Directed Multilateral Matching in a Monetary Economy*

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Abstract

We consider a monetary economy with directed multilateral matching between buyers and sellers. A buyer chooses how much money to hold, observes the location of all sellers, and decides which seller to visit. The number of buyers that arrive at a particular seller is random due to lack of coordination. Every seller has a single indivisible good and all buyers have the same valuation for the good, though they may hold different amounts of money. The good is allocated according to a second price auction where buyers bid with their money rather than valuations. We show that in equilibrium ex ante identical buyers choose different money holdings: carrying more money is costly but it increases the probability of winning the auction. The unique equilibrium distribution of money holdings is analytically characterized. The entry of sellers is efficient at the Friedman rule but is suboptimal for higher inflation rates.

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1 Introduction

In recent years a large literature has developed that models trading frictions explicitly to deliver environments with micro-foundations for the existence of fiat money.[1] While successful in addressing many issues, some of the assumptions that are commonly used to describe decentralized trade have attracted criticism. In particular, trade is usually modeled to occur in random bilateral meetings between the agents in the economy, where 'random' refers to the fact that an agent may or may not meet someone with whom he would like to trade. However, the randomness of the matching process is at odds with the presumption that economic agents are generally aware, or can easily learn, where some commodity is traded.[2] Moreover, the assumption of pairwise meetings results in a bilateral monopoly which typically leads to bargaining as a way of determining the terms of trade.[3] Since the outcome of the bargaining game becomes intractable in the presence of private information, it is usually assumed that the money holdings of the agents is common knowledge. This is a particularly stringent and unnatural condition on the environment's information structure.

In this paper we try to address these two issues by modeling decentralized trade in a very different way which yields new insights while keeping with the spirit of the money search literature. First, we assume that matching between agents is directed: in every period the agents who want to consume (buyers) know where each productive agent (seller) is located and hence the actual process of finding a suitable match is deterministic.[4] Frictions are introduced by assuming that buyers can only visit one seller at a time and they cannot coordinate their decisions of which location to go to. Hence they randomly choose which seller to visit which results in stochastic local demand realizing at each seller's location. To simplify matters, we assume that the supply at each location is fixed and, in particular, that

[1] See Kiyotaki and Wright (1993) for the canonical model.

[2] For instance, see Howitt (2005) for a formulation of this critique.

[3] See Trejos and Wright (1995) or Shi (1995). Exceptions include Green and Zhou (1998, 2002).

[4] We should emphasize that our use of the term 'directed' rests on the fact that a buyer knows where he can find the agents with whom he would like to trade and he does not have to randomly search for them. This definition is different from the directed *search* literature (e.g. Burdett, Shi, and Wright (2001)) where a buyer's visit is directed by the (publicly observable) posted price of a seller.

each seller has a single indivisible good for sale. The additional frictions of anonymity and lack of double coincidence of wants make fiat money essential for trade in these meetings (see Kocherlakota (1998) and Wallace (2001)).

The second innovation of our paper is to use an auction to allocate the good among the, potentially multiple, buyers. Since there is a single unit for sale at each location, the price has to adjust to balance supply and demand. A natural way to model the price formation process is to have buyers compete in price for the good, i.e. to have Bertrand competition from the side of the *buyers.* This is equivalent to an ascending bid, or second price, auction where agents bid with their money holdings (as opposed to their valuations). The results is that the 'wealthiest' buyer purchases the good and pays the second highest money holdings to the seller. It is important to note that a buyer may hold less money than his actual valuation for the good, i.e. he may be willing to spend more but unable to do so. To summarize, we model the allocation of the good as a second price auction where buyers bid with their money holdings, rather than their valuations.[5]

This mechanism allows the money holdings of the agents to be revealed in a natural way. The terms of trade do not depend on how much money the winning buyer has but on how much he has to pay in order to outbid all competing buyers. It is worth stressing that agents in our model face randomness in consumption because local demand conditions are stochastic: a buyer can always visit a seller, but it may turn out that the seller's price is above the buyer's liquidity due to high demand. This is in contrast to models of random search where consumption uncertainty is typically due to the randomness of the matching process, i.e. due to whether a suitable match is found or not. This difference has important implications for the incentives to hold fiat money, as we elaborate below.

To introduce the explicit choice of liquidity for the agents and to keep our model tractable we embed the idea presented above in the framework of Lagos and Wright (2005), henceforth LW. That is, agents have periodic access to a centralized Walrasian market where they trade

[5]This is also known as a budget constrained auction. See Che and Gale (1998).

in a competitive way and they can re-balance their portfolios of fiat money. Furthermore, as in LW, we assume that the agents' preferences are quasi-linear which implies that there are no wealth effects in the demand for money. This means that the trading history of an agent does not affect his choice of money holdings, though, as we describe in the next paragraph, the resulting distribution of money holdings is non-degenerate in our set-up.

The main trade-off facing a buyer when he chooses how much money to carry is the following. Bringing more money allows a buyer to outbid more of his potential competitors, leading to a higher probability of consuming. On the other hand, holding fiat money is costly because the value of any unspent balances depreciates due to inflation and discounting. It turns out that even if buyers have the same valuation for the sellers' good, in equilibrium they choose to bring different amounts of money to the decentralized market. The distribution of money holdings is unique and it is characterized analytically. The intuition behind this result is not hard to see: if all buyers brought the same balances, then a deviant with infinitesimaly more money would win all the auctions and enjoy discretely higher probability of consuming for negligible additional cost. As a result, identical buyers are indifferent in equilibrium between holding a range of potential money balances.[6] Dispersion in money leads to dispersion in prices since the sale price depends on how many buyers visit a particular seller.[7]

To evaluate the welfare properties of our model we introduce an entry decision on the side of the sellers, as in Rocheteau and Wright (2005).[8] The main result is that entry is suboptimal except when the money supply contracts at the rate of time preference – the Friedman rule. The reason is that for inflation rates above the Friedman rule the value of fiat money depreciates over time and hence buyers bring less money than their real valuation for the sellers' good. This means that sellers receive less on average than their social value

[6]Dispersion of money holdings is typically a feature of models where an agent's liquidity *does* depend on his history of trades, e.g. Molico (2006), Green and Zhou (1998, 2002), or Camera and Corbae (1999).

[7]Price dispersion is also an equilibrium outcome of the monetary model of Head and Kumar (2005) where it results from informational asymmetries among buyers.

[8]Entry is the only efficiency margin because every meeting between a seller and some buyer(s) results in a trade, which is the efficient outcome.

to a match and therefore fewer sellers enter to the market than is optimal. At the Friedman rule, holding fiat money is costless leading buyers to bring balances equal to their valuation of the good because the intra-buyer competition dominates. It is interesting to note that in Rocheteau and Wright (2005) the number of sellers can be below *or* above the optimal.

The paper of Corbae, Temzelides, and Wright (2003) also introduces directed matching, though in a different cooperative way. Furthermore, they focus on bilateral meetings. The paper of Camera and Selcuk (2005) is close in spirit to ours. They consider a non-monetary environment with posted prices which are subject to renegotiation depending on realized local demand.

The rest of the paper is organized as follows. Section 2 describes the model and proves some preliminary results. The following section solves the buyer's problem and derives the equilibrium distribution of money balances, while section 4 describes the entry decision of sellers. Section 5 examines the efficiency properties of inflation and section 6 considers a number of extensions. Section 7 concludes.

2 The Model

Time is discrete and runs forever. Each period is divided in two subperiods, following LW: a centralized market ('day') characterized by Walrasian trading and a decentralized search market ('night') characterized by trading frictions that are modeled explicitly. There is a continuum of infinitely-lived agents who belong to one of two different types, called *buyers* and *sellers* (types b and s, respectively). The difference is that while both types produce and consume in the Walrasian market, in the search market a buyer can only consume and a seller can only produce. Meetings in the decentralized market occur between subsets of the population in a way described in detail below. There are two main frictions in the decentralized market. First, all meetings are assumed to be *anonymous*. This means that all trades have to be *quid pro quo* since anonymity precludes credit. Second, it is clear from the assumptions on agents' types that there there is no double coincidence of wants: some agents can only produce while others can only consume. Hence, the agents cannot use barter to exchange goods. These frictions mean that a medium of exchange is essential for trade.[9]

There is a single storable object, fiat money, which can be used as a medium of exchange in the decentralized market. The stock of money at time t is given by M_t and it is perfectly divisible. The money stock changes at gross rate γ, so that $M_{t+1} = \gamma\ M_t$, and new money is introduced, or withdrawn if $\gamma < 1$, via lump sump transfers to all agents in the Walrasian market. Concentrating on lump sum transfers ensures that money injections do not affect the agents' behavior. We focus on policies with $\gamma \geq \beta\ \delta$, where $\beta\ \delta$ is the discount factor as discussed below, as it is easy to check that there is no equilibrium otherwise. Furthermore, to examine what happens when the rate of money growth is exactly equal to the discount factor (the Friedman rule) we take the limit of equilibria as $\gamma \to \beta\ \delta$.

We denote the measure of buyers and sellers by B and S, respectively, and introduce a participation choice to the decentralized market for sellers. In particular, a seller can pay an entrance cost of K units of utility to enter the night market. We interpret this as the fixed cost of going to the market and preparing for production. We normalize the measure of buyers to 1 and let λ be the buyer-seller ratio (i.e. the measure of sellers that enter in the night market is $1/\lambda$). We continue most of the analysis with a fixed λ, and we endogenize it in section 4.[10] We assume that S is large enough such that entry is given by an indifference condition for sellers.

Let $W_t^j(m)$ be the value of an agent of type $j \in \{b, s\}$ who enters the Walrasian market at time t holding m units of money. His instantaneous utility depends on consumption, x, and hours of work, h. We assume that preferences are quasi-linear and take the form $U(x) - h$, where an hour of work produces a unit of the consumption good x. Furthermore, we assume that $U'(x) > 0$ and $U''(x) < 0$ for all x and the Inada conditions $\lim_{x \to 0} U'(x) = \infty$,

[9]See Kocherlakota (1998) and Wallace (2001).

[10]Alternatively, one can think of a fixed number of agents who decide every period whether to become a buyer or a seller in the decentralized market. This generalization is straightforward in our setup.

$\lim_{x\to\infty} U'(x) = 0$. Let the discount rate between day and night be β and denote the value of carrying m' dollars to the night market of period t by $V_t^j(m')$. The agent's value function in the day market at time t is

$$\begin{aligned} W_t^j(m) &= \max_{x,h,m'} \{U(x) - h + \beta\, V_t^j(m')\} \\ \text{s.t.} \quad x &\leq h + \phi_t(\hat{T}_t + m - m') \end{aligned} \tag{1}$$

where ϕ_t is the value of money in consumption terms,[11] and $\hat{T}_t$ is the nominal monetary transfers to (or from) the agent, i.e. $\hat{T}_t = (\gamma - 1)\ M_{t-1}/(B + S)$. Throughout the paper, nominal variables carry a hat superscript.

It will prove useful to solve some of the simple non-monetary choices of the agents at this stage so that we can concentrate on the the more interesting decisions relating to money holdings. Substituting the constraint into equation (1) with equality gives

$$W_t^j(m) = \phi_t\ (m + \hat{T}_t) + \max_{x,m'} \{U(x) - x - \phi_t\ m' + \beta\ V_t^j(m')\}$$

Note that the quasi-linearity of preferences simplifies the optimal choice of the agent significantly by eliminating wealth effects: current balances, m, do not have any marginal effect on the decision of consumption or future money balances. Furthermore, the optimal choice of x and m' dichotomizes. This means that the optimal consumption of the general good can be immediately calculated by setting the first order conditions with respect to x equal to zero. Our assumptions on $U(\cdot)$ ensure that the condition $U'(x^*) = 1$ is both necessary and sufficient for the optimal choice of x. As a result the problem can be further simplified to

$$W_t^j(m) = \phi_t\ (m + \hat{T}_t) + U^* + \max_{m'} [-\phi_t\ m' + \beta\ V_t^j(m')] \tag{2}$$

for $j \in \{b, s\}$, where $U^* = U(x^*) - x^*$ and $U'(x^*) = 1$.

[11] That is, the price of x is normalized to 1.

The choice of future money balances is more involved and we first need to describe the decentralized market in order to see the relevant incentives.

The search market operates as follows. There is a continuum of locations, and every location is populated by one seller. Each seller has the capacity to produce a single indivisible good at zero marginal cost.[12] Buyers get utility $u > 0$ from consuming the good. Matching between buyers and sellers occurs in a different way from most of the literature. Instead of searching at random, we assume that buyers know the exact location of all sellers. Therefore, they can direct their search to a particular location and meet with a seller for sure. However, buyers cannot coordinate with each other about which location to visit due to the fact that this is a large market. We capture this inherent lack of coordination by assuming that every buyer randomly chooses which seller to visit. This is crucial because it implies that the number of buyers that visit a particular seller is a random variable (hence multiple buyers may visit the same seller) and there is a single good for sale at each location. Therefore, the good may get rationed and some of the buyers may end up not consuming. Before describing the allocation process, note that we have urn-ball matching and so the number of buyers follows a Poisson distribution with parameter λ.[13] This matching function exhibits constant returns to scale and therefore the buyer-seller ratio is the only relevant statistic.

The way the good is allocated is a further innovation of this paper: we assume that an auction takes place, as opposed to the more commonly used bargaining. As already mentioned, each seller has a single good to be allocated to one of the potentially multiple buyers that visit his location. All buyers have the same valuation for the good, but they may hold different amounts of money.[14] We assume that the good is allocated according to

[12]Having a positive cost of production does not significantly change the results but it does complicate the analysis. It is therefore relegated to section 6.1.

[13]Suppose k buyers are allocated randomly across l sellers. The number of buyers that visit a particular seller follows a binomial distribution with probability $1/l$ and sample size k. As $k, l \to \infty$ keeping $k/l = \lambda$ the distribution converges to a Poisson distribution with parameter λ.

[14]This is a budget constrained auction as in Che and Gale (1998). One contribution of our paper is that the budget constraint of buyers is determined endogenously.

a second price auction with zero reserve price. This is equivalent to the buyers bidding up against each other (as in an ascending bid auction) with the seller accepting any non-negative bids since his reservation value for the good is 0. As a result, the buyer with most money (or, one of the buyers with the highest money holdings in the case of a tie) buys the good and the price that he pays is equal to the money holdings of his 'richest' competitor. Of course, if a single buyer appears he acquires the good for the price of zero, which is the reservation price of sellers.

The incentives to hold money are now clear: holding more money increases the probability of consuming because it allows the buyer to outbid more of his potential competitors; on the other hand it is costly because the value of any unspent money balances depreciates over time because of discounting and inflation.

It is worth noting that the reason why an agent may not spend his fiat money is very different than in most of the monetary search literature. In this paper, the amount of money a buyer spends depends on how many other buyers visit the same location and on how much money they hold (essentially, local demand). Hence, despite the fact that a buyer is matched with some seller with probability one he does not spend all of his money, in general. In most of the literature, the cost of holding money arises from the fact that the money holder may fail to meet someone whose good he wants to buy due to the randomness of the matching process.

We now turn to characterizing the buyers' value function analytically. The probability (and price) of a purchase of a particular buyer depends on how much money other buyers hold. Hence we need to introduce some more notation to describe the money holdings of all buyers: aggregating the choices of m' across buyers gives the distribution of money holdings at the end of centralized trading, or equivalently the beginning of the search market, which we denote by $\hat{F}_t(\cdot)$.

Let $V_t^b(m, n)$ denote the expected payoff of a buyer who carries m dollars and meets n other buyers at the location that he visits. Poisson matching implies that the probability that he meets exactly n competitors is given by $P_n = (\lambda^n \, e^{-\lambda})/n!$. The value of entering the decentralized market with m dollars is thus given by

$$V_t^b(m) = \sum_{n=0}^{\infty} P_n \, V_t^b(m,n) \tag{3}$$

To calculate $V_t^b(m,n)$, let $m_{(n)}$ be the highest money holdings among the n competitors that the current buyer faces. If $m < m_{(n)}$ the buyer with m dollars does not transact and he keeps all his money for the next Walrasian market. If $m > m_{(n)}$ the buyer with m dollars buys the good and pays $m_{(n)}$ to the seller. If $m = m_{(n)}$ there is a tie and the good is allocated at random to one of the buyers with $m_{(n)}$ dollars, who then transfers his full money holdings to the seller. Therefore, $m_{(n)}$ is the only statistic needed in order to calculate the buyer's payoff when paired with n competitors.

Since buyers are allocated at random across sellers each buyer's money holdings is a random draw from $\hat{F}_t(\cdot)$. Hence, the highest money holdings among the n other buyers is the nth order statistic among n *iid* draws from $\hat{F}_t(\cdot)$ which is distributed according to $\hat{F}_t(\cdot)^n$.[15] Furthermore, observe that two randomly chosen buyers hold exactly the same amount of money only if $\hat{F}_t(\cdot)$ has a mass point at that level. To denote this possibility, we define $\mu_t(m) \equiv \hat{F}_t(m) - \hat{F}_t(m^-)$, where $\hat{F}_t(m^-) = \lim_{\tilde{m} \nearrow m} \hat{F}_t(\tilde{m})$. This definition implies that $\mu_t(m) > 0$ if and only if there is a mass point at m. Conditional on all competitors holding weakly less than m dollars, the number of competing buyers (out of n) who have exactly m dollars follows a binomial distribution with sample size n and probability $\mu_t(m)/\hat{F}_t(m)$.[16] Let $q_{nk}(m)$ denote the probability that k out of the other n buyers hold exactly m dollars conditional on none of them having more than m dollars. If there is no mass point at m, then $q_{n0}(m) = 1$ and $q_{nk}(m) = 0$ for $k \geq 1$.

[15]This is standard result from statistics. For instance, see Hogg and Craig (1994).

[16]Conditional on all buyers holding weakly less than m dollars, the money holdings of an agent is a random draw from $\hat{F}_t(\cdot)$ truncated at m. Hence, the probability that the result of any draw is exactly equal to m is given by $\mu_t(m)/\hat{F}_t(m)$. The binomial distribution follows since there are n draws.

The value of meeting n competitors at time t when holding m dollars is given by

$$\begin{aligned} V_t^b(m,n) &= [1-\hat{F}_t(m)^n]\ \delta\ W_{t+1}^b(m) + \\ & \int_{r_t}^{m^-} [u+\delta\ W_{t+1}^b(m-\tilde{m})]\ d\hat{F}_t(\tilde{m})^n + \\ & \hat{F}_t(m)^n \sum_{k=1}^{n} q_{nk}(m)\ [\frac{u+\delta\ W_{t+1}^b(0)}{k+1} + \frac{k\ \delta\ W_{t+1}^b(m)}{k+1}] \end{aligned} \tag{4}$$

where δ is the discount factor between night and day. The term in the first square brackets gives the probability that at least one competitor holds *strictly more* money than m dollars, which means that the current buyer does not purchase the good and he keeps his money for next period's centralized market. The second term denotes the expected payoff when all other buyers hold *strictly less* money and hence the buyer with m dollars gets the good and pays the amount that his 'richest' competitor holds. The integral gives the instantaneous utility from consuming the good (u) plus the continuation value after accounting for the capital loss due to the payment.

Last, if none of the competitors brings more than m dollars but $k \geq 1$ of them hold exactly m dollars, then the current buyer gets the good with probability $1/(k+1)$, consumes, and continues to the next Walrasian market without any money; with probability $k/(k+1)$ he does not purchase and he keeps all his money for the next day. It should be clear from this discussion that the probability of purchase is discontinuous at m if $\hat{F}_t(\cdot)$ has a mass point at that level. Moreover, as mentioned above, the last term of equation (4) drops out if there is no mass point at m.

We now turn to the sellers. First, note that sellers can derive no benefit from holding money and therefore they choose to carry no money at night. A seller can choose whether to enter the decentralized market or not. If he does, he gives up K units of utility and may earn some revenues which he can spend in the following centralized market. Let $\hat{\Pi}_t$ be the seller's expected revenues if he enters the search market at time t. Note that $\hat{\Pi}_t$ is a sufficient statistic for next period's value due to the linearity of $W_t^s(\cdot)$. If the seller chooses not to enter he continues to the following Walrasian market without money. As a result, the night value to the (potential) seller is given by

$$V_t^s = \max\{-K + \delta\ W_{t+1}^s(\hat{\Pi}_t),\ \delta\ W_{t+1}^s(0)\}$$

In equilibrium sellers are indifferent between the two options which means that, using equation (2), the following condition has to hold for all t:

$$\delta\ \phi_{t+1}\ \hat{\Pi}_t = K \tag{5}$$

To determine $\hat{\Pi}_t$, note that the price a seller receives is equal to the second highest money holdings among the buyers that show up in his location. When n buyers visit a particular seller, the $(n-1)$ order statistic is distributed according to $\hat{F}_t^{(n-1,\ n)}(m) = n\ \hat{F}_t(m)^{n-1}\ [1-\hat{F}_t(m)] + \hat{F}_t(m)^n$.[17] The probability that n buyers show up is given by P_n, and we define the distribution of prices by $\hat{G}_t(m) \equiv \sum_{n=1}^{\infty}\ P_n\ \hat{F}_t^{(n-1,\ n)}(m)$. In other words, $\hat{G}_t(m)$ denotes the probability that a seller receives no more than m dollars in the decentralized market at time t, after summing over all the possible number of buyers. Therefore, the expected revenues of a seller at t are given by $\hat{\Pi}_t = \int_0^\infty\ \tilde{m}\ d\hat{G}_t(\tilde{m})$.

Last, we need to define market clearing in the market for money. Since sellers have zero balances, the money demand at time t is given by the amount that buyers want to hold, i.e. $MD_t = \int_0^\infty\ \tilde{m}\ d\hat{F}_t(\tilde{m})$. Hence, the money market is in equilibrium at time t if

$$MD_t = M_t. \tag{6}$$

Now that the value functions are well-specified, we can define an equilibrium.

[17] See Hogg and Craig (1994).

Definition 2.1 *An equilibrium is a list* $\{W_t^j, V_t^j, \hat{F}_t, \phi_t, \lambda_t^*\}$ *where* W_t^j *and* V_t^j *are the value functions,* $\hat{F}_t$ *is the distribution of money holdings at the beginning of the night market at* t*,* ϕ_t *are the prices,* λ_t^* *is the buyer-seller ratio such that the following conditions are satisfied for all* t*.*

1. *Optimality: given prices, any* $m' \in supp\hat{F}_t$ *solves (2),*

2. *Market Clearing: equation* (6) *holds.*

3. *Free entry: Equation* (5) *holds.*

Throughout this paper we concentrate on monetary equilibria, so $\phi_t > 0 \,\forall\, t$. Furthermore, we only examine stationary equilibria, in the sense that the real value of average money balances remains constant over time. In other words, we look at equilibria where $\phi_t \; MD_t = \phi_{t+1} \; MD_{t+1}$. Since the money supply grows at a constant rate γ and $MD_t = M_t$ we have that $\phi_t = \gamma \; \phi_{t+1}$.

At this stage we should remark that there are two important decisions that agents make in our environment: buyers choose how much money to hold and sellers choose whether to enter. More specifically, a buyer takes as given λ, ϕ_t, and other buyers' decisions in order to pick his optimal holdings. Aggregating across buyers, this results in $\hat{F}_t(\cdot)$ as a function of λ and ϕ_t. In equilibrium, ϕ_t is such that money demand equals M_t, which pins down prices as a function of market tightness and money supply. Last, free entry of sellers gives λ as a function of the cost of entry. We proceed to characterize the buyers' decisions in the next section. Section 4 looks at sellers.

3 The Buyer's Problem

In this section we examine the choice of money holdings of the buyer. We characterize the resulting distribution of balances for a given buyer-seller ratio, λ. We then redefine the problem in terms of real variables.

At the beginning of every period the problem of the individual buyer is to choose the optimal money holdings, taking as given the choices of all other buyers and prices. It is immediate that the utility of consumption puts an upper bound on the range of the optimal decision. More specifically, let m_t^* be such that a buyer is indifferent between spending m_t^* to consume or keeping the full amount for the next Walrasian market. This amount exists since $u < \infty$, and it is defined by the equation: $u + \delta\, W_{t+1}^b(0) = \delta\, W_{t+1}^b(m_t^*) \Rightarrow m_t^* = u/(\delta\, \phi_{t+1})$. It is easy to verify that in equilibrium a buyer never brings more than m_t^* to the search market, since any additional amount is not spent and hence it simply depreciates. Letting $\underline{m}_t$ and $\overline{m}_t$ denote the infimum and supremum, respectively, of the support of $\hat{F}_t(\cdot)$ this discussion implies that $0 \leq \underline{m}_t \leq \overline{m}_t \leq m_t^*$.

We can reformulate the buyer's problem on a period-by-period basis as

$$\max_{m \in [0, m_t^*]} \; -\phi_t\, m + \beta\, V_t^b(m) \tag{7}$$

taking $\hat{F}_t(\cdot)$ and ϕ_t as given (stationarity implies that knowing the price of money for some t pins down the whole path of prices). The first proposition follows.[18]

Proposition 3.1 *In equilibrium $\hat{F}_t(\cdot)$ is non-atomic on $[0, m_t^*)$, the support of $\hat{F}_t(\cdot)$ is connected, and the infimum of the support is at 0.*

Proof: Suppose that $\hat{F}_t(\cdot)$ has a mass point at some $\check{m} \in [0, m_t^*)$ and recall that equation (4) implies that the probability of buying is discontinuous at $\check{m}$. Purchasing the good for $\check{m}$ dollars gives positive net utility (since $\check{m} < m_t^*$) and hence $V_t^b(\check{m}) < V_t^b(\check{m}^+)$. Since the cost of bringing infinitesimally more money is negligible it is clear that bringing $\check{m} + \epsilon$ yields strictly higher payoff than $\check{m}$ and therefore in equilibrium a buyer never brings $\check{m}$ yielding a contradiction.

[18] The logic of this proposition is similar to Burdett and Judd (1983), though the context is very different.

Suppose that there is no buyer whose money holdings belong to some interval (m_1, m_2), with $\underline{m}_t \leq m_1 < m_2 \leq \overline{m}_t$. We now show that the buyer with m_1 dollars is strictly better off. The reason is that the m_1-buyer trades in exactly the same events as the buyer with m_2 dollars since they both 'beat' exactly the same competitors (except when there is a mass point at m_2 which can only occur if $m_2 = m_t^*$; however in that case the additional trades that the m_2-buyer can perform do not yield any extra utility since he is indifferent between keeping his money or consuming). It is therefore easy to verify that $V_t^b(m_2) = V_t^b(m_1) + \phi_{t+1}\ (m_2 - m_1)$. Examining the initial decision of how much money to hold, we have that $-\phi_t\ m_1 + \beta\ V_t^b(m_1) - [-\phi_t\ m_2 + \beta\ V_t^b(m_2)] = (m_2 - m_1)\ [\phi_t - \beta\ \delta\ \phi_{t+1}]$ which is strictly positive since $\phi_t = \gamma\ \phi_{t+1}$ and $\gamma > \beta\ \delta$. This means that choosing to carry m_1 dollars gives higher value than holding m_2 which cannot hold in equilibrium.

Last, a buyer bringing $\underline{m}_t$ dollars can only transact when he does not meet any competitors, in which case the price he pays is equal to 0. This means that $V_t(\underline{m}_t) = V_t(0) + \delta\ \phi_{t+1}\ \underline{m}_t$, which implies that $\underline{m}_t > 0$ cannot occur in equilibrium for the same reason as above. *QED*

The reason why $\hat{F}_t(\cdot)$ is non-atomic in its interior is straightforward to see. If there is a mass point in the distribution of money holdings, then it is very likely to meet a competitor holding exactly that amount of money. In that case, a buyer who brings infinitesimally more money has discretely higher probability of winning the auction for a negligible additional cost. Therefore, this buyer enjoys a higher expected payoff which cannot happen in equilibrium. Later on, we prove that there cannot be a mass point at m_t^* either.

One important implication of this result is that the optimal decision of buyers is correspondence valued: there is a range of values of m that, in equilibrium, yield the same expected payoff and therefore buyers are willing to randomize over them. Furthermore, equation (7) means that $V_t(m)$ is *not* strictly concave, but rather it has to be linear in the domain of solutions.

For the remainder of the paper we only consider $\hat{F}_t(\cdot)$ that are continuous on $[0, m_t^*)$ with $\hat{F}_t(0) = 0$ and $supp\hat{F}_t = [0, \overline{m}_t]$. We can now rewrite equation (3) as

$$V_t^b(m) = \delta\ W_{t+1}^b(m) + \sum_{n=0}^{\infty} P_n\ \{u\ \hat{F}_t(m)^n - \delta\ \phi_{t+1} \int_0^m\ \tilde{m}\ d\hat{F}_t(\tilde{m})^n\} \tag{8}$$

This expression is very intuitive: the first term is the value that the buyer can guarantee himself without a purchase; inside the curly brackets, the first term is the probability of buying the good times the instantaneous utility of consumption while the second term is the expected capital loss from a purchase.

Note that equation (8) holds even if there is a mass point at m_t^*. The reason is that the payoff to a buyer with m dollars from meeting a competitor holding m_t^* dollars is equivalent to not purchasing regardless of m. This is obvious when $m < m_t^*$; when $m = m_t^*$, the price is bid up to m_t^* which means that the buyer is indifferent between buying the good or continuing with all his money. As a result, whether he actually buys the good or not does not change his utility and we can disregard the possibility of buying the good at that, maximum, price.

We now turn to the explicit characterization of the solution to the buyer's problem. In equilibrium, $-\phi_t\ m + \beta\ V_t^b(m)$ has to be constant on $[0, \overline{m}_t]$. Our strategy is to construct $\hat{F}_t(\cdot)$ so that this condition holds.

Proposition 3.2 *In equilibrium, the distribution of money holdings is uniquely defined by* $\hat{F}_t(m) = \frac{1}{\lambda}\ \log\{1 - e^{\lambda}\ [\gamma/(\beta\ \delta) - 1]\ \log[1 - \frac{\delta\ \phi_t\ m}{\gamma\ u}]\}$. *Furthermore,* $\overline{m}_t < m_t^*$.

Proof: Equation (7) implies that $V_t^{b\prime}(m) = \phi_t/\beta$ for $m \in [0, \overline{m}_t]$. For $V_t^b(\cdot)$ to be differentiable, any equilibrium $\hat{F}_t(\cdot)$ has to be differentiable on $(0, \overline{m}_t)$. We start by assuming differentiability and we then verify that our solution fulfills this property.

Taking the derivative of (8) with respect to m we get (using Leibniz's rule and noting that m does not enter the integrant)

$$\begin{aligned} V_t^{b\prime}(m) &= \delta\ \phi_{t+1} + \sum_{n=0}^{\infty} P_n\ \{u\ n\ \hat{F}_t(m)^{n-1}\ \hat{F}_t'(m) - \delta\ \phi_{t+1}\ m\ n\ \hat{F}_t(m)^{n-1}\ \hat{F}_t'(m)\} \\ &= \delta\ \phi_{t+1} + (u - \delta\ \phi_{t+1}\ m)\ \hat{F}_t'(m) \sum_{n=0}^{\infty} P_n\ n\ \hat{F}_t(m)^{n-1} \\ &= \delta\ \phi_{t+1} + (u - \delta\ \phi_{t+1}\ m)\ \hat{F}_t'(m)\ \lambda\ e^{-\lambda\ (1-\hat{F}_t(m))} \end{aligned} \tag{9}$$

where the last step follows from the fact that $n \sim Po(\lambda)$ and some algebra.

Equating (9) with ϕ_t/β and rearranging yields the following differential equation:

$$\lambda\ \hat{F}_t'(m)\ e^{\lambda\ \hat{F}_t(m)} \quad = \quad e^{\lambda}\ \frac{\delta\ \phi_{t+1}\ i_t}{u - \delta\ \phi_{t+1}\ m}$$

where $i_t \equiv \phi_t/(\phi_{t+1}\ \beta\ \delta) - 1$ is the nominal interest rate at t. We can integrate both sides over m and use the initial condition $\hat{F}_t(0) = 0$ to arrive at the explicit characterization of the distribution of money balances in equilibrium:

$$\hat{F}_t(m) = \frac{1}{\lambda}\ \log\{1 - e^{\lambda}\ i_t\ \log[1 - \frac{\delta\ \phi_{t+1}\ m}{u}]\} \tag{10}$$

Recalling that $\phi_t = \phi_{t+1}\ \gamma$ yields the desired expression.

The maximum money balances, $\overline{m}_t$, can be calculated by using the fact that $\hat{F}_t(\overline{m}_t) = 1$:

$$\overline{m}_t = \frac{u}{\delta\ \phi_{t+1}}(1 - e^{-\frac{1-e^{-\lambda}}{i_t}}) \tag{11}$$

Since $m_t^* = u/(\delta\ \phi_{t+1})$ it is clear that all buyers bring less money than m_t^* and hence there is no mass point in the distribution of money holdings. QED

The next step is to close the buyers' side of the model by equating the demand of money with exogenous supply M_t in order to find the equilibrium 'price of money' ϕ_t.

Proposition 3.3 *There is a unique ϕ_t^* such that $MD_t = M_t$.*

Proof: Using the expressions derived in the previous proposition, we can define money demand at t as a function of ϕ_t, $MD_t(\phi_t)$. We first prove that money demand decreases monotonically in ϕ_t by showing that the $\hat{F}_t(\cdot)$ that results from a low ϕ_t first order stochastically dominates the money distribution that results from high ϕ_t. Using equation (10), some algebra shows that

$$\partial \hat{F}_t(m)/\partial \phi_t \;=\; \frac{e^{\lambda}\; i_t\; \delta\; m}{\lambda^2\; \{1 - e^{\lambda}\; i_t\;\; \ln[1 - (\delta\; \phi_t\; m)/(\gamma\; u)]\}\; [\gamma\; u - \delta\; \phi_t\; m]} > 0$$

which implies that the proportion of buyers holding no more than m dollars increases with ϕ_t and hence $\partial MD_t/\partial \phi_t < 0$.

To complete the proof we need to show that $MD_t(\infty) < M_t < MD_t(0)$ for some arbitrary M_t. Note that $\lim_{\phi_t \to \infty} \overline{m}_t = 0 \Rightarrow \lim_{\phi_t \to \infty} MD_t(\phi_t) = 0$. Also, $\lim_{\phi_t \to 0}\; \overline{m}_t = \infty$ and $\lim_{\phi_t \to 0}\; \hat{F}_t(m) = 0,\; \forall m < \overline{m}_t$ imply that $\lim_{\phi_t \to 0}\; MD_t(\phi_t) = \infty$. *QED*

To simplify notation, we now denote all variables in real terms. We express a dollar in terms of the consumption utility that it can bring *in the next centralized market*. That is, m dollars in the decentralized market of time t are worth $z = \delta\; \phi_{t+1}\; m$ units of utility in the next centralized market. Furthermore, we let $G_t(\cdot)$ denote the distribution of real revenues for the seller and note that the expected real revenues are given by $\Pi_t = \delta\; \phi_{t+1}\; \hat{\Pi}_t$. Since $\phi_{t+1} = \phi_t/\gamma$, from now on we dispense with the time subscript. Using real balances, we rewrite the value functions for the two types of agents:

$$W^b(z) \;=\; (z + T)\; \gamma/\delta + U^* + \max_{z'}\; \{-z'\; \gamma/\delta + \beta\; V^b(z')\} \tag{12}$$

$$V^b(z) \;=\; \delta\; W^b_{+1}(z) + \sum_{n=0}^{\infty} P_n\; \{u\; F(z)^n - \int_0^z\; \tilde{z}\; dF(\tilde{z})^n\} \tag{13}$$

$$W^s(z) \;=\; (z + T)\; \gamma/\delta + U^* + \beta\; V^s \tag{14}$$

$$V^s \;=\; \max\{W^s(\Pi) - K,\; W^s(0)\} \tag{15}$$

where $T \equiv \phi\ \hat{T}$ denotes real transfers. Note that the real money balances that the buyer chooses are in terms of what the money is worth in the next period and hence $W_{+1}(z) = (z/\gamma)\ (\gamma/\delta) + W_{+1}(0) = z/\delta + W_{+1}(0)$, where the real balances are divided by γ *today* because next period's inflation is taken into account.

We can also rewrite the distribution of real balances as

$$F(z) = \frac{1}{\lambda}\ \ln\{1 - e^{\lambda}\ i\ \ln(1 - \frac{z}{u})\} \tag{16}$$

where $i = \gamma/(\delta\ \beta) - 1$. This also means that the highest money holdings are given by

$$\overline{z} = u\ (1 - e^{-\frac{1-e^{-\lambda}}{i}}) \tag{17}$$

Note that $\overline{z} < u$ as long as $i > 0$. As $\gamma \to \beta\ \delta$ and the rate of money growth approaches the Friedman rule, $i \to 0$. This implies that $F(z) \to 0$ for any $z < \overline{z}$ and the distribution of real balances collapses to a single mass point at u. Moreover, at the Friedman rule $\overline{z} = u$ which means that the real balances of *every* buyer equal his valuation for the good.

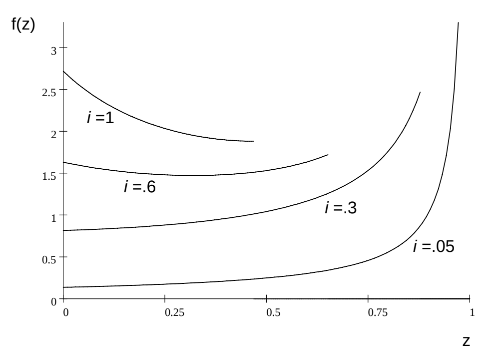

Figure 1: Density of real money balances for different levels of the interest rate i and $u = 1$, $\lambda = 1$.

Figure 1 shows the density of real money holdings for different levels of the interest rate and $\lambda = 1$. At very high interest rates the density is decreasing. In the intermediate range it is U-shaped. For low interest rate is is increasing.

4 Sellers: Entry Decision

We proceed to evaluate the entry decision by sellers. We prove that equilibrium is unique and it exists if and only if the inflation rate is below some threshold value.

For equilibrium, we need to determine λ^*, the buyer-seller ratio, such that $\Pi = K$. The first proposition establishes the uniqueness of equilibrium.

Proposition 4.1 *The expected profits of sellers increase in λ.*

Proof: We show that $\partial\Pi/\partial\lambda > 0$. To do that, we prove that the distribution of prices for high λ first order stochastically dominates the one for a low λ, or $\partial G(z)/\partial\lambda < 0$. As a result, the expected price is strictly higher when there are more buyers per seller.

To get the exact equation for $G(z)$ note that

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} P_n \; F^{(n-1)}(z) \\ &= \frac{1-F(z)}{F(z)} \sum_{n=0}^{\infty} P_n \; n \; F(z)^n + \sum_{n=0}^{\infty} P_n \; F(z)^n \\ &= (1+\lambda-\lambda \; F(z)) \; e^{-\lambda \; (1-F(z))} \end{aligned}$$

Using equation (16), it is easy to show that

$$\begin{aligned} \frac{\partial F(z)}{\partial \lambda} &= -\frac{F(z)}{\lambda} + \frac{-e^{-\lambda} \; i \; \ln(1-z/u)}{\lambda \, [1-e^{-\lambda} \; i \; \ln(1-z/u)]} \\ &= \frac{1}{\lambda} \, [1-F(z)-e^{-\lambda \; F(z)}] \end{aligned} \tag{18}$$

The last step is to note that with some algebra

$$\begin{aligned} \frac{\partial G(z)}{\partial \lambda} &= -\lambda \; (1-F(z)) \; [1-F(z)-\lambda \; \frac{\partial F(z)}{\partial \lambda}] \; e^{-\lambda \; (1-F(z))} \\ &= -\lambda \; (1-F(z)) \; e^{-\lambda} < 0 \end{aligned} \tag{19}$$

where the second equality results from inserting equation (18). This completes the proof. *QED*

It is now easy to see that $\lim_{\lambda \to 0} \Pi = 0$. Therefore, if $\lim_{\lambda \to \infty} \Pi > K$ the (unique) equilibrium exists.

Proposition 4.2 *An equilibrium exists if and only if* $K < \bar{K}(\gamma)$ *where* $\bar{K}(\gamma) = u\ (1 - e^{-1/i})$.

Proof: Note that as $\lambda \to \infty$ a seller is visited by some buyer for sure. Furthermore, the seller's revenues converge to the highest money holdings. Recalling that $\overline{z} = u\ (1 - e^{-\frac{1-e^{-\lambda}}{i}})$,

$$\lim_{\lambda \to \infty} \Pi = \lim_{\lambda \to \infty} \overline{z} = u\ (1 - e^{-1/i})$$

which completes the proof. *QED*

Observe that $\bar{K}(\gamma)$ is decreasing in the inflation rate. As a result, it is 'more difficult' to have an equilibrium with high inflation, in the sense that the set of costs that can support it is smaller. Therefore, for given market fundamentals (K), when the inflation rate is too high the decentralized market closes down. The value of fiat money depreciates too fast and buyers are not able to transfer enough real resources to make it worthwhile for seller to enter.

5 Efficiency and Inflation

In this section we examine the effects of inflation on efficiency. We show that efficiency is attained only at the Friedman rule, when the stock of money decreases at the rate of time preference.

The relevant margin for efficiency is the entry of sellers. Since every meeting between a seller and some buyers results in a purchase, the question of interest is whether the efficient number of sellers enter into the market. Furthermore, the rate of inflation affects the amount of money that buyers are willing to hold, which affects the sellers' incentives to enter.

We start by solving the planner's problem and we then show that efficiency is attained only when the inflation rate approaches the Friedman rule. A planner chooses the level of λ to maximize the surplus in a given decentralized market. In other words, he maximizes the following objective function:

$$\mathcal{W} = \frac{(1 - e^{-\lambda})\ u}{\lambda} - \frac{K}{\lambda} \tag{20}$$

The first term gives the total number of sellers $(1/\lambda)$, times the probability that a seller trades $(1 - e^{-\lambda})$, times the surplus that is generated from a trade (u). The second term gives the total entry cost of $1/\lambda$ sellers.

Setting the first order conditions with respect to λ to zero yields

$$(1 - e^{-\lambda^P} - \lambda^P\ e^{-\lambda^P})\ u \quad = \quad K$$

It is easy to check that the second derivative is negative, hence the first order condition is necessary and sufficient. The planner's optimal buyer-seller ratio is given by λ^P.

This expression suggests that the market should operate (i.e. $\lambda < \infty$) whenever $u > K$. This implies that if the inflation rate is so high that the market closes down, this leads to an immediate inefficiency. Furthermore, suppose that an equilibrium exists The buyer-seller ratio is determined by the free entry condition $\Pi = K$. This means that efficiency is attained only if $\Pi = (1 - e^{-\lambda} - \lambda\ e^{-\lambda})\ u$. Recall that as $\gamma \to \beta\ \delta$, all buyers bring real balances that are equal to their valuation of the good, u. In that case the seller enjoys the full surplus of any match as long as two or more buyers appear, which happens with probability $1 - P_0 - P_1 = 1 - e^{-\lambda} - \lambda\ e^{-\lambda}$. If zero or one buyer show up, then the seller receives zero. Therefore, in that case the revenues of the seller are equal to the left hand side and entry is efficient. Last, if the inflation rate is more than the Friedman rule, then the sellers appropriate a strictly lower part of the surplus. As a result, entry is suboptimal for any inflation rate that is above the Friedman rule.

6 Extensions

We now consider three extensions to the basic framework developed above: positive marginal cost for sellers, and buyers that are heterogeneous in the utility they derive from consumption and belong to either discrete or continuous types.

6.1 Positive Marginal Cost

We start with the case where sellers incur marginal cost $c > 0$ when they produce a good for some buyer, in addition to the fixed entry cost K. The main difference is that now the sellers have a positive reservation price. This makes it costly for buyers to participate in the decentralized market because they need to bring a minimum amount of money in order to have a chance of purchasing (as opposed to being able to buy even if they bring zero dollars, which was the case in section 3). Since that amount may remain unspent, and hence lose value, this introduces a positive participation cost. We proceed the analysis by first fixing the number of sellers that are in the market to some $\bar{S}$ and analyzing the participation decision of buyers. We then look at the entry problem of sellers.

Let r be the minimum price that the seller is willing to receive for the good and note that it is defined by $\delta\, W^s_{+1}(0) = -c + \delta\, W^s_{+1}(r) \Rightarrow r = c/\gamma$. It is immediate that bringing balances in $(0, r)$ is dominated by bringing zero, since a positive (but insufficient) amount cannot be used for any purchase and hence it simply depreciates. We label the buyers that bring r real dollars or more to the market as *effective* buyers and denote their measure by $\bar{B}$. The effective buyer-seller ratio is then given by $\lambda_E = \bar{B}/\bar{S}$. The decision problem of effective buyers is almost identical to the one in section 3 and is characterized in the following proposition.

Proposition 6.1 *If $\lambda_E > 0$, the distribution of money holdings of effective buyers in equilibrium is non-atomic in $[r, z^*)$ and it is given by* $F(z) = \frac{1}{\lambda_E}\ \ln\{1 - e^{\lambda_E}\ [\gamma/(\beta\ \delta) - 1]\ \ln[\frac{u-z}{u-c}]\}$.

Also, $\overline{z} < z^*$.

Proof: The proof is identical to propositions 3.1 and 3.2 and is therefore omitted. *QED*

We turn to the buyers' decision to participate in the search market. We want to compare the expected payoffs of participating in the decentralized market versus staying out. Since all effective buyers earn the same expected payoffs regardless of how much money they carry, a simple way to characterize their value is to consider a buyer holding r. This buyer only purchases if there is no competitor at the location he visits (which occurs with probability $e^{-\lambda_E}$) in which case he spends all of his money. Otherwise, he continues to the next centralized market with r dollars. Therefore, this buyer prefers participating in the market only if

$$\begin{aligned}
\beta\ \delta\ W_{+1}(0) &\leq -r\ \gamma/\delta + \beta\ [e^{-\lambda_E}\ (u + \delta\ W_{t+1}(0)) + (1 - e^{-\lambda_E})\ \delta\ W_{t+1}(r)] \Rightarrow \\
i\ c &\leq u\ e^{-\lambda_E} \qquad (21)
\end{aligned}$$

recalling that $W_{+1}(r) = r\ \gamma/\delta + W_{+1}(0)$, $i = \gamma/(\beta\ \delta) - 1$, and $r = c/\gamma$. This condition puts an upper bound on the buyer-seller ratio, as a function of the inflation rate:

$$\bar{\lambda}(\gamma) \leq -\log(c\ i/u) \qquad (22)$$

It is obvious that as the inflation rate increases, the buyers are willing to participate in the market only if the face less competition from each other. Note that when $\gamma \geq \beta\ \delta\ (u/c + 1)$, the right-hand side of (22) is non-positive, hence there is no trade in the decentralized market. This occurs because it is too costly for buyers to bring even the minimum amount required by sellers to produce (this effect is similar to the case where no sellers enter into the decentralized market, as described in section 4). If the inflation rate is below that threshold, the number of buyers is determined by the indifference condition that results from setting (21) to equality, for a given number of sellers $\bar{S}$.

Turning to the entry decision of sellers, note that their profits are still characterized by the expressions derived in section 4. The only difference is that now the upper bound for profits is given by $\Pi(\bar{\lambda})$ which is strictly lower than $\lim_{\lambda \to \infty} \Pi(\lambda)$. As a result, the set of Ks that can support trading in the decentralized market is strictly smaller than in section 4.

6.2 Heterogeneous Buyers: Discrete Types

Consider the case of two types of buyers who differ in how much they enjoy consuming the good of the night market. In particular, share α_H are high type buyers and they receive u_H when consuming; the complementary proportion, $\alpha_L \equiv 1 - \alpha_H$, are low types and receive $u_L < u_H$. We show that every low type buyer hold less money than any high type buyer. We then characterize the distributions of money balances of each type.

Let $F_i(\cdot)$ denote the distribution of real balances and $Z_i \equiv suppF_i(\cdot)$ denote the support of that distribution for an agent of type $i \in \{L, H\}$. Then, $F(z) = \alpha_H \ F_H(z) + \alpha_L \ F_L(z)$ gives the unconditional distribution of money balances. Let $\overline{z}$ be the highest balances of any agent. An argument similar to proposition 3.1 shows that $suppF = [0, \overline{z}]$ and $F(\cdot)$ is non-atomic on $[0, \overline{z}]$, if $\overline{z} < z^*$. While we cannot guarantee that $F(\cdot)$ is differentiable, an argument similar to the one in section 3 shows that $F_i(\cdot)$ is differentiable in the interior of Z_i. As a result we can meaningfully evaluate the first order conditions of buyers.

Buyer optimization implies that $V_i'(z) = \gamma/(\beta \ \delta)$ when $z \in Z_i$ for both types. The derivative of $V_i(\cdot)$ can be evaluated on the interiors of Z_L and Z_H. Therefore, for $z \in \text{int}(Z_L) \cup \text{int}(Z_H)$

$$\begin{aligned} V_i'(z) &= \frac{\gamma}{\delta} + \sum_{n=0}^{\infty} P_n \{u_i \ n \ F'(z) \ F^{n-1}(z) - n \ z \ F'(z) \ F^{n-1}(z)\} \\ &= \frac{\gamma}{\delta} + (u_i - z) \ \lambda \ F'(z) \ e^{-\lambda \ (1-F(z))} \end{aligned} \tag{23}$$

It is now easy to see that $V_H'(z) > V_L'(z)$ for $z \in suppF(\cdot)$. As a result, all low type buyers hold less money that any high type buyer. Furthermore, the support of the two distributions are adjacent, which implies that $F(\cdot)4$ is not differentiable at their common point.

The above analysis means that $Z_L = [0, \overline{z}_L]$ and $Z_H = [\overline{z}_L, \overline{z}]$. To replicate the analysis of proposition 3.2, note that the decentralized market value functions for the two types of agents are given by

$$
\begin{aligned}
V_L(z) &= \delta\, W^L_{+1}(z) + e^{-\lambda_H} \sum_{n=0}^{\infty} P^L_n\, \{u_L\, F_L(z)^n - \int_0^z \tilde{z}\, dF_L(\tilde{z})^n\} && (24)\\
V_H &= \delta\, W^H_{+1}(z) + \sum_{n=0}^{\infty} P^H_n\, \{u_H\, F_H(z)^n - \int_0^z \tilde{z}\, dF_H(\tilde{z})^n\} && (25)
\end{aligned}
$$

where P^i_N is the probability the n buyer of type i show up. Note that a low type buyer has a chance to buy only if no high type buyers appear at the location he visits, which occurs with probability $e^{-\lambda_H}$. Also, it does not matter to a high type whether any low types are visiting his location, since they hold less money with probability one.

We can now replicate the analysis of section 3 to arrive at the explicit characterization of the distributions of real money balances for the two types. The initial conditions for the two distributions are $F_L(0) = 0$ and $F_H(\overline{z}_L) = 0$, where $F_L(\overline{z}_L) = 1$. Therefore we have:

$$
\begin{aligned}
F_L(z) &= \frac{1}{\lambda_L}\, \log\{1 - e^{\lambda}\, i\, \ln(1 - \frac{z}{u_L})\}, \quad \forall\, z \in [0, \overline{z}_L]\\
\overline{z}_L &= u_L\, (1 - e^{-e^{-\lambda_H}(1-e^{-\lambda_L})/i})\\
F_H(z) &= \frac{1}{\lambda_H}\, \log\{1 - e^{\lambda_H}\, i\, \ln(\frac{u_H - z}{u_H - \overline{z}_L})\} \quad \forall\, z \in [\overline{z}_L, \overline{z}_H]\\
\overline{z}_H &= u_H\, (1 - e^{-(1-e^{-\lambda_H})/i}) + \overline{z}_L\, e^{-(1-e^{-\lambda_H})/i}
\end{aligned}
$$

6.3 Heterogeneous Buyers: Continuous Types

Turning to continuous types, suppose that the buyers' utility is distributed according to $u \sim H(\cdot)$ which is non-atomic, with $\underline{u} \equiv \inf_u suppH(u) > 0$. To solve this case, we assume that there exists a bidding function $z(u)$ such that the optimal strategy for a buyer of type $\hat{u}$ is to bring $z(\hat{u})$, when all other buyers follow the same bidding function. The discussion in the previous section shows that a buyer who values the good more brings more money to the market and we therefore restrict attention to bidding functions that are strictly increasing. It is also straightforward to show that $z(u)$ has to be continuous in any equilibrium, for the same reasons why $F(\cdot)$ is continuous. We now proceed to characterize $z(u)$ which in turn pins down $F(\cdot)$.

First, observe that $F(z(u)) = H(u)$ since $z(u)$ is strictly increasing. As a result, in equilibrium a buyer 'beats' all competitors with lower valuations and loses from buyers who value the good more. Therefore, when examining potential deviations for the buyer, we look at the cases where he 'pretends' to be of a different type. Hence, given $z(u)$, the problem of a type-u buyer when choosing his money holdings is

$$\max_{\hat{u}} \; -z(\hat{u})\; \gamma/\delta + \beta\; [\delta\; W_{+1}(z(\hat{u})) + \sum_{n=0}^{\infty}\; P_n\{u\; H(\hat{u})^n - \int_0^{\hat{u}}\; z(\tilde{u})\; dH(\tilde{u})^n\}] \tag{26}$$

The first order conditions of this problem have to equal to zero at $\hat{u} = u$ for the buyer to bring the amount prescribed by the bidding function. In other words

$$\begin{aligned} -z'(u)\; \gamma/\delta + \beta\; [z'(u) + \sum_{n=0}^{\infty}\; P_n\{u\; n\; H'(u)\; H(u)^{n-1} - z(u)\; n\; H'(u)\; H(u)^{n-1}\}] &= 0 \Rightarrow \\ -z'(u)\; i + (u - z(u))\; \lambda\; H'(u)\; e^{-\lambda\; (1-H(u))} &= 0 \end{aligned} \tag{27}$$

Furthermore, equation (27) has to hold for all u. This means that $z(u)$ is defined by the following first order linear differential equation:

$$z'(u) + z(u)\; \lambda\; H'(u)\; e^{-\lambda\; (1-H(u))}/i = u\; \lambda\; H'(u)\; e^{-\lambda\; (1-H(u))}/i \tag{28}$$

We can multiply both side with the integrating factor $e^{v(u)}$, where

$$v(u) = \int \lambda\; H'(\tilde{u})\; e^{-\lambda\; (1-H(\tilde{u}))}/i\; d\tilde{u} \tag{29}$$

$$= e^{-\lambda\; (1-H(u))}/i \tag{30}$$

The left-hand side is then given by $d[z(u)\ e^{v(u)}]/du$, and we can use the fundamental theorem of calculus to arrive at the explicit formulation for the equilibrium bidding function:

$$z(u) = u - \underline{z}\ e^{e^{-\lambda\ (1-H(u))/i}} - \int_{\underline{u}}^{u} e^{e^{-\lambda(H(u)-H(\tilde{u}))/i}}\ d\tilde{u} \tag{31}$$

While this equation does not look particularly attractive, it gives a unique and explicit solution for the bidding function, given $H(\cdot)$.

7 Conclusions

This model extends recent monetary theory by considering environment with directed matching and auctions. Fiat money is still essential for trade due to the assumptions of anonymity and lack of double coincidence of wants. However most of characterization results for the equilibrium are novel. The fact that buyers compete directly with each other to consume the good leads to very different incentives when choosing their money holdings. In particular, we show that there is money dispersion in all equilibria, which leads to price dispersion. Despite the non-degeneracy of the distribution of money holdings, the equilibrium characterization remains tractable and admits an analytical solution. Furthermore, we show that only the Friedman rule allows for full transferable utility between buyer and seller, and only in this case are sellers able to recover the social surplus of their investment, yielding an efficient outcome.

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