Investment, Consumption and Hedging under Incomplete Markets*

Jianjun Miao[†] and Neng Wang[‡]

February 7, 2006

Abstract

Entrepreneurs often face undiversifiable idiosyncratic risks from their business investments. Motivated by this observation, we extend the standard real options approach to investment to an incomplete markets environment and analyze the joint decisions of business investments, consumption-saving and portfolio selection. We show that precautionary saving motive affects the investment timing decision in an important way. When the investment payoffs are given in lump sum, risk aversion accelerates investment. For an agent with sufficiently strong precautionary motive, an increase in volatility may accelerate investment, opposite to the standard real options analysis. When the agent can trade the market portfolio to partially hedge against the investment risk, the systematic volatility is compensated via the standard CAPM argument, and the idiosyncratic volatility generates a private equity premium. Finally, for the flow payoff case, the agent's idiosyncratic risk exposure alters both the implied option value and the implied project value, causing the reversal of the results in the lump sum payoff case.

Keywords: real options, idiosyncratic risk, hedging, risk aversion, precautionary saving, incomplete markets
JEL Classification: G11, G31, E2

*We thank Andy Abel, Dave Backus, Jonathan Berk, Ricardo Caballero, Darrell Duffie, Jan Eberly, Lorenzo Garlappi, Robert Geske, Simon Gilchrist, Steve Grenadier, John Heaton, Vicky Henderson, Bob Hodrick, Hong Liu, Debbie Lucas, Michael Manove, Robert McDonald, Mitch Petersen (WFA discussant), Tim Riddiough, Tom Sargent, Eduardo Schwartz, Suresh Sundaresan, Fernando Zapatero (FEA discussant), and seminar participants at Berkeley (Haas), Boston College, Boston University, Columbia, FEA, Kellogg School, NYU Stern, UCLA Anderson School, UIC, UIUC Business School, University of Rochester, USC (Marshall School), Washington University in St. Louis, University of Wisconsin at Madison, and WFA for helpful discussions. First version: July 2004.

[†]Department of Economics, Boston University, 270 Bay State Road, Boston, MA 02215. Email: miaoj@bu.edu. Tel.: 617-353-6675.

[‡]Columbia Business School, 3022 Broadway, Uris Hall 812, New York, NY 10027. Email: neng.wang@columbia.edu; Tel.: 212-854-3869.

1 Introduction

Real investment activities play a fundamental role in the economy. A real investment often has three important characteristics. First, it is often partially or completely irreversible. Second, its future rewards are uncertain. Finally, the investment time is to some extent flexible. In the last three decades, a voluminous literature has developed that aims to study the implications of these three characteristics for the investment decision.[1] A key insight of this literature is to view making an investment decision as exercising an American style call option, where "American style" refers to the flexibility of choosing the time of option exercise. Based on this analogy and the seminal contribution on option pricing by Black and Scholes (1973) and Merton (1973), one can apply financial option theory to analyze the irreversible investment decision. This real options approach to investment has become a workhorse in modern economics and finance.

This real options approach relies on one of the following assumptions: (*i*) the real investment opportunity is tradable; (*ii*) its payoff can be spanned by existing traded assets; or (*iii*) the agent is risk neutral. However, these assumptions are violated in many applications. For example, consider entrepreneurial activities. Entrepreneurs combine their business investment opportunities and ideas with their skills to generate economic profits. While entrepreneurs may have valuable projects, these projects may not be freely traded or their payoffs may not be spanned by existing assets because of liquidity restrictions or the lack of liquid markets. These capital market imperfections may be due to moral hazard, adverse selection, transactions costs, or contractual restrictions.[2] Thus, the investment opportunities may have substantial undiversifiable idiosyncratic risks. Owning them exposes entrepreneurs to these undiversifiable risks. Consequently, entrepreneurs' well-being depends heavily on the outcome of their investments. Moreover, entrepreneurs' attitudes towards risk should play an important role in determining their interdependent consumption-saving, portfolio selection, and investment decisions.[3]

While entrepreneurial activities have other important dimensions such as how much to invest, and how to finance the investment project, we focus on the investment timing aspect of entrepreneurial activities. We extend the standard real options approach to analyze the implications of uninsurable idiosyncratic risk for this decision. We use a utility maximization

[1] Arrow (1968) and Bernanke (1983) are among early contributions on irreversible investment. For early stochastic continuous-time models, see Brennan and Schwartz (1985), McDonald and Siegel (1986), Pindyck (1988) and Bertola and Caballero (1994). Abel and Eberly (1994) provide a unified model of (incremental) investment under uncertainty. Dixit and Pindyck (1994) provide a textbook treatment of important contributions to this literature.

[2] Grenadier and Wang (2005) analyze a real options model with agency issues.

[3] There is a fast growing literature on empirical evidence for entrepreneurship. See Gentry and Hubbard (2004), Heaton and Lucas (2000), and Moskowitz and Vissing-Jorgensen (2002), among others.

framework in which an agent chooses his consumption and portfolio allocations, as well as undertakes an irreversible investment.

To facilitate the discussions of our model and results, consider real estate development as an example. The value of the vacant land may be viewed as the option value of developing the real estate.[4] Suppose that a land owner is also the one who knows the best use of his land. For example, the owner has superior knowledge about the local market conditions and knows the most profitable property to construct due to his inalienable human capital. However, the owner cannot sell this yet-to-be-developed property without incurring significant value discount due to transactions costs or asymmetric information such as moral hazard and adverse selection. Therefore, it may be of interest for the owner to keep the land (option) and to be the developer even though owning the land exposes himself to uninsurable idiosyncratic risks of the most profitable property (underlying asset). It is worth noting that land is primarily held by noninstitutional investors such as individuals and private partnerships (Williams (2001)). In addition, individuals and private partnerships are subject to undiversifiable idiosyncratic risks more than institutional investors like pension fund firms and life insurance companies.

While a real estate entrepreneur owns the land and will choose the time to build the property, he may either sell the property or continue to manage the property after developing it. Of course, choosing whether to sell or manage the property is of itself a decision. We assume this decision is exogenous in the paper in order to focus on the effect of idiosyncratic risks on the development decision.[5] When he pays the construction cost and sells the property upon the completion of development, he then receives a lump-sum sale price. We dub this situation the lump-sum payoff case. Alternatively, the real estate entrepreneur may not only be the developer, but also the manager. The entrepreneur may be the most qualified manager, because he can locate the tenants with the highest willingness to pay and maintain the property at the lowest operating expenses. Therefore, it may still make economic sense for the developer to manage the property after construction is complete, even though he will face additional undiversifiable idiosyncratic property risks after development. Under this setting, the developer receives a perpetual stream of uninsurable rental payments (in excess of operating expenses) from managing the property after development. We dub this scenario the flow payoff case.

[4] See Titman (1985), Williams (1991), and Grenadier (1996) for applications of the real options approach to real estate development.

[5] We may extend our model to endogenize sale/no sale decision. Essentially, the sale situation is one where the bidder with the highest valuation of the property is someone else who may have comparative advantages in management. This fits reasonably well into the description of merchant builders. The no-sale scenario corresponds to the case where the developer may also be the best manager in that he can find the tenants with highest willingness to pay and manage the property with lowest operating expenses.

Standard real options analysis (under complete markets) assumes that an agent can fully diversify the idiosyncratic property risks. One can then take the risk-adjusted present discounted value of future cash flows as the market sale value, and thus there is no distinction between the preceding two scenarios. However, when the investment opportunity is not tradable and not spanned by existing traded assets, the standard replicating and no arbitrage argument does not apply. We thus follow the certainty equivalent approach in the literature on the pricing of nontraded assets to value cash flows by analyzing the entrepreneur's utility maximization problem.[6] We show that the lump-sum and flow payoff cases deliver different economic predictions, and hence the equivalence between these two cases no longer holds.

We start with the lump-sum payoff case. We first analyze the effect of risk attitude. By adopting the constant absolute risk aversion (CARA) utility specification, we derive intuitive semi-closed-form solutions which greatly simplify our analysis.[7] For this utility, the risk aversion parameter also measures the precautionary saving motive (captured by the convexity of marginal utility (Kimball (1990)). We show that a stronger precautionary saving motive results in lower certainty equivalent wealth associated with the investment opportunity, which is also the implied option value. Thus, risk aversion speeds up investment.

We next turn to the effect of risk. An important prediction of our model is that the idiosyncratic project volatility has two opposing effects on the implied option value and hence on the investment timing decision. On one hand, the standard real options model states that volatility increases the option value due to its asymmetric convex payoff. On the other hand, idiosyncratic volatility lowers the certainty equivalent wealth and consumption because of the entrepreneur's precautionary saving motive and the interdependence of consumption and investment under incomplete markets. Hence, the net effect of volatility on the option value is ambiguous. When the entrepreneur has sufficiently strong precautionary motive or the idiosyncratic volatility is sufficiently large, the precautionary saving effect may dominate the standard option effect. If the volatility does not directly affect the investment payoff as in the lump-sum payoff case (for example via sale to diversified buyers such as real estate investment trusts (REITs) investors in our real estate example), then idiosyncratic volatility under incomplete markets encourages the entrepreneur to invest earlier, opposite to the standard real options analysis. Going back

[6] See Carpenter (1998), Detemple and Sundaresan (1999), Hall and Murphy (2000), Kahl, Liu and Longstaff (2003), among others, on nontraded asset valuation such as employee stock options. See Section 2 for further discussions.

[7] While power utility is more commonly used in economics, this utility will complicate our anlaysis since it will lead to a two dimensional free-boundary problem, which is hard to analyze. See Section 2.2 for a further discussion. We should emphasize that our key insight of precautionary saving effect still carries over for the power utility.

to our real estate development example, our model predicts that the entrepreneur may exercise his development option early when he is exposed to uninsurable idiosyncratic shocks to his investment opportunity, particularly if he plans to sell his property upon the completion of construction. The entrepreneur's urge to avoid the certainty equivalent wealth discount due to idiosyncratic shocks encourages him to invest earlier, *ceteris paribus*.

When the entrepreneur can hedge against the project risk by trading a risky asset such as the market portfolio, the total volatility may be decomposed into idiosyncratic and systematic volatility. As a result, the entrepreneur's precautionary saving demand (due to idiosyncratic volatility) is then mitigated, which in turn makes the investment option more valuable, *ceteris paribus*. When the investment payoff is independent of the idiosyncratic volatility (for example via sale to diversified buyers), the model then predicts that the entrepreneur invests sooner under incomplete markets than under complete markets, because the option value is lower in the presence of idiosyncratic shocks.

We finally analyze the case where the investment payoffs are given in flow terms. In our previous real estate development example, this case corresponds to the one where the developer also manages the real estate after its completion. Because the developer still faces undiversifiable idiosyncratic risk from the payoff stream after exercising the investment option, he values this payoff stream as certainty equivalent wealth lower than the level, if the payoff steam were marketable. Thus, the previously discussed precautionary saving effect also influences the certainty equivalent value of the project payoffs after exercising the investment option. Because of this additional effect, many results obtained in the lump-sum payoff case are reversed.

In addition to contributing to the investment (real options) literature, our paper also contributes to the portfolio choice literature. Building on the insights behind the Black-Merton-Scholes analysis, we study hedging against endogenously timed income under incomplete markets.[8] We show that the hedging demand increases with the investment option delta.[9] Since the option delta increases in the underlying project payoff value, our model predicts that the developer's hedging demand increases when his development option gets closer to being "in the money." With regard to the consumption-saving literature, we extend the standard incomplete markets analysis to allow the agent to endogenously determine the timing of his income process. We show that volatility not only has a negative effect on consumption, but also a positive option effect due to the endogeneity of the income timing choice.

[8] See Svensson and Werner (1993), Duffie et al. (1997), Koo (1998), Viceira (2001), Heaton and Lucas (2000), Davis and Willen (2002), and Chapter 6 in Campbell and Viceira (2002), among others, on dynamic consumption and portfolio choice when an investor is endowed with nontraded stochastic income.

[9] Delta is defined as the change in the investment option value for a unit increase of the underlying project payoff value.

Two recent papers, Henderson (2005) and Hugonnier and Morellec (2005), are related to ours. Henderson (2005) assumes that the agent maximizes expected wealth at the time of investment. Hugonnier and Morellec (2005) assume that the manager trades off his incentives to exercise an option under incomplete markets pre-maturely in order to lower the idiosyncratic risk exposure and the cost of increasing the likelihood of control challenge due to efficiency loss and firm value destruction. While both papers study real options models under incomplete markets, neither paper studies an agent's consumption decision and its interaction with investment and portfolio choice decisions. As in our lump-sum payoff case, both papers show that market incompleteness encourages an agent to exercise the investment option earlier. Importantly, we show that investment may also be delayed due to market incompleteness when investment payoffs are delivered over time in flows rather than delivered once in lump-sum payment. Our results demonstrate that the timing of payoffs after investment is important in determining the investment timing decision.

The remainder of the paper proceeds as follows. Section 2 analyzes a self-insurance model when the payoff from real investment is given in lump sum. Section 3 generalizes the model in Section 2 to allow for the hedging opportunity. Section 4 extends the models in Sections 2 and 3 to settings in which the real investment payoffs are given in flows. Section 5 concludes. Technical details are relegated to appendices.

2 A Self-Insurance Model with Lump-sum Payoff

This section provides a simple model that allows us to develop intuition for how the agent's attitude towards risk affects his investment decisions when he cannot fully insure himself against the idiosyncratic shocks from investment. In order to achieve this objective in the simplest possible setting, we integrate a canonical consumption/saving model with a standard real options based irreversible investment model.[10]

2.1 Model Setup

Time is continuous and the horizon is infinite. There is a single perishable consumption good (the numeraire). The agent derives utility from a consumption process C according to

$$E\left[\int_0^\infty e^{-\beta t} U\left(C_t\right) dt\right], \tag{1}$$

[10]See Leland (1968) for early studies on precautionary savings. See Zeldes (1989), Caballero (1991b), and Deaton (1991) for dynamic incomplete markets consumption models. See Brennan and Schwartz (1985), McDonald and Siegel (1986), and Dixit and Pindyck (1994) for standard real options models.

where U is an increasing and concave function and $\beta > 0$ is his discount rate. For expositional convenience, we assume that β is equal to r, the risk-free interest rate.[11]

The agent owns an investment project and can undertake this project irreversibly at some endogenously chosen time τ. Note that the investment time τ is stochastic from today's perspective. The investment costs $I > 0$. The agent pays this cost only at the investment time τ. This cost is financed from the agent's own wealth. If there is a shortage of fund, the agent may borrow at the risk-free rate r. In order to focus on the effect of market incompleteness in the simplest possible setting, we do not consider borrowing constraints or costly external financing. Instead, we impose the conventional transversality condition for the agent to rule out Ponzi games. After the agent exercises the investment option at time τ, the project generates a lump-sum payoff X_τ. We also assume that the payoff process X is governed by an arithmetic Brownian motion process

$$dX_t = \alpha_x \, dt + \sigma_x \, dZ_t, \;\; X_0 \text{ given}, \tag{2}$$

where α_x and σ_x are positive constants and Z is a standard Brownian motion.[12] This process implies that payoffs may take negative values. We interpret negative values as losses. We choose the arithmetic Brownian motion purely for analytical convenience and for being in line with further analysis in Section 4 when the payoffs are given in flow terms over time. We may obtain essentially the same insights by using a geometric Brownian motion process to model the payoffs.

As discussed earlier, investing in the project is analogous to exercising a perpetual American call option, in the sense that the agent has the right but not the obligation to invest at some future time of his choosing. Importantly, unlike for financial options, the underlying asset for the real option may not be traded in the market. For example, the building (the underlying asset in the real estate development example) before it is set up is not traded in the market. If we further assume that existing financial assets do not completely span the payoffs for the underlying asset (the building), then we cannot apply the dynamic replication argument in the standard option pricing theory such as the Black-Merton-Scholes model. In this section, the only financial asset available for the agent to trade and to smooth his consumption is the risk-free asset. Hence, the agent inevitably bears the project risk, which are all undiversifiable.

[11] It is straightforward to extend our analysis to allow for differences between the agent's subjective discount rate and the interest rate. We choose not to, however, because no additional insight will be gained for the issue that we are after.

[12] Unlike the often adopted geometric Brownian motion process, the specification in (2) proves more convenient within our setup. Wang (2005) derives a closed-form consumption-saving rule using affine processes and exponential utility.

Let $\{W_t : t \geq 0\}$ denote a wealth process. Then the wealth dynamics are given by

$$dW_t = (rW_t - C_t)\,dt,\ W_0 \text{ given.} \tag{3}$$

That is, the agent accumulates wealth at the rate of $(rW_t - C_t)$, the difference between the interest income rW_t and consumption rate C_t. At the investment time τ, the agent pays the investment cost I and obtains the lump-sum payoff X_τ, and hence his wealth is raised by the amount $(X_\tau - I)$. That is, the agent's wealth jumps by a discrete amount $(X_\tau - I)$ at τ, in that $W_\tau = W_{\tau-} + X_\tau - I$, where $W_{\tau-}$ and W_τ denote the agent's wealth just before and immediately after the agent exercises the investment option, respectively. The agent's optimization problem is to choose both his investment timing strategy τ and consumption process C to maximize his utility given in (1) subject to (3) and a transversality condition specified later.

2.2 Optimality Conditions

We solve the agent's decision problem by working backwards using dynamic programming. We consider first the problem after the agent exercises the investment option. In this case, the agent's optimization problem is a standard deterministic consumption-saving problem without income. Let $V^0(w)$ be the corresponding value function. By a standard argument, $V^0(w)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:[13]

$$rV^0(w) = \max_{c\in\mathbb{R}} U(c) + (rw - c)\,V_w^0(w)\,. \tag{4}$$

Under the deterministic setting, the agent's consumption is constant over time and is equal to the annuity value rw of his wealth, and therefore, his wealth remains constant at w at all times. This is the familiar consumption smoothing result.[14] It is immediate to conclude that the value function is thus given by $V^0(w) = U(rw)/r$.

We next consider the case before the option is exercised. It is worth noting that the agent's value function depends on both his wealth w and the current value x of his investment opportunity. Let $V(w,x)$ denote the corresponding value function. The standard dynamic programming argument implies that $V(w,x)$ satisfies the following HJB equation:

$$rV(w,x) = \max_{c\in\mathbb{R}} U(c) + (rw - c)\,V_w(w,x) + \alpha_x V_x(w,x) + \frac{\sigma_x^2}{2} V_{xx}(w,x)\,. \tag{5}$$

The above HJB equation is similar to an asset pricing equation. It states that the agent chooses his consumption optimally by setting the return $rV(w,x)$ of his value function to equal the sum

[13] The transversality condition $\lim_{T\to\infty}\left[e^{-rT}J(W_T)\right] = 0$ must also be satisfied.

[14] This result follows from two steps: (i) the equality between the agent's discount rate and the interest rate implies that the marginal utility is constant at all times $(U'(C_t) = U'(C_s))$; (ii) The strict concavity of the utility function further implies that $C_t = C_s$.

of his instantaneous utility $U(c)$ and the total expected changes of his value function (due to the change in wealth and also in the investment opportunity).

We now specify boundary conditions. First, the no-bubble condition $\lim_{x\to-\infty} V(w,x) = V^0(w)$ must be satisfied. This condition states that when the investment payoff goes to negative infinity, the agent will never exercise the investment option and his value function is equal to that without the investment option. Next, as is standard in the optimal stopping problems, at the instant of investment, the following value-matching condition must hold:

$$V(w,x) = V^0(w + x - I). \tag{6}$$

This equation implicitly defines an investment boundary $x = \overline{x}(w)$. In general, this boundary $\overline{x}(w)$ depends on the agent's wealth level w. Finally, because this boundary is chosen optimally, the following smooth-pasting condition is satisfied:[15]

$$\left.\frac{\partial V(w,x)}{\partial x}\right|_{x=\overline{x}(w)} = \left.\frac{\partial V^0(w+x-I)}{\partial x}\right|_{x=\overline{x}(w)}, \tag{7}$$

$$\left.\frac{\partial V(w,x)}{\partial w}\right|_{x=\overline{x}(w)} = \left.\frac{\partial V^0(w+x-I)}{\partial w}\right|_{x=\overline{x}(w)}. \tag{8}$$

The first smooth-pasting condition (7) states that the marginal change of the investment opportunity has the same marginal effect on the agent's value functions just before and immediately after exercising the option. Similarly, the second smooth-pasting condition (8) states that the marginal effect of wealth must be the same on the agent's value functions just before and immediately after exercising the option. Unlike the conventional irreversible investment models (Dixit and Pindyck (1994)), here the agent's wealth enters as an additional state variable, which gives rise to the second smooth-pasting condition (8).

2.3 Model Solution for CARA Utility

We have now formulated the agent's optimization problem as a combined control (consumption) and stopping (investment) problem, which is generally difficult to solve. Our objective is to understand the economic effects of uninsurable idiosyncratic risk and the attitude towards risk on investment and consumption decisions. In order to achieve this objective in the simplest possible way, we assume that the agent has CARA utility $U(c) = -e^{-\gamma c}/\gamma$, where the parameter $\gamma > 0$ is the coefficient of absolute risk aversion. It is also equal to the coefficient of absolute prudence $-U'''(c)/U''(c)$, which captures the precautionary saving motive (Kimball (1990)). While CARA utility does not capture the wealth effect, we emphasize that the main results and

[15] See, for example, Krylov (1980), Dumas (1991) and Dixit and Pindyck (1994).

insights of this paper do not rely on the choice of this utility function. As we will see below, the driving force of the paper is the precautionary saving, which can be captured by any utility function having convex marginal utility. We leave generalization to incorporate wealth effects for future research.

Given CARA utility, we may immediately conclude that the value function after investment is given by

$$V^0(w) = -\frac{1}{\gamma r} \exp(-\gamma r w). \tag{9}$$

Next, we conjecture that the value function before the option exercise takes the following form:

$$V(w,x) = -\frac{1}{\gamma r} \exp\left[-\gamma r \left(w + G(x)\right)\right], \tag{10}$$

where $G(x)$ is a function to be determined. One can interpret $G(x)$ as the certainty equivalent wealth derived from the agent's investment opportunity. Specifically, we follow the consumption literature to define certainty equivalent wealth as the value w_{ce} satisfying the equation $V^0(w + w_{ce}) = V(w,x)$; that is, the agent is indifferent between the situation where he receives stochastic income in the future and the situation where he has no income but a total wealth level of $(w + w_{ce})$. Using the explicit functional forms of $V^0(w)$ and $V(w,x)$, we have $w_{ce} = G(x)$.

The boundary conditions (6)-(8) and the additive separability of wealth w and certainty equivalent wealth $G(x)$ in the exponent of the value function $V(w,x)$ indicate that the investment boundary is flat, in that $\overline{x}(w)$ is independent of wealth w. This property substantially simplifies our analysis. The following proposition summarizes the solution to the agent's combined consumption and investment problem.

Proposition 1 *The agent exercises the investment option the first time the process X hits the threshold $\bar{x}$ from below. After exercising the option, the agent's value function and consumption rule are given by (9) and $\overline{c}(w) = rw$, respectively. Before exercising the option, his value function and consumption rule are, respectively, given by (10) and*

$$\overline{c}(w,x) = r\left(w + G(x)\right), \tag{11}$$

where $(G(x), \bar{x})$ is the solution to the following free boundary problem:

$$rG(x) = \alpha_x G'(x) + \frac{\sigma_x^2}{2} G''(x) - \frac{\gamma r \sigma_x^2}{2} G'(x)^2, \tag{12}$$

subject to the no-bubble condition $\lim_{x \to -\infty} G(x) = 0$, and the boundary conditions:

$$G(\bar{x}) = \bar{x} - I, \tag{13}$$

$$G'(\bar{x}) = 1. \tag{14}$$

Moreover, G is increasing.

We now analyze the intuition behind this proposition and discuss its implications.

2.4 Interdependence of Investment and Consumption

As in the standard real options approach, the agent trades off between holding the investment option to obtain an implied option value of waiting and exercising this option to obtain investment payoffs. The key to our analysis is to derive the implied option value. We show below that, unlike the standard real options approach, risk aversion and consumption play an important role in the determination of the option value under incomplete markets.

Implied Option Value. Proposition 1 demonstrates that the certainty equivalent wealth $G(x)$ solves a free-boundary problem (12)-(14). These equations are similar to, but different from, the valuation equations and boundary conditions in the standard real option models of McDonald and Siegel (1986) and Dixit and Pindyck (1994). Based on this similarity, we interpret x as the project value and the certainty equivalent wealth $G(x)$ as the implied option value to invest in the underlying project. More formally, we follow the literature on the pricing of nontraded assets by defining the implied option value Q of the project as the solution to the equation $V(w-Q,x) = V^0(w)$; that is, the agent is indifferent between the situation where he has no investment opportunity and the situation where he pays the price Q and obtains the investment opportunity. Given the functional form of V^0 and V in (9) and (10), we see that $Q = G(x)$.

The two interpretations of $G(x)$ – the certainty equivalent wealth and the implied option value – are the same in our setup. This is due to the absence of the wealth effect under CARA utility. We will thus use certainty equivalent wealth (from the consumption literature perspective) and implied option value (from the investment literature perspective) interchangeably throughout the remainder of the paper.

Proposition 1 nests the standard (risk neutral) real options problem as a special case. Setting $\gamma = 0$ in equation (12) enables us to derive the following explicit solutions for the option value $G(x)$ and the investment threshold $\bar{x}$:

$$G(x) = \frac{1}{\lambda_0} e^{\lambda_0 (x-\bar{x})}, \quad \text{for } x \leq \bar{x}\,, \text{ and } \bar{x} = I + \frac{1}{\lambda_0}\,, \tag{15}$$

where $\lambda_0 = -\sigma_x^{-2}\alpha_x + \sqrt{\sigma_x^{-4}\alpha_x^2 + 2r\sigma_x^{-2}}$ for $\sigma_x > 0$, and $\lambda_0 = r/\alpha$ for $\sigma_x = 0$. It is straightforward to verify that both the option value $G(x)$ and the investment threshold $\bar{x}$ increase in volatility σ_x of the payoff. These are the main results of the real options literature. The agent can capture the upside gains by investing and limit the downside losses by simply waiting until the option is sufficiently "in the money." This asymmetric convex payoff generates the positive effect of volatility on the option value and investment threshold.

The main difference between our model and the standard (risk neutral) real options model is that option value $G(x)$ depends not only on the parameters describing the risk-free rate r, drift α_x and volatility σ_x, but also depends on the agent's precautionary motive. The latter dependence captures the notion that the agent's risk attitude matters not only for consumption decisions, but also for investment decisions when markets are incomplete. The last nonlinear term on the right side of (12) captures the agent's precautionary savings motive. It confirms the intuition that the implied option value $G(x)$ is lower when the precautionary motive is stronger, *ceteris paribus*. Since the project payoff value x does not depend on the agent's risk attitude, the net effect of an increase in γ is to encourage earlier investment. Figure 1 plots the implied option value $G(x)$ versus the value of the underlying investment opportunity x for two values of γ. Note that the payoff line $(x - I)$ is independent of risk aversion γ, this figure clearly illustrates that the investment threshold decreases with the agent's precautionary motive or risk aversion γ.

[Insert Figure 1 Here]

Investment Threshold. To gain further intuition, we use the asymptotic approximation method to compute approximate solutions for the implied option value $G(x)$ and the investment threshold $\bar{x}$.[16] We expand the option value $G(x)$ and the investment threshold $\bar{x}$ to the first order of σ_x^2, in that $G(x) \approx G_0(x) + G_1(x)\sigma_x^2$ and $\bar{x} \approx \bar{x}_0 + \delta_1\sigma_x^2 \equiv \bar{x}_1$. Plugging these expansions in (12)-(14), we show in the appendix that $\bar{x}_0 = I + \alpha_x/r$ and

$$\bar{x}_1 = \bar{x}_0 + \left(\frac{1}{\alpha_x} - \gamma\right)\frac{\sigma_x^2}{2}. \tag{16}$$

This approximate solution indicates that, to a first-order approximation with respect to σ_x^2, a stronger precautionary motive (higher γ) lowers the investment threshold, consistent with our earlier discussions based on the non-linear ODE (12) and the boundary conditions (13)-(14).

The above approximate solution also helps us to understand the effect of volatility on investment threshold. An increase in volatility σ_x has two opposing effects. On one hand, a higher volatility increases option value and hence encourages waiting, as in standard real option models. On the other hand, an increase in σ_x also raises the precautionary savings demand

[16] See Judd (1998). Kogan (2001) applies this method to solve an irreversible (incremental) investment model.

and hence lowers the certainty equivalent wealth $G(x)$, and hence lowers the threshold, *ceteris paribus*. Both effects are reflected in the last term on the right side of (16). When γ is sufficiently small, the option effect dominates the precautionary saving effect. Thus, an increase in volatility σ_x raises the implied option value and delays investment, same as the predictions in the standard real options models. By contrast, when γ is sufficiently large, the precautionary saving effect may dominate the option effect. Therefore, an increase in σ_x lowers the certainty equivalent wealth $G(x)$, and hence encourages the agent to exercise his option sooner, opposite to the standard real options result.

Finally, we use numerical solutions to confirm our intuition. We apply the projection method detailed in the appendix to solve the free boundary problem characterized by (12)-(14). We find that, for a small σ_x, our preceding approximate solution is very close to the "true" solution delivered by the projection method. For a large range of parameter values, Figure 2 plots the investment threshold as a function of the volatility σ_x and the parameter γ. This figure demonstrates that our preceding results and intuition extend to general parameter values.

[Insert Figure 2 Here]

Consumption. We now turn to the agent's consumption policies. After exercising the option, the agent solves a deterministic consumption smoothing problem. As noted earlier, his wealth remains constant and consumption is equal to the interest income at all times. Before exercising the option, the agent's consumption rule (11) is given by the annuity value of the sum of his financial wealth w and his certainty equivalent wealth $G(x)$.

Even though the agent does not receive payoff x before exercising the option, he rationally anticipates that he will exercise his investment option some time in the future. Thus, the future investment payoff matters not only for his future consumption, but also for his current consumption. Our model captures the forward-looking consumption smoothing intuition in an incomplete markets setting with endogenous stochastic income.

The standard intuition in the consumption literature is that volatility lowers consumption because of precautionary motive. Here, we show that consumption may potentially increase in volatility because the option effect may dominate the precautionary saving effect on $G(x)$. This effect is not present in the consumption literature, because almost all models in the consumption literature take stochastic income as *exogenously* given and hence rules out the option effect of income volatility on consumption.

In summary, the uninsurable idiosyncratic risk alters results in the standard real options and consumption literature. When idiosyncratic risk is large or the precautionary motive is strong, the option value and the investment threshold may decrease in volatility, contrary to the standard real option results. Therefore, applying the real options analysis and ignoring consumption smoothing motive to settings where the idiosyncratic risk is likely to matter such as entrepreneurial investments, is potentially misleading and incorrect.

3 Lump-sum Payoff Case with Hedging Opportunities

In the previous section, the agent can trade only a risk-free asset to partially insure himself against the project risk. The restriction that the agent can only insure via the risk-free asset is obviously strong. We now generalize the setting by allowing the agent to trade a risky asset to partially hedge against the project risk. We may interpret this financial asset as the market portfolio. Unlike the self-insurance model in the previous section where all risks are idiosyncratic and uninsurable, investing in the risky asset allows the agent to partially hedge and hence separate systematic volatility from idiosyncratic volatility. We show that distinguishing idiosyncratic volatility from systematic volatility is of fundamental importance, because these two volatilities play *different* roles in determining the option value and the exercising decisions. Our analysis nests the standard complete-markets analysis as a special case.

3.1 The Model

Let $\{P_t : t \geq 0\}$ denote the risky asset's price process and assume that the return is governed by the following process:

$$dP_t/P_t = \mu_e dt + \sigma_e dB_t, \tag{17}$$

where μ_e and σ_e are positive constants, and B is a standard Brownian motion correlated with the Brownian motion Z, which drives the innovations of the project payoff as given in (2). Let $\rho \in [-1, 1]$ be the correlation coefficient between the return on the risky asset and the agent's project payoff, and let $\eta = (\mu_e - r)/\sigma_e > 0$ denote the Sharpe ratio of the market portfolio.

One can alternatively rewrite the observed payoff process $\{X_t : t \geq 0\}$ given in (2) as follows:

$$dX_t = \alpha_x dt + \rho\sigma_x dB_t + \epsilon_x \, d\widetilde{B}_t, \tag{18}$$

where B and $\widetilde{B}$ are two independent standard Brownian motions, and

$$\epsilon_x = \sqrt{1-\rho^2}\, \sigma_x. \tag{19}$$

One may think of B as the Brownian motion describing the systematic (market) risk, and thus $\rho\sigma_x$ is the systematic component of the volatility for the project payoff. One may then interpret $\widetilde{B}$ as the Brownian motion describing the idiosyncratic project risk, and thus ϵ_x is the idiosyncratic volatility. A higher absolute value of the correlation coefficient $|\rho|$ implies that systematic volatility has a larger weight, *ceteris paribus*.

Let π_t be the amount allocated to the risky asset at time t, measured in units of the consumption good. The agent's problem is to choose a consumption process C, a portfolio allocation rule π, and an investment timing strategy τ to maximize his utility (1) subject to his wealth dynamics:

$$dW_t = \left(rW_t + \pi_t\left(\mu_e - r\right) - C_t\right)dt + \pi_t\sigma_e dB_t,\ W_0 \text{ given.} \tag{20}$$

Similar to Section 2, the agent's wealth jumps immediately after he invests, in that $W_\tau = W_{\tau-} + X_\tau - I$, where $W_{\tau-}$ and W_τ are his wealth just before and immediately after his investment at time τ, respectively. Note that (20) is the same both before and after the option exercise.

We use the same dynamic programming method as in Section 2 to solve the agent's problem and summarize the results below.

Proposition 2 *The agent exercises the investment option the first time the process X hits the threshold $\bar{x}$ from below. After exercising the option, the optimal consumption and portfolio rules are given by*

$$\bar{c}\left(w\right) = r\left(w + \frac{\eta^2}{2\gamma r^2}\right), \tag{21}$$

$$\bar{\pi}\left(w\right) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r}. \tag{22}$$

Before exercising the option, the optimal consumption and portfolio rules are given by

$$\bar{c}\left(w,x\right) = r\left(w + G\left(x\right) + \frac{\eta^2}{2\gamma r^2}\right), \tag{23}$$

$$\bar{\pi}\left(w,x\right) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r} - \frac{\rho\sigma_x}{\sigma_e}G'\left(x\right), \tag{24}$$

where $(G,\bar{x})$ is the solution to the following free boundary problem:

$$rG(x) = \left(\alpha_x - \rho\sigma_x\eta\right)G'(x) + \frac{\sigma_x^2}{2}G''(x) - \frac{\gamma r\epsilon_x^2}{2}G'(x)^2, \tag{25}$$

subject to the no-bubble condition $\lim_{x\to-\infty} G\left(x\right) = 0$, and also the boundary conditions

$$G(\bar{x}) = \bar{x} - I, \tag{26}$$

$$G'(\bar{x}) = 1. \tag{27}$$

Moreover, G is increasing.

We next discuss the implications of this proposition and analyze the role of hedging.

3.2 Undiversifiable Idiosyncratic Risk and Implied Option Value

Similar to the self-insurance model in Section 2, we may interpret $G(x)$ either as the certainty equivalent wealth, or as the implied option value. Before discussing the option value $G(x)$, we first sketch out the standard complete markets model when the idiosyncratic risk is fully diversifiable. Let $\Phi(x)$ denote the option value under complete markets. Given complete markets, standard finance theory implies that the option value and the investment threshold are independent of preferences. Indeed, we may apply the martingale method to rewrite the dynamic budget constraint as a static Arrow-Debreu budget constraint.[17] Appendix B shows that $\Phi(x)$ satisfies the following differential equation:

$$r\Phi(x) = (\alpha_x - \rho\sigma_x\eta)\,\Phi'(x) + \frac{\sigma_x^2}{2}\Phi''(x)\,, \tag{28}$$

and the boundary conditions $\lim_{x\to-\infty}\Phi(x) = 0$, $\Phi(x^*) = x^* - I$, and $\Phi'(x^*) = 1$.

Equation (28) resembles a standard valuation equation in dynamic asset pricing models.[18] After correcting for risk, traded securities such as the option earn the risk-free rate of return r, as seen from the left side of (28). The right side of (28) gives the instantaneous expected changes of the option value with respect to the underlying asset value x. The risk correction is reflected by the drift change from α_x to $(\alpha_x - \rho\sigma_x\eta)$ in the first term on the right side of (28). This risk correction may be obtained from a CAPM argument and is consistent with standard dynamic asset pricing theories, which state that only systematic risk demands a premium.

We turn to the differential equation (25) for the option value $G(x)$. Re-writing (25) gives

$$rG(x) = \left(\alpha_x - \rho\sigma_x\eta - \frac{\gamma r\epsilon_x^2}{2}G'(x)\right)G'(x) + \frac{\sigma_x^2}{2}G''(x)\,. \tag{29}$$

First, we note that the standard convexity effect of volatility on option value depends on the total volatility σ_x, same as the one in (28) under complete markets. This is reflected by the last (quadratic) term in (29). Also similar to the differential equation (28) for $\Phi(x)$, the change of drift from α_x to $(\alpha_x - \rho\sigma_x\eta)$ in the first term on the right side of (29) accounts for the effect of systematic risk on valuation, the standard CAPM argument. Importantly, unlike the differential equation (28) for $\Phi(x)$, the third component in the bracket of the drift term on the right side of (29), $\gamma r\epsilon_x^2 G'(x)/2$ reflects the effect of the idiosyncratic risk on the implied option value $G(x)$. We may dub this term as the *idiosyncratic* risk premium.

[17] See Cox and Huang (1989), and Karatzas, Lehoczky, and Shreve (1987) on the martingale method. This method and the dynamic programming method deliver the same solution. See Duffie (2001) for a textbook treatment.

[18] See Duffie (2001) for a textbook treatment.

Intuitively, when idiosyncratic risks cannot be fully diversified, the agent naturally demands a higher risk premium for a larger idiosyncratic volatility ϵ_x, *ceteris paribus.* A more prudent agent (with a larger coefficient of risk aversion γ) also demands a higher risk premium. Finally, a higher option delta $G'(X)$ indicates that the option value is more sensitive to the change of the underlying investment opportunity set and hence requires a higher idiosyncratic risk premium. Moskowitz and Vissing-Jorgensen (2002) find that the private equity premium is low in the U.S. given the amount of idiosyncratic risks that entrepreneurs face. While our model is not designed to address this quantitative private equity premium issue, our model responds to urgent needs to develop theories which capture the role of the idiosyncratic risk on the interdependent consumption, investment and portfolio choices for entrepreneurs, as suggested by Gentry and Hubbard (2004), Heaton and Lucas (2000), and Moskowitz and Vissing-Jorgensen (2002).

We now turn to the effects of idiosyncratic volatility ϵ_x and risk aversion coefficient γ on the investment threshold $\bar{x}$. First, note that as in the self-insurance model of Section 2, the payoffs upon option exercising are given by $(x - I)$. Hence, neither idiosyncratic volatility nor risk aversion γ matters for the project payoff values. Second, a direct comparison between (28) and (29) implies that a larger idiosyncratic volatility ϵ_x or a higher risk aversion coefficient γ lowers the option value $G(x)$, holding the systematic risk constant. Taking the two effects together, we may conclude that a higher idiosyncratic volatility ϵ_x and a larger risk aversion coefficient γ lowers the investment threshold $\bar{x}$, *ceteris paribus.* This result also implies that the agent hastens investment under incomplete markets than under complete markets since the solution for the latter is effectively obtained by setting $\gamma = 0$.

3.3 Consumption and Portfolio Rules

The consumption rule (21) and the portfolio rule (22) after the option exercise are solutions to the standard Merton style consumption-portfolio choice problem with CARA utility (Merton (1969)). After exercising the option, the agent has no more hedging demand since the lump-sum project payoff has been realized at the exercising time τ. Equation (22) gives the standard mean-variance efficient rule for CARA utility. The agent's ability to invest in the risky asset to explore the risk premium makes him better off relative to the self insurance setting in Section 2. This is reflected by $\eta^2/(2\gamma r^2)$, the second term in the consumption rule (21).

Next, consider the agent's consumption decision before the option exercise. Equation (23) states that the agent's consumption is equal to the annuity value of the sum of three terms: (*i*) financial wealth w, (*ii*) certainty equivalent wealth $G(x)$, and (*iii*) the constant $\eta^2/(2\gamma r^2)$. The forward looking agent rationally finances a certain fraction of his current consumption via the certainty equivalent wealth $G(x)$ for his investment opportunity. Moreover, investing in the risky asset makes him better off and yields a higher current consumption, *ceteris paribus*. This is reflected by the third component in the consumption rule (23), same as the argument for the after-investment consumption rule (21).

We now turn to the agent's portfolio rule (24) before investment. In addition to the standard Merton's mean-variance term, the agent also has a hedging demand, because his investment project payoff is correlated with the market portfolio. First, hedging demand is greater when the degree of correlation $|\rho|$ is higher, the standard and well known result. Second, the portfolio rule (24) also suggests that hedging demand is greater when $G'(x)$, the option Δ, is higher. This result is less known, but is intuitive. Before the investment decision is made, the agent holds a valuable option on a non-tradable underlying asset. Hence, the agent naturally hedges more against the fluctuations of the option value of his investment, if this option value is more sensitive to the change of the underlying asset (a higher option delta), *ceteris paribus*.

4 Models with Flow Payoffs

While some real world examples may fit in the lump-sum payoff setting that we have just analyzed, there are many situations under which the investment payoffs are given as cash flows over time, rather than as a lump-sum payment. We emphasize that unlike the lump-sum payoff case where the project payoff is exogenously given, one has to derive the implied value or the certainty equivalent value of the cash flows by solving the agent's consumption decision after the option exercise. Intuitively, idiosyncratic volatility also lowers the implied project value or the certainty equivalent wealth after option exercise. Hence, the overall impact of idiosyncratic volatility on investment decision and implied option value is less obvious. Indeed, we show that the predictions for the flow payoff case may be reversed compared to those for the lump-sum payoff case.

In the flow payoff case, after the agent irreversibly exercises his investment option at some time τ, he obtains a perpetual stream of payoffs $\{Y_t : t \geq \tau\}$. Assume that the flow payoff process Y is governed by an arithmetic Brownian motion process:

$$dY_t = \alpha_y \, dt + \sigma_y \, dZ_t, \ Y_0 \text{ given}, \tag{30}$$

where α_y and σ_y are positive constants and Z is a standard Brownian motion. As will be clear below, the arithmetic Brownian motion process allows us to obtain explicit solutions after investment so that the problem before investment is easier to analyze. Using a geometric

Brownian motion process to model the cash flow process will complicate the analysis without adding many new insights.

We present our analysis in three subsections. First, we analyze the self-insurance case in which the agent can trade only a risk-free asset and hence all risk is idiosyncratic, similar to Section 2. Then, we allow the agent to trade a market portfolio to partially hedge against the flow payoff risk and hence to separate idiosyncratic volatility away from systematic volatility, similar to Section 3. Finally, we discuss empirical implications of the models in both the lump-sum and flow payoff cases.

4.1 Self-Insurance

When the agent can trade only a risk-free asset, the agent's wealth $\{W_t : t \geq 0\}$ after the option exercise ($\tau \leq t$) evolves according to

$$dW_t = (rW_t + Y_t - C_t)\, dt. \tag{31}$$

This equation resembles that in a standard incomplete markets consumption-savings model with a stream of labor income $\{Y_t : t \geq \tau\}$. At the investment time τ, the agent pays the cost I and hence wealth is lowered from $W_{\tau-}$, the level just prior to investment, to W_τ, the level immediately after the option exercise, in that $W_\tau = W_{\tau-} - I$. Before exercising the option ($0 \leq t < \tau$), the agent does not receive flow payoffs and thus his wealth evolves according to (3) as in the lump-sum case. The agent's decision problem is to choose both an investment timing strategy τ and a consumption process C so as to maximize his utility (1) subject to wealth accumulation equations (31) and (3) and a transversality condition specified in the appendix.

We solve the agent's decision problem backward by dynamic programming. Let $J(w, y)$ be the value function after the option exercise. Unlike the lump-sum payoff case, the payoff value y is an additional state variable for J. By the standard argument, $J(w, y)$ satisfies the following HJB equation:

$$rJ(w, y) = \max_{c \in \mathbb{R}} U(c) + (rw + y - c)\, J_w(w, y) + \alpha_y J_y(w, y) + \frac{\sigma_y^2}{2} J_{yy}(w, y)\,. \tag{32}$$

Let $V(w, y)$ denote the value function before the option exercise.[19] Similar to Section 2, $V(w, y)$ satisfies the following HJB equation:

$$rV(w, y) = \max_{c \in \mathbb{R}} U(c) + (rw - c)\, V_w(w, y) + \alpha_y V_y(w, y) + \frac{\sigma_y^2}{2} V_{yy}(w, y)\,. \tag{33}$$

[19]Note that we use the same notation for the value function before investment as that for the lump-sum payoff case.

We now briefly discuss the boundary conditions for the flow payoff case and relate to the lump-sum payoff case analyzed earlier. Similar to the lump-sum payoff case, the no-bubble condition $\lim_{y\to-\infty} V(w,y) = V^0(w)$ must be satisfied. Similar to, but different from the lump-sum payoff case, we have the following value matching condition:

$$V(w,y) = J(w-I,y)\,. \tag{34}$$

This equation determines an investment boundary $\overline{y}(w)$. Moreover, the agent's optimality further requires the following smooth pasting conditions to hold:

$$\left.\frac{\partial V(w,y)}{\partial y}\right|_{y=\overline{y}(w)} = \left.\frac{\partial J(w-I,y)}{\partial y}\right|_{y=\overline{y}(w)}, \tag{35}$$

$$\left.\frac{\partial V(w,y)}{\partial w}\right|_{y=\overline{y}(w)} = \left.\frac{\partial J(w-I,y)}{\partial w}\right|_{y=\overline{y}(w)}. \tag{36}$$

Both smoothing-pasting conditions are similar to, but different from those for the lump-sum case, because the cash flow payoff y enters as an additional state variable even after the agent makes the investment.

We use a procedure similar to that in Section 2 to solve the above problem and then show that the investment threshold $\bar{y}(w)$ is independent of wealth w for CARA utility agents.

Proposition 3 *The agent exercises the investment option the first time the process Y hits the threshold $\bar{y}$ from below. After exercising the option, the optimal consumption rule is given by*

$$\overline{c}(w,y) = r(w+f(y))\,, \tag{37}$$

where $f(y)$ is given by

$$f(y) = \left(\frac{y}{r} + \frac{\alpha_y}{r^2}\right) - \frac{\gamma\sigma_y^2}{2r^2}\,. \tag{38}$$

Before exercising the option, the optimal consumption rule is given by

$$\overline{c}(w,y) = r(w+g(y))\,, \tag{39}$$

where $(g,\bar{y})$ is the solution to the following free boundary problem:

$$rg(y) = \alpha_y g'(y) + \frac{\sigma_y^2}{2}g''(y) - \frac{\gamma r\sigma_y^2}{2}g'(y)^2, \tag{40}$$

subject to the no-bubble condition $\lim_{y\to-\infty} g(y) = 0$ and the boundary conditions

$$g(\bar{y}) = f(\bar{y}) - I, \tag{41}$$

$$g'(\bar{y}) = f'(\bar{y}) = \frac{1}{r}\,. \tag{42}$$

Moreover, g is increasing.

Comparing Propositions 1 and 3, we see that the valuation equation for the implied option value is similar. However, the consumption rule after investment and the boundary conditions are different. We next analyze the implications of these differences.

4.1.1 Implied Project Value and Consumption

When the payoffs are given in terms of cash flows over time, the agent continues to face the undiversifiable idiosyncratic cash flow risk after exercising his investment option. Therefore, the idiosyncratic risk lowers both the implied option value and also the certainty equivalent project payoff value. After option exercise, the agent's optimization problem is a standard incomplete-markets consumption-savings problem with stochastic income $\{Y_t : t \geq \tau\}$. Because of the CARA utility and the arithmetic Brownian motion process specifications, we are able to derive the explicit expression for the consumption rule given in (37)-(38).[20]

To understand the consumption rule (37), we define human wealth $h(y)$ as the present discounted value of all investment cash flows following Friedman (1957) and Hall (1978). For the arithmetic Brownian motion income process (30), this gives

$$h(y) \equiv E\left(\int_0^\infty e^{-rt} Y_t dt \,\middle|\, Y_0 = y\right) = \frac{y}{r} + \frac{\alpha_y}{r^2}\,. \tag{43}$$

Note that this traditional definition of human wealth does not incorporate the effect of risk. Using $h(y)$, we may rewrite the consumption rule given in (37) and (38) as follows:

$$\overline{c}(w, y) = r\left(w + h(y) - \frac{\gamma\sigma_y^2}{2r^2}\right). \tag{44}$$

When $\gamma = 0$ or $\sigma_y = 0$, consumption is the annuity value of the sum of financial wealth w and human wealth $h(y)$, in that $\overline{c}(w, y) = r(w + h(y))$. This is Friedman's permanent-income hypothesis. In terms of time series, this implies that consumption is a martingale, in that $C_t = E_t(C_{t+1})$, Hall's random walk consumption model (Hall (1978)).

Importantly, when the agent has precautionary motive ($\gamma > 0$), a precautionary savings demand arises in the presence of uninsurable idiosyncratic shocks. This demand after the option exercise is reflected by the term $\gamma\sigma_y^2/\left(2r^2\right)$ in (38). We may interpret $f(y)$ as the certainty equivalent (risk-adjusted) human wealth or the implied project value, following essentially the same analysis in Section 2. Since $f(y) = h(y) - \gamma\sigma_y^2/\left(2r^2\right)$, the certainty equivalent human

[20] Caballero (1991b) derives this consumption rule in discrete time. Wang (2004) extends Caballero (1991b) to more general bi-variate income processes with partial observability and hence increases precautionary savings demand due to estimation risk. Wang(2003) shows that general equilibrium restriction eliminates the precautionary saving demand in Caballero (1991b).

wealth $f(y)$ decreases in risk aversion coefficient γ and also in income volatility σ_y. This differs from the lump-sum payoff case where option exercising gives a complete exit from incomplete markets and hence precautionary motive and volatility do not affect the value of payoffs from exercising.

Now consider consumption before investment. Equation (39) implies that the rational forward looking agent finances his consumption partially out of his future payoffs from his real investment opportunities. More formally, consumption is given by the annuity value of the sum of financial wealth w and $g(y)$, before the investment is made. Following our analysis in the lump-sum payoff case, we may interpret $g(y)$ as the certainty equivalent wealth for the investment opportunity before investment is made, or equivalently, the implied option value on the investment opportunity. We next turn to the analysis of $g(y)$.

4.1.2 Implied Option Value and Investment Threshold

The implied option value $g(y)$ and the investment threshold $\bar{y}$ are determined jointly by the the differential equation (40) and the corresponding boundary conditions (41) and (42). The differential equation (40) is similar to its counterpart (12) for the lump-sum payoff case. However, the boundary conditions for the flow payoff case are different from those for the lump-sum payoff case in Proposition 1 in that the agent values the stream of payoffs after option exercise with the certainty equivalent wealth $f(y)$ given in (38). These boundary conditions jointly suggest that the investment threshold is determined by trading off between the option value of waiting $g(y)$ and the certainty equivalent wealth $f(y)$ for the stochastic income stream (after netting out the fixed investment cost I).

Unlike the lump-sum payoff case, the total payoff volatility σ_y and the precautionary motive also lower the implied project value $f(y)$, because the agent is exposed to idiosyncratic shocks after making his investment decision, and hence values the cash flow at a value lower than $h(y)$, the present discounted value of his future incomes.

It is transparent to analyze the impact of risk aversion coefficient γ and income volatility σ_y on the investment threshold $\bar{y}$ via the approximation method. We approximate $g(y)$ and $\bar{y}$ simultaneously to the first order of σ_y^2. We then obtain the approximate investment threshold:

$$\bar{y}_1 = \bar{y}_0 + \frac{1}{2\alpha_y}\sigma_y^2, \tag{45}$$

where $\bar{y}_0 = rI$ is the exactly solved investment threshold in the deterministic case ($\sigma_y = 0$). Therefore, to the first-order approximation, the investment threshold $\bar{y}_1$ increases in volatility σ_y, and moreover, the agent's risk attitude does not affect investment timing. This prediction is thus qualitatively the same as in the standard real option models to the first order.

The intuition for this result is as follows. In the flow payoff case, the agent receives a stream of uninsurable incomes after the option exercise. Therefore, the agent's precautionary motive lowers both the implied project value $f(y)$ and the implied option value $g(y)$. It turns out that the precautionary effect on $g(y)$ and $f(y)$ offsets each other to the first-order approximation. Thus, it has little impact on the investment timing since the investment threshold $\bar{y}$ is determined by the relative magnitudes of the implied option value $g(y)$ and the project payoff $f(y)$. This result differs from the lump-sum payoff case where precautionary motive only affects the implied option value, not the project payoff value. As a result, the investment threshold is lowered by the agent's precautionary motive to the first-order approximation in the lump-sum payoff case. Unlike the lump-sum payoff case, exercising the option does not eliminate the effect of uninsurable idiosyncratic shocks, when payoffs are given in flow terms over time.

To further understand the impact of the agent's precautionary motive γ on the investment decision, we use the second-order approximation with respect to σ_y^2 and obtain the following approximate investment threshold:

$$\bar{y}_2 = \bar{y}_1 + \frac{1}{\alpha_y^2}\left(\gamma - \frac{r}{2\alpha_y}\right)\sigma_y^4, \tag{46}$$

where $\bar{y}_1$ is given in (45). Equation (46) indicates that, to the second-order approximation, the investment threshold *increases* in γ, *opposite* to the prediction for the lump-sum payoff case. While the precautionary saving effect is present both before and after the option exercise as argued earlier, the precautionary saving effect, to the second-order approximation, has a larger impact on $f(y)$ than on $g(y)$. The intuition is as follows. Before exercising the option, the agent may time when to invest in the risky investment. While the volatility effect on the implied option value $g(y)$ and the implied project value $f(y)$ to the first order washes out, this additional flexibility of timing the investment decision on the margin implies that the precautionary saving effect is stronger after exercising the option than before. This suggests that an increase in the precautionary motive γ lowers $f(y)$ more than $g(y)$, thereby delaying the exercise of the option. We emphasize that the effect of γ on the investment decision is of the second order.

Finally, we use numerical solutions to conduct further analysis. Figure 3 plots the investment threshold as a function of volatility σ_y and the parameter γ. This figure confirms our preceding approximation results. Moreover, it illustrates that the effects of volatility σ_y on the investment threshold are stronger when the agent is more precautionary, i.e., when γ is higher.

[Insert Figure 3 Here]

Figure 4 illustrates the effect of changes in γ. An increase in γ raises precautionary savings both after and before the option exercise, thereby lowering both the implied project value $f(y)$ and the implied option value $g(y)$. This figure confirms our earlier analysis that $f(y)$ is lowered more than $g(y)$, so that the agent delays exercising the investment option.

[Insert Figure 4 Here]

4.2 Hedging

We now turn to the flow payoff case with hedging opportunity. Based on our previous analysis, we anticipate that the model contains the following two features: (i) the hedging opportunity allows the separation of idiosyncratic volatility from the systematic volatility, and hence captures the different effects of these two forms of volatility on investment and consumption decisions; (ii) the flow payoff implies that idiosyncratic volatility continues to matter after option exercise and hence lowers the certainty equivalent payoff value, similar to the self-insurance model for the flow payoff case.

Let π_t denote the amount allocated in the risky asset with returns given in (17) at time t. As in Section 3, we may denote ϵ_y as the idiosyncratic volatility, in that

$$\epsilon_y = \sqrt{1-\rho^2}\,\sigma_y. \tag{47}$$

We may rewrite the observed flow payoff process $\{Y_t : t \geq 0\}$ given in (30) as follows:

$$dY_t = \alpha_y dt + \rho\sigma_y dB_t + \epsilon_y\, d\widetilde{B}_t, \tag{48}$$

where B describes the systematic (market) risk and $\widetilde{B}$ describes the idiosyncratic project risk.

Before the agent exercises the investment option at time τ, his wealth accumulation is the same as (20). After time τ, his wealth evolves as follows:

$$dW_t = \left[rW_t + \pi_t\left(\mu_e - r\right) + Y_t - C_t\right] dt + \pi_t \sigma_e dB_t. \tag{49}$$

Note that the flow payoff Y appears in (49), not in (20). As before, the agent's wealth immediately after his investment W_τ is given by $W_\tau = W_{\tau-} - I$, where $W_{\tau-}$ denotes his wealth level just prior to his investment at time τ. The following proposition characterizes the solution.

Proposition 4 *The agent exercises the investment option the first time the process Y hits the threshold $\bar{y}$ from below. After exercising the option, the optimal consumption and portfolio rules* *are given by*

$$\bar{c}\,(w,y) = r\left(w+f\,(y)+\frac{\eta^2}{2\gamma r^2}\right), \tag{50}$$

$$\bar{\pi}\,(w,y) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r}-\frac{\rho\sigma_y}{\sigma_e r}\,, \tag{51}$$

where $f\,(y)$ is given by

$$f(y)=\left(\frac{1}{r}y+\frac{\alpha_y-\rho\sigma_y\eta}{r^2}\right)-\frac{\gamma\epsilon_y^2}{2r^2}. \tag{52}$$

Before exercising the option, the optimal consumption and portfolio rules are given by

$$\bar{c}\,(w,y) = r\left(w+g\,(y)+\frac{\eta^2}{2\gamma r^2}\right), \tag{53}$$

$$\bar{\pi}\,(w,y) = \frac{\eta}{\gamma\sigma_e}\frac{1}{r}-\frac{\rho\sigma_y}{\sigma_e}g'(y)\,, \tag{54}$$

where $(g,\bar{y})$ is the solution to the following free boundary problem:

$$rg(y)=(\alpha_y-\rho\sigma_y\eta)\,g'(y)+\frac{\sigma_y^2}{2}g''(y)-\frac{\gamma r\epsilon_y^2}{2}g'(y)^2\,, \tag{55}$$

subject to the no-bubble condition $\lim_{y\to-\infty} g\,(y)=0$, *and the boundary conditions*

$$g(\bar{y}) = f(\bar{y})-I, \tag{56}$$

$$g'(\bar{y}) = f'(\bar{y})=\frac{1}{r}\,. \tag{57}$$

Moreover, g is increasing.

As in the previous subsection, we interpret $f\,(y)$ as the implied project value and $g\,(y)$ as the implied option value. Unlike the lump-sum payoff model with hedging opportunities in Section 3, hedging affects not only the implied option value $g\,(y)$, but also the implied project value $f\,(y)$. In particular, hedging reduces the agent's exposure to idiosyncratic volatility from σ_y to $\epsilon_y=\sqrt{1-\rho^2}\sigma_y$. Thus, the precautionary savings demand after the option exercise is reduced from $\gamma\sigma_y^2/\left(2r^2\right)$ to $\gamma\epsilon_y^2/\left(2r^2\right)$. In addition, the portfolio rule (51) after the option exercise consists of the standard mean-variance term and the hedging demand term.[21]

To compare with the complete markets solution, we assume that the agent can trade an additional risky asset to diversify the idiosyncratic risk as in Section 3.2. Appendix B shows that the market value of the investment option satisfies the differential equation

$$r\Psi(y)=(\alpha_y-\rho\eta\sigma_y)\,\Psi'(y)+\frac{\sigma_y^2}{2}\Psi''(y)\,, \tag{58}$$

[21] See Svensson and Werner (1993) and Davis and Willen (2002) for the consumption and portfolio choice problem under incomplete markets in a continuous-time setting and a discrete-time setting, respectively.

and the boundary conditions $\lim_{y\to-\infty}\Psi(y)=0$, $\Psi(y^*)=F(y^*)-I$, and $\Psi'(y^*)=1/r$, where

$$F(y)=\frac{1}{r}y+\frac{\alpha_y-\rho\eta\sigma_y}{r^2} \tag{59}$$

is the market value of the cash flow process Y. The above two equations reveal that both the option value and the project value under complete markets are independent of preferences and effectively are the solutions in (52) and (55) for $\gamma=0$. In addition, both values are higher than under incomplete markets. Similar to our analysis in Section 4.1.2, the net effect of incomplete hedging on the investment timing depends on the relative magnitudes of changes in the implied option value and the project value. Similar to the insights from the self-insurance model with flow payoffs, the impact of idiosyncratic shocks on the project value is greater than on the option value to the second order. Thus, unlike in the lump-sum payoff case analyzed in Section 3.2, incomplete hedging raises the investment threshold and delays investment, compared to the complete markets benchmark. This result demonstrates that the timing of payoffs matters for investment decision under incomplete markets, which is different from a complete markets setting where the timing of payoffs does not matter as shown in Appendix B.

4.3 Empirical Implications

Our analysis has empirical implications. For example, the model in Section 2 suggests that unlike a standard real options analysis, a positive investment-uncertainty relationship may potentially arise for entrepreneurial activities, when the idiosyncratic risk is sufficiently large. Thus, one may be cautious in interpreting some conflicting results found in empirical studies.[22] Our analysis also suggests that the investment behavior of undiversified individuals is different from that of well-diversified individuals or institutions. In particular, risk attitude plays an important role under incomplete markets. Consider again the real estate development example. Suppose that we have a sample containing both undiversified individual developers and publicly traded REITs. Suppose that both individual entrepreneurs and REITs specialize in development of and not management of the properties. That is, we may take the sales value of the property upon completion of constructions as given. Then our model in Section 3 predicts that the individual entrepreneurs are more likely to develop earlier than the publicly traded REITs, because the idiosyncratic risk lowers the implied option value of waiting for individual developers. However, if they also manage the properties after completion of development, then our models in the previous two subsections suggest that the preceding prediction may be reversed because the properties are also less valuable to the undiversified individual developers.

[22]See Quigg (1993), Berger et al. (1996), Leahy and White (1996), and Moel and Tufano (1998) for empirical works. See Caballero (1991a) for a theoretical analysis.

5 Conclusions

Entrepreneurs' business investment opportunities are often nontradable and their payoffs cannot be spanned by existing traded assets due to reasons such as incentives and informational asymmetries. These features invalidate the standard real options approach to investment. Extending this approach, we develop a utility-based real options model to analyze an agent's interdependent real investment, consumption, and portfolio choice decisions.

We show that project volatility has not only a positive option effect, but also a negative effect on the implied option value. The latter effect is induced by the precautionary saving motive. For the lump-sum payoff case, risk aversion accelerates investment. Unlike the standard real options analysis, an increase in project volatility may accelerate investment if the agent has a sufficiently strong precautionary motive. We further extend our model to allow for the opportunity to hedge. We show that hedging reduces the agent's exposure to idiosyncratic risk, and hence raises the option value. In addition, hedging allows the decomposition of total project volatility into systematic volatility and idiosyncratic volatility. The latter volatility generates an idiosyncratic risk premium. We finally analyze settings where investment payoffs are given in flow terms over time. Unlike the standard real options analysis, the lump-sum and flow payoff cases have different implications. Because the precautionary saving effect matters both before and after investment in the flow payoff case, many predictions in this case differ from and may even be opposite to those in the lump-sum payoff case.

In order to analyze the effect of uninsurable idiosyncratic risk on investment in the simplest possible setting, we have intentionally ignored the wealth effect by adopting the CARA utility. However, the wealth effect may potentially play an important role in settings such as entrepreneurship. We extend our analysis to incorporate the wealth effect on entrepreneurial investment in Miao and Wang (2005a). Finally, when entrepreneurs invest in nontradable projects, they often need to make financing decisions jointly. For the real estate example, often the majority part of the construction and operating expenses is financed by mortgages. We analyze the interaction between investment and financing decisions in Miao and Wang (2005b).

Appendices

A Proofs

Proof of Proposition 1: >From the first-order condition $U'(c) = V_w(w, x)$, we can derive the consumption policy before the option exercise given in (11). Substituting it into the HJB equation (5), we can show that $G(x)$ satisfies the ODE (12). Given the functional forms of the value functions, we can also show that the no-bubble condition, the value-matching and the smooth-pasting conditions become the boundary conditions in Proposition 1. By a standard dynamic programming argument, one can show that V satisfies

$$V(w, x) = \max_{(\tau, C)} E\left[\int_0^\tau e^{-rt} U(C_t)\, dt + e^{-r\tau} V^0(W_\tau + X_\tau - I) \,\middle|\, (W_0, X_0) = (w, x)\right]. \tag{A.1}$$

Consider $x < x'$. For $X_0 = x'$, let τ' be the optimal investment time and $\{C_t' : 0 \leq t \leq \tau'\}$ be the optimal consumption process before investment. Since V^0 is an increasing function and, given any sample path,

$$X_{\tau'} \equiv x + \alpha_x \tau' + \sigma_x W_{\tau'} < X_{\tau'}' \equiv x' + \alpha_x \tau' + \sigma_x W_{\tau'},$$

we have

$$\int_0^{\tau'} e^{-rt} U\left(C_t'\right) dt + e^{-r\tau'} V^0(W_{\tau'} + X_{\tau'} - I) < \int_0^{\tau'} e^{-rt} U\left(C_t'\right) dt + e^{-r\tau'} V^0\left(W_{\tau'} + X_{\tau'}' - I\right).$$

Taking conditional expectations yields

$$\begin{aligned} & E\left[\int_0^{\tau'} e^{-rt} U\left(C_t'\right) dt + e^{-r\tau'} V^0(W_{\tau'} + X_{\tau'} - I) \,\middle|\, (W_0, X_0) = (w, x)\right] \\ < \quad & E\left[\int_0^{\tau'} e^{-rt} U\left(C_t'\right) dt + e^{-r\tau'} V^0\left(W_{\tau'} + X_{\tau'}' - I\right) \,\middle|\, (W_0, X_0) = \left(w, x'\right)\right] = V\left(w, x'\right). \end{aligned}$$

Given the wealth dynamics described in Section 2.1, $\{C_t' : 0 \leq t \leq \tau'\}$ and τ' are also feasible for $X_0 = x$. Thus, the left side of the above equation is less or equal to $V(w, x)$ by (A.1). So, $V(w, x) < V(w, x')$ and V is increasing in x. Q.E.D.

Proof of Proposition 2: Without risk of confusion, we still use $V^0(w)$ and $V(w, x)$ to denote the value function after and before the option exercise, respectively, when the agent can trade a risky asset. By a standard argument, V^0 satisfies the following HJB equation:

$$rV^0(w) = \max_{(c, \pi) \in \mathbb{R}^2} U(c) + \left[rw + \pi(\mu_e - r) - c\right] V_w^0(w) + \frac{(\pi \sigma_e)^2}{2} V_{ww}^0(w). \tag{A.2}$$

The transversality condition $\lim_{T\to\infty} E\left[e^{-rT}V^0\left(W_T\right)\right] = 0$ must also be satisfied. Given CARA utility, one can follow Merton (1969) to derive the consumption and portfolio rules in (21)-(22) and

$$V^0\left(w\right) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + \frac{\eta^2}{2\gamma r^2}\right)\right]. \tag{A.3}$$

Before the option exercise, the value function $V\left(w,x\right)$ satisfies the following HJB equation:

$$\begin{aligned} rV\left(w,x\right) &= \max_{(c,\pi)\in\mathbb{R}^2} U\left(c\right) + \left[rw + \pi\left(\mu_e - r\right) - c\right]V_w\left(w,x\right) + \alpha_x V_x\left(w,x\right) \\ &\quad + \frac{\sigma_x^2}{2}V_{xx}\left(w,x\right) + \frac{\left(\pi\sigma_e\right)^2}{2}V_{ww}\left(w,x\right) + \pi\sigma_e\sigma_x\rho V_{wx}\left(w,x\right). \end{aligned} \tag{A.4}$$

We conjecture that the value function V takes the form

$$V(w,x) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + G(x) + \frac{\eta^2}{2\gamma r^2}\right)\right], \tag{A.5}$$

where $G(x)$ is a function to be determined. Using the first-order conditions,

$$U'\left(c\right) = V_w\left(w,x\right),\ \pi = \frac{-V_w\left(w,x\right)}{V_{ww}\left(w,x\right)}\frac{\mu_e - r}{\sigma_e^2} + \frac{-V_{wx}\left(w,x\right)}{V_{ww}\left(w,x\right)}\frac{\rho\sigma_x}{\sigma_e}, \tag{A.6}$$

one can derive the optimal consumption and portfolio policies before exercising the option given in (23)-(24). Plugging these expressions back into the HJB equation gives (25). As in Section 2, the boundary conditions are given by the no-bubble, value-matching, and smooth-pasting conditions similar to (6)-(8). Using these boundary conditions, one can derive the boundary conditions in Proposition 2. The rest of the proof follows a similar argument to that in Proposition 1. Q.E.D.

Proof of Proposition 3: We conjecture that the value function after the option exercise J takes the following form:

$$J(w,y) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + f(y)\right)\right], \tag{A.7}$$

where $f(y)$ is a function to be determined. To solve for this function, we use the first-order condition $U'\left(c\right) = J_w\left(w,y\right)$ to derive the optimal consumption rule given in (37). Substitute it back into the HJB equation (32) to derive the following ODE:

$$0 = \left(y - rf\left(y\right)\right) + \alpha_y f'\left(y\right) + \frac{\sigma_y^2}{2}\left[f''(y) - \gamma r f'(y)^2\right]. \tag{A.8}$$

It can be verified that its solution is given by (38). Moreover, it is such that the value function satisfies the transversality condition $\lim_{T\to\infty} E\left[e^{-rT}J\left(W_T,Y_T\right)\right] = 0$.

We conjecture that the value function before the option exercise, $V(w,y)$, takes the form:

$$V(w,y) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w+g(y)\right)\right], \tag{A.9}$$

where $g(y)$ is a function to be determined. From the first-order condition $U'(c) = V_w(w,y)$, we can derive the consumption policy before investment given in (39). Substituting it into the HJB equation (33), we can show that $g(y)$ satisfies the ODE (40). By a standard dynamic programming argument, one can show that V satisfies

$$V(w,y) = \max_{\tau,C} E\left[\int_0^\tau e^{-rt}U(C_t)\,dt + e^{-r\tau}J(W_\tau - I, Y_\tau)\,\middle|\,(W_0,Y_0)=(w,y)\right]. \tag{A.10}$$

Since it follows from (A.7) that J is increasing and concave in y, one can show that V is also increasing and concave in y. The rest of the proof follows from a similar argument to that in Proposition 1. Q.E.D.

Proof of Proposition 4: Without risk of confusion, we still use $J(w,y)$ and $V(w,y)$ to denote the value function after and before the option exercise, respectively, when the agent can also trade a risky asset. By a standard argument, $J(w,y)$ satisfies the HJB equation

$$\begin{aligned} rJ(w,y) &= \max_{(c,\pi)\in\mathbb{R}^2} U(c) + \left[rw + \pi(\mu_e - r) + y - c\right]J_w(w,y) + \alpha_y J_y(w,y) \\ &\quad + \frac{\sigma_y^2}{2}J_{yy}(w,y) + \frac{(\pi\sigma_e)^2}{2}J_{ww}(w,y) + \pi\sigma_e\sigma_y\rho J_{wy}(w,y). \end{aligned} \tag{A.11}$$

The transversality condition $\lim_{T\to\infty} E\left[e^{-rT}J(W_T,Y_T)\right] = 0$ must also be satisfied. We conjecture that $J(w,y)$ takes the following form:

$$J(w,y) = -\frac{1}{\gamma r}\exp\left[-\gamma r\left(w + f(y) + \frac{\eta^2}{2\gamma r^2}\right)\right], \tag{A.12}$$

where the function f is to be determined. By the first-order conditions,

$$U'(c) = J_w(w,y), \ \pi = \frac{-J_w(w,y)}{J_{ww}(w,y)}\frac{\mu_e - r}{\sigma_e^2} + \frac{-J_{wy}(w,y)}{J_{ww}(w,y)}\frac{\rho\sigma_y}{\sigma_e}, \tag{A.13}$$

one can derive the optimal consumption and portfolio policies after investment given in (53)-(54). Substituting them back into the HJB equation (A.11), one can derive the solution for $f(y)$ given in (52). It can be verified that this solution satisfies the transversality condition.

The value function before the option exercise, V, satisfies the following HJB equation:

$$\begin{aligned} rV(w,y) &= \max_{(c,\pi)\in\mathbb{R}^2} U(c) + \left[rw + \pi(\mu_e - r) - c\right]V_w(w,y) + \alpha_y V_y(w,y) \\ &\quad + \frac{\sigma_y^2}{2}V_{yy}(w,y) + \frac{(\pi\sigma_e)^2}{2}V_{ww}(w,y) + \pi\sigma_e\sigma_y\rho V_{wy}(w,y). \end{aligned} \tag{A.14}$$

We conjecture that the value function V takes the following form:

$$V(w,y) = -\frac{1}{\gamma r} \exp\left[-\gamma r \left(w + g(y) + \frac{\eta^2}{2\gamma r^2}\right)\right], \tag{A.15}$$

where $g(y)$ is a function to be determined. Using the first-order conditions,

$$U'(c) = V_w(w,y), \; \pi = \frac{-V_w(w,y)}{V_{ww}(w,y)} \frac{\mu_e - r}{\sigma_e^2} + \frac{-V_{wy}(w,y)}{V_{ww}(w,y)} \frac{\rho\sigma_y}{\sigma_e}, \tag{A.16}$$

one can derive the optimal consumption and portfolio policies before investment given in (50)-(51). Plugging these expressions into the HJB equation gives a differential equation for $g(\cdot)$. The rest of the proof follows from a similar argument to that in Propositions 1 and 3. Q.E.D.

B Complete Markets Solution

To derive the market markets solution, we the agent can trade an additional risky asset which spans the idiosyncratic risk generated by the Brownian motion $\widetilde{B}$. Specifically, let the return of the second risky asset be given by $dS_t/S_t = rdt + \sigma_S d\widetilde{B}_t$, where σ_S is a positive constant. Since the idiosyncratic risk is by definition independent of the market risk, this risky asset yields an expected rate of return r and does not demand a risk premium by the CAPM. Therefore, the implied unique stochastic discount factor ξ is given by $-d\xi_t/\xi_t = rdt + \eta dB_t$ with $\xi_0 = 1$, where η is the Sharpe ratio of the market portfolio.

The agent's joint consumption, investment and asset allocation decision problem can then be formulated as a two-stage problem with the agent (i) choosing an investment policy to maximize the option value so that the agent's total wealth is maximized; and (ii) choosing optimal consumption given this total wealth.

We first derive the solution for the lump-sum payoff case. Using the unique stochastic discount factor ξ, we can write the option value maximization problem as follows:

$$\Phi(x) = \max_{\tau} E\left[\xi_\tau (X_\tau - I) \,\middle|\, X_0 = x\right]. \tag{B.1}$$

By a standard argument, we can derive explicit expressions for the option value and the investment threshold,

$$\Phi(x) = \frac{1}{\lambda_x} e^{\lambda_x (x - x^*)}, \tag{B.2}$$

$$x^* = I + \frac{1}{\lambda_x}, \tag{B.3}$$

where $\lambda_x = -\sigma_x^{-2}(\alpha_x - \rho\eta\sigma_x) + \sqrt{\sigma_x^{-4}(\alpha_x - \rho\eta\sigma_x)^2 + 2r\sigma_x^{-2}} > 0$.

For the flow-sum payoff case, we can similarly write the market option value as

$$\Psi(y) = \max_{\tau} E\left[\int_{\tau}^{\infty} \xi_t Y_t \, dt - \xi_\tau I \middle| Y_0 = y\right]. \tag{B.4}$$

By a standard argument, we derive the following explicit expressions for the option value and the investment threshold:

$$\Psi(y) = \frac{1}{r\lambda_y} e^{\lambda_y (y - y^*)}, \tag{B.5}$$

$$y^* = rI - \frac{\alpha_y - \rho\eta\sigma_y}{r} + \frac{1}{\lambda_y}, \tag{B.6}$$

where $\lambda_y = -\sigma_y^{-2}(\alpha_y - \rho\eta\sigma_y) + \sqrt{\sigma_y^{-4}(\alpha_y - \rho\eta\sigma_y)^2 + 2r\sigma_y^{-2}} > 0$.

We observe that, under complete markets, the lump-sum and flow payoff formulations are mathematically equivalent, since we may discount cash flows using the unique stochastic discount factor ξ. Specifically, by defining $X_t = \xi_t^{-1} E_t\left(\int_t^\infty \xi_s Y_s ds\right) = F(Y_t)$, we can show that the problems (B.1) and (B.4) are equivalent. Thus, they deliver the same option value $\Phi(x) = \Psi(y)$ and investment timing strategy. However, this equivalence fails when the investment opportunity is not tradable and not spanned by the existing traded assets.

C Approximation Method

In this appendix, we provide our approximation solution methodology. We sketch out the procedure for the self insurance model with a lump-sum payoff. Essentially identical procedures may be applied to models in Section 3 and 4. We may divide the procedure into four steps.

Step 1. Solve for the case with deterministic payoff $\left(\sigma_x^2 = 0\right)$. With $\sigma_x = 0$, risk attitude (γ) does not affect the investment threshold. The implied option value $G_0(x)$ and the investment threshold $\bar{x}_0$ are both known in closed form and are given by

$$G_0(x) = \frac{\alpha_x}{r} \exp\left[\frac{r}{\alpha_x}(x - \bar{x}_0)\right], \quad x \leq \bar{x}_0, \tag{C.1}$$

$$\bar{x}_0 = I + \frac{\alpha_x}{r}. \tag{C.2}$$

Step 2. Consider small σ_x^2. Conjecture that the approximate option value and the investment threshold are

$$G(x) \approx G_0(x) + G_1(x)\,\sigma_x^2, \tag{C.3}$$

$$\bar{x}_1 = \bar{x}_0 + \delta_1 \sigma^2, \tag{C.4}$$

where $G_0(x)$ and $\bar{x}_0$ are solved in Step 1, and $G_1(x)$ and δ_1 are the coefficient function and the coefficient to be determined.

Step 3. Plugging the approximate solution (C.3) into the ODE (12) and boundary conditions (13)-(14) and keeping the terms up to σ_x^2, we have the following:

$$\alpha_x G_1'(x) + \frac{1}{2} G_0''(x) - \frac{\gamma r}{2} G_0'(x)^2 = r G_1(x) \;, \tag{C.5}$$

subject to $G_1(\bar{x}_1) = 0$ and $G_1'(\bar{x}_1) = -r\delta_1/\alpha_x$. Note that unlike the original nonlinear ODE (12) for $G(x)$, we now have a free boundary problem defined by a first-order ODE (C.5) for $G_1(x)$ with certain boundary conditions.

Step 4. Solving the above differential equation gives our reported solution in (16) and

$$G_1(x) = \frac{r}{2\alpha_x^2}(\bar{x}_0 - x)\, e^{-\frac{r}{\alpha_x}(\bar{x}_0 - x)} - \frac{\gamma}{2}\left[e^{-\frac{r}{\alpha_x}(\bar{x}_0 - x)} - e^{-\frac{2r}{\alpha_x}(\bar{x}_0 - x)}\right], x \le \bar{x}_1.$$

D Computation Method

We describe the solution method to the free boundary problem described in Proposition 3. The problems described in other propositions can be solved similarly. We use the projection method implemented with collocation (Judd (1998)). We do not use the traditional shooting method or finite difference method because these methods are inefficient for our nonlinear problem and extensive simulations.

We first rewrite the second order ODE (40) as a system of first-order ODEs. Let $\Delta(y) = g'(y)$. Then (40) can be rewritten as

$$\Delta'(y) = \frac{2}{\sigma_y^2}\left(r g(y) - \alpha_y \Delta(y)\right) + \gamma r \Delta(y)^2. \tag{D.1}$$

The boundary conditions are

$$\lim_{y \to -\infty} g(y) = 0, \tag{D.2}$$

$$g(\overline{y}) = f(\overline{y}) - I, \tag{D.3}$$

$$\Delta(\overline{y}) = 1/r. \tag{D.4}$$

Note that condition (D.2) states that when y goes to minus infinity, the agent never exercises the investment option, and hence the implied option value is equal to zero.

The idea of the algorithm is to first ignore the smooth-pasting condition (D.4) and then to solve a two point boundary value problem with a guessed threshold value $\overline{y}_0$. Since the boundary condition (D.2) is open ended, we pick a very small negative number $\underline{y}$ and set $g(\underline{y}) = 0$. The true value of the threshold is found by adjusting $\overline{y}_0$ so that the smooth-pasting condition (D.4) is satisfied. We then adjust $\underline{y}$ so that the solution is not sensitive to this value. The algorithm is outlined as follows.

Step 1. Start with a guess $\overline{y}_0$ and a preset order n.

Step 2. Use Chebyshev polynomial to approximate g and Δ :

$$g(y;a) = \sum_{i=0}^{n} a_i T_i(y), \ \Delta(y;b) = \sum_{i=0}^{n} b_i T_i(y), \tag{D.5}$$

where $T_i(y)$ is the Chebyshev polynomial of order i, and $a = (a_0, a_1, ..., a_n)$ and $b = (b_0, b_1, ..., b_n)$ are $2n+2$ constants to be determined. Substitute the above expressions into the preceding system of ODEs and evaluate it at n roots of $T_n(y)$. Together with the two boundary conditions, we then have $2n+2$ equations for $2n+2$ unknowns $a = (a_0, a_1, ..., a_n)$ and $b = (b_0, b_1, ..., b_n)$. Let the solution be $\widehat{a}$ and $\widehat{b}$.

Step 3. Search for $\overline{y}_0$ such that the smooth-pasting condition, $\Delta\left(\overline{y}_0; \widehat{b}\right) = 1/r$, is approximately satisfied.

References

Abel, A. B. and J. C. Eberly, 1994, A unified model of investment under uncertainty, *American Economic Review* 84, 1369-1384.

Arrow, K. J., 1968, *Optimal Capital Policy with irreversible investment,* in James N. Wolfe, ed., Value, Capital, and Growth, Essays in Honor of Sir John Hicks, Edinburgh: Edinburgh University Press.

Berger, P. G., E. Ofek, and I. Swary, 1996, Investor valuation of the abandonment option, *Journal of Financial Economics* 42, 257-287.

Bernanke, B. S., 1983, Irreversibility, uncertainty, and cyclical investment, *Quarterly Journal of Economics,* 98, 85-106.

Bertola, G. and R. J. Caballero, 1994, Irreversibility and aggregate investment, *Review of Economic Studies,* 61, 223-246.

Black, F. and J. M. Scholes, 1973, The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637-659.

Brennan, M. J., and E. S. Schwartz, 1985, Evaluating natural resource investments, *Journal of Business*, 58, 135-57.

Caballero, R. J., 1991a, On the sign of the investment-uncertainty relationship, *American Economic Review* 81(1), 279-88.

Caballero, R. J., 1991b, Earnings uncertainty and aggregate wealth accumulation, *American Economic Review* 81(4), 859-71.

Campbell, J. and L. Viceira, 2002, *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors,* Oxford University Press, Oxford.

Carpenter, J., 1998, The exercise and valuation of executive stock options, *Journal of Financial Economics*, 48, 127-58.

Cox, J. and C.-F. Huang, 1989, Optimal consumption and portfolio rules when asset prices follow a diffusion process, *Journal of Economic Theory* 49, 33-83.

Detemple, J. and S. Sundaresan, 1999, Nontraded asset valuation with portfolio constraints: A binomial approach, *Review of Financial Studies*, 12, 835-72.

Davis, S. J. and P. Willen, 2002, Occupation-level income shocks and asset returns: their covariance and implications for portfolio choice, working paper, University of Chicago.

Deaton, A. S., 1991, Saving and liquidity constraints, *Econometrica*, 59, 1221-48.

Dixit, A. and R. Pindyck, 1994, *Investment Under Uncertainty*, Princeton University Press, Princeton, NJ.

Duffie, D., 2001, *Dynamic Asset Pricing Theory*, 3rd Ed., Princeton University Press, Princeton, NJ.

Duffie, D., W. H. Fleming, H. M. Soner, and T. Zariphopoulou, 1997, Hedging in incomplete markets with HARA utility, *Journal of Economic Dynamics and Control* 21, 753-782.

Dumas, B., 1991, Super contact and related optimality conditions, *Journal of Economic Dynamics and Control* 4, 675-685.

Friedman, M., 1957, *A Theory of the Consumption Function*, Princeton University Press, Princeton, NJ.

Gentry, W. M., and R. G. Hubbard, 2004, Entrepreneurship and household saving, working paper, Columbia Business School.

Grenadier, S., 1996, The strategic exercise of options: development cascades and overbuilding in real estate markets, *Journal of Finance* 51, 1653-1679.

Grenadier, S. and N. Wang, 2005, Investment timing, agency and information, *Journal of Financial Economics*, 75, 493-533.

Hall, B. J. and K. J. Murphy, 2000, Optimal exercise prices for executive stock options, *American Economic Review* 90, 209-214.

Hall, R. E., 1978, Stochastic implications of the life cycle-permanent income hypothesis: theory and evidence, *Journal of Political Economy* 91, 249-65.

Heaton, J. and D. Lucas, 2000, Portfolio choice and asset prices: The importance of entrepreneurial risk, *Journal of Finance* 55, 1163-1198.

Henderson, V., 2005, Valuing the option to invest in an incomplete market, working paper, Princeton University.

Hugonnier, J. and E. Morellec, 2005, Investment under uncertainty and incomplete markets, working paper, University of Lausanne.

Judd, K. L., 1998, *Numerical Methods in Economics,* MIT Press, Cambridge, MA.

Kahl, M., J. Liu, and F. A. Longstaff, 2003, Paper millionaires: how valuable is stock to a stockholder who is restricted from selling it? *Journal of Financial Economics* 67, 385-410.

Karatzas, I., J. Lehoczky, and S. Shreve, 1987, Optimal portfolio and consumption decisions for a 'small investor' on a finite horizon, *SIAM Journal of Control and Optimization* 25, 1157-86.

Kimball, M. S., 1990, Precautionary saving in the small and in the large, *Econometrica* 58, 53-73.

Kogan, L., 2001, An equilibrium model of irreversible investment, *Journal of Financial Economics,* 62 201-45.

Koo, H.-K., 1998, Consumption and portfolio selection with labor income: A continuous-time approach, *Mathematical Finance,* 8, 49-65.

Krylov, N. V., 1980, *Controlled Diffusion Processes*, Springer, Berlin.

Leahy, J. V. and T. M. Whited, 1996, The effect of uncertainty on investment: Some stylized facts, *Journal of Money, Credit and Banking* 28, 64-83.

Leland, H., 1968, Saving and uncertainty: the precautionary demand for savings, *Quarterly Journal of Economics* 82, 465-73.

McDonald, R. and D. Siegel, 1986, The value of waiting to invest, *Quarterly Journal of Economics* 101, 707-728.

Merton, R. C., 1969, Lifetime portfolio selection under uncertainty: the continuous-time case, *Review of Economics and Statistics 51*, 247-257.

Merton, R. C., 1973, Theory of rational option pricing, *Bell Journal of Economics and Management Science* 4, 141-183.

Miao, J. and N. Wang, 2005a, Learning, investment, and entrepreneurial survival, working paper, Boston University and Columbia University.

Miao, J. and N. Wang, 2005b, Entrepreneurial investment and financing decisions, work in progress, Boston University and Columbia University.

Moel, A. and P. Tufano, 2002, When are real options exercised? An empirical study of mine closings, *Review of Financial Studies* 15, 35-64.

Moskowitz, T. J. and A. Vissing-Jorgensen, 2002, The returns to entrepreneurial investment: A private equity premium puzzle?, *American Economic Review* 92(4), 745-778.

Quigg, L., 1993, Empirical testing of real option-pricing models, *Journal of Finance* 48, 621-640.

Pindyck, R., 1988, Irreversible investment, capacity choice, and the value of the firm, *American Economic Review,* 78(5), 1110-48.

Svensson, L. E. O. and I. M. Werner, 1993, Nontraded assets in incomplete markets: Pricing and portfolio choice, *European Economic Review* 37, 1149-1168.

Titman, S., 1985, Urban land prices under uncertainty, *American Economic Review*, 75, 505-14.

Viceira, L., 2001, Optimal portfolio choice for long-horizon investors with nontradable labor income, *Journal of Finance,* 56, 433-70.

Wang, N., 2003, Caballero meets Bewley: The permanent-income hypothesis in general equilibrium, *American Economic Review*, 93(3), 927-936.

Wang, N., 2004, Precautionary saving with partially observed income, *Journal of Monetary Economics*, 51(8), 1645-1681.

Wang, N., 2005, Generalizing the permanent-income hypothesis: Revisiting Friedman's conjecture on consumption, forthcoming, *Journal of Monetary Economics.*

Williams, J. T., 1991, Real estate development as an option, *Journal of Real Estate Finance and Economics* 4, 191-208.

Williams, J. T., 2001, Agency, ownership, and returns on real assets, working paper, Professors Capital.

Zeldes, S. P., 1989, Optimal consumption with stochastic income: Deviations from certainty equivalence. *Quarterly Journal of Economics*, 104, 275-298.

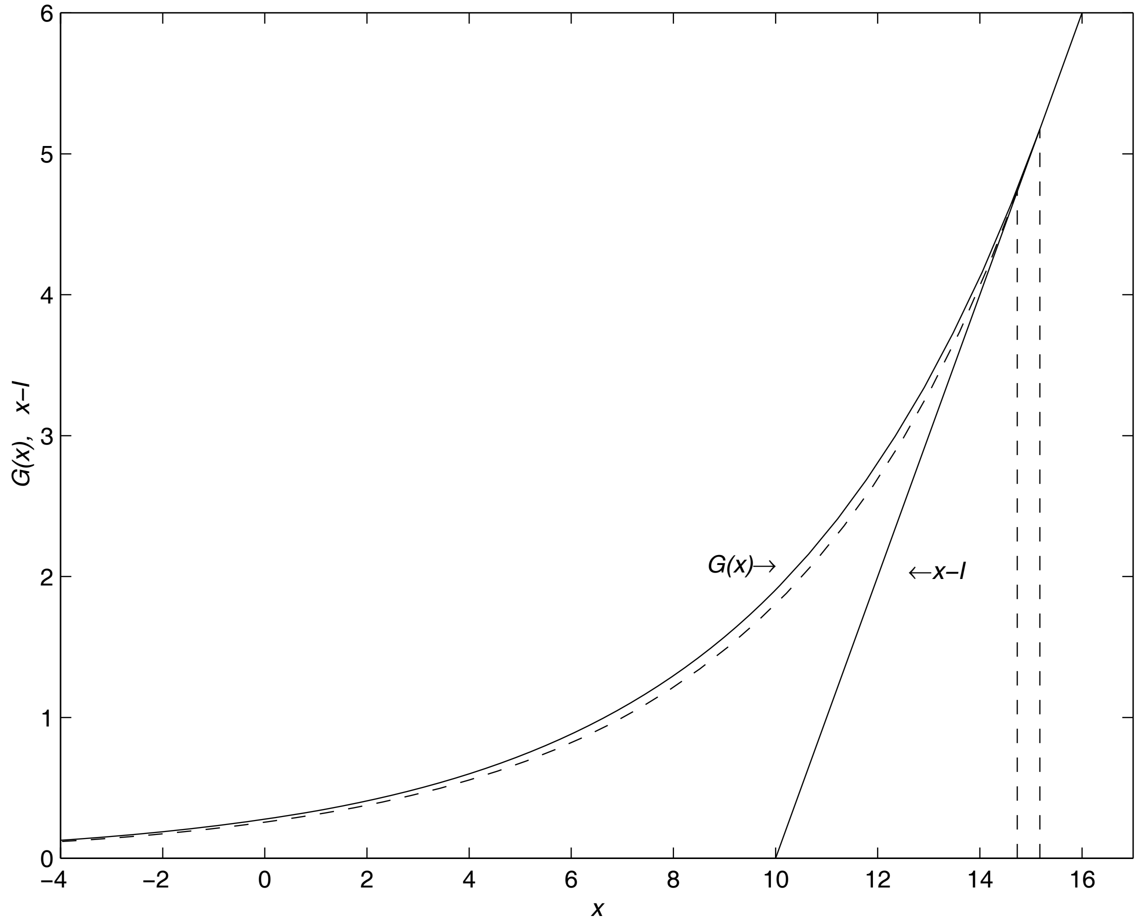

Figure 1: **Implied option value** $G(x)$**.** This figure plots the functions $x - I$ and $G(x)$ for the model in Section 2. The parameter values are set as follows: $r = 2\%$, $\alpha_x = 0.1$, $\sigma_x = 20\%$, and $I = 10$. The solid curve is for $\gamma = 1$, and the dashed curve is for $\gamma = 25$.

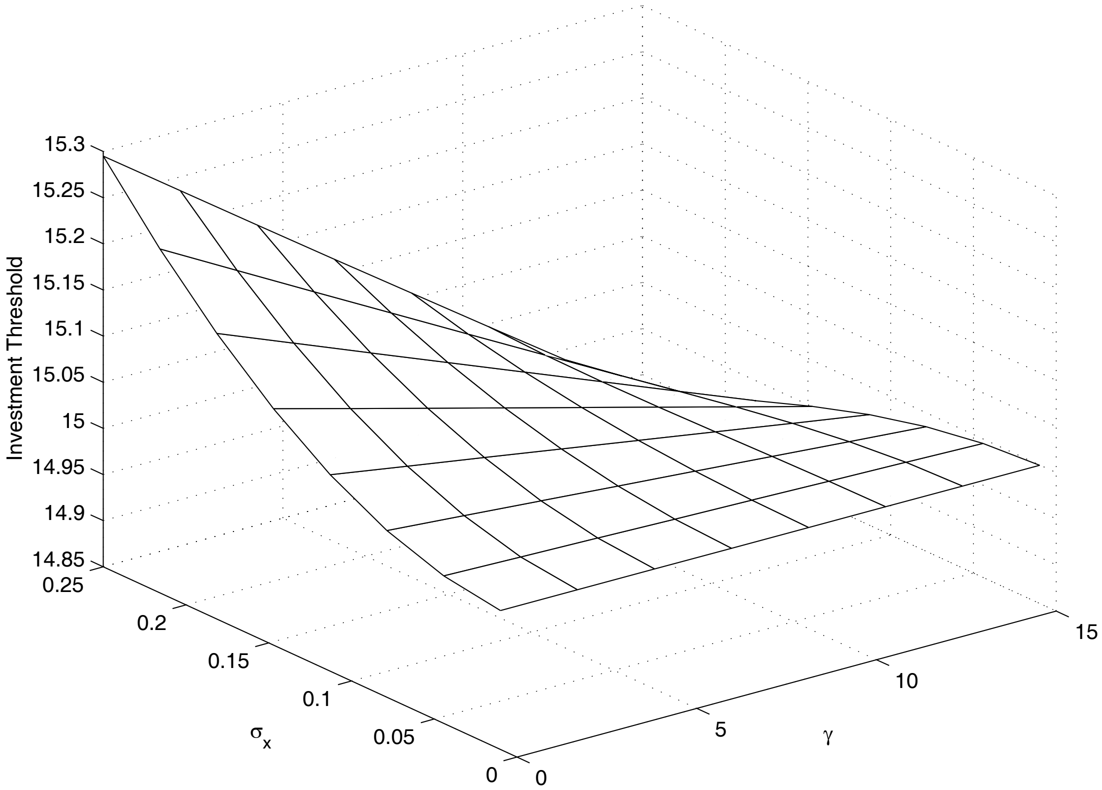

Figure 2: **Investment threshold, risk aversion, and project volatility.** This figure plots the investment threshold at varying levels of γ and σ_x for the lump sum payoff case. Other parameter values are set as $r = 2\%$, $\alpha_x = 0.1$, and $I = 10$.

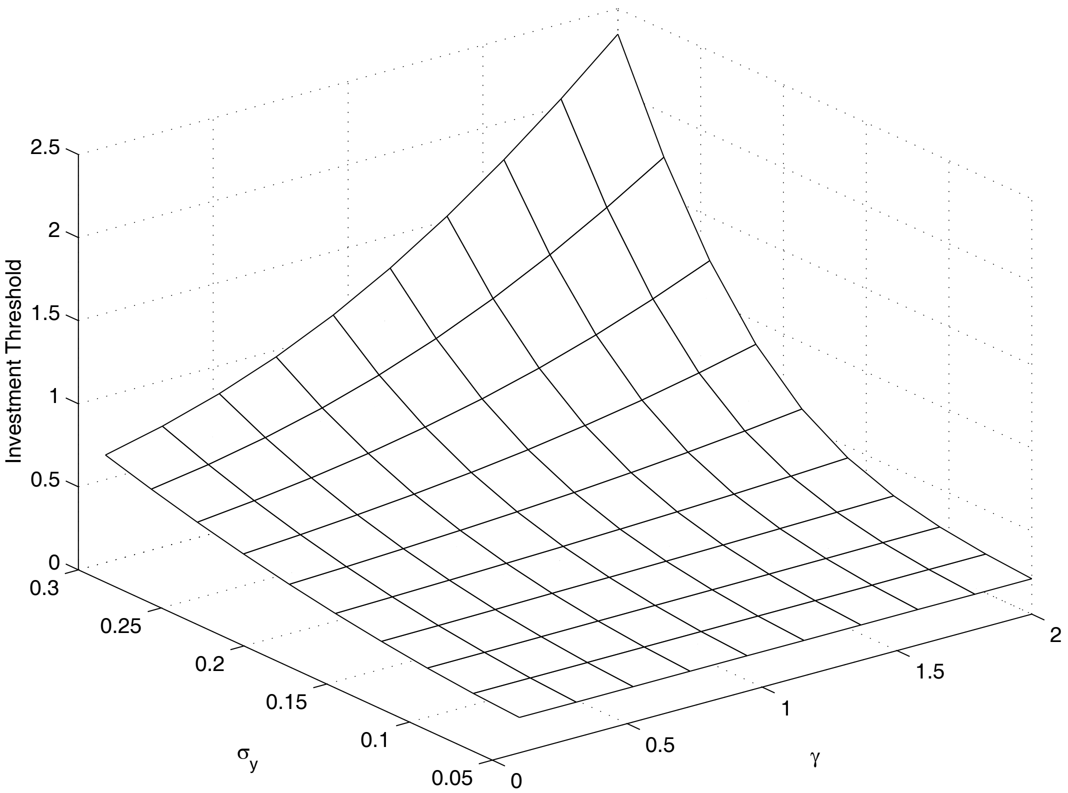

Figure 3: **Investment threshold, risk aversion, and project volatility.** This figure plots the investment threshold at varying levels of γ and σ_y for the flow payoff case. Other parameter values are set as $\beta = r = 2\%$, $\alpha_y = 0.1$, and $I = 10$

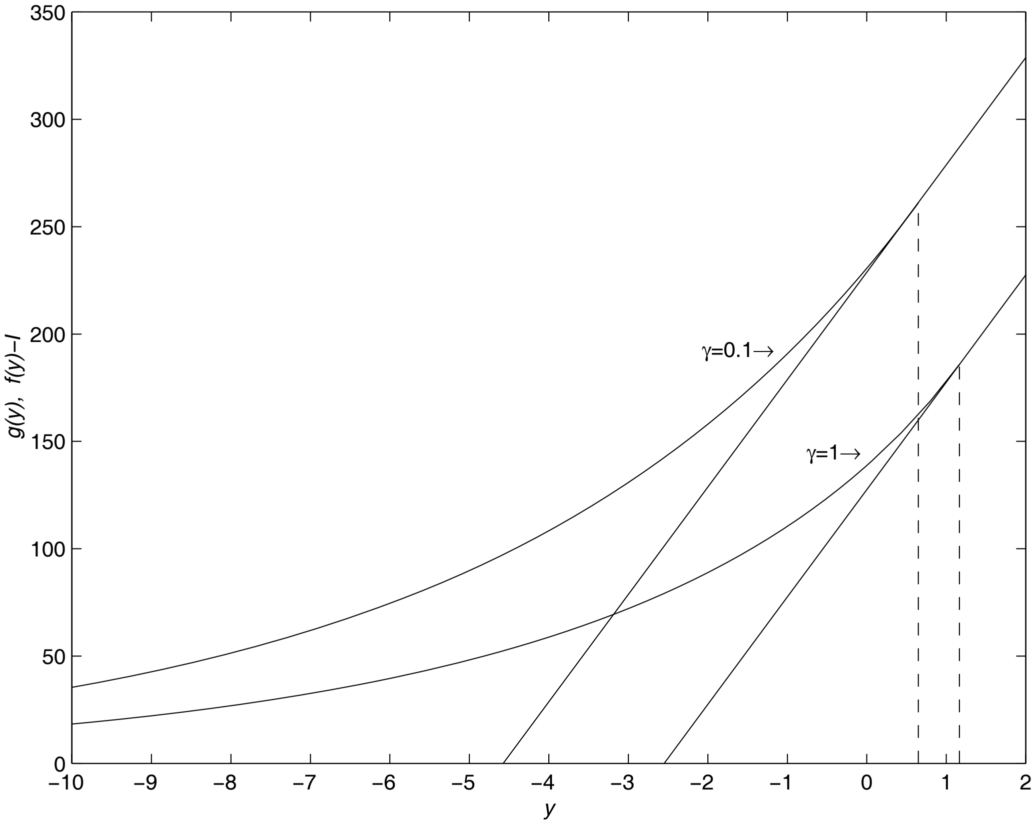

Figure 4: **Impact of changes in γ in the flow payoff case.** This figure plots the impact on $g(y)$ and $f(y)$ when the value of γ is increased from 1 to 2. Other parameter values are set as $\beta = r = 2\%$, $\alpha_y = 0.1$, $\sigma_y = 30\%$, and $I = 10$.