The distribution of wealth and redistributive policies *

Jess Benhabib
Department of Economics
New York University

Alberto Bisin
Department of Economics
New York University

This draft: February 2006-Preliminary

Abstract

In this paper we study theoretically the dynamics of the distribution of wealth in an Overlapping Generation economy with bequest and various forms of redistributive taxation. We characterize the transitional dynamics of the wealth distribution and as well as the stationary distribution.

We show that, in our economy, the stationary wealth distribution is a power law, a Pareto distribution in particular. Wealth is less concentrated (the Gini coefficient is lower) for both higher capital income taxes and estate taxes, but the marginal effect of capital income taxes is much stronger than the effect of estate taxes.

Finally, we characterize optimal redistributive taxes with respect to an utilitarian social welfare measure. Social welfare is maximized short of minimal wealth inequality and with zero estate taxes.

*We gratefully acknowledge conversations with Alberto Bressan, Mariacristina De Nardi, Xavier Gabaix, Leslie Greengard, Frank Hoppensteadt, Gianluca Violante, and Ed Wolff. Thanks to Nicola Scalzo and Eleonora Patacchini for help with "impossible" Italian references. This paper is part of the Polarization and Conflict Project CIT-2-CT-2004-506084 funded by the European Commission-DG Research Sixth Framework Programme.

1 Introduction

Rather invariably across a large cross-section of countries and time periods income and wealth distributions are skewed to the right and display a heavy upper tail (slowly declining top wealth shares). These observations have lead Vilfredo Pareto, in the *Cours d'Economie Politique* (1897), to introduce the distributions which take his name[1] and to theorize about the possible economic and sociological factors generating wealth distributions of such form. The results of Pareto's investigations take the form of the "Pareto's Law," enunciated e.g., by Samuelson (1965) as follows:

> In all places and all times, the distribution of income remains the same. Neither institutional change nor egalitarian taxation can alter this fundamental constant of social sciences.

Since Pareto, economists have lost confidence in "fundamental constant(s) of social sciences". Nonetheless distributions of income and wealth which are very concentrated and skewed to the right have been well documented over time and across countries. E.g., Atkinson (2001), Moriguchi-Saez (2005), Picketty (2001), Piketty-Saez (2003), and Saez-Veall (2003) document large top income shares consistently over the last century, respectively, in the U.K., Japan, France, the U.S., and Canada. Large top wealth shares in the U.S. since the 60's are documented e.g., by Wolff (1987, 2004).[2] Also, heavy upper tails (power law behavior) of the distributions of income and wealth is a well documented empirical regularity; see e.g., Nirei-Souma (2004) for income in the U.S. and Japan from 1960 to 1999, Clementi-Gallegati (2004) for Italy from 1977 to 2002, and Dagsvik-Vatne (1999) for Norway in 1998.

While Pareto argued that "egalitarian taxation" did not have any significant effect on the distribution of income, many have later concluded that the redistributive taxation regimes introduced after World War II did in fact significantly reduce income and wealth inequality; notably, e.g., Lampman (1962) and Kuznets (1955). Most recently, Piketty (2001) has argued that redistributive taxation may have prevented large income shares from recovering after the shocks that, he documents, they experienced during World War II in France.[3]

[1] Pareto distributions are power laws. They display heavy tails, in the sense that the frequency of events in the tails of the distribution declines more slowly than e.g., in a Normal distribution. They represent a subset of the class of stable Levy distributions, that is, of the distributions which are obtained from the version of the Central Limit Theorem which does not impose finite mean and variance; see e.g., Nolan (2005).

[2] While income and wealth are correlated and have qualitatively similar distributions, wealth tends to be more concentrated than income. For instance the Gini coefficient of the distribution of wealth in the U.S. in 1992 is .78, while it is only .57 for the distribution of income (Diaz Gimenez-Quadrini-Rios Rull, 1997); see also Feenberg-Poterba (2000).

[3] This line of argument has been extended to the U.S., Japan, and Canada, respectively, by Piketty-Saez (2003) and Moriguchi-Saez (2005), Saez-Veall (2003).

In this paper we study theoretically the dynamics of the distribution of wealth in an Overlapping Generation economy with bequest and various forms of redistributive taxation. We characterize the transitional dynamics of the wealth distribution as well as the stationary distribution.

More specifically, our economy is populated by a continuum of age structured overlapping generations of agents with a constant probability of death as in Blanchard (1985) and Yaari (1965). The population is stationary and each agent who dies is substituted by his/her child. A subset of the agents has ("joy of giving") preferences for bequests. Agents are born with an initial wealth which is composed of the bequests of their parents (for those born from parents with preferences for bequests) and, if they qualify, welfare subsidies from the government. Agents face a constant interest rate. They choose an optimal consumption-savings plan, which includes the allocation of their wealth between annuities and assets (which become bequests at their death). The government taxes capital income and estates to redistribute wealth in the form of welfare subsidies. The government budget is balanced.

While this economy is very stylized, the stationary distribution of wealth we obtain has the main qualitative properties which, we have argued, characterize wealth distributions: skewedness and fat tails. We show in fact that the stationary wealth distribution in our economy is a power law, a Pareto distribution in particular.[4] The level of concentration and of inequality of wealth at the stationary distribution depends on the demographic characteristics of the economy, its structural parameters, as well as on the endogenous growth rate of the economy. Most specifically, for instance, wealth is less concentrated (the Gini coefficient is lower) the lower is the growth rate of individual wealth accumulation and the higher is the growth rate of aggregate wealth. We study analytically the dependence of the distribution of wealth, of wealth inequality in particular, on various redistributive fiscal policy instruments like capital income taxes, estate taxes, and the form and extent of welfare subsidies. In particular, wealth is less concentrated (the Gini coefficient is lower) for higher capital income taxes and estate taxes. Furthermore we show that the marginal effect of capital income taxes is much stronger than the effect of estate taxes.

Finally, we characterize optimal redistributive taxes with respect to an utilitarian social welfare measure. We show that, even with such an "egalitarian" welfare measure,

[4] Importantly, we obtain this result without the help of any specific assumptions regarding the distribution of income or earnings. Of course the specific quantitative properties of the distribution of wealth are instead closely related to the underlying distribution of earnings. For instance, Castaneda-Diaz Gimenez-Rios Rull (2003) show that a detailed model of the stochastic process of skills calibrated to the U.S. distribution of earnings accounts quantitatively well for the U.S. wealth distribution in equilibrium; see also De Nardi (2000). Cagetti-De Nardi (2000, 2003) stress instead the importance of entrepreneurship and of borrowing constraints to account quantitatively for the wealth distribution of the U.S.

maximizing social welfare is not equivalent to minimizing the concentration or inequality of wealth. This is because minimizing wealth inequality would require excessively (and hence inefficiently) reducing the economy's growth rate. Most interestingly we find that social welfare is maximized with zero estate taxes. Social welfare maximizing capital income taxes, on the contrary, are positive and, in the simulation we have run, close to the value which minimizes the Gini coefficient.

A large literature has studied theoretically models of the dynamics of individual wealth which result in power laws and particularly in Pareto distributions.[5] Notably, Champernowne (1953), Rutherford (1955), Simon (1955), Wold-Whittle (1957) and most of the subsequent literature study dual accumulation models, that is, models in which different stochastic processes drive wealth accumulation for low and high wealth ranges. Champernowne (1953), Rutherford (1955), and Simon (1955) obtain Pareto distributions from multiplicative wealth accumulation processes with stochastic rates of return. Wold-Whittle (1957) study instead a birth and death process with population growth and exogenous exponential wealth accumulation and bequests. Most recently, the analysis of stochastic processes generating power laws in the distribution of wealth has become an important subject in Econophysics (see Mantegna-Stanley, 2000) and many such processes, often along the lines of the cited pioneering studies of the 50's, have been analyzed.[6] The characteristic feature of this literature is that the stochastic processes which generate power laws are exogenous. In particular they are not the result of agents' optimal consumption-savings decisions and hence they are not related to the deep structural parameters of the economy nor to any policy parameter of interest. It is then impossible in the context of these models to study for instance the dependence of the distribution

[5] More general power laws have also been obtained. E.g., Mandelbrot (1960) introduces stochastic processes to obtain Pareto-Levy distributions; Reed-Jorgensen (2003) obtain instead Double Pareto-Lognormal distributions. An early alternative mechanism to produce skewed (but not Pareto) distributions of wealth has been proposed by Roy (1950). If wealth is proportional to talent and talent is composed of many independent attributes interacting multiplicatively, then a simple application of the law of large numbers implies that wealth would be distributed lognormally, hence skewed to the right. The independence of attributes need not be taken literally in practice: Haldane (1942) has shown that, in the case of three correlated but normally distributed variables, the distribution under the independence assumption provides a close approximation as long as the coefficients of variations (ratios of standard deviations to the means) of the individual components are close enough.

[6] For instance, Nirei-Souma (2004) study multiplicative wealth accumulation models with stochastic rates of return and a reflective lower barrier (Kesten processes); Levy (2003) studies the implications of differential rate of return across groups; Solomon (1999) and Malcai et. al. (2002) study similar processes in which the rate of return on wealth accumulation is interdependent across different groups of individuals (Generalized Lotka-Volterra models); Levy (2003) shows that different rates of returns across non-interdependent groups generate wealth distributions which are Pareto only in the tail. Also, Das-Yargaladda (2003) and Fujihara-Ohtsuki-Yamamoto (2004) study stochastic processes in which individuals randomly interact and exchange wealth, and Souma-Fujiwara-Aoyama (2001) add network effects to such random interactions.

of wealth on fiscal policy. To our knowledge this paper is the first to derive analytically the distribution of wealth from an equilibrium economy with optimizing agents and a well-defined government sector responsible for fiscal policies.

2 Wealth accumulation in an OLG economy with bequests

Consider the Overlapping Generation (OLG) economy in Yaari (1965) and Blanchard (1985).[7] Each agent at time t has a probability of death $\pi(t) = pe^{-pt}$.[8] Let $c(s,t)$ and $w(s,t)$ denote, respectively, consumption and wealth at t of an agent born at s. All agents have identical momentary utility from consumption $u(c(s,t))$ satisfying the standard monotonicity and concavity assumptions. Agents may care about the bequest they leave to their children. We assume that agents have a single heir. At any time t an agent allocates his wealth between an asset and an annuity. The asset pays a return r (constant for simplicity), gross of taxes. In perfect capital markets, by no-arbitrage, the annuity pays a return $p + r$, where p is the probability of death. Let $\omega(s,t)$ denote the amount invested in the asset at time t by an agent born at s, with wealth $w(s,t)$. Therefore $w(s,t) - \omega(s,t)$ denotes the amount that an agents invests in the annuity. If the agent dies at time t the amount bequeathed is $\omega(s,t)$. Letting b denote the estate tax on bequeathed wealth, the agent's heir inherits $(1-b)\omega(s,t)$. The agent's utility from bequests is $\chi\phi((1-b)\omega(s,t))$, where ϕ denotes an increasing bequest function. We assume that a subset of agents have no preferences for bequests, that is, have $\chi = 0$. An agent born at time s receives, at birth, initial wealth $w(s,s)$ (and, again for simplicity, no labor income). We let τ denote the capital income tax[9].

The maximization problem of an agent born at time s is:

$$\max \int_t^\infty e^{(\theta+p)(t-v)} \left(u\left(c\left(s,t\right)\right) + p\phi\left(\left(1-b\right)\omega\left(s,v\right)\right)\right) dv \tag{1}$$

subject to:

$$w\left(s,t\right) = w\left(s,s\right) + \int_s^t \left(\left(r+p-\tau\right) w\left(s,v\right) - p\omega\left(s,v\right) - c\left(s,v\right)\right) dv \tag{2}$$

In the interest of closed form solutions we assume

$$u(c) = \ln(c), \quad \phi\left(\omega\right) = \ln\omega$$

[7] More specifically, we consider the formulation with endogenous bequests in Yaari (1965).

[8] Therefore, an agent lives t periods with probability $\int_t^\infty pe^{-pt}dt = e^{-pt}$, and his expected life at any time t is $\int_t^\infty (s-t)\, pe^{-(s-t)p} ds = p^{-1}$.

[9] We assume for simplicity that the tax τ is imposed on both the asset and the annuity.

The characterization of the optimal consumption-savings path is then straightforward[10].

Proposition 1 *The consumption-savings path which solves the agent's maximization problem (1) is characterized by:*

$$c = \eta w, \quad \omega = \chi \eta w, \tag{3}$$

with $\eta = \frac{(p+\theta)}{p\chi+1}$*; and*

$$\dot{w}(s,t) = (r - \theta - \tau)\, w(s,t) \tag{4}$$

Notably, the growth rate of an agent's wealth, $g = r - \theta - \tau$, is independent of the preference parameter for bequests χ. Agents who care about leaving bequests to their children consume a smaller fraction of wealth than agents who do not (and invest all their wealth in annuities), but grow at the same rate g. As a consequence, g decreases with the capital income tax τ but is independent of estate taxes b.

2.1 The aggregate economy

To study the dynamics of the aggregate economy we need to specify its demographics. We assume that the population is constant, and normalized to 1. As a consequence, for any agent who dies at any time there is a new agent born. Since each agent in the economy dies with probability p, at any time s p agents die.[11] Of the p agents dying at any s, only $q < p$ leave an inheritance, while $p - q$ die with no estate, e.g., because they have no preferences for bequests, $\chi = 0$.[12] Correspondingly, $p - q$ agents are born with no wealth and q with the inherited wealth.[13]

Let the aggregate economy's growth rate be denoted g'. Aggregate wealth is defined as:

$$W(t) = \int_{\infty}^{t} w(s,t) p e^{p(s-t)} ds$$

Let $W(s,t)$ denote the aggregate wealth at time t of all agents born at time s. Then

$$\dot{W(t)} = W(t,t) - pW(t) + \int_{-\infty}^{t} \frac{dW(s,t)}{dt} p e^{p(s-t)} ds$$

[10] We restrict parameters so that interior solutions obtain.

[11] At any time t the size of the cohort born at 0 is pe^{-pt}. The total population of the economy, at any time t is therefore 1: $\int_{-\infty}^{t} pe^{(s-t)p} ds = e^{(s-t)p} \mid_{-\infty}^{t} = 1$.

[12] In other words, a fraction $\frac{p-q}{p}$ of the agents have no preferences for bequests.

[13] It is straightforward to show that the analysis of the distribution of wealth in our economy is is equivalent to the analysis of the distribution of *pre-capita* wealth in an economy in which all agents have preferences for bequests, and hence all agents in the economy inherit, but at any time s there is an inflow of $p - q$ agents with minimal wealth from outside, e.g., from immigration. Notably, in this case the population of the economy is not constant but rather grows at a constant rate.

Since the growth rate of wealth is constant across all agents in our economy, $\frac{dW(s,t)}{dt} = (r - \tau - \theta) W(s,t)$ and

$$\dot{W}(t) = W(t,t) - pW(t) + (r - \tau - \theta) W(t) \tag{5}$$

The growth rate of $W(t)$ is determined once we specify the initial wealth of all newborn agents at each time t, $W(t,t)$. In our economy the distribution initial wealth $w(t,t)$ across agents is determined by *i)* any component of wealth that is inherited in addition to the financial wealth inherited from parents, notably e.g., some component of human capital, and *ii)* wealth subsidies due to the government welfare policy, notably fiscal subsidies to support a minimal wealth at birth. For simplicity and without loss of generality we assume that no initial human capital component is present. Instead we study different welfare policies. All such policies have the property that aggregate subsidies constitute a fixed proportion, say γ, of aggregate wealth. This is to asure a constant aggregate economy's growth rate g'. Let the fraction of wealth invested in annuity be denoted μ. The fraction of assets carried by an agent as a fraction of total assets, and inherited upon death, is then denoted by $1 - \mu$ and, from (1), it can be written as:

$$1 - \mu = \frac{\omega}{w} = \frac{(p + \theta)\chi}{p\chi + 1}; \tag{6}$$

As a consequence the aggregate wealth of newborn at t is comprised of the aggregate inherited wealth and of the government subsidies: $W(t,t) = q(1-\mu)(1-b)W(t) + \gamma W(t)$. It follows that the dynamics of aggregate wealth is

$$\dot{W}(t) = (r - \tau - \theta - p) W(t) + q(1-\mu) W(t) + \gamma W(t)$$

and

$$g' = r - \tau - \theta - p + q(1-\mu)(1-b) + \gamma \tag{7}$$

The wealth distributed in welfare subsidies, γW, is the main liability of the government. Its revenues are derived from *i)* the revenues of the capital income tax, τW; *ii)* the revenues of the estate tax, $pb(1-\mu)W$. The requirement that government run a balanced budget at any period t determines the proportion γ of wealth that it can distribute as welfare subsidies:[14]

$$\gamma = \tau + qb(1 - \mu)$$

As a consequence, under a balanced budget, the growth rate of the economy g' is:

[14] We should note at this point that we could also allow some of the tax collections to finance exogenous government expenditures or a public good that enters the preferences of agents separably and does not influence their other decisions. For example we could specify that the government has expenditures proportional to wealth, μW.

$$g' = r - \theta - p + q(1 - \mu)$$

It follows from (6) that μ is independent of estate taxes b. Then the aggregate economy's growth rate g' is decreasing in capital income taxes τ and is independent of estate taxes b. Capital income taxes in fact depress the savings rate by reducing the net interest rate on savings. Estate taxes, on the other hand, have no effect on savings when bequests are optimally chosen under logarithmic preferences, as we have assumed.

It will be important in the following to restrict parameters so that individual wealth accumulates faster than aggregate wealth, that is:

$$g - g' = p - q(1 - \mu) - \tau > 0 \tag{8}$$

Note also that $g - g'$ is independent of b, and decreases with τ.

2.2 Welfare policy

The growth rate of aggregate wealth, g', does not depend on the specifics of the welfare policy, but only on the proportion of wealth distributed as subsidies in the aggregate. The distribution of wealth, on the other hand, does depend on the welfare policy. We shall study two main welfare policies which are distinct in terms of their redistributive means. Both policies guarantee that all agents born at any time t with no inheritance receive a transfer of wealth to bring them to a minimum wealth level $\underline{w}(t)$ which grows at the aggregate economy's rate g', that is $\underline{w}(t) = \underline{w}e^{g't}$. The two welfare policies differ instead on how they support the wealth of tagents born with an inheritance:

Lump-sum subsidies. All agents born at any t with an inheritance receive a lump-sum subsidy equal to $x(t)$ which grows at the aggregate economy's rate g': $x(t) = xe^{g't}$.[15]

Means-tested subsidies. All agents born at any t with inheritance less than $\underline{w}(t)$ get a transfer of wealth to bring them to $\underline{w}(t)$.

In the case of lump-sum subsidies, the total amount of subsidies paid by the government at any time t is independent of the distribution of wealth at t and is a constant fraction of wealth at each time t:

$$(p - q)\underline{w} + qx$$

[15] We assume for simplicity that

$$(1 - \mu)(1 - b)\underline{w}(t) + x(t) \geq \underline{w} \tag{9}$$

so that no inheriting agent has initial wealth smaller that the minimal wealth.

A fiscal policy (τ, b) determines the set of feasible welfare policies $(\underline{w}, x)$, which satisfies

$$(p-q)\underline{w}(t) + qx(t) = \tau W(t) + qb(1-\mu)W(t)$$

In the case of means-tested subsidies the total amount of subsidies paid by the government at any time t depends on the distribution of wealth at t. In particular, the policy subsidizes the wealth of those newborn whose parents are relatively poor at death, that is, have wealth between $\underline{w}(t)$ and $((1-b)(1-\mu))^{-1}\,\underline{w}(t)$. Let $f(w,t)$ denote the distribution of wealth at time t. Total subsidies (government expenditures) at time t are:

$$(p-q)\underline{w}(t) + q\int_{\underline{w}(t)}^{((1-b)(1-\mu))^{-1}\underline{w}(t)} \left(\underline{w}(t) - (1-b)(1-\mu)w\right) f(w,t)dw \qquad (10)$$

It is important to note that such subsidies can be supported by a stationary tax policy (with constant rates τ, b, as we have assumed) only if the distribution of wealth is stationary (independent of t) or if we allow the government to run fiscal deficits and surpluses and only require a balanced budget intertemporally, rather than for all t.[16]

3 The distribution of wealth in the OLG economy

We study the dynamics of the distribution of wealth of the OLG economy with inheritance and estate taxes introduced in the previous section. We solve for both the transitional dynamics and the stationary distribution. We study conditions under which the stationary distribution is Pareto.[17]

Let the distribution of wealth at time t be denoted $f(w,t)$. Its dynamics are described by a linear partial differential equation (PDE) with variable coefficients, an initial condition for the initial wealth distribution, and a boundary condition that reflects the injection of wealth to newborns under ourwelfare policies.

[16] Recall that we have assumed that agents value net bequests, $(1-b)(1-\mu)w$. Importantly, they do not value the subsidies received by their children through the welfare state. This is just for analytical tractability so that μ remains constant for all wealth levels. An alternative numerically tractable formulation for the utility of bequests the under means-tested subsidies could be $\chi \max\{0, \ln\left((1-b)(1-\mu)\frac{w}{\underline{w}}\right)\}$, which guarantees that agents will not give any bequests if they die with discounted wealth smaller than $((1-b)(1-\mu))^{-1}\,\underline{w}$. For lump-sum subsidies, where all agents receive x at birth and start life with $w \geq \underline{w}$, the utility function could be $\ln(\left((1-b)(1-\mu)\frac{w+x}{\underline{w}}\right)$. In either case however μ would depend on wealth.

[17] Wold-Whittle (1957) pioneered the methods of analysis of the dynamics of the distribution of wealth that we adopt in this paper. They studied an economy with dual accumulation. Below a cut-off wealth is assumed to simply grow exponentially. The distribution of wealth above the cut-off is instead determined by a birth-death process. While Wold-Whittle (1955) assume full inheritance and do not study any fiscal policies, population growth in their economy dilutes wealth across children and hence its effect are related to the effects of partial inheritance and estate taxes in our economy.

Let $\sigma(w)$ denote the wealth a parent needs to have at time of death t for his heir born at $t+\Delta$ to inherit wealth w. The expression $\sigma(w)$ takes a different form for the two specification of welfare policies that we study.[18] At time 0 the distribution of wealth $w \in (\underline{w}, \infty)$ is exogenous. Let it be denoted $h(w)$. We assume for simplicity that at time $t=0$ all agents have wealth greater than minimal wealth:

$$h(w) = 0 \text{ for any } w \geq \underline{w}$$

The PDE describing the evolution of the distribution of wealth is obtained as the Chapman-Kolmogorov equation which governs the dynamics of $f(w,t)$ (its derivation is detailed in Appendix A):

$$\frac{\partial f\left(w,t\right)}{\partial t} = -\left(p+g\right) f\left(w,t\right) + q\frac{\partial \sigma(w)}{\partial w} f\left(\sigma(w),t)\right) - gw\ \frac{f\left(w,t\right)}{\partial w} \tag{11}$$

with initial condition is

$$f(w,0) = h(w) \tag{12}$$

The distribution of wealth at time t must also satisfy the boundary condition (derived in Appendix A):

$$f\left(\underline{w}(t),t\right) = \frac{p-q}{g}\frac{1}{\underline{w}(t)} + q\int_{\underline{w}(t)}^{\sigma(\underline{w}(t))} f(w,t)dw \tag{13}$$

This boundary condition guarantees that, at each t, the population size is constant and normalized to 1; that is, $\int f(w,t)dw = 1$. Note that $f\left(\underline{w}(t),t\right)$, the density of wealth at $w = \underline{w}(t)$, is composed of the density of wealth corresponding to the $p-q$ agents who do not receive any inheritance, $\frac{p-q}{g-g'}\frac{1}{\underline{w}}$, and of the the agents whose inheritance at t is below $\underline{w}(t)$, $q\int_{\underline{w}(t)}^{\sigma(\underline{w}(t))} f(w,t)dw$. Recall that under our assumptions this last component is positive only with a welfare policy characterized by means-tested subsidies, and is zero with lump-sum subsidies.

Formally, our problem is the following: Find a density $f(w,t)$ which satisfies the PDE (11) for all $w > \underline{w}(t)$, the initial condition (12), and the boundary condition (13).It will, however, be much more convenient to work in variables discounted by the aggregate economy's growth rate g'. For this purpose define $z = we^{-g't}$. Note that the support

[18] With lump-sum subsidies

$$\sigma(w) = \frac{w-x}{(1-\mu)(1-b)};$$

while with means-tested subsidies

$$\sigma(w) = \frac{w}{(1-\mu)(1-b)}.$$

of z is stationary and equal to $(\underline{w}, \infty)$. The PDE which we obtain after the necessary transformations is:

$$\frac{\partial f(z,t)}{\partial t} = -(p+g-g')\, f(z,t) + q\frac{\partial \sigma(z)}{\partial z} f(\sigma(z),t)) - (g-g')\, z\, \frac{f(z,t)}{\partial w} \tag{14}$$

with initial condition:

$$f(z,0) = h(z) \tag{15}$$

and boundary condition:

$$f(\underline{w},t) = \frac{p-q}{g-g'}\frac{1}{\underline{w}} + q\int_{\underline{w}}^{\sigma(\underline{w})} f(z,t) : dz \tag{16}$$

To solve (11) under (15) and (16) we apply the "method of characteristics" as detailed in Appendix C.

Lemma 1 *There exists a distribution of discounted wealth $f(z,t)$ which satisfies PDE as well as (15). It is characterized by:*

$$f(z,t) =$$

$$\begin{cases} \left(\frac{z}{\underline{w}}\right)^{-\frac{p}{g-g'}-1} f(\underline{w}, t-\tau(z,\underline{w})) + \\ \quad +q\int_{\underline{w}}^{z} \frac{\partial\sigma(y)}{\partial y} f(\sigma(y), t-\tau(z,y))\,(y)^{\left(\frac{p}{g-g'}\right)} (g-g')^{-1} (z)^{-\left(\frac{p}{g-g'}+1\right)} dy & \textit{for } z \in \left(\underline{w}, \underline{w}e^{(g-g')t}\right) \\ e^{-(p+g-g')t} h\left(ze^{(g-g')t}\right) \\ \quad +q\int_{\underline{w}}^{z} \frac{\partial\sigma(y)}{\partial y} f(\sigma(y), t-\tau(z,y))\,(y)^{\left(\frac{p}{g-g'}\right)} (g-g')^{-1} (z)^{-\left(\frac{p}{g-g'}+1\right)} dy & \textit{for } z \geq \underline{w}e^{(g-g')t} \end{cases} \tag{17}$$

where $\tau(z,y) = \frac{\ln z}{\ln y}\frac{1}{g-g'}$

Proof. See Appendix D. ■

This characterization has an interesting economic interpretation. Notice that $\tau(z,y) = \frac{\ln z}{\ln y}\frac{1}{g-g'}$ represents the age of an agent who has wealth z at time t and was born with wealth y. The age of an agent who has wealth z at time t and was born with wealth $\underline{w}$ is then $\tau(z,\underline{w})$. Consider the density of any discounted wealth level $z \in \left(\underline{w}, \underline{w}e^{(g-g')t}\right)$. The first component of the density $f(z,t)$ in (17) is $\left(\frac{z}{\underline{w}}\right)^{-\frac{p}{g-g'}-1} f(\underline{w}, t-\tau(z,\underline{w}))$. It represents the density of agents who have entered the economy with wealth $\underline{w}$, have never died since, and have reached wealth z at t. It is determined by the boundary condition at time $t-\tau(z,\underline{w})$. Similarly, the second component of the density $f(z,t)$ in (17) is $q\int_{\underline{w}}^{z} \frac{\partial\sigma(y)}{\partial y} f(\sigma(y), t-\tau(z,y))\,(y)^{\left(\frac{p}{g-g'}\right)} (g-g')^{-1} (z)^{-\left(\frac{p}{g-g'}+1\right)} dy$. It represents the density of agents who have entered the economy with some wealth y, have never died since, and have reached wealth z at t. Consider instead the density of discounted wealth levels z at time t greater than $\underline{w}e^{(g-g')t}$. The only agents who can possess such a discounted wealth level are: *i)* those agents who were born at time 0 and have never died , *ii)* the children of those agents who have died at some time $t' < t$ and left inheritance larger than $\underline{w}e^{(g-g')t'}$. The density of these agents is represented by the second line of (17).

The distribution of wealth $f(z,t)$ must then satisfy (17) as well as (16). It is in general impossible to find a closed form solution unless the boundary condition (16) has the property that $f(\underline{w},t)$ is constant in t, which in fact is the case if no agent leaves any inheritance. We will discuss this as a special case in Section 3.

We can nonetheless study the limit distribution of the dynamics of $f(z,t)$. First of all we can show that (see the proof of Proposition 2 in Appendix A) the density of of discounted wealth levels z at time t greater than $\underline{w}e^{(g-g')t}$, represented by the second line of (17), declines with time. It is in fact bounded above by $e^{-(p-q+g-g')t}h\left(ze^{-(g-g')t}\right)$. It therefore declines at a rate (greater than) $p-q+g-g'$, due to the rate $p-q$ at which agents die with no inheritance and the rate at which the density "spreads" on account of growth, $g-g'$.

Proposition 2 *The distribution of wealth $f(z,t)$ which satisfies (17) as well as (16) has a stationary distribution, $f(z)$, which solves the following integral equation:*

$$f(z)=\left(\frac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)}f(\underline{w})+q\int_{\underline{w}}^{z}\frac{\partial\sigma(y)}{\partial y}f\left(\sigma(y)\right)(y)^{\left(\frac{p}{g-g'}\right)}\left(g-g'\right)^{-1}(z)^{-\left(\frac{p}{g-g'}+1\right)}dy \tag{18}$$

for

$$f(\underline{w})=\frac{p-q}{g-g'}\frac{1}{\underline{w}}+q\int_{\underline{w}}^{\sigma(\underline{w})}f(z)dz. \tag{19}$$

The integral equation (18) can be solved for quite generally. To provide intuition, we proceed by studying various special cases first.

No inheritance We first study the special case in which agents have no preferences for bequests, $\chi=0$. In this case agents only invest in annuities and leave no bequests, $\mu=1$. All p newborns at time t receive $\underline{w}$ $(q=0)$. Furthermore, from (8), $g-g'=p-\tau$. In this economy, the density of wealth at the boundary $\underline{w}$ is constant over time and the boundary condition is reduced to:

$$f(\underline{w},t)=\frac{p}{g-g'}\frac{1}{\underline{w}}, \tag{20}$$

while the initial condition is the same as in (15).

Proposition 3 *The economy without bequests has the following distribution of discounted wealth at each time t:*

$$f(z,t) = \begin{cases} \frac{p}{p-\tau}\, \underline{w}^{\frac{p}{p-\tau}}\, z^{-\left(\frac{p}{p-\tau}+1\right)} & \text{for } z \in \left(\underline{w}, \underline{w}e^{(p-\tau)t}\right) \\ e^{-(p+p-\tau)t} h\left(ze^{-(p-\tau)}\right) & \text{for } z \geq \underline{w}e^{(p-\tau)t} \end{cases} \tag{21}$$

$f(z,t)$ is a truncated Pareto distribution in the range $\left(\underline{w}, \underline{w}e^{(p-\tau)t}\right)$. The ergodic distribution of discounted wealth is

$$f(z) = \frac{p}{p-\tau}\, \underline{w}^{\frac{p}{p-\tau}}\, z^{-\left(\frac{p}{p-\tau}+1\right)}$$

which is a Pareto distribution with finite mean.[19]

Full inheritance, no estate taxes We now study another special case, in which agents leave all of their wealth as inheritance to their heirs and no estate taxes are imposed. This requires that χ be large enough and $b = 0$. Recall however that at each time t, nonetheless, $p - q$ agents die without heirs and $p - q$ agents are born with minimal wealth $\underline{w}$. Furthermore, from (8), $g - g' = p - q - \tau$. If $\mu = 0, x = 0$, it follows immediately that the boundary condition (16) requires:

$$f\left(\underline{w}, t\right) = \frac{p-q}{g-g'} \frac{1}{\underline{w}},$$

Proposition 4 *The economy with full inheritance and no estate taxes has the following distribution of discounted wealth at each time t:*

$$f(z,t) \begin{cases} \frac{p-q}{p-q-\tau}\, \underline{w}^{\frac{p-q}{p-q-\tau}}\, z^{-\left(\frac{p-q}{p-q-\tau}+1\right)} & \text{for } z \in \left(\underline{w}, \underline{w}e^{(p-q-\tau)t}\right) \\ e^{-(p+p-\tau)t} h\left(ze^{-(p-q-\tau)t}\right) & \text{for } z \geq \underline{w}e^{(p-q-\tau)t} \end{cases} \tag{22}$$

It is a truncated Pareto distribution in the range $\left(\underline{w}, \underline{w}e^{(p-q-\tau)t}\right)$. The ergodic distribution of discounted wealth is

$$f(z) = \frac{p-q}{p-q-\tau}\, \underline{w}^{\frac{p-q}{p-q-\tau}}\, z^{-\left(\frac{p-q}{p-q-\tau}+1\right)}$$

which is a Pareto distribution with finite mean.[20]

Note that in fact this economy is observationally equivalent to an economy without bequest in which all agents die without heirs with probability $p - q$. (The fraction q of agents who die at any t leaving full inheritance to the offspring effectively do not die).

[19] The mean is finite since $\frac{p}{p-\tau} > 1$.

[20] The mean is finite since $\frac{p-q}{p-q-\tau} > 1$.

Means-tested subsidies We are now ready to study the case in which in which agents leave part of their wealth as inheritance to their heirs and estate taxes are imposed; that is, the case in which $0 < \mu, b < 1$. We first study the economy in which welfare policies are characterized by means-tested subsidies.

We study directly the stationary distribution as in this case we cannot analyticaly solve (17) for the transititional dynamics of $f(z,t)$. We can therefore look for a function $f(z)$ which satisfies the integral equation (18) and the boundary condition (19). For this economy (see footnote 18), we have

$$\sigma(y) = \frac{y}{(1-\mu)(1-b)}$$

We use the transformation $j = \sigma(y) = \frac{y}{(1-\mu)(1-b)}$ and obtain, from (18):

$$\begin{aligned} f(z) &= \left(\frac{z}{\frac{\underline{w}}{(1-\mu)(1-b)}}\right)^{-\left(\frac{p}{g-g'}+1\right)} f(\underline{w}) \\ &+ q(g-g')^{-1} \int_{\frac{\underline{w}}{(1-\mu)(1-b)}}^{\frac{z}{(1-\mu)(1-b)}} f(j) \left[((1-\mu)(1-b)j)^{\left(\frac{p}{g-g'}\right)} \; z^{-\left(\frac{p}{g-g'}+1\right)}\right] dj \end{aligned} \tag{23}$$

Recall that, from (8), $g - g' = p - q(1-\mu) - \tau$ We proceed by guessing a Pareto distribution for $f(z)$:

$$f(z) = \frac{p - aq(1-\mu)(1-b)}{p - q(1-\mu) - \tau} \, \underline{w}^{\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}} \; z^{-\left(\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}+1\right)} \tag{24}$$

and then solve for the parameters a to satisfy, respectively, (23) and the boundary condition (19).

After some algebra, we can show that the guess (24) satisfies (23) if and only if a solves the fixed point equation:

$$a = ((1-\mu)(1-b))^{\left(\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}-1\right)} \tag{25}$$

It is straightforward to show that (25) has a unique fixed point, which we denote a^*, and that $0 < a^* < 1$. The boundary condition (19) is also satisfied and $\int_{\underline{w}}^{\infty} f(z)dz = 1$. The quantity $a(1-\mu)(1-b) = q((1-\mu)(1-b))^{\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}}$ is one minus the value of the cumulative Pareto distribution at $\underline{w}((1-\mu)(1-b))^{-1}$ and represents the fraction of the q agents that inherit wealth greater than $\underline{w}$, and receive no subsides to supplement their wealth at birth.

We summarize this analysis with the following result.

Proposition 5 *The economy with inheritance, estate taxes, and means-tested subsidies has a stationary distribution of discounted wealth*

$$\begin{aligned} f(z) = \frac{p-a^*q(1-\mu)(1-b)}{p-q(1-\mu)-\tau} \, \underline{w}^{\frac{p-a^*q(1-\mu)(1-b)}{p-q(1-\mu)-\tau}} \; z^{-\left(\frac{p-a^*q(1-\mu)(1-b)}{p-q(1-\mu)-\tau}+1\right)}, \\ \text{for } 0 < a^* < 1 \text{ satisfying } (25) \end{aligned} \tag{26}$$

which is a Pareto distribution with finite mean.[21]

Lump-sum subsidies We now study the economy for which $0 < \mu, b < 1$ where welfare policies support a minimal discounted wealth $\underline{w}$ and provide all agents with discounted wealth greater that or equal to $\underline{w}$ with discounted lump-sum subsidies x. Under our assumptions it follows immediately that the boundary condition (20) holds, that is $f(\underline{w}, t) = \frac{p-q}{g-g'} \frac{1}{\underline{w}}$.

The stationary distribution satisfies the integral equation (18). For this economy (see footnote 15), we have

$$\sigma(y) = \frac{y-x}{(1-\mu)(1-b)}, \text{ and hence } \sigma(\underline{w}) = \underline{w}, \quad \frac{\partial \sigma(z)}{\partial z} = \frac{1}{(1-b)(1-\mu)}$$

We operate the transformation $j = \sigma(y) = \frac{y-x}{(1-\mu)(1-b)}$ and obtain, from (18):

$$\begin{aligned} f(z) = & \left(\frac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)} f(\underline{w}) \\ & + q\left(g-g'\right)^{-1} \int_{\underline{w}}^{\frac{z-x}{(1-\mu)(1-b)}} f(j) \left[\left((1-\mu)(1-b)j + x\right)^{\left(\frac{p}{g-g'}\right)} (z)^{-\left(\frac{p}{g-g'}+1\right)} \right] dj \end{aligned} \tag{27}$$

While we do not have of a closed form solution to this integral equation, a unique solution exists (see Appendix D). Moreover we can show that, for large z, the distribution of discounted wealth is approximately Pareto. We summarize the analysis with the following result.

Proposition 6 *The economy with inheritance, estate taxes, and welfare policies with minimal wealth support and lump-sum subsidies has a stationary distribution of discounted wealth with the following properties:*

i) *for any z, it is bounded below by a Pareto distribution with exponent*

$$\frac{p}{p - q(1-\mu) - \tau} \tag{28}$$

and it is bounded above by a Pareto distribution with exponent

$$\frac{p - a^* q(1-\mu)(1-b)}{p - q(1-\mu) - \tau} > 1, \quad \textit{for } 0 < a^* < 1 \textit{ satisfying } (25)$$

ii) *for large z, it is approximated by a Pareto distribution with exponent*

$$\frac{p - a^* q(1-\mu)(1-b)}{p - q(1-\mu) - \tau} > 1, \quad \textit{for } 0 < a^* < 1 \textit{ satisfying } (25)$$

[21] The mean is finite since $\frac{p-a^*q(1-\mu)(1-b)}{p-*q(1-\mu)-\tau} > 1$.

3.1 Fiscal Policy Effects

In this section we study the effects fiscal policy changes, that is changes in estate taxes b and capital income taxes τ on the aggregate growth rate of the economy and on the stationary distribution of discounted wealth. Furthermore we characterize optimal redistributive taxes with respect to an utilitarian social welfare measure. We restrict our analysis to welfare policies with means-tested subsidies.

Positive effects of fiscal policies We have shown in the previous section that, without lump-sum transfers, the stationary distribution of discounted wealth is a Pareto distribution with finite mean whose exponent depends on the policy parameters, on the deep preference parameters, on the demographics, and on the interest rate. More specifically, for parameters such that $0 < \mu < 1$ the Pareto exponent of the stationary distribution, denoted by P, is[22]

$$P = \frac{p - a^* q(1-\mu)(1-b)}{p - q(1-\mu) - \tau}, \quad \text{with } a^* = (1-\mu)^{\left(\frac{p-a^* q(1-\mu)(1-b)}{p-q(1-\mu)-\tau} - 1\right)}$$

For a Pareto distribution the Gini coefficient, the standard measure of inequality, is inversely related to the Pareto exponent. In particular, as noted by Chipman (1974), letting G denote the Gini coefficient,

$$G = \frac{1}{2P - 1}$$

We proceed by characterizing the effects of policy variables $b \in [0, 1]$ and $\tau \in [0, p - q(1-\mu)]$ on P. The upper bound on τ is required so that $g - g'$, and therefore P, remain non-negative.

Proposition 7 *The Pareto coefficient of the economy's stationary distribution of discounted wealth is increasing in capital income taxes* τ, $\frac{\partial P(\tau,b)}{\partial \tau} > 0$, *and non-decreasing in estate taxes* b, $\frac{\partial P(\tau,b)}{\partial b} \geq 0$. *Perfect equality* ($G = 0, P = \infty$) *is attained for* $\tau = p - q(1-\mu)$ *for any* b.

To better illustrate the effects of fiscal policies on the Pareto coefficient we calibrate a simple economy. We choose the following parameter values:

$$p = 0.016,\ q = 0.013,\ \theta = 0.04,\ \chi = 10,\ r = 1.08 \tag{29}$$

We choose p for an expected productive life of $p^{-1} = 62$ years, and $\chi = 10$ implying that agents with a positive bequest motive hold 0.49% of their wealth in inheritable,

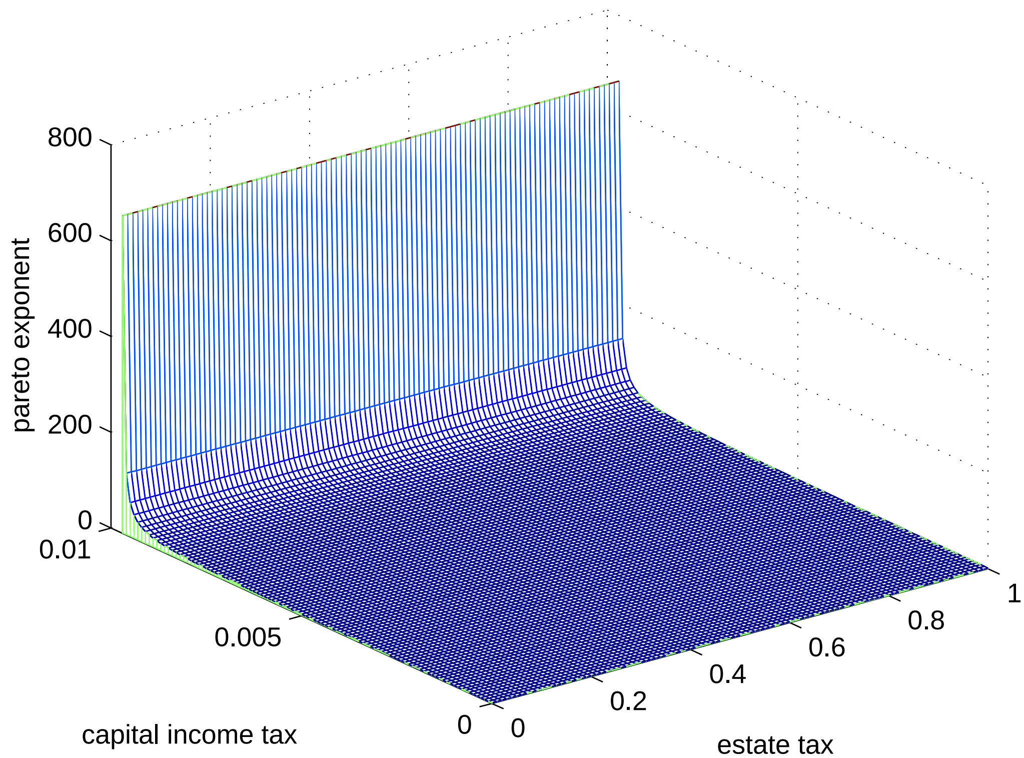

Figure 1:

non-annuitized assets. The fraction of the population that leave bequests to their heirs is $\frac{q}{p} = 0.812\,5$. Figure 1, shows the the relationship between P and the taxes (b, τ).

The effect of capital income taxes on P is essentially due to their effect on the differential growth rate $g - g' = p - q(1-\mu) - \tau$. As τ rises towards its upper bound $p - q\,(1-\mu)$, the Pareto exponent becomes large and tends towards infinity. Consequently the Gini coefficient is reduced, and the wealth distribution becomes more equal. As the distribution becomes more highly peaked, the expression $a^*\,(1-\mu)\,(1-b) = ((1-\mu)\,(1-b))^P$, representing the fraction of the q agents that inherit wealth above $\underline{w}$, declines. Consequently, the effect of estate taxes b decline as well: with small a^* the effect of b on $P = \frac{p - a^* q(1-\mu)(1-b)}{p - q(1-\mu) - \tau}$ becomes negligible. It follows that the higher is the value of τ, the more insignificant is the effect of the estate taxes b on the Pareto and Gini coefficients.[23]

[22]Note that $P = \frac{p}{p-\tau}$ if $\mu = 1$, while, if $\mu = 0$, $P = \frac{p-q}{p-q-\tau}$.

[23]Interestingly, Castaneda-Diaz Gimenez-Rios Rull (1993) also find small effects of estate taxes on the distribution of wealth in an equilibrium economy calibrated to match the U.S. distribution of earnings.

This can be seen from Figure 2 which plots the effect of b on P for various values of τ.

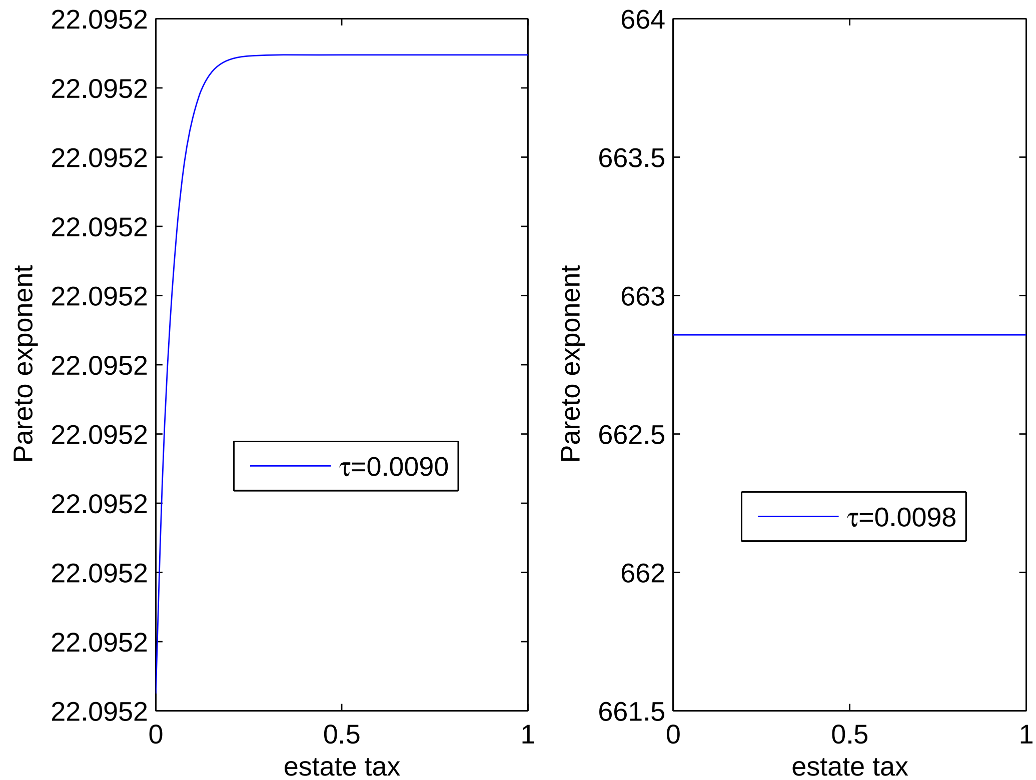

Figure 2:

It is also of interest to note the effect of bequests on the Pareto coefficient. An increase of the preference for bequest, χ, (or of the fraction of agents with such preference, q) increases the fraction of wealth left as inheritance, $1-\mu$. As a consequence, the aggregate growth rate of the economy increases without raising the growth rate of individual wealth, and the Pareto coefficient rises, decreasing wealth inequality.

Fiscal policies (b, τ) do not only affect the Gini coefficient, but also the minimal wealth that can be supported by welfare, $\underline{w}$. Since tax collections finance subsidies so that the government budget remains balanced, discounted mean wealth, M, which we normalize to unity in our simulations, remains constant over time. At the stationary Pareto distribution, (26), we have

$$\underline{w} = \frac{P-1}{P} M \tag{30}$$

Thus as $P \to \infty$, $G \to 0$, perfect equality is reached where the minimum wealth is equal to mean wealth: $\underline{w} = M$.[24]

Normative effects of fiscal policies Instead of focusing on inequality, we may take social welfare to be the main target of fiscal policy. This of course requires the choice of a social welfare function.[25] Chipman (1974), restricting his attention to Pareto distributions, showed that with additively separable social welfare functions, increasing the Pareto coefficient (and thus decreasing the Gini coefficient) does indeed increase social welfare if the mean (rather than the lower bound) of the distribution is kept constant. These results however are derived in a static context and cannot be applied directly to our model, as discussed below.

In the context of an additively separable (utilitarian) welfare criterion, we can inquire into the welfare properties of the stationary distribution of wealth $f(z)$. We can in fact express the social welfare of the agents alive at an arbitrary time t as a function of the Pareto exponent P. Consider a representative agent who solves the maximization problem (1-2). Her optimal consumption-savings choice path is characterized in Section 2. Given an arbitrary discounted wealth z at time t, this time t discounted utility along the optimal path can be written as (see the derivation in Appendix A):

$$U(z) = \frac{1}{\theta + p}\left(\frac{g\,(1+p\chi)}{\theta + p} + \ln \eta + p\chi \ln(\eta\chi)\,(1-b)\right) + \frac{1+p\chi}{\theta+p}\ln z \tag{32}$$

It is independent of t. Recall that a fraction $\frac{p-q}{p}$ of the agents have no preferences for bequests, that is, they have $\chi = 0$. For these agents, given an arbitrary discounted wealth z at time t, their time t discounted utility along the optimal path can be written as:

$$U_0(z) = \frac{1}{\theta + p}\left(\frac{g}{\theta + p} + \ln(p+\theta)\right) + \frac{1}{\theta+p}\ln z$$

[24] At the stationary distribution (26), the government budget constraint, under which tax collections exactly finance subsidies each period, can be written as (see the derivation in Appendix A):

$$\underline{\mathrm{w}} = \frac{\tau + bq(1-\mu)M}{p - q(1-\mu)(1-b)\left(a^* + \frac{P}{P-1}(1-a^*)\right)} \tag{31}$$

where a^* solves (25). Of course, (30) and (31) are equivalent. This can be easily verified: substitute for M from (30) into (31), eliminate $\underline{\mathrm{w}}$, solve for $\frac{P}{P-1}$, and verify that the solution is consistent with the definition of the Pareto exponent.

[25] A large literature has explored the properties of social welfare functions, in particular those that are additively separable in individual utilities and that are increasing in the mean of the distribution of income and decreasing in a measure of its dispersion for all possible income or wealth distributions; see Samuelson (1965) for an early contribution to the subject. Atkinson (1970) and Newbery (1970) demonstrated that if individual utilities are strictly concave there exists no additively separable social welfare function that satisfies these properties; and later Sheshinski (1972) demonstrated that a Rawlsian welfare criterion would indeed satisfy them.

The utilitarian social welfare of the agents alive at an arbitrary time, at the stationary wealth distribution $f(z)$ defined by (26), a Pareto distribution with mean M and exponent P, is:

$$\Omega\left(\underline{w}, P\right) = \frac{q}{p}\int_{\underline{w}}^{\infty} U(z)f(z)dz + \frac{p-q}{p}\int_{\underline{w}}^{\infty} U_0(z)f(z)dz$$

However, we set $\underline{w}$ so that the government budget remains balanced. As discussed in footnote 3.1 above, the mean wealth M remains constant, and $\underline{w} = M\frac{P-1}{P}$. It is straightforward to show then that

$$\Omega\left(M, P\right) = \frac{1}{p+\theta}\left(\frac{g\left(1+p\chi\right)}{p+\theta} + \ln\eta + p\chi\ln\left(\eta\chi\left(1-b\right)\right) + \frac{\left(1+p\chi\right)}{p+\theta}\left(\ln\left(\frac{P-1}{P}M\right) + P^{-1}\right)\right)$$

where $g = r - \theta - \tau$. Therefore,

$$\frac{\partial\Omega\left(M, P\right)}{\partial P} = \frac{\left(1+p\chi\right)}{\left(p+\theta\right)^2}\frac{P^{-2}}{P-1} > 0 \tag{33}$$

We can now consider the welfare effects of different fiscal policies, that is, of different combinations of estate taxes b and capital income taxes τ which satisfy government budget balance, (31). A policy (b, τ) affects on the Pareto exponent P of the stationary distribution $f(z)$ as P depends on τ and b. In a static framework without growth and without a bequest motive, the utilities of agents and the social welfare function does not directly depend on b or on τ except through the Pareto coefficient. Maximizing social welfare would then be equivalent to maximizing P, and given the egalitarian social welfare function, not surprisingly, it follows from (33) that social welfare would be maximized under complete equality: $P = \infty$ and $G = 0$. However this is no longer the case in a dynamic context because both τ and b enter the social welfare function through g and through the bequest motive, in addition to entering through the Pareto coefficient. The derivatives of the social welfare function with repect to τ and b now become

$$\begin{aligned}
\frac{\partial\Omega\left(M, b, \tau\right)}{\partial\tau} &= \left(p+\theta\right)^{-2}\left(\left(1+p\chi\right)\frac{P^{-2}}{P-1}\frac{\partial P}{\partial\tau} - 1\right) \qquad (34)\\
&= \left(p+\theta\right)^{-2}\left(\frac{\left(P-1\right)^{-1}\left(1+p\chi\right)}{p - q\left\{\left(\left(1-\mu\right)\left(1-b\right)\right)^{P}\left(1+P\ln\left(\left(1-\mu\right)\left(1-b\right)\right)\right)\right\}} - 1\right)
\end{aligned}$$

$$
\begin{aligned}
& \frac{\partial \Omega (M, b, \tau)}{\partial b} && (35)\\
&= (p+\theta)^{-1} \left(\frac{(1+p\chi)}{p+\theta} \frac{P^{-2}}{P-1} \frac{\partial P}{\partial b} - p\chi (1-b)^{-1} \right) \\
&= (p+\theta)^{-1} \left[\begin{array}{c} \frac{(1+p\chi)}{(P-1)(p+\theta)} \frac{\chi^{P-1}(p+\theta)^{P}(1+p\chi)^{-1}(1-b)^{P-1} q}{p-q\left\{((1-\mu)(1-b))^{P}(1+P\ln((1-\mu)(1-b)))\right\}} \\ -p(1-b)^{-1} \end{array} \right] \chi
\end{aligned}
$$

where $\Omega (M, b, \tau)$ is the social welfare function expressed explicitely as a function of the policy parameters b, τ; also $\frac{\partial P}{\partial \tau}$ and $\frac{\partial P}{\partial b}$ are defined by (49) and (50) in the proof of Proposition 7 in Appendix A. From Proposition 7 we know that when the Pareto exponent is maximized at $\tau = p - q(1-\mu)$, we have $\frac{\partial P}{\partial b} = 0$, so for $\chi > 0$ social welfare would decline in b due to the bequest motive, as is clear from(35). Consequently, the optimal b would be zero. If however τ has an interior solution so that $\frac{\partial P}{\partial b} > 0$, we cannot determine whether or not b will be interior.[26] In fact it is clear from inspecting (34) that the value of τ that maximizes social welfare has to be less than $p - q(1-\mu)$ because for $\tau \rightarrow p - q(1-\mu)$ we have $(P-1) \rightarrow \infty$, $((1-\mu)(1-b))^{P} P \rightarrow 0$ and $\frac{\partial \Omega(M,b,\tau)}{\partial \tau} < 0$.

Another interesting feature of social welfare function is that for small values of the bequest parameter χ we have $\frac{\partial \Omega(M,b,\tau)}{\partial b} < 0$, so that the maximizing social welfare requires setting $b = 0$. The reason is that for small values of χ, the agent sets a high μ and therefore leaves a small bequest. The negative effect of b on social welfare through its reduction of bequests, given by $-p\chi (1-b)^{-1}$, dominates the positive effect of b on social welfare through the Pareto exponent. This is because as $\chi \rightarrow 0$, $(1-\mu)^{P} \rightarrow 0$ as χ^{P} with $P > 1$, so that for small χ the expression in square brackets in the last two lines of (35) is negative.

For the parameters given by (29), Figure 3 shows the plot of the social welfare function.

Welfare is maximized at $(b, \tau) = (0, 0.0095)$ where the maximum value of τ is $p - q(1-\mu) = 0.0097$, so τ is indeed interior. Figure 4a below shows that social welfare does decline with b for $\tau = 0.0095$ for $\chi = 10$. The Pareto exponent is $P = 71.3846$, the lower bound on wealth is $\underline{w} = 0.986$, the fraction of the q agents who inherit more than $\underline{w}$ is $(1-\mu)(1-b)a^{*} = (1-\mu)^{71.3846} = (2.658582224432109)(10)^{-23}$ and the fraction of wealth that the q agents hold in non-inheritable annuitized form is $\mu = 0.5172$. Figure 4b shows the same, but for for a much smaller bequest parameter, $\chi = 0.01$. Despite the small χ, welfare still declines with b for the reasons discussed above, and it is maximized at $(b, \tau) = (0, 0.0158)$ where the maximum allowed value of τ is $p - q(1-\mu) = 0.0159987$. Now however the welfare maximizing capital tax is higher[27] at $\tau = 0.0157$, the Pareto exponent is lower at $P = 53.4631$, the lower bound

[26] Note that even if χ goes to zero, $(1-\mu) = \frac{(p+\theta)\chi}{1+p\chi}$ goes to zero as well.

[27] The capital taxes are higher despite the direct effect of a lower bequest motive χ because a low χ implies a higher μ and a lower pareto exponent, which tends to make wealth distribution more unequal.

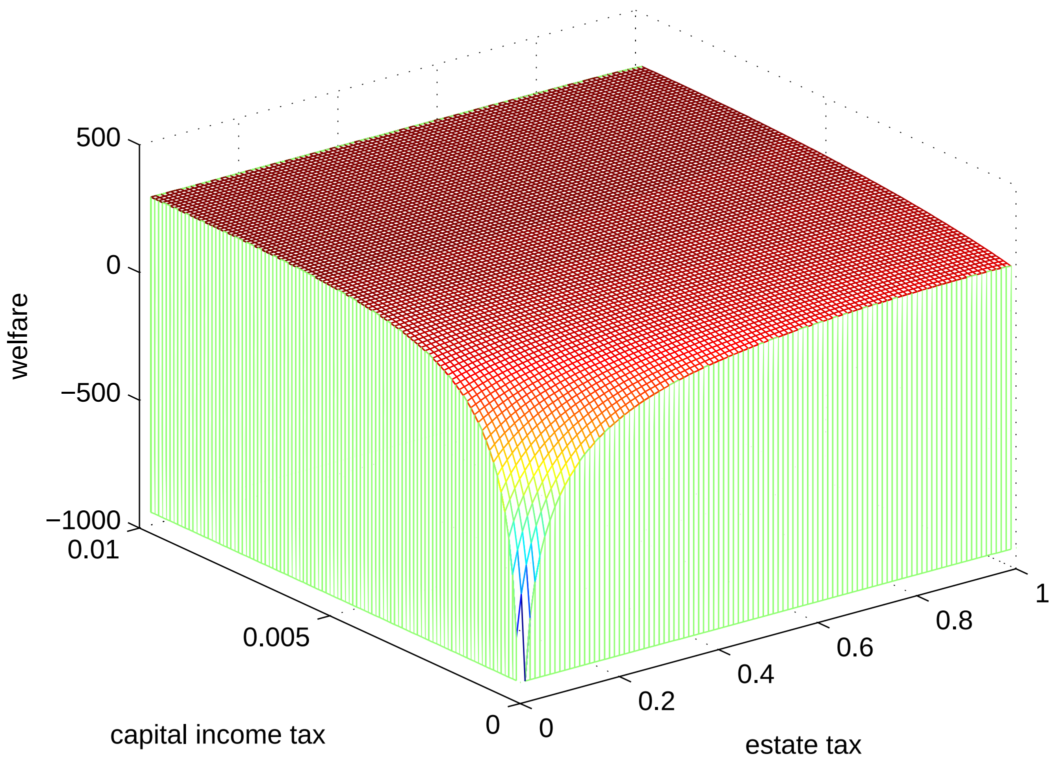

Figure 3:

on wealth is $\underline{w} = 0.9813$, the fraction of the q agents who inherit more than $\underline{w}$ is $(1-\mu)(1-b)a^* = (1-\mu)^{53.4631} = 4.8385\,(10)^{-228}$ and the fraction of wealth that the q agents hold in non-inheritable annuitized form is $\mu = 0.9999$.

Thus for both $\chi = 10$ and $\chi = 0.001$, at the social welfare optimum of for the stationary distribution $f(z)$ estate taxes $b = 0$, capital taxes are interior but close to their maximum allowed value of $p - q(1-\mu)$, and in both cases almost all the population is concentrated just below the mean wealth of 1. However, the egaltarianism implicit in the social welfare function is implemented through capital rather than through estate taxes. Depending on the bequest motive χ, this comes at the expense of growth of almost 1% to 1.5%.

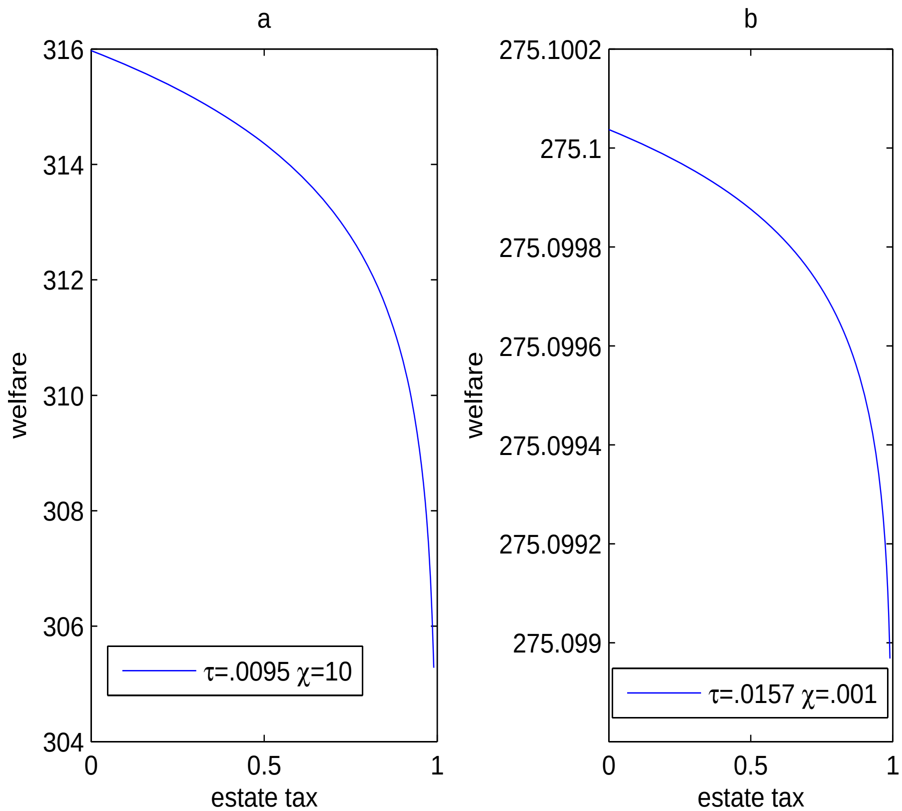

Figure 4: Figure 4

3.2 References

A.B. Atkinson (1970): 'On the Measurment of Inequality,' *Journal of Economic Theory*, 2, 244-63.

A.B. Atkinson (2001): 'Top Incomes in the United Kingdom over the Twentieth Century,' Mimeo, Nuffield College, Oxford.

K. Bjerke (1970): 'Income and Wage Distributions-Part I: A Survey of the Literature,' *Review of Income and Wealth*, 16(3), 235-52.

O.J. Blanchard (1985): 'Debt, Deficits, and Finite Horizons,' *Journal of Political Economy*, 93, 223-47.

F.D. Blau and L.M. Kahn (2005): 'Do Cognitive Test Scores Explain Higher U.S. Wage Inequality?,' *The Review of Economics and Statistics*, 87, 1, 184-93.

M. Cagetti and M. De Nardi (2000): ‘Entrepreneurship, Bequests and the Distribution of Wealth,’ mimeo, University of Chicago.

M. Cagetti and M. De Nardi (2003): ‘Entrepreneurship, Frictions, and Wealth,’ mimeo, University of Chicago.

F.P. Cantelli (1921): ‘Sulla Deduzione delle Leggi di Frequenza da Considerazioni di Probabilita’,’ *Metron*, 83-91.

A. Castaneda, J. Diaz-Gimenez, and J.V. Rios-Rull (2003): ‘Accounting for the U.S. Earnings and Wealth Inequality,’ *Journal of Political Economy*, 111, 4, 818-57.

D.G. Champernowne (1953): ‘A Model of Income Distribution,’ *Economic Journal*, 63, 318-51.

J.S. Chipman (1974): ’The Welfare Ranking of Pareto Distributions,’ *Journal of Economic Theory,* 9, 275-82.

J.S. Chipman (1974): *The Paretian Heritage,* manuscript, University of Minnesota, May 1976.

F. Clementi and M. Gallegati (2004): ‘Power Law Tails in the Italian Personal Income Distribution,’ forthcoming, *Physica A*.

J.K. Dagsvik and B.H. Vatne (1999): ‘Is the Distribution of Income Compatible with a Stable Distribution?’

A. Das and S. Yarlagadda (2003): ‘A Distribution Function Analysis of Wealth Distribution, arXiv:con-mat/0310343.

M. De Nardi (2000): ‘Wealth Inequality and Intergenerational Links,’ mimeo, University of Chicago.

A. Deaton, P.O. Gourinchas, and C. Paxson (2000): ‘Social Security and Inequality over the Life Cycle,’ NBER Working Paper 7570.

J. Diaz Gimenez, V. Quadrini, and J.V. Rios Rull (1997): ‘Dimensions of Inequality: Facts on the U.S. Distributions of Earnings, Income, and Wealth,’ *Federal Reserve Bank of Minneapolis Quarterly Review*, 21, 2, 3-21.

S.J. Farlow (1982): *Partial Differential Equations for Scientists and Engineers,* New York, Dover.

D. Feenberg and J. Poterba (2000): ‘The Income and Tax Share of Very High Income Household: 1960-1995,’ *American Economic Review*, 90, 264-70.

E. Fermi (1949): 'On the Origin of Cosmic Radiation,' *Physics Review*, 75, 1169-74.

A. Fujihara, T. Ohtsuki, and H. Yamamoto (2004): 'Power-Law Tails in Nonstationary Stochastic Processes with Asymmetrically Multiplicative Interactions,' *Physical Review E*, 70, 031106.

I.I. Gikhman and A.V. Skorokhod (1969): *Introduction to the Theory of Random Processes,* Philadelphia, W.B. Sauders.

J.B.S. Haldane, (1942), "Moments of the Distributions of Powers and Products of Normal Variates," *Biometrika*, 42, 226-242.

J. Heckman and E. Vytlacil (2001): 'Identifying the Role of Cognitive Ability in Explaining the Level of and Chance in the Return to Schooling,' *The Review of Economics and Statistics*, 83, 1, 1-12.

R. Herrnstein and C. Murray (1994): *The Bell Curve,* New York, Free Press.

A.W. Hood (2003): *Partial Differential Equations (10 Lectures)*, School of Mathematical and Computational Sciences, University of St Andrews, mimeo.

K. Kawamura and N. Hatano (2003): 'Models of Universal Power-Law Distributions,' arXiv:con-mat/0303331.

S. Kuznets (1955): 'Economic Growth and Economic Inequality,' *American Economic Review*, 45, 1-28.

R.J. Lampman (1962): *The Share of Top Wealth-Holders in National Wealth, 1922-1956*, Princeton, NJ, NBER and Princeton University Press.

M. Levy (2003): 'Are Rich People Smarter?' *Journal of Economic Theory*, 110, 42-64.

O. Malcai, O. Biham, P. Richmond, and S. Solomon (2002): 'Theoretical Analysis and Simulations of the Generalized Lotka-Volterra Model,' *Physical Review E*, 66, 031102.

R.N. Mantegna and H.E. Stanley (2000): *An Introduction to Econophysics*, Cambridge, U.K., Cambridge University Press.

C. Moriguchi and E. Saez (2005): 'The Evolution of Income Concentration in Japan, 1885-2002: Evidence from Income Tax Statistics,' mimeo, University of California, Berkeley.

C. Morisson (2000): 'Historical Perspectives on Income Distribution: The Case of Europe,' in A. Atkinson and F. Bourguignon (eds.), *Handbook of Income Distribution*, Amsterdam, North Holland.

D. Newbery (1970): 'A Theorem on the Measurment of Inequality,' *Journal of Economic Theory*, 2, 264-66.

M. Nirei and W. Souma (2004): 'Two Factor Model of Income Distribution Dynamics,' mimeo, Utah State University.

J.P. Nolan 2005): *Stable Distributions; Models for Heavy Tailed Data*, mimeo, Math/Stat Department, American University, Washington, D.C.

V. Pareto (1897): *Cours d'Economie Politique*, Vol. II, F. Rouge, Lausanne.

T. Piketty (2001): 'Income Inequality in France, 1901-1998,' *Journal of Political Economy.*

T. Piketty and E. Saez (2003): 'Income Inequality in the United States, 1913-1998,' *Quarterly Journal of Economics*, CXVIII, 1, 1-39.

A.D. Polyanin and A.V. Manzhirov (1998): *Handbook of Integral Equations,* Boca Raton, Florida, CRC Press.

W.J. Reed and M. Jorgensen (2003): 'The Double Pareto-Lognormal Distribution: A New Parametric Model for Size Distributions,' mimeo, University of Victoria.

Roy, A. D. (1950) 'The Distribution of Earnings and Individual Output," *The Ecoomic Journal*,' 60, 489-505.

R.S.G. Rutherford (1955): 'Income Distribution: A New Model,' *Econometrica*, 23, 277-94.

E. Saez and M. Veall (2003): 'The Evolution of Top Incomes in Canada,' NBER Working Paper 9607.

P.A. Samuelson (1965): 'A Fallacy in the Interpretation of the Pareto's Law of Alleged Constancy of Income Distribution,' *Rivista Internazionale di Scienze Economiche e Commerciali*, 12, 246-50.

N. Scafetta, S. Picozzi, and B. West (2002): 'Pareto's Law: A Model of Human Sharing and Creativity,' arXiv:cond-mat/0209373 v1.

E. Sheshinski (1972): 'Relation between a Social Welfare Function and the Gini Index of Income Inequality,' *Journal of Economic Theory*, 4, 98-100.

H.A. Simon (1955): 'On a Class of Skew Distribution Function,' *Biometrika*, 52, 425-40.

S. Solomon (1999): 'Generalized Lotka-Volterra (GLV) Models,' arXiv:con-mat/9901250 v1.

G. Solon (1999): 'Mobility Within and Between Generations,' mimeo

D. Sornette (2000): *Critical Phenomena in Natural Sciences,* Berlin, Springer Verlag.

W. Souma, Y. Fujiwara, and H. Aoyama (2001): 'Small-World Effects in Wealth Distribution,' arXiv:con-mat/0108482.

J.C. Strikwerda (2004): *Finite Difference Schemes and Partial Differential Equations,* Philadelphia, SIAM.

F.G. Tricomi (1957): *Integral Equations,* New York, Interscience (Dover edition, 1985).

M. Yaari (1965): 'Uncertain Lifetime, Life Insurance, and the Theory of the Consumer,' *Review of Economic Studies*, 32, No. 2, 137-150.

H.O.A. Wold and P. Whittle (1957), 'A Model Explaining the Pareto Distribution of Wealth,' *Econometrica*, 25, 4, 591-5.

E. Wolff (1987): 'Estimates of Household Wealth Inequality in the U.S., 1962-1983,' *The Review of Income and Wealth*, 33, 231-56.

E. Wolff (2004): 'Changes in Household Wealth in the 1980s and 1990s in the U.S.,' mimeo, NYU.

3.2.1 Appendix A: Proofs - for completeness

Proof of Prop. 1. The dynamic equation for wealth accumulation is

$$\frac{dw\left(s,t\right)}{dt}=\left(r+p-\tau\right)w\left(s,v\right)-p\omega\left(s,v\right)-c\left(s,v\right)$$

First order conditions include

$$\omega\left(s,t\right)=\chi c\left(s,t\right) \tag{36}$$

$$\dot{c}\left(s,t\right)=\left(r-\tau-\theta\right)c\left(s,t\right) \tag{37}$$

The aggregate dynamics for the agent can then be written as:

$$\dot{w}\left(t,s\right)=\left(r+p-\tau\right)w\left(t,s\right)-\left(p\chi+1\right)c\left(s,v\right)$$

Postulating $c=\eta w$, after some algebra,

$$\frac{dc\left(s,t\right)}{dt}=\left(\left(r+p-\tau\right)-\eta\left(p\chi+1\right)\right)c\left(s,v\right) \tag{38}$$

So that, equating 37 and 38 we verify that in fact

$$c=\eta w,\quad \text{with } \eta=\frac{\left(p+\theta\right)}{p\chi+1} \tag{39}$$

Furthermore, by (36),

$$\omega=\left(1-b\right)^{-1}\chi\eta w,\quad \text{with } \eta=\frac{\left(p+\theta\right)}{p\left(1-b\right)^{-1}\chi+1}$$

Finally, using

$$\dot{w}\left(t,s\right)=\left(r+p-\tau\right)w\left(t,s\right)-\left(p\chi+1\right)\eta w\left(t,s\right)$$

and

$$\eta=\frac{\left(p+\theta\right)}{p\chi+1}$$

we can solve for the growth of the agent's wealth, which we denote g:

$$g=r-\tau-\theta \tag{40}$$

Derivation of the PDE, equation (11). Consider the Chapman-Kolmogorov equation which governs the dynamics of $f(w,t)$. Let $w_1>w_l(t)$. The mass of wealth in the interval (w_1,w) at time $t+\Delta$ is $\int_{w_1}^{w}f\left(w,t+\Delta\right)dw$. At a first order approximation this mass has two components. First, since individual wealth grows at rate g, it contains the mass of agents who have wealth in the interval $((1-g\Delta)\,w_1,(1-g\Delta)\,w)$ at time t and are alive at $t+\Delta$. Secondly, through the boundary condition it contains the contribution of those newborns who inherit a fraction of their parents' wealth: the newborns at time t who do not inherit from their parents, or whose inheritance fall below $w_l\,(t)$ add to the density at $w_l(t)$).

Summarizing, the Chapman-Kolmogorov equation can then formally be written as:

$$\int_{w_1}^{w} f\,(w,t+\Delta)\,dw = (1-p\Delta)\int_{(1-g\Delta)w_1}^{(1-g\Delta)w} f\,(w,t)\,dw + q\Delta\int_{\sigma(w_1)}^{\sigma(w)} f\,(w,t)\,dw \quad + o\,(\Delta)$$

Differentiating with respect to w and ignoring second-order terms (terms in Δ^2),

$$f\,(w,t+\Delta) = (1-p\Delta)\,(1-g\Delta)\,f\,((1-g\Delta w),t) + q\Delta\frac{\partial\sigma(w)}{\partial w}f\,(w\sigma(w),t))$$

Rearranging,

$$\frac{f\,(w,t+\Delta)-f\,(w,t)}{\Delta} =$$

$$= \quad \frac{f\,((1-g\Delta w),t)-f\,(w,t)-(\Delta p+\Delta g)\,f\,((1-g\Delta w),t)+q\Delta\frac{\partial\sigma(w)}{\partial w}f\,(w\sigma(w),t))}{\Delta}$$

and, letting $\Delta\to 0$,

$$\frac{\partial f\,(w,t)}{\partial t} = -\,(p+g)\,f\,(w),t) + q\frac{\partial\sigma(w)}{\partial w}f\,(w\sigma(w),t)) - g\frac{\partial f\,(w,t)}{\partial w}$$

Derivation of the boundary condition, (13). The two terms of (13) are, respectively, the density of the newborns with no inheritance and the density of the newborn with inheritance lower that $\underline{w}$.

The first term of (13) can be derived from the age distribution. In particular the density of newborn agents (agents of age $a=0$) with no inheritance is $p-q$. The wealth $w(a)$ of an agent of age a born with wealth $\underline{w}$ is $w(a)=\underline{w}e^{ga}$. Operating the appropriate change of variable to obtain the distribution of wealth from the distribution of age, and evaluating at $w=\underline{w}$, we obtain $\frac{p-q}{g}\frac{1}{\underline{w}}$. The second term is straightforwardly derived.

Proof of Lemma 1 To solve (11) under (16) and (15) we apply the "method of characteristics" as detailed in Appendix C. Let the characteristic space (τ,t) be defined by

$$\frac{dz}{d\tau}=(g-g')z,\quad \frac{dt}{d\tau}=1.$$

Let $z(0)=m$ and $t(0)=0$. In the characteristic space the PDE (11) is then reduced to the following differential equation:

$$\frac{d\left(f\left(z(\tau),\tau\right)\right)}{d\tau}=-\left(p+g-g'\right)f\left(z(\tau),\tau\right)+q\frac{\partial\sigma(z)}{\partial z}f\left(\sigma(z(\tau)),\tau\right) \tag{41}$$

It can be verified that (41) has solution:

$$f\left(z\left(\tau\right),\tau\right)=e^{-(p+g-g')\tau}f\left(m,0\right)+\int_0^{\tau}q\frac{\partial\sigma(z)}{\partial z}f\left(\sigma(z(\eta))\ ,\eta\right)e^{(p+g-g')(\eta-\tau)}d\eta \tag{42}$$

The characteristic space is split along the characteristic $z=\underline{w}e^{(g-g')\tau}$. In particular, for $z\geq\underline{w}e^{(g-g')\tau}$ the solution to the PDE is determined by the initial condition, while for $z<\underline{w}e^{(g-g')\tau}$ the solution is instead determined by the boundary condition through the inverse transformation $\tau(z,y)=\ln\frac{z}{y}\frac{1}{(g-g')}$. Then, substituting back into the original space (z,t), we obtain

$$f\left(z,t\right)=$$

$$\begin{cases}\left(\frac{z}{\underline{w}}\right)^{-\frac{p}{g-g'}-1}f\left(\underline{w},t-\tau\left(z,\underline{w}\right)\right)+ \\ \quad +q\int_{\underline{w}}^{z}\frac{\partial\sigma(y)}{\partial y}f\left(\sigma(y)\,,t-\tau\left(z,y\right)\right)\left(y\right)^{\left(\frac{p}{g-g'}\right)}\left(g-g'\right)^{-1}\left(z\right)^{-\left(\frac{p}{g-g'}+1\right)}dy \quad \text{for } z\in\left(\underline{w},\underline{w}e^{(g-g')t}\right) \\ e^{-(p+g-g')t}h\left(ze^{(g-g')t}\right) \\ \quad +q\int_{\underline{w}}^{z}\frac{\partial\sigma(y)}{\partial y}f\left(\sigma(y)\,,t-\tau\left(z,y\right)\right)\left(y\right)^{\left(\frac{p}{g-g'}\right)}\left(g-g'\right)^{-1}\left(z\right)^{-\left(\frac{p}{g-g'}+1\right)}dy \quad \text{for } z\geq\underline{w}e^{(g-g')t}\end{cases}$$

Proof of Prop. 2. Consider the dynamics of $f(z,t)$ as characterized by (17) in Lemma 1. Consider discounted wealth levels $z\geq\underline{w}e^{(g-g')t}$. In this region, the density an any time t is

$$e^{-(p+g-g')t}h\left(ze^{(g-g')t}\right)+q\int_{\underline{w}}^{z}\frac{\partial\sigma(y)}{\partial y}f\left(\sigma(y)\,,t-\tau\left(z,y\right)\right)\left(y\right)^{\left(\frac{p}{g-g'}\right)}\left(g-g'\right)^{-1}\left(z\right)^{-\left(\frac{p}{g-g'}+1\right)}dy$$

Notice that, if $\sigma(y)=y$, the density in the region $z\geq\underline{w}e^{(g-g')t}$ at time t is larger than in the case $\sigma(y)>y$. But, when $\sigma(y)=y$ (17) can be easily solved to obtain that

$$f(z,t)=e^{-(p-q+g-g')t}h\left(ze^{-(g-g')t}\right),\qquad \text{for } z\geq\underline{w}e^{(g-g')t}.$$

It is now straightforward to notice that $=e^{-(p-q+g-g')t}h\left(ze^{-(g-g')t}\right)$ vanishes for $t\to\infty$.

We conclude that at the stationary distribution the whole mass is in the region $z\in\left(\underline{w},\underline{w}e^{(g-g')t}\right)$. As a consequence, then, from (17),

$$f(z)=\left(\frac{z}{\underline{w}}\right)^{-\frac{p}{g-g'}-1}f\left(\underline{w}\right)++q\int_{\underline{w}}^{z}\frac{\partial\sigma(y)}{\partial y}f\left(\sigma(y)\right)\left(y\right)^{\left(\frac{p}{g-g'}\right)}\left(g-g'\right)^{-1}\left(z\right)^{-\left(\frac{p}{g-g'}+1\right)}dy$$

Proof of Prop. 3. Substituting (20), (8), and $\mu = 1$, $q = 0$ into (17) reduces it to

$$f(z,t)\begin{cases} \frac{p}{p-\tau}\frac{1}{\underline{w}}\left(\frac{z}{\underline{w}}\right)^{-\left(\frac{p}{p-\tau}+1\right)} & \text{for } z \in \left(\underline{w}, \underline{w}e^{(p-\tau)t}\right) \\ e^{-(p+p-\tau)t}h\left(z\right) & \text{for } z \geq \underline{w}e^{(g-g')t} \end{cases} \tag{43}$$

Proof of Prop. 4 Consider $z \in \left(\underline{w}, \underline{w}e^{(g-g')t}\right)$. In this range, substituting (20), and $b = 0$, in (17), it follows that $f(z,t)$ is stationary (independent of t), and hence it satisfies the integral equation (18), which in this case takes the form:

$$f\left(z\right) = \left(\frac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)} f(\underline{w}) + q\int_{\underline{w}}^{z} \left(y\right)^{\left(\frac{p}{g-g'}\right)} \left(g-g'\right)^{-1} \left(z\right)^{-\left(\frac{p}{g-g'}+1\right)} f\left(y\right) dy \tag{44}$$

This is a Volterra integral equation of the second type, with separable kernel, for which a closed form solution exists and is discussed in Appendix D. Applying this solution, we obtain,

$$\begin{aligned} f\left(z\right) &= \left(\tfrac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right) \\ &\quad + q\left(g-g'\right)^{-1}\int_{\underline{w}}^{z} z^{-\left(\frac{p}{g-g'}+1-q(g-g')^{-1}\right)} j^{\left(\frac{p}{g-g'}-q(g-g')^{-1}\right)} \left(\tfrac{j}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right) dj \end{aligned} \tag{45}$$

Straightforward algebraic manipulations, together (20), are now enough to produce the result.

Proof of Prop. 5 The integral equation in this case, after the transformation $j = \sigma(y) = \frac{y}{(1-\mu)(1-b)}$ is reduced to:

$$\begin{aligned} f\left(z\right) &= \\ &\left(\frac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right) + q\left(g-g'\right)^{-1}\cdot \\ &\cdot\int_{\frac{\underline{w}}{(1-\mu)(1-b)}}^{\frac{z}{(1-\mu)(1-b)}} f\left(j\right)\left[\left((1-\mu)(1-b)j\right)^{\left(\frac{p}{g-g'}\right)} z^{-\left(\frac{p}{g-g'}+1\right)}\right] dj \end{aligned} \tag{46}$$

where $g - g' = p - q\left(1-\mu\right) - \tau$. We guess:

$$f(z) = \frac{p - aq\left(1-\mu\right)\left(1-b\right)}{p - q\left(1-\mu\right) - \tau}\, \underline{w}^{\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}} z^{-\left(\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}+1\right)} \tag{47}$$

and substitute into the integral equation. Let $f(\underline{w}) = \frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}\,\frac{1}{\underline{w}}$. We obtain

$$\frac{z}{\underline{w}}^{-\left(\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}+1\right)} f(\underline{w}) =$$

$$\frac{z}{\underline{w}}^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right) + q\left(g-g'\right)^{-1} \left(z\right)^{-\left(\frac{p}{g-g'}+1\right)} \cdot$$

$$\cdot \int_{((1-\mu)(1-b))^{-1}\underline{w}}^{((1-\mu)(1-b))^{-1}z} \left(\frac{j}{\underline{w}}\right)^{-\frac{p}{g-g'}+1-aq(g-g')^{-1}(1-\mu)(1-b)} f\left(\underline{w}\right)\left(1-\mu\right)\left(1-b\right) j^{\frac{p}{g-g'}} dj$$

and, after some algebraic manipulations,

$$\frac{z}{\underline{w}}^{-\left(\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}+1\right)} f(\underline{w}) = \frac{z}{\underline{w}}^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right) +$$

$$+q\left(g-g'\right)^{-1}\left(\left(1-\mu\right)\left(1-b\right)\right)^{\frac{p}{g-g'}} \frac{z}{\underline{w}}^{-\left(\frac{p}{g-g'}+1\right)} \underline{w}^{-aq(g-g')^{-1}((1-\mu)(1-b))} f\left(\underline{w}\right) \cdot$$

$$\cdot \int_{((1-\mu)(1-b))^{-1}\underline{w}}^{((1-\mu)(1-b))^{-1}z} j^{-1+aq(g-g')^{-1}(1-\mu)(1-b)} dj$$

and hence

$$\frac{z}{\underline{w}}^{-\left(\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu)-\tau}+1\right)} f(\underline{w}) =$$

$$= \frac{z}{\underline{w}}^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right) \left(\begin{array}{c} 1 + a^{-1}\underline{w}^{-aq(g-g')^{-1}(1-\mu)(1-b)} \left(\left(1-\mu\right)\left(1-b\right)\right)^{\frac{p-aq(1-\mu)(1-b)}{g-g'}-1} \\ \cdot\left(z^{aq(g-g')^{-1}(1-\mu)(1-b)} - \underline{w}^{aq(g-g')^{-1}(1-\mu)(1-b)}\right) \end{array} \right)$$

$$= \frac{z}{\underline{w}}^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right) \left(\begin{array}{c} 1 + a^{-1}\left(\left(1-\mu\right)\left(1-b\right)\right)^{\frac{p-aq((1-\mu)(1-b))}{g-g'}-1} \\ \left(\frac{z}{\underline{w}}^{aq(g-g')^{-1}(1-\mu)(1-b)} - 1\right) \end{array} \right)$$

Let $a^{-1}\left(\left(1-\mu\right)\left(1-b\right)\right)^{\frac{p-aq(1-\mu)(1-b)}{g-g'}-1} = 1$, or

$$a = \left(\left(1-\mu\right)\left(1-b\right)\right)^{\frac{p-aq(1-\mu)(1-b)}{g-g'}-1}$$

This is fixed point equation which has a unique solution, $a^* < 1$. In fact, it is easily checked that $\left(\left(1-\mu\right)\left(1-b\right)\right)^{\frac{p-aq((1-\mu)(1-b))}{g-g'}-1} = \left(\left(1-\mu\right)\left(1-b\right)\right)^{\frac{p-aq(1-\mu)(1-b)}{p-q((1-\mu))-\tau}-1}$ is strictly positive for $a = 0$, it has a negative derivative with respect to a, and it is less than 1 for

$a = 1$. Consequently,

$$\frac{z}{\underline{w}}^{-\left(\frac{p-a^{*}q((1-\mu)(1-b))}{g-g'}+1\right)} f\left(\underline{w}\right) =$$

$$= \frac{z}{\underline{w}}^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right)\left(1+\left(\frac{z}{\underline{w}}^{a^{*}q(g-g')^{-1}(1-\mu)(1-b)}-1\right)\right)$$

$$= \frac{z}{\underline{w}}^{-\left(\frac{p-a^{*}q((1-\mu)(1-b))}{g-g'}+1\right)} f\left(\underline{w}\right)$$

and the guess is verified.

Proof of Prop. 6 The integral equation in this case, after the transformation $j = \sigma(y) = \frac{y-x}{(1-\mu)(1-b)}$ is reduced to:

$$\begin{aligned} f\left(z\right) = {} & \left(\tfrac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right) \\ & +q\left(g-g'\right)^{-1}\int_{\underline{w}}^{\frac{z-x}{(1-\mu)(1-b)}} f\left(j\right)\left[\left((1-\mu)(1-b)j+x\right)^{\left(\frac{p}{g-g'}\right)}\left(z\right)^{-\left(\frac{p}{g-g'}+1\right)}\right] dj \end{aligned}$$

A lower bound on $f(z)$, $l(z)$ is obtained by the solution to

$$\begin{aligned} l\left(z\right) = {} & \left(\tfrac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)} l\left(\underline{w}\right) \\ & +q\left(g-g'\right)^{-1}\int_{\underline{w}}^{\underline{w}} f\left(j\right)\left[\left((1-\mu)(1-b)j+x\right)^{\left(\frac{p}{g-g'}\right)}\left(z\right)^{-\left(\frac{p}{g-g'}+1\right)}\right] dj = \\ = {} & \left(\tfrac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)} f\left(\underline{w}\right) \end{aligned}$$

$l(z)$ is a power function with exponent $\frac{p}{p-q(1-\mu)-\tau}$, which is a Pareto distribution integrating to unity if defined over $w \geq f\left(\underline{w}\right)\left(\frac{p}{p-q(1-\mu)-\tau}\right)^{-1}$.

An upper bound on $f(z)$, $u(z)$ is obtained by the solution to

$$\begin{aligned} u\left(z\right) = {} & \left(\tfrac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)} u\left(\underline{w}\right) \\ & +q\left(g-g'\right)^{-1}\int_{\underline{w}}^{\frac{z}{(1-\mu)(1-b)}} u\left(j\right) j^{\frac{p}{g-g'}} z^{-\left(\frac{p}{g-g'}+1\right)} dj \end{aligned}$$

since $1-\mu)(1-b)j + x \leq j$ by construction. Adapting the proof of Prop. 5, we can show that $u(z)$ is a power function with exponent

$$\frac{p-a^{*}q\left(1-\mu\right)\left(1-b\right)}{p-q\left(1-\mu\right)-\tau} > 1, \quad \text{for } 0 < a^{*} < 1 \text{ satisfying (25)}$$

which is a Pareto distribution integrating to one if defined for $w \geq f\left(\underline{w}\right)\left(\frac{p-a^{*}q(1-\mu)(1-b)}{p-q(1-\mu)-\tau}\right)^{-1}$. The distribution $f(z)$ for $z \geq f\left(\underline{w}\right)\left(\frac{p-a^{*}q(1-\mu)(1-b)}{p-q(1-\mu)-\tau}\right)^{-1}$ lies in between $u(z)$ and $l\left(z\right)$ [28], both of which converge to zero for large z and therefore it is approximated by

$$f\left(z\right)=\left(\frac{z}{\underline{w}}\right)^{-\left(\frac{p}{g-g'}+1\right)}f\left(\underline{w}\right) \\ +q\left(g-g'\right)^{-1}\int_{\underline{w}}^{\frac{z}{(1-\mu)(1-b)}}f\left(j\right)\left[\left((1-\mu)(1-b)j\right)^{\left(\frac{p}{g-g'}\right)}\left(z\right)^{-\left(\frac{p}{g-g'}+1\right)}\right]dj$$

which has the solution derived in the proof of Prop. 5: a Pareto distribution with exponent

$$\frac{p-a^{*}q\left(1-\mu\right)\left(1-b\right)}{p-q\left(1-\mu\right)-\tau}>1,\quad \text{for } 0<a^{*}<1 \text{ satisfying (25)}$$

Proof of Proposition 7: Since

$$P=\frac{p-\left(\left(1-\mu\right)\left(1-b\right)\right)^{P}q}{p-q(\left(1-\mu\right))-\tau} \tag{48}$$

The right side 48 is increasing in P, for $P=1$ it is larger than 1, and for $P\rightarrow\infty$ it is finite. Therefore we focus on the unique solution of $P\geq 1$. Since $p-q((1-\mu))-\tau\geq 0$, and $0\leq\left(\left(1-\mu\right)\left(1-b\right)\right)^{P}\leq 1$ it follows that $lim_{\tau\rightarrow(p-q((1-\mu))}P=\infty$. Computing the derivatives of P, and substituting for $p-q((1-\mu)$ from 48 we get

$$\frac{dP}{d\tau}=\frac{P^{2}}{p-q\left\{\left(\left(1-\mu\right)\left(1-b\right)\right)^{P}\left(1+P\ln\left(\left(1-\mu\right)\left(1-b\right)\right)\right)\right\}}>0 \tag{49}$$

$$\frac{dP}{db}=\frac{P^{2}\left(\left(1-\mu\right)\left(1-b\right)\right)^{P-1}q\left(1-\mu\right)}{p-q\left\{\left(\left(1-\mu\right)\left(1-b\right)\right)^{P}\left(1+P\ln\left(\left(1-\mu\right)\left(1-b\right)\right)\right)\right\}}\geq 0 \tag{50}$$

Note that $\frac{dP}{db}=0$ would obtain only if $P\rightarrow\infty$ and $\left(1-\mu\right)\left(1-b\right)<1$. As shown, this is indeed the case only if $\tau\rightarrow(p-q((1-\mu))$ and may also be ascertained directly by applying L'Hopital's rule to 50. To do this first apply L'Hopital's rule twice to $\frac{P^{2}}{((1-\mu)(1-b))^{-P-+1}}$ and once to $\frac{P}{((1-\mu)(1-b))^{P}}$ by differenting with respect to τ, and show that both expressions converge to zero as $\tau\rightarrow(p-q((1-\mu))$ because $lim_{\tau\rightarrow(p-q((1-\mu))}P=\infty$. Then substitute into the expression $\frac{dP}{db}$ to see that $lim_{\tau\rightarrow p-q((1-\mu)}\frac{dP}{db}=\frac{0}{p}$.

Derivation of the government budget constraint, (31).

[28] Alternatively the solution of $f(z)$ may be explicitly written as the limiting solution obtained by the successive approximation method (see Polyanin and Manhzirov, section 9.9). The it is possible to show that the iterated kernels of $f(z)$ lie below the iterated kernels of $u(z)$.

Consider the government expenditures (10), written in terms of discounted wealth and evaluated at the stationary distribution $f(z)$:

$$(p-q)\underline{w}+q\int_{\underline{w}}^{((1-b)(1-\mu))^{-1}\underline{w}}\left(\underline{w}-(1-b)(1-\mu)z\right)f(z)dz \tag{51}$$

Furthermore, the stationary distribution is

$$\begin{aligned}
f(z) &= f\left(\underline{w}\right)\frac{z}{\underline{w}}^{-\left(\frac{p-a^{*}q((1-\mu)(1-b))}{(p-q((1-\mu))-\tau)}+1\right)} = \\
&= \frac{p-aq\left(\left(1-\mu\right)\left(1-b\right)\right)}{\left(p-q((1-\mu))-\tau\right)}\,\underline{w}^{\frac{p-a^{*}q((1-\mu)(1-b))}{p-q((1-\mu))-\tau}}\,z^{-\left(\frac{p-a^{*}q((1-\mu)(1-b))}{p-q((1-\mu))-\tau}+1\right)}
\end{aligned}$$

We proceed first by computing $\int_{\underline{w}}^{((1-b)(1-\mu))^{-1}\underline{w}}(1-b)(1-\mu)zf(z)dw$:

$$\begin{aligned}
&\frac{p-aq\left(\left(1-\mu\right)\left(1-b\right)\right)}{\left(p-q((1-\mu))-\tau\right)}\underline{w}^{\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}}\cdot \\
&\cdot\int_{\underline{w}}^{((1-\mu)(1-b))^{-1}\underline{w}} z^{-\left(\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}+1\right)}\left(1-\mu\right)\left(1-b\right)zdz \\
= &\frac{p-aq\left(\left(1-\mu\right)\left(1-b\right)\right)}{\left(p-q((1-\mu))-\tau\right)}\underline{w}^{\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}}\cdot \\
&\cdot\left(-\frac{p-aq\left(\left(1-\mu\right)\left(1-b\right)\right)}{p-q((1-\mu))-\tau}+1\right)^{-1}\left(1-\mu\right)\left(1-b\right) \\
&\cdot\left(\left(\left(1-\mu\right)\left(1-b\right)\right)^{\left(\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}-1\right)}-1\right)\underline{w}^{-\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}+1} \\
= &\left(\frac{p-aq\left(\left(1-\mu\right)\left(1-b\right)\right)}{-p+aq\left(\left(1-\mu\right)\left(1-b\right)\right)+p-q((1-\mu))-\tau}\right)\underline{w} \\
&\left(\left(\left(1-\mu\right)\left(1-b\right)\right)^{\left(\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}\right)}-1\right)\left(1-\mu\right)\left(1-b\right) \\
= &\left(1-\mu\right)\left(1-b\right)\frac{p-aq\left(\left(1-\mu\right)\left(1-b\right)\right)}{q\left(1-\mu\right)\left(1-a\left(1-b\right)\right)+\tau} \\
&\left(1-\left(\left(1-\mu\right)\left(1-b\right)\right)^{\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}-1}\right)\underline{w}
\end{aligned}$$

Furthermore we compute $\int_{\underline{w}}^{((1-b)(1-\mu))^{-1}\underline{w}} f(z)dw$:

$$\begin{aligned}
&\frac{p-aq\left((1-\mu)(1-b)\right)}{(p-q((1-\mu))-\tau)}\underline{w}^{\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}}\int_{\underline{w}}^{((1-\mu)(1-b))^{-1}\underline{w}}\left(z^{-\left(\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}+1\right)}\right)dz\\
=&\left\{\begin{array}{c}\left(\frac{p-aq((1-\mu)(1-b))}{(p-q((1-\mu))-\tau)}\right)\underline{w}^{\left(\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}\right)}\left(-\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}\right)^{-1}\\ \cdot\left(\left((1-\mu)(1-b)\right)^{\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}}-1\right)\underline{w}^{-\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}}\end{array}\right\}\\
=&\left(1-\left((1-\mu)(1-b)\right)^{\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}}\right)
\end{aligned}$$

Substituting the computations in (51), we conclude that government expenditures are:

$$(p-q)\underline{w}+q\underline{w}\left(\begin{array}{c}\left(1-\left((1-\mu)(1-b)\right)^{\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}-1}\right)\\ -(1-\mu)(1-b)\frac{p-aq(1-\mu)(1-b)}{q(1-\mu)(1-a(1-b))+\tau}\\ \left(1-\left((1-\mu)(1-b)\right)^{\frac{p-aq(1-\mu)(1-b)}{p-q(1-\mu))-\tau}-1}\right)\end{array}\right)$$

and therefore that the government budget constraint can be written as:

$$\begin{aligned}
&\underline{w}\left(\begin{array}{c}(p-q)+q\left(1-\left((1-\mu)(1-b)\right)^{\frac{p-aq((1-\mu)(1-b))}{p-q((1-\mu))-\tau}-1}\right)\\ \left(1-(1-\mu)(1-b)\frac{p-aq(1-\mu)(1-b)}{q(1-\mu)(1-a(1-b))+\tau}\right)\end{array}\right)\\
=&\left(\tau+bq(1-\mu)\right)W(0)
\end{aligned}$$

where without loss of generality we set $M=W(0)=1$.

Derivation of the discounted utility along the optimal path, (32). In our economy, the optimal consumption-savings path of an arbitrary agent is characterized by (3). Along this path, it is straightforward to compute

$$U(z)=\int_t^{\infty} e^{(\theta+p)(t-\nu)}\left(\ln\eta w(t,\nu)+p\chi\ln(1-b)\chi\eta w(t,\nu)\right)d\nu$$

where $w(t,\nu)=ze^{g(\nu-t)}$; or,

$$U(z)=\int_t^{\infty} e^{(\theta+p)(t-\nu)}\left(\ln\eta+\ln z+g(\nu-t)+p\chi\ln(1-b)\chi\eta+p\chi\ln z+p\chi g(\nu-t)\right)d\nu$$

We proceed to analyze separately three components of $U(z)$:

i) $\int_t^{\infty} e^{(\theta+p)(t-\nu)}\left(\ln\eta+p\chi\ln(1-b)\chi\eta\right)d\nu$;

ii) $\int_t^{\infty} e^{(\theta+p)(t-\nu)}\left(1+p\chi\right)g(\nu-t)d\nu$; and finally,

iii) $\int_t^{\infty} e^{(\theta+p)(t-\nu)}\left(1+p\chi\right)\ln z d\nu$.

i) Integrating,

$$\int_t^{\infty} e^{(\theta+p)(t-\nu)} \, (\ln \eta + p\chi \ln(1-b)\chi\eta) \, d\nu = \frac{1}{\theta+p} (\ln \eta + p\chi \ln(1-b)\chi\eta)$$

ii) Integrating by parts,

$$\int_t^{\infty} e^{(\theta+p)(t-\nu)} \, (1+p\chi) \, g(\nu-t) d\nu =$$

$$= g\,(1+p\chi) \left(-\frac{1}{\theta+p} \left[e^{(\theta+p)(t-\nu)} (\nu - t) \right]_t^{\infty} + \frac{1}{\theta+p} \int_t^{\infty} e^{(\theta+p)(t-\nu)} d\nu \right) =$$

$$= g\,(1+p\chi) \left(-\frac{1}{\theta+p} \left[e^{(\theta+p)(t-\nu)(\nu-t)} \right]_t^{\infty} - \frac{1}{(\theta+p)^2} \left[e^{(\theta+p)(t-\nu)} \right]_t^{\infty} \right) =$$

$$= \frac{g\,(1+p\chi)}{(\theta+p)^2}$$

iii) Integrating,

$$\int_t^{\infty} e^{(\theta+p)(t-\nu)} \, (1+p\chi) \ln z d\nu = \frac{1+p\chi}{\theta+p} \ln z$$

Adding up,

$$U(z) = \frac{1}{\theta+p} \left(\frac{g\,(1+p\chi)}{\theta+p} + \ln \eta + p\chi \ln \,(1-b)\chi\eta \right) + \frac{1+p\chi}{\theta+p} \ln z$$

3.2.2 Appendix B: On the mechanisms possibly underlying a Pareto distribution of wealth

Various stochastic processes for individual wealth are known to aggregate into a Pareto distribution of wealth in the population; see Sornette (2000) for a technical review and Chipman (1976) for a careful and outstanding account of the historical contributions of this subject; see also Levy (2003).

One such process is exemplified here; its mathematical formulation first appears in Cantelli (1921).[29] Suppose a variable determining wealth (e.g., talent, age), which we denote α, is exponentially distributed. That is the number of people with $\alpha = \alpha_0$ is

$$N(\alpha_0) = pe^{-p\alpha_0}$$

Suppose wealth increases exponentially with α:

$$w = ae^{g\alpha}, \quad a > 0, \quad g \geq 0 \tag{52}$$

Therefore, we can solve for $\alpha = g^{-1} \ln \frac{w}{a}$, operate a change of variables and express the distribution of wealth as

$$N(w) = N\left(g^{-1} \ln \frac{w}{a}\right) \frac{d\alpha}{dw}$$

that is,

$$N(w) = \frac{p}{g} a^{-\frac{p}{g}} w^{-\left(\frac{p}{g}+1\right)}$$

This is a Pareto distribution with the exponent $\frac{p}{g}$.[30]

The underlying mechanism which makes wealth Pareto distributed in our basic model is a similar one in which the factor α is represented by age. This is clearly illustrated by considering the simple economy with no bequests. At any time t, in this economy, the distribution of the population by age $t - s$ implied by the demographic structure of the economy is in fact

$$N(t-s) = pe^{-p(t-s)}$$

Moreover, abstracting from the complications of inheritance, each optimal consumption-savings choices imply a wealth accumulation process results in wealth increasing exponentially with age.

[29] See also Fermi (1949)'s study of cosmic rays.

[30] A notable literature appeared in Italian in the first decades of the twentieth century which studies the wealth distribution resulting from different assumptions regarding the distribution of the generating factor we called α and on the functional dependence of wealth on this factor; see Chipman (1976) for a detailed discussion of these contributions.

3.2.3 Appendix C: On the basic PDE and its solution by the "method of characteristics"

We illustrate in this Appendix the "method of characteristics" for the solution of partial differential equation (PDE's) by applying to a linear PDE with variable coefficients, a simple form of the PDE we solve in the paper. Consider the following PDE:

$$\frac{\partial f}{\partial t} = -af - bz\frac{\partial f}{\partial z} \tag{53}$$

with initial condition

$$f(z, 0) = h(z)$$

Suppose first of all that the PDE is to be solved for $z \in \Re$, that is, that there is no boundary condition. The Method of Characteristics (see e.g., Farlow (1982), Ch. 27) requires solving the PDE in the characteristic space, (τ, t), implicitly constructed as follows:

$$\frac{dz}{d\tau} = bz, \quad \frac{dt}{d\tau} = 1$$

that is,

$$z(\tau) = c_1 e^{-b\tau}, \quad t(\tau) = \tau + c_2 \tag{54}$$

Let $z(0) = m$ and $t(0) = 0$, so that $c_1 = m$ and $c_2 = 0$. This construction has the property that the chain rule

$$\frac{df}{d\tau} = \frac{\partial f}{\partial z}\frac{\partial z}{\partial \tau} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial \tau}$$

and (53) imply

$$\frac{df}{d\tau} = -af \tag{55}$$

a simple ordinary differential equation. The initial condition in characteristic space is $f(m, 0) = h(m)$. The differential equation, together with the initial condition has solution

$$f(z(\tau), \tau) = h(m)e^{-a\tau}$$

Substituting back into the original space (z, t), using (54):

$$f(z, t) = h\left(ze^{-bt}\right)\, e^{-at} \tag{56}$$

In words: the density on z at time t is the same density that at time 0 was on ze^{-bt} dampened at a rate a.

Suppose now that the PDE is to be solved for $z \geq \underline{z}$, and that there is a boundary condition

$$f(\underline{z}, t) = B,$$

The Method of Characteristics applies to this class of problems, boundary value problems, as follows (see e.g., Hood (2003) and Strickwerda (2004), Ch. 1.2). The characteristic space is split along the characteristic $z = \underline{z}e^{b\tau}$. In particular, for $z \geq \underline{z}e^{b\tau}$ the solution to the PDE is determined by the initial condition, and

$$f(z,t) = h\left(ze^{-bt}\right)\ e^{-at}$$

For $z < \underline{z}e^{b\tau}$ the solution is instead determined by the boundary condition through the inverse transformation $\tau(z,y) = \ln \frac{z}{y}\frac{1}{b}$ and

$$f(z,t) = B\frac{1}{b}\left(\frac{z}{\underline{z}}\right)^{\frac{a}{b}}$$

Summarizing, the solution to the boundary value problem is:

$$f(z,t) = \begin{cases} B\frac{1}{b}\left(\frac{z}{\underline{z}}\right)^{\frac{a}{b}} & \text{for } z < \underline{z}e^{bt} \\ h\left(z\,e^{-bt}\right)\ e^{-at} & \text{for } z \geq \underline{z}e^{bt} \end{cases} \tag{57}$$

3.2.4 Appendix D: On Volterra-Fredholm integral equations of the second type

In this Appendix we report some results for the class of integral equations that we study in the paper. We consider Volterra-Fredholm integral equations of the second type with separable kernel:

$$f(z) = h(z) + \lambda \int_a^{\sigma(z)} K(y)H(z)f(y)dy$$

where the real maps h, σ, K, and H are continuously differentiable. It is convenient to study the following equivalent equation:

$$f(z) = h(z) + \lambda \int_a^{\infty} \tilde{K}(z,y)H(z)f(y)dy, \quad \tilde{K}(z,y) = K(y)I_{[a,\sigma(z)]}(z,y) \tag{58}$$

where $I_{[a,\sigma(z)]}(z,y)$ is the indicator function of the interval $[a,\sigma(z)]$, $I_{[a,\sigma(z)]}(y) = \begin{cases} 1 & \text{for } y \in [a,\sigma(z)] \\ 0 & \text{otherwise} \end{cases}$. Note that $\tilde{K}(z,y)$ is not continuous. The theory of Volterra-Fredholm integral equations is, however, developed for square integrable kernels (see Tricomi (1957)), a condition which is obviously satisfied by $\tilde{K}(z,y)H(z)$. For the uniqueness of such solutions (excluding solutions that are zero almost everywhere) see Tricomi (1957), p. 10 and Chapter II and also p.63.

A simple explicit solution is reported by Polyanin-Manzhirov (1998), Ch. 2.1-7 (equation 50), for the following integral equation:

$$f(z) = h(z) + \lambda \int_a^{z} y^{\alpha_1} z^{\alpha_2} f(y)dy, \quad \text{for } \alpha_1 + \alpha_2 = -1$$

It corresponds to a special case of (58) in which:

$$\sigma(z) = z,\ \tilde{K}(z,y) = K(y) = y^{\alpha_1},\ H(z) = z^{-\alpha_1 - 1}$$

Its solution is:

$$f(z) = h(z) + \int_a^{z} R(z,y)h(y)dy, \quad \text{for } R(z,y) = a z^{\alpha_2 - \lambda} y^{\alpha_1 + \lambda}$$