opposed to standard intertemporal conditions for models without default, here that cost is only experienced if in the following period the government choose to repay. That is, the cost of repaying is realized for the set of y's, $A(B') = (y^*(B'), \overline{y}]$, for which repayment is optimal when the economy has assets equal to B'.

Given that default sets are such that only the upper bound depends on the level of debt, the equilibrium price function $q(B)$ can then be characterized by the following condition:

$$q(B) = \frac{1}{1+r} [1 - F(y^*(B))] \tag{13}$$

where F is the cumulative probability distribution of the stochastic endowments.

Equilibrium bond prices will fall in three ranges. For asset levels above $\overline{B}$ prices are equal to the inverse of the risk free rate. For asset levels less than $\underline{B}$, prices are zero. For intermediate asset levels, $B \in [\underline{B}, \overline{B}]$ prices will be increasing in the level of assets

[10] In countinuous time optimal stopping problems, it has been shown that under special cases not only the function is continuous at the boundary, but present a 'smooth pasting' condition. Which would imply that the derivatives with respect to y of the continuation and default values are equal at the boundary.

because $y^*(B)$ is decreasing in this range. Note that the bond price function will be crucially dependent upon the probability distribution of the endowment shocks.

If initial bond values are B then the probability of default at every point of the state space is given by $F(y^*(B))$, which is greater than zero in for $B \in [-\infty, \overline{B}]$. The next issue to be addressed is whether the economy would ever choose a B' such that $D(B') \neq \emptyset$. That is if in the ergodic distribution of assets a point exists where default has a positive probability, such that if initial bond holdings are larger than $\overline{B}$, the model can have default as an equilibrium outcome. To clarify this issue, imagine the economy happens to start at $B \leq \underline{B}$, then in that period the economy would default with probability one. But given that discount prices are zero in this range, if the economy's initial bond holdings are greater than $\overline{B}$, the economy would never choose as optimal asset holdings levels of $B' \leq \underline{B}$, because it would get nothing this period, and would incur a liability the following period. The range of B' for which default can potentially be an equilibrium outcome is limited to $(\underline{B}, \overline{B}]$, because here is where default sets are non-empty and equilibrium prices are different from zero.

Intuitively, the necessary condition for default to be an equilibrium outcome of the model is that the equilibrium price function does not decrease "too fast" as assets decrease. For default to potentially be an equilibrium outcome, the economy must be able to find it optimal to borrow bonds less than or equal to $\overline{B}$ such that the economy is exposed to the risk of default. And for this to ever be an optimal decision, the economy should be able to increase total resources borrowed $q(B)B$, that is, have a higher level of consumption, by choosing a lower level of assets at a lower price. This means that the equilibrium total resources borrowed $q(B)B$ needs to be increasing for some $B \in (\underline{B}, \overline{B})$. Figure (2) helps visualize this issue.

The slope of $q(B)B$ for $B > \overline{B}$ equals $(1+r)^{-1}$ because bond prices are constant and default probabilities are zero. The slope of $q(B)B$ for $B < \underline{B}$ is equal to zero, because bond prices are zero. For the intermediate range $(\underline{B}, \overline{B})$ the slope at equilibrium prices is:

$$\frac{\partial[q(B)B]}{\partial B} = \frac{1}{(1+r)} \left\{ [1 - F(y^*(B))] - f((y^*(B)) \cdot B \frac{\partial y^*}{\partial B} \right\} \tag{14}$$

Note that the sign of this derivative is ambiguous because bond positions $B \in (\underline{B}, \overline{B})$ are negative, and $y^*(B)$ is decreasing in B.[11] In fact if there exists some $B^* \in (\underline{B}, \overline{B})$ for which $\dfrac{\partial[q(B^*)B^*]}{\partial B^*} = 0$ this level of assets corresponds to the endogenous borrowing limit in the model. The government would never find optimal to choose a level of assets below B^* because it can always find some other contract such that consumption the current period

[11] In general $q(B)B$ may not differentiable at the points $\underline{B}$ and $\overline{B}$.

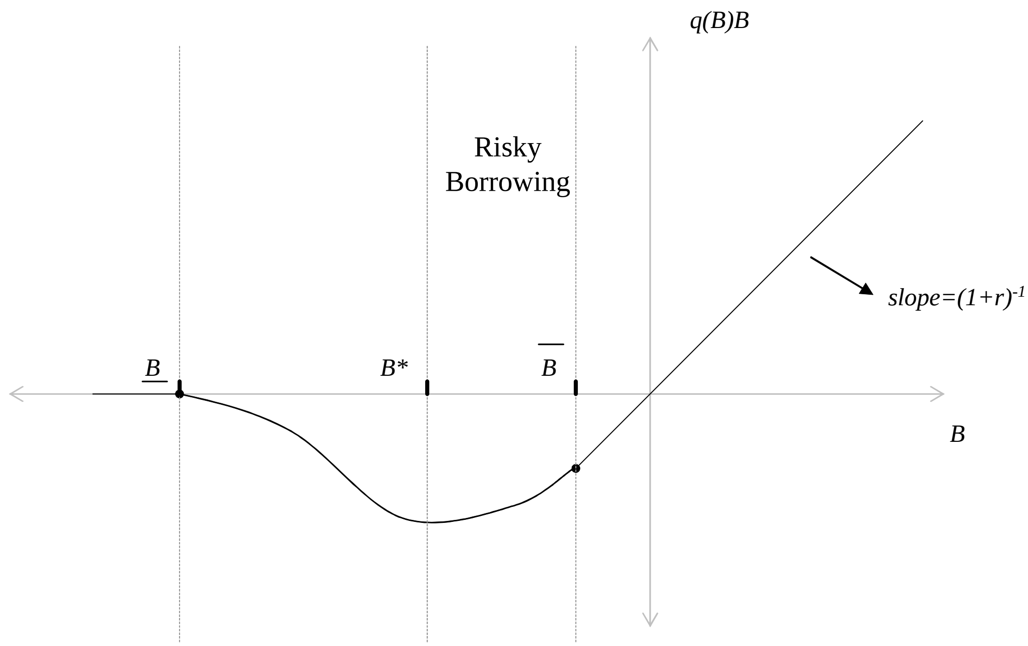

Figure 2: Total Resources Borrowed

increases by the same amount while incurring a smaller liability for next period. The region that is relevant for risky borrowing and thus for default to be an equilibrium outcome of the model is then $B \in (B^*, \overline{B})$.

The necessary condition for the above derivative to be positive within a range depends on the hazard function (i.e. $\frac{f(y)}{[1-F(y)]}$) of the probability distribution in the neighborhood of $\underline{y}$ relative to how fast the upper bound on the default sets increase with debt. The following proposition summarizes these findings:

Proposition 4. *Default can be an equilibrium outcome of the model for all probability distributions over* $[\underline{y}, \overline{y}]$ *satisfying the property:*

$$\lim_{B \to \overline{B}^-} \frac{1}{h(y^*(B))} > \lim_{B \to \overline{B}^-} B \frac{\partial y^*(B)}{\partial B}$$

where $h(y)$ *is the hazard function of the distribution.*

Proof. When the above condition holds, $\frac{\partial [q(B)B]}{\partial B} > 0$ in the neighborhood of $\overline{B}$ from the left. Given that $y^*(B)$ is continuous by Proposition 3, total resources borrowed increase for lower levels of assets in the region where assets carry a default premium. □

The hazard function $h(y^*(B))$ represents the instantaneous probability of default at B

for $B \leq \overline{B}$. The above condition requires that the instantaneous probability of default as B approaches $\overline{B}$ from the left is sufficiently small such that as B decreases (debt increases), the total resources borrowed increase. Due to Proposition 3, $y^*(\overline{B}) = \underline{y}$, and thus the condition is a restriction on the probability of the bad endowment shock. The government might then be willing to take on a risky loan, because in periods of low endowments it can increase consumption.

A sufficient condition for default to be an equilibrium outcome of the model is the following:

Corollary 4.1. *Default can be an equilibrium outcome of the model for all probability distributions over* $[\underline{y}, \overline{y}]$ *satisfying the property:*

$$\lim_{y \to \underline{y}} h(y) = 0$$

where $h(y)$ *is the hazard function of the distribution.*

Proof. See Appendix.

For all probability distributions satisfying the above property, the model will present a region in the state space where engaging in risky borrowing can increase consumption or $\dfrac{\partial[q(B)B]}{\partial B} > 0$, making default a positive probability event.

3.2 Case of *i.i.d* Nontradable Endowments

Now the role of volatile nontradable endowment is explored and its effect on default incentives. Here it is assumed that nontradable endowments follow an *i.i.d.* stochastic process, and without loss of generality that $\lambda = 1$, and $\theta = 0$.

Proposition 5. *Default incentives are stronger for low nontradable endowments if the cross derivative of the utility function is negative. For all* $y_1^N \leq y_2^N$ *, if* $y_2^N \in D(B)$*, then* $y_1^N \in D(B)$ if $u_{c^T c^N} < 0$.

Proof. See Appendix.

Given that assets are tradable denominated only, nontradable fluctuations affect default decisions by their effect on the utility of tradable consumption. When $u_{c^T c^N} < 0$, a low nontradable shock will tend to increase the marginal utility of tradable consumption, giving the government incentives for borrowing. In the region of default the economy experiences capital outflows and the fact that households cannot borrow as much as desired because of high interest rates and debt limits is relatively more costly for households who experience a low nontradable endowment if the above condition holds. Thus, default is more likely in the low nontradable states because repayment of tradable denominated loans is more costly.

For CES preferences it is well known that sign of this cross derivative depends on the relative magnitudes of the elasticity of intratemporal substitution between tradables and nontradables versus the intertemporal elasticity of substitution (see for example Obstfeld and Rogoff 1996 textbook). When the elasticity of intratemporal substitution between tradables and nontradables is greater the intertemporal elasticity of substitution the cross derivative is negative, and otherwise its positive. Intuitively, decisions on optimal asset holdings depend on nontradable output fluctuations due to the effect on the relative price of nontradables. The intertemporal elasticity through time and intratemporal elasticity between tradable and nontradable consumption pull the time path of nontradable prices in opposite directions. When the intratemporal elasticity is greater than the intertemporal elasticity a low nontradable shock today tends to produce a decreasing time path in the nontradable price, which gives incentives for borrowing in the current period.

The analytical characterization of the equilibrium in this section is for i.i.d. shocks, with bond prices not depending on the endowments' realization and only increasing in the level of assets demanded. However given that debt is used for insurance purposes, the policy function for assets is increasing in the endowments as in Hugget (1993), so that when the economy is hit by negative endowment shocks, more debt is demanded. This generates in the time series a negative correlation between interest rates and endowments even for i.i.d. shocks because higher debt is associated with higher interest rates. The following section analyzes the relation between interest rates and output for a persistent endowment process and where the negative relation between output and interest rates remains.

4 Simulations

4.1 Data for Argentina

Argentina in December of 2001 experienced one of the largest defaults in recent history, defaulting on \$100 billion dollars of their external government debt. It also experienced a severe economic crises with output decreasing about 20% percent at the time of the default. This section documents some statistics of the Argentinian economy corresponding to the period of default.

Table 2 presents business cycle statistics for Argentinian data that are quarterly real series taken from the Ministry of Finance (MECON) from 1993 to 2001[12]. The table also presents the percent deviation of the variables in Q1 2002, the default period. Output and consumption data are log, and the current account data are reported as a percentage of output. Real exchange rates are constructed from dollar nominal exchange rates (dollar per peso) using the ratio of the consumer price index for Argentina and the US. All data are filtered with a linear trend.

Aggregate output, tradable output and nontradable output are all negatively correlated with interest rates. This negative relations are much stronger in the default episode because in the crisis output plummeted and interest rates skyrocketed. Consumption is as volatile as nontradable output, and negatively correlated with interest rates. The negative relation is also magnified in the default episode. Real exchange rates prior to the default episode were extremely stable in this period, and uncorrelated with interest rates. On the other hand, in the default episode real exchange rates collapsed from 1 dollar per peso, to 0.4 dollars per peso or over 70% while interest rates spike by 20% giving a negative relation between devaluations an interest rates. The current account prior to default was moderately procyclical, but during the crisis Argentina experienced a sharp current account reversal because foreign credit dried up. As the table shows all variables experienced dramatic deviations at the time of the default.

Table 1. Statistics for Argentina

	Default episode	Prior default Q1 1993 - Q4 2001		
	$x : Q1 - 2002$	$std(x)$	$corr(x, y)$	$corr(x, r)$
Interest Rates	20.55	2.8	-0.3626	
Consumption	-20.79	4.8	0.8842	-0.6067
Tradable Output	-23.34	6.62	0.8375	-0.3809
Nontradable Output	-20.72	5.03	0.8851	-0.5478
Aggregate Output	-20.80	5.11		-0.3626
Current Account	2.45	1.5	-0.21	0.5033
Real Exchange Rate	-72.97	2.39	0.5115	0.0454

[12]The series are constructed such that all variables are consistent with the model's statistics. Specifically, aggregate output is denominated in terms of tradables and sectoral output and aggregate consumption is denominated in real terms deflated by the appropriate price deflator.

4.2 Calibration and functional forms

In this section the model is solved numerically to asses its quantitative implications regarding business cycle properties of interest rates, real exchange rates and consumption together with default episodes' dynamics. The main issue of interest in the quantitative analysis is to address whether adding an endogenous default decision to a very simple endowment model can help account for the real dynamics observed in emerging markets in times of defaults and crises. The benchmark model is calibrated to match certain features of the Argentinian economy.[13]

The following utility function is used in the numerical simulations:

$$u(c_t^T, c_t^N) = \frac{c(c_t^T, c_t^N)^{1-\sigma}}{1-\sigma},$$

where $c(c_t^T, c_t^N)$ is the constant elasticity of substitution aggregator

$$\left[\omega\left(c_t^T\right)^{-\eta} + (1-\omega)\left(c_t^N\right)^{-\eta}\right]^{-\frac{1}{\eta}}.$$

The parameters of the benchmark model are calibrated to mimic some of the empirical regularities in the Argentinian economy or taken from other emerging markets studies. For the preference parameters, the risk aversion coefficient is set to 5 which is a common value use in real business cycles studies for emerging markets. The elasticity of substitution between tradable and nontradable consumption $1/(1+\mu)$ is taken from Gonzales and Neumeyer (2003), where they estimate the elasticity for Argentina to be equal to 0.48.

To calibrate the relative sizes of the tradable and nontradable sectors in Argentina, we follow the standard methodology of assessing the degree of tradability of goods by computing the share of total trade (exports plus imports) of each sector as a percentage of total sectoral gross output. We find that the agricultural, manufacturing and energy sectors have a high degree of tradability, with an average share in this period of 0.38 , 0.78 and 0.34 respectively. Nontradable output includes construction and all service sectors, which have a degree of tradability of less than 5%. An interesting fact to note is that in Argentina the size of the tradable sector is small, constituting only 26% of output. We normalize mean $y^T = 1$, and then set mean y^N equal to 2.78.

The weight on tradable consumption in the CES aggregator, ω, is set to normalize the

[13] The model is solved by a value function iteration algorithm that allows for the bond price vector to be endogenous. Specifically, endogenous and exogenous states are discretized, and the model is solved by iterating on the value function for an intial guess of the bond price vector. The bond price vector is updated with a Gauss-Seidel algorithm, using the creditors equilibrium zero profit condition. The procedure is repeated until the convergence criterion is met for the bond price vector.

relative price of nontradables to be equal to one in the steady state when the economy is in autarky. The probability of reentering financial markets after default, θ is set to 0.5 which is in line to the estimates of Gelos et. al. (2002). They find that during the default episodes of the 1990's economies were excluded from the credit markets on average less than 1 year. For the benchmark calibration, the fraction of output lost in times of default, $(1-\lambda)$, is set equal to 0.02, which is the percent in output contraction estimated by Puhan and Sturzenegger (2002) following the default episodes in the 1980's in Latin America. In the sensitivity analysis we explore the effects of other default penalties.

The time preference parameter β is set to 0.87 which is lower than standard business cycle studies. We need a relatively low β for default to arise in equilibrium. Although lower β's allows somewhat higher default probabilities, the relation is not monotonic. If for example $\beta = 0$, no debt could be allowed in equilibrium and thus in the limiting distribution the default probability will be equal to zero.

Table 2.
Parameter Values

Elasticity of Substitution	$1/(1+\eta) = 0.48$	Gonzales and Neumeyer (2003)
Weight on CES	$\omega = 0.1061$	Normalization $p^N = 1$ in autarky steady state
Tradable Share	$y^N/y^T = 2.78$	$y^N/y^T = 2.78$ Argentina
Risk Free Interest Rate	$r = 0.01$	US quarterly interest rate
Output loss in default	$1-\lambda = 0.02$	Sturzenegger (2003)
Probability of Re-entry	$\theta = 0.5$	Gelos et al. (2002)
Discount Factor	$\beta = 0.87$	
Risk Aversion	$\sigma = 5$	

The stochastic processes for the sectoral output are estimated from the Argentinian linearly detrended data and they are assumed to be AR(1) processes where the innovations to the shocks are allowed to be correlated.

$$y_t^T = \rho^T y_{t-1}^T + \varepsilon_t^T$$

$$y_t^N = \rho^N y_{t-1}^N + \varepsilon_t^N$$

$E[\varepsilon^T] = E[\varepsilon^N] = 0$ and the variance -covariance matrix of the error terms is the following:

$$E[\varepsilon^{T\prime}\varepsilon^{N}] = \begin{bmatrix} \sigma^{T} & \sigma^{TN} \\ \sigma^{TN} & \sigma^{N} \end{bmatrix}$$

The above VAR process is estimated by a seemingly unrelated regression method using a two step procedure to get the GLS estimates. The following are the estimated coefficients: $\rho^N = 0.5546$, $\rho^T = 0.0878$, $\sigma^T = .0062$, $\sigma^N = .0028$ $\sigma^{TN} = .0038$. Each shock is then discretized into a 11 state Markov chain by using a quadrature based procedure (Hussey and Tauchen 1991) from their joint distribution. Table 2 summarizes the parameter values.

4.3 Simulation Results

Table 3 presents business cycle statistics for the benchmark calibration of the model. The business cycle statistics are mean values from 100 simulations of 38 observations each. The model was simulated for 100,000 periods and the time series statistics chosen were the ones containing a default to compare the model with the data in Argentina. The model does not have predictions for interest rates in times of default, because the economy is assumed to be excluded from the market. But the model predicts that in expectation of a default interest rates should compensate investors for a positive default probability. Thus the time series chosen were the 38 observations right before a default occurred and the model statistics should be compared with the data for Argentina just prior the default which is the one provided in Table 1. The simulated data are log and filtered equally as the Argentinian data. In addition, the denomination of the statistics in the model is consistent with that of the data.

Overall the model can match several features of the Argentinian economy. Interest rates are negatively correlated with aggregate, tradable and nontradable output, and consumption in the business cycle. The correlations between consumption and sectoral and aggregate output are positive and in line with the data. On the downside, the model does not match the correlations of the current account with output and interest rates observed . Debt is used in the model to smooth output variations. Households generally want to run down their assets in periods of low output, and engage in precautionary savings in periods of relatively high output. Thus, as in any insuring type model of debt without investment, current account and output should be positively correlated. However, this model produces a very low positive correlation, once the risk of default is modeled. This is because bonds in this model are sometimes not very good insurance assets as interest rates are high exactly in periods of low output. As it will be shown below, in the region of default the model presents a negative relation between current account and output.

Aggregate output is defined in terms of tradables and is more volatile than sectoral output. Consumption (CES) is as volatile as nontradable output. The volatility of interest rates is lower than the one observed for Argentina. Note though, that interest rate volatility in this model is endogenous and it is only due to volatile default premium. Risk neutral pricing of the endogenous risk of default can explain little of the volatility of country interest rates. The only mechanism in the model for volatile interest rates are varying default probabilities due to volatile endowments. The model does not address other sources of interest rate volatility such as the volatility in the international interest rate and the feedback and magnifying effects that volatile interest rates can have on output.

The real exchange rate is the most volatile variable in the model, being more volatile than aggregate output and consumption. The volatility of the real exchange rate comes from the volatility and covariation of the exogenous nontradable output and the endogenous tradable consumption through equation (4). Endogenous time varying interest rates and debt limits make tradable consumption much more volatile in this model, which is an important driving mechanism of the high volatility of the real exchange rate. The comparison of the behavior of real exchange rates with the Argentinian data prior default should be done with caution because in this period Argentina was under the convertibility plan, thus the low volatility and zero correlation with interest rates observed. When looking at the default period we see that real exchange rates collapse and interest rates spike. The negative correlation between interest rates and exchange rates seems a regularity in the data for emerging economies[14].

[14] The contemporanous correlation of real exchange rates and interest rates for the same time period is -0.35 in Korea, -0.84 in Mexico, and -0.07 in Brazil.

Table 3.
Statistics for the Benchmark Model

	Default Episode	$std(x)$	$corr(x,y)$	$corr(x,r^c)$
Interest Rates	0.66	0.06	-0.772	
Consumption Tradable	-7.15	5.36	0.9223	-0.8169
Consumption CES	-5.79	5.46	0.6416	-0.6767
Nontradable Output	-5.73	5.63	0.5153	-0.5946
Tradable Output	-6.18	7.7	0.7576	-0.6204
Aggregate Output	-6.00	6.77		-0.772
Current Account	7.71	1.17	0.2314	-0.1153
Real Exchange Rate	-0.51	7.65	0.5369	-0.2568
Debt Limit (% tradable output)	0.35			
Default Probability	0.2 %			
Mean Spread	0.2 %			

The endogenous borrowing limit $\underline{B}$ for this economy is equal to -35% percent of tradable output or 9% percent of total output. The price of loans above this limit would be zero because default would be a certain outcome. The risk free borrowing limit $\overline{B}$ is equal to -29% of tradable output. This limit corresponds to the maximum level of debt that gives the economy incentives for repayment in all states. The mean level of assets in the distribution conditional on not defaulting, is -27% of tradable output. Thus the economy on average is a net debtor. These endogenous borrowing constraints affect equilibrium allocations because they limit the ability of the economy to share risk. The borrowing limits are tight in this economy because the relative benefit from participating in the international financial markets are small[15].

Default is a rare outcome in the model. Annualized default probabilities are 0.2%. This is lower than the case for Argentina, that has experienced 3 default episodes in the past 100 years. Nevertheless, default is a positive probability event that affects equilibrium allocations and prices. The reason why default is so rare has to do with the intuition of proposition 4. The upper sloping portion of the function $q(B)B$ in the region with positive default probabilities is very small, with B^* equal to -33% of tradable output. This means that the set of y for which the economy finds it optimal to default expands fast once the economy hits the risk free debt limit. That is for a given level of negative assets (debt) B the difference between

[15] The fact that the costs in terms of lifetime utility from being in autarky is small is related to the small costs of business cycles in Lucas' 1987. Gourinchas et. al. (2002) document on the marginal benefits of financial integration relative to autarky.

$v^c(B, y)$ and $v^d(y)$ across exogenous states y does not change too much. In the case analyzed in the section 3.1 for i.i.d. stochastic tradable shocks, this is equivalent than saying that the threshold $y^*(B)$ is very steep.

The mean annual spread (defined as the difference between the country interest rate and the risk free interest rate, $r^c - r$) in the limiting distribution of assets conditional on not defaulting is 0.2%. Thus equal to the average default probability of the model. The average spread for Argentina is equal to 8.7% for the period 1993-2001. Thus the model falls short in this regard. Note that in the model there is a one to one matching from default probabilities to interest rates due to risk neutral lenders, and thus without some other specification for lenders the model cannot obtain these two variables unequal as in the data.

However, the fact that default probabilities do not account for all the yield spread in bonds is a well known puzzle in the finance literature on corporate defaultable bonds, even in the presence of risk averse lenders. Huang and Huang (2003) find that in investment grade bonds, default probabilities and credit risk account only for 19% of the yield observed in defaultable bonds. In fact, for shorter maturity bonds, they find that default risk accounts for almost nothing of the yield.[16]

The upper panel of figure (3) plots the price schedule faced by the economy for the highest and lowest tradable endowments, holding constant the nontradable output, and for the highest and lowest nontradable endowments, holding the tradable endowment constant. Bond prices are an increasing function of assets. That is, larger levels of debt are associated with higher interest rates. For values of debt of up to 29% of mean tradable output, the economy faces low interest rates. Effectively, it is charged the risk free interest rate for these loans. At this level, default incentives start to increase in the economy giving rise to higher interest rates. At debt levels of about 35% of tradable output, households refuse to pay any liabilities for all endowment shocks and thus the probability of default for this debt level or higher is 1.

A feature of the model is that it produces a narrow range for asset positions that carry positive but finite risk premia (i.e. the range $B \in (\underline{B}, \overline{B}]$). This is because the range of assets that have positive risk premia is determined by the $B's$ that make the default value and the contract value equal across states and this difference does not vary significantly. The narrow range of risky assets is part of the reason why default is rare in the model. Below we explore how alternative punishment specifications can alter the range for risky lending.

In this benchmark calibration the economy is more likely to default in high nontradable

[16]The majority of the finance studies on defaultable bonds use reduce form models that take as exogenous the process for the default probabilities.

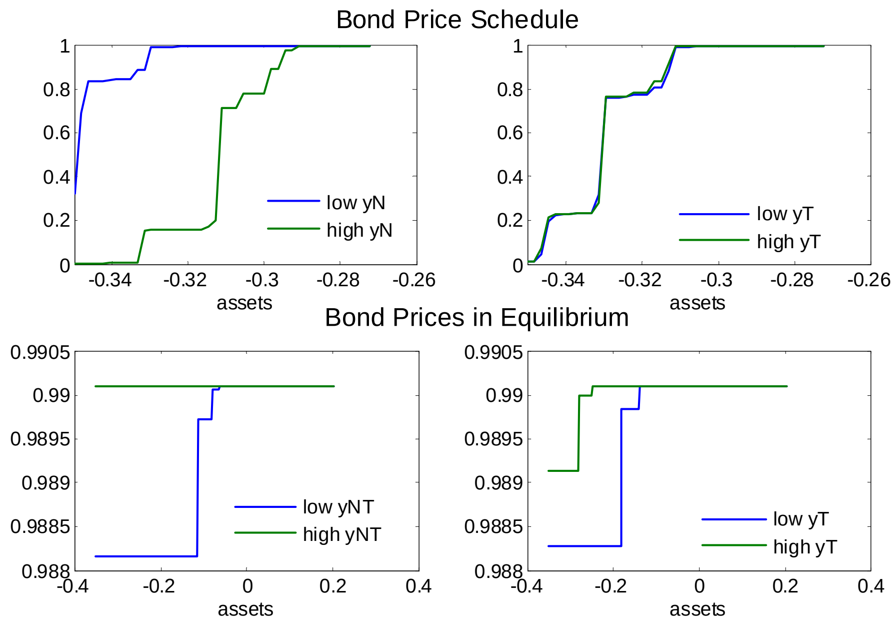

Figure 3: Bond Prices

endowments and thus for a given level of borrowing B' it faces a higher interest rate premium in these states. For *i.i.d.* shocks with CES preferences the relation between default incentives and nontradable endowments depends only on the relative elasticities. Thus given our elasticities parameters default occurs in low nontradable endowments with i.i.d. shocks. Once shocks deviate from the *i.i.d.* case, this is no longer the case, and in fact for this calibration default occurs first for high nontradable endowments. In terms of tradable endowments, as assets decrease the economy defaults first for intermediate levels of tradable endowments, although the difference in the bond price schedule is almost negligable given the low persistence level in this sector.

The persistence of shocks is an important determinant of the relation between default and output. Intuitively, even though the costs of defaulting are larger in recession because outflows are more costly in bad times with concave utility, the relative benefits from defaulting, i.e. autarky, increase also with persistent shocks.

The lower panel of figure (3) shows the actual bond price the economy pays along the equilibrium path by its choice of borrowing. Along the equilibrium when the economy is in a recession it chooses higher levels of debt, and thus the bond price is lower. This is why the simulations produce the negative correlation between output and interest rates.

An interesting feature of the model is that capital outflows (*i.e.* $y^T - c^T$) can be occur

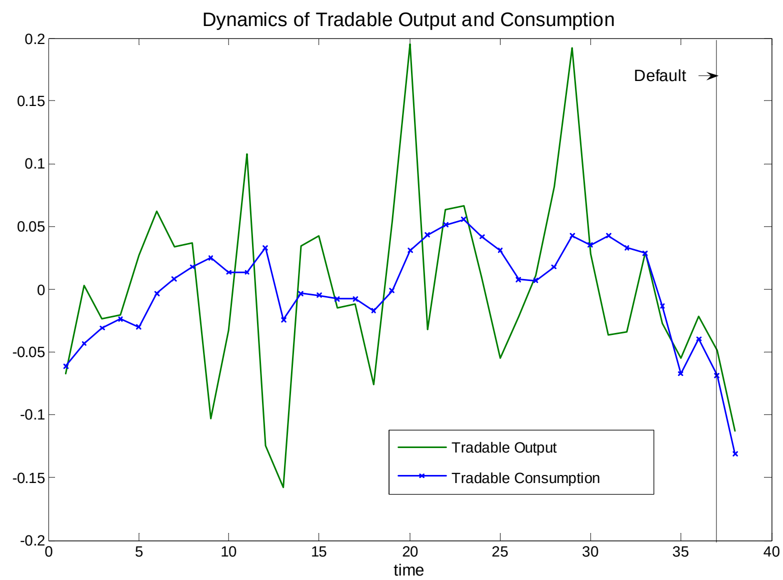

Figure 4: Capital Outflows

in a recession because default probabilities are high. When interest rates are low, debt is a good insurance instrument: capital outflows are large in good times because the economy saves, and are low in bad times because the economy borrows. However, when the economy is highly indebted and interest rates are high due to high default probabilities, debt becomes a less good insurance instrument. In fact, given that interest rates are countercyclical, the economy can face capital outflows in a recession. This result is similar to Atkeson's (1991) result, where he shows that in an insurance model of debt that features moral hazard and unenforceability of debt contracts, the optimal debt contract will feature capital outflows in a recession. Here, what drives the result is the incompleteness of assets and the fact that default is an equilibrium outcome.

Figure (4) plots the time series for tradable output and consumption. In normal times, the asset available to the economy helps with consumption smoothing. However when the economy has large levels of debt the economy experiences capital outflows in recessions (periods 35 and 36). And it precisely for this reason that the economy prefers to default (period 37). This feature of the model matches the empirical regularity that emerging markets in crises are not able to use the international markets for smoothing and experience net capital outflows.

Real exchange rates are determined by equation (4). The economy faces real exchange rate devaluations if tradable consumption is low, or if nontradable consumption is high. The

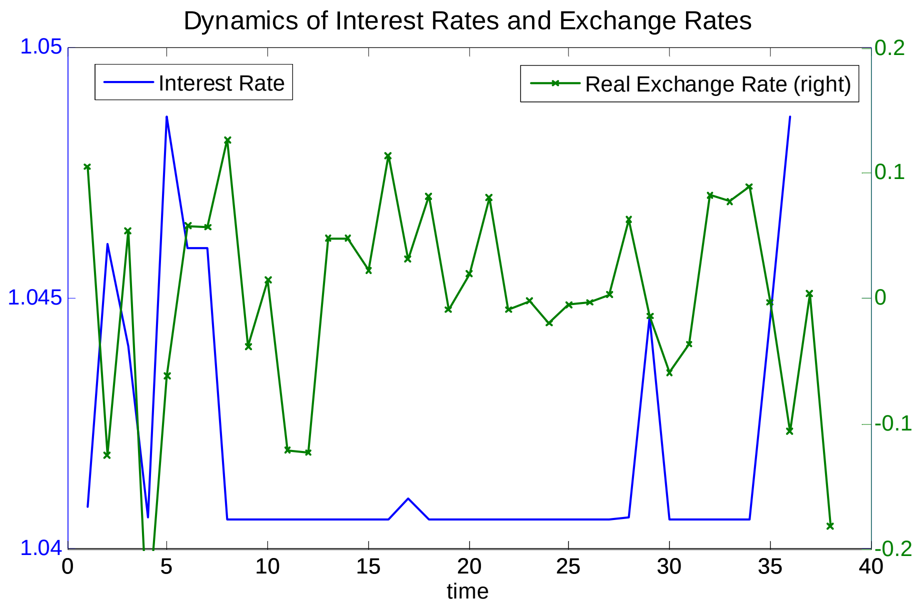

Figure 5: Real Exchange Rates

reason why real exchange rates are weakly negatively related to interest rates is because along the equilibrium path interest rates are negatively related to both tradable and nontradable consumption. Figure (5) shows the dynamics of interest rates and real exchange rates with the economy choosing default in period 37. In this example the period of high interest rates prior to default was due to low tradable consumption, and thus real exchange rates and interest rates move in opposite directions.

Figure (6) plots the dynamics for interest rates and aggregate consumption (CES) for the same simulation of shocks than the previous graphs. The figure shows graphically the negative correlation between consumption and interest rates, which is a feature of the model that is robust across simulations. Not all interest rates spikes follow by a default as it is evident from period 7 of the shown simulations, but it is generally true that spikes in interest rates are accompanied by periods of low consumption and output.

4.4 Experiments

4.4.1 One Sector Model

In this experiment the benchmark model is compared with a one sector endowment model where all output is composed of tradable goods. The mean and volatility of output in the one sector model is such that it equals the two sector benchmark aggregate output. We find

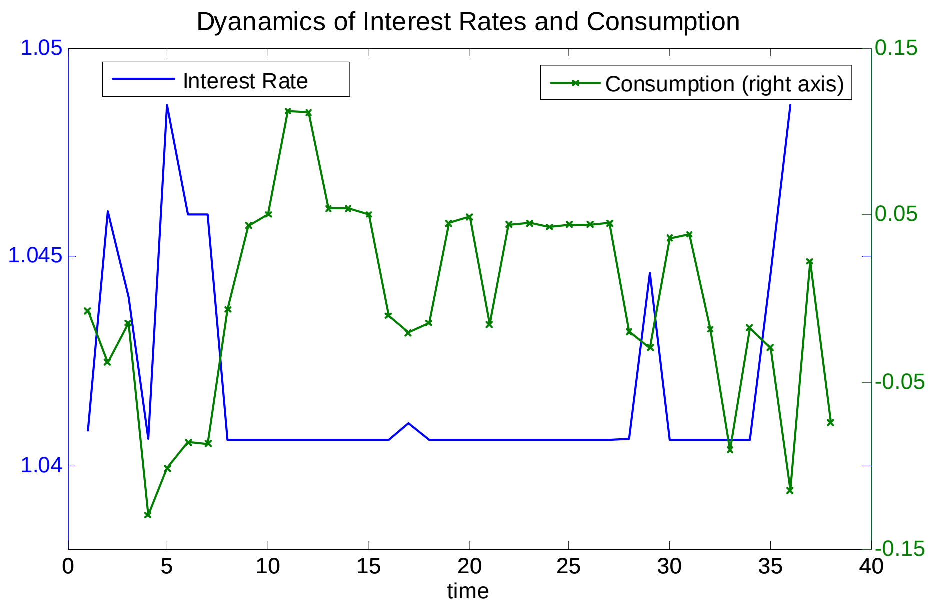

Figure 6: Countercyclical Interest Rates

that this economy would have no default in the long run distribution with the benchmark calibration. However, an interesting result from this experiment is that this economy would face a looser debt limit of 37% of aggregate output, whereas in the benchmark calibration the debt limit is 9% of aggregate output. That is, the economy is able to access greater credit when output is composed of tradable goods only. The intuition of this result is that, from an incentives perspective, an economy with larger tradable sectors will benefit more from greater access to international borrowing to smooth fluctuations. This tends to suggest that economies with relatively larger tradable sectors would have greater access to international credit markets.

4.4.2 Trade Default Penalties

It has been documented by Rose (2002) that another reason why countries repay their debts is because trade decreases by 8% a year over and above any decrease in output after default. In this final section we explore how trade penalties might affect incentives to default within the context of our model. For this purpose we modify the tradable consumption to account for an importable and exportable sector.

$$c^T = \left[\alpha \left(c^H\right)^{-\eta} + (1-\alpha)\left(c^F\right)^{-\eta}\right]^{-\frac{1}{\eta}}$$

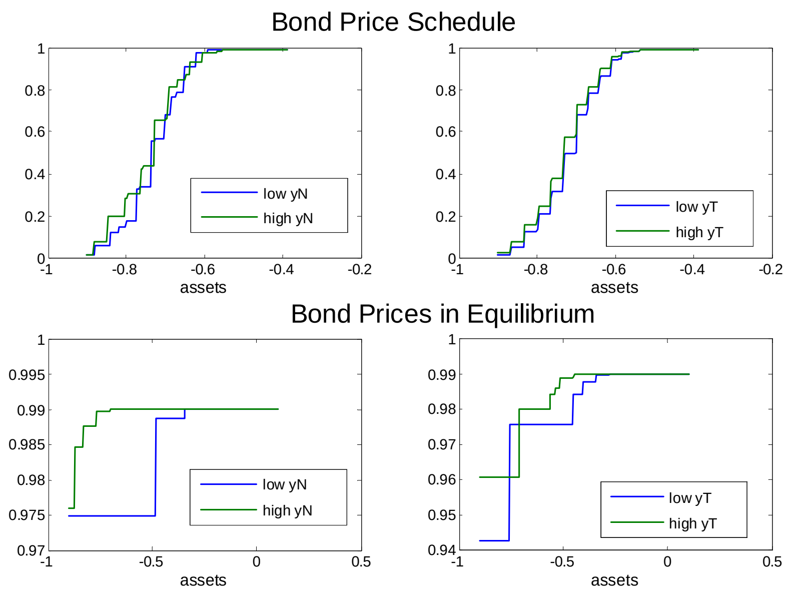

Figure 7: Bond Prices with Trade Penalties

We assume that the terms of trade are constant and equal to 1 and thus absent of default this specification is exactly equal than the specification presented for the benchmark model. It is assumed that $\alpha = 0.5$ and the elasticity of consumption equals the elasticity of tradable and nontradable goods above. All other parameters are equal to the benchmark model. To model trade penalties it is assumed that c^F decreases by 8% a year in the periods when the country is in financial autarky after a default.

The main feature of this economy is that it increases the range for risky borrowing $(\underline{B}, \overline{B})$ as it is evident from figure (7). The bond price schedule that the economy faces in this case is a much more smooth function of debt.

The reason why trade penalties increase the range of risky borrowing is that the difference between the value of staying in the contract and the value of autarky is more sensitive to the shock. Autarky is essentially more equal across states for the case of trade penalties because the drop in c^F is independent of the shock as c^F decreases by 8% from its mean level prior default.

Table 4 shows that these type of penalties for defaulting can have significant effects on default probabilities and equilibrium interest rates. With the benchmark calibration, the default probability increases to 1.2% , interest volatility increases to 2.7 , debt limits are much looser and equal about 100% of tradable output, and the correlation of output and interest rates is negative. Overall, this specification for penalties gives the model greater flexibility in being able to capture the high spreads observed in emerging markets foreign debt.

Table 4.
Trade Penalties

Mean$(r^c - r^*)$	Default Rate	Debt limit	$std(r^c)$	$corr(y,r^c)$	$corr(c,r^c)$	$corr(p^c,r^c)$
1.37	0.012	0.9875	2.7	-0.7727	-0.8416	-0.2822

This experiment suggests that more evidence is needed regarding the type of punishments economies encounter after defaulting on their international debt. In addition more theoretical work is needed in understanding the interdependence of credit and trade relations.

5 Conclusion

This paper models endogenous default risk in a stochastic dynamic framework of a small open economy that has two sectors and that features liability dollarization. The paper presents a model where interest rates respond to output fluctuations through endogenous time-varying default probabilities. The main contributions of the paper are three. First, it studies analytically the relationship between default and output in an environment of incomplete markets and provides conditions for default to be an equilibrium outcome. Second, it explores quantitatively the predictions of the model in explaining the real dynamics observed in default episodes. The model matches several features of the data in Argentina such as the negative correlation between output and consumption with interest rates. In times of default the model presents sharp declines in output and consumption and significant devaluations in real exchange rates. Third, it explores the effects that fluctuations and relative sizes of nontradable sectors have on incentives to default on tradable denominated debt. The model matches the data in that real exchange rate devaluations are correlated with high interest rates. An empirical implication of the model is that economies with relatively small tradable sectors have higher incentives to default on tradable denominated debt and thus face higher interest rates.

Even though the model is able to explore default incentives and can match interest rate counter-cyclicality, it cannot match the magnitude of the interest rate spread and volatility observed in the data of emerging markets. This is because the model assumes that creditors are risk neutral and it lacks a feedback mechanism through which volatile and high interest rates affect output. Interesting extensions of the model would be addressing these issues by exploring alternative specifications for creditors, modeling default penalties as a bargaining outcome (see Yue 2004 for some recent work studying this issue), and adding a feedback from interest rates to production.

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Appendix A

Proposition 1. *Default is more likely the higher the level of debt: For* $B^1 < B^2$ *, if default is optimal for* B^2*, in some states* y*, then default will be optimal for* B^1 *for the same states* y*., that is* $D(B^2) \subset D(B^1)$.

This result is similar than Eaton and Gersovitz (1981) and Corbae et at (2001).

For all $\{y^T, y^N\} \in D(B^2)$, $u(y^T, y^N) + \beta E v^d(y') > u(y^T + B^2 - qB', y^N) + \beta E v^o(B', y')$. Since $y^T + B^2 - qB' > y^T + B^1 - qB'$ for all B', thus $u(y^T + B^2 - qB', y^N) + \beta E v^o(B', y') > u(y^T + B^1 - qB', y^N) + \beta E v^o(B', y')$. That is the value of the contract under no default is increasing in foreign asset holdings. Hence $u(y^T, y^N) + \beta E v^d(y') > u(y^T + B^1 - qB', y^N) + \beta E v^o(B', y')$, which implies that $\{y^T, y^N\} \in D(B^1)$ ■

The Case of i.i.d. Endowment Shocks

Proposition 2. *Default incentives are stronger the lower the tradable endowment. For all* $y_1^T < y_2^T$ *, if* $y_2^T \in D(B)$*, then* $y_1^T \in D(B)$.

In order to prove proposition 2, we will first prove the Lemma 2.1

Lemma 2.1 . *If for some* B*, the default set is non empty* $D(B) \neq \emptyset$*, then there are no contracts available for the economy* $\{q(B'), B'\}$ *such that the economy can experience capital inflows,* $B - q(B')B' > 0$

This is a proof by contradiction.

Suppose there are contracts $\{q(B'), B'\}$ available to the economy such that $B - q(B')B' > 0$. But that the economy choose under the contract utility some $\widehat{B}$ to maximize utility such that $B - q(\widehat{B})\widehat{B} < 0$ and then finds default to be the optimal option because $u(y^T, y^N) + \beta E v^d(y') > u(y^T + B - q(\widehat{B})\widehat{B}, y^N) + \beta E v^o(\widehat{B}, y')$.

Now note that under all contracts $\{q(B'), B'\}$ such that $B - q(B')B' > 0$ staying in the contract is always preferable to default because $E v^o(B', y') \geq E v^d(y')$, and $u(y^T + B - q(B')B', y^N) > u(y^T, y^N)$. This implies that $\widehat{B}$ cannot be the maximizing level of assets and then find default to be optimal, because it is a contradiction.

Thus, if $D(B) \neq \emptyset$, then $\exists$ some $y \in Y$, such that $v^d(y) \geq v^c(B, y)$, or equivalently, $u(y) + \beta E v^d(y') \geq u(y + B - q(B')B') + \beta E v^o(B', y')$.

Given that B' is chosen to maximize the value of the contract, then if default is preferable, it must be the case that not only $B - q(B')B' < 0$, but that $\nexists$ a contract available $\{q(B'), B'\}$such that $B - q(B')B' > 0$ □

Now we prove proposition 2.

If $y^2 \in D(B)$ then by definition $u(y_2^T, y^N) + \beta E v^d(y') \geq u(y_2^T + B - q(B')B', y^N) + \beta E v^o(B', y')$

If

$$\begin{aligned} & u(y_2^T + B - q(B^2)B^2, y^N) + \beta E v^o(B^2, y') - \\ & \left\{ u(y_1^T + B - q(B^1)B^1, y^N) + \beta E v^o(B^1, y') \right\} > \\ & u(y_2^T, y^N) + \beta E v^d(y') - \left\{ u(y_1^T, y^N) + \beta E v^d(y') \right\} \end{aligned} \tag{15}$$

then $y^2 \in D(B)$, implies $y^1 \in D(B)$,.

Now it is necessary to show that expression (15) holds.

Given that shocks are iid, the right hand side of equation (15) simplifies to $\left[u(y_2^T, y^N)\right] - \left[u(y_1^T, y^N)\right]$

Because of utility maximization:

$$u(y_2^T + B - q(B^2)B^2, y^N) + \beta E v^o(B^2, y') \geq u(y_2^T + B - q(B^1)B^1, y^N) + \beta E v^o(B^1, y')$$

Thus if

$$\begin{aligned} & u(y_2^T + B - q(B^1)B^1, y^N) + \beta E v^o(B^1, y') - \\ & \left\{ u(y_1^T + B - q(B^1)B^1, y^N) + \beta E v^o(B^1, y') \right\} > \\ & \left\{ \left[u(y_2^T, y^N)\right] - u(y_1^T, y^N) \right\} \end{aligned} \tag{16}$$

holds then through transitivity expression (15)holds.

Simplifying (16):

$$u(y_2^T + B - q(B^1)B^1, y^N) - u(y_1^T + B - q(B^1)B^1, y^N) > \left[u(y_2^T, y^N)\right] - \left[u(y_1^T, y^N)\right]$$

Now, note that due to Lemma 2.1, if $y^2 \in D(B)$ then $B - q(B')B' < 0$ for all available $\{q(B'), B'\}$ thus $B - q(B^1)B^1 < 0$.

Hence, given that utility is increasing and strictly concave in both arguments, then (16) holds, which implies that $y^1 \in D(B)$.$\square$

Proposition 3. *If default sets are non-empty, then they are closed intervals where only the upper bound depends on the level of debt:*

$$D(B) = \ [\underline{y}, y^*(B)] \qquad\qquad \text{for } B \leq \overline{B}$$

where $y^*(B)$ *is a continuous, non-increasing function of* B, *such that:*

$$y^*(B) = \left\{ \begin{array}{lr} y^*(B) : v^d(y^*(B)) = v^c(B, y^*(B)) & \text{for } \ \underline{B} \leq B \leq \overline{B} \\ \overline{y} & \text{for } \quad B < \underline{B} \end{array} \right\}$$

Proof:

For $B < \underline{B}$, $\ D(B) = Y$, so that $y^*(B) = \overline{y}$

For $\underline{B} \leq B \leq \overline{B}$ let $\Psi(B, y) \equiv v^d(y) - v^c(B, y)$

$$\Psi(B, y) = u(y) + \beta E v^d(y') - u(y^T + B - q(B')B'(B, y), y^N) - \beta E v^o(B'(y, B), y')$$

$$\begin{aligned} \frac{\partial \Psi}{\partial y^T} \ &= \ u_{c^T}(y^T, y^N) - u_{c^T}(y^T + B - q(B')B'(B, y), y^N) + \\ &\frac{\partial B'(B, y)}{\partial y^T} \left[\frac{\partial [q(B')B']}{\partial B'} u_{c^T}(y^T + B - q(B')B'(B, y), y^N) - \beta \frac{\partial \left[E v^o(B'(y, B), y') \right]}{\partial B'} \right] \end{aligned}$$

The term in brackets is exactly the first order condition of the government's problem and thus it is equal to zero. Thus,

$$\frac{\partial \Psi}{\partial y^T} = u_{c^T}(y^T, y^N) - u_{c^T}(y^T + B - q(B')B'(B, y), y^N)$$

The sign of the above derivative depends on whether $B - q(B')B'$ is greater or less than zero. In general, this can be positive or negative because of the insurance type use of debt. But for all $B \in [\underline{B}, \overline{B}]$, default sets are non-empty, and so $B - q(B')B' < 0$ due to Lemma 2.1

Thus for $B \in [\underline{B}, \overline{B}]$, $\dfrac{\partial \Psi(B, y)}{\partial y^T} < 0$.

Given that default sets are non-empty and strictly less than the entire endowment space for $B \in (\underline{B}, \overline{B}]$, then for some y default is preferable, and for some y repayment is preferable within this range. But given that $\Psi(B, y)$ is monotonically decreasing in y for all $B \in [\underline{B}, \overline{B}]$, then there exists a unique y^* such that for value $y \leq y^*(B)$ default is preferable, and for $y > y^*(B)$ repayment is preferable, where $v^c(B, y^*(B)) = v^d(y^*(B))$. And thus default sets can be characterized by closed intervals where only the upper bound depends on the level of debt.

Now using the implicit function theorem :

$$\frac{\partial y^T}{\partial B} = \frac{v^c_B(B, y)}{v^d_{y^T}(y) - v^c_{y^T}(B, y)} = \frac{v^c_B(B, y)}{u_{c^T}(y^T, y^N) - u_{c^T}(y^T + B - q(B')B'(B, y), y^N)} < 0.$$

which says that for $B \in [\underline{B}, \overline{B}]$, $y^*(B)$ is decreasing in B.$\square$

Corollary 4.1. *Default can be an equilibrium outcome of the model for all probability distributions over* $[\underline{y}, \overline{y}]$ *satisfying the property:*

$$\lim_{y \to \underline{y}} h(y) = 0$$

where $h(y)$ *is the hazard function of the distribution.*

Proof.

The condition that the slope of $q(B) \cdot B$ be positive stated in terms of the hazard function of the distribution:

$$\frac{\partial [q(B)B]}{\partial B} = [1 - F(y^*(B))] - f((y^*(B)) \ B\frac{\partial y^*}{\partial B} > 0$$

or

$$\frac{1}{h(y^*(B))} > B\frac{\partial y^*}{\partial B}$$

Note that

$$\lim_{B \to \overline{B}^-} y^*(B) = \underline{y}$$

due to Proposition 3. Thus for distributions satisfying the above condition:

$$\lim_{B \to \overline{B}^-} \frac{1}{h(y^*(B))} = \infty$$

The only thing we need to prove now is that the $\lim\limits_{B \to \overline{B}^-} \dfrac{\partial y^*(B)}{\partial B}$ is finite

$$\lim_{B \to \overline{B}^-} \frac{\partial y^*(B)}{\partial B} = \lim_{B \to \overline{B}^-} \frac{v_B^c(B, y)}{u'(y^T, y^N)c_{c^T} - u'(y^T + B - q(B')B'(B, y), y^N)c_{c^T}}$$

The numerator of the above expression is finite and positive for finite B. Note that the only y which we need to consider the limit, is $\underline{y}$. And specifically that

$\lim_{B \to \overline{B}^-} B - q(B'(B))B'(B, \underline{y}) < 0.$

which holds by continuity due to Lemma 2.1. □

Proposition 5. *Default incentives are stronger for low nontradable endowments if the cross derivative of the utility function is negative. For all* $y_1^N \leq y_2^N$ *, if* $y_2^N \in D(B)$*, then* $y_1^N \in D(B)$ if $u_{c^T c^N} < 0$.

Using the exact same strategy than the one used in proposition for proposition 2 the condition needed to prove the proposition simplifies to depending only on period utility. That is if :

$$\left[u(y^T + B - q(B^1)B^1, y_2^N)\right] - \left[u(y^T + B - q(B^1)B^1, y_1^N)\right] > \left[u(y^T, y_2^N)\right] - \left[u(y^T, y_1^N)\right] \quad (17)$$

then for all $y_1^N < y_2^N$, if $y_2^N \in D(B)$, *then* $y_1^N \in D(B)$

Rearranging:

$$\left[u(y^T, y_1^N)\right] - \left[u(y^T + B - q(B^1)B^1, y_1^N)\right] > \left[u(y^T, y_2^N)\right] - \left[u(y^T + B - q(B^1)B^1, y_2^N)\right] \quad (18)$$

where $B - q(B')B' < 0$ for $B \in [\underline{B}, \overline{B}]$.

Let:

$$\Psi(y, B) = \left[u(y^T, y^N)\right] - \left[u(y^T + B - q(B^1)B^1, y^N)\right]$$

$$\frac{\partial \Psi(y, B)}{\partial y^N} = \frac{\partial u(y^T, y^N)}{\partial y^N} - \frac{\partial u(y^T + B - q(B^1)B^1, y^N)}{\partial y^N}$$

If $\dfrac{\partial^2 u(c^T, c^N)}{\partial c^T \partial c^N} < 0$ then $\dfrac{\partial \Psi(y, B)}{\partial y^N} < 0$, which then makes equation (18) hold. □