Optimal Fiscal Policy over the Business Cycle

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Abstract

How should taxes, government expenditures, the fiscal and primary surpluses and government liabilities be set over the business cycle? We assume that the government's objective is to maximize the welfare of a representative household, government expenditures increase the utility of the representative household, only distortionary labor income taxes are available, and the cycle is driven by exogenous technology shocks. We first consider the commitment case, and characterize the Ramsey equilibrium. In the case that the utility function is separable in leisure and constant elasticity of substitution between private and public consumption, taxes, government expenditures and the primary surplus should all be constant positive fractions of production, and both government liabilities and the fiscal surplus should be pro-cyclical. Then, we relax the commitment assumption, and we show numerically that, for a realistic value of the preferences discount factor, there is a sustainable equilibrium with the same outcome and value as the Ramsey equilibrium.

Keywords: Fiscal policy, Commitment, Time-consistency, Ramsey equilibrium, Markov perfect equilibria, Sustainable equilibria.
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1 Introduction

How should taxes, government expenditures, the fiscal and primary surpluses and government liabilities be set over the business cycle? We assume that the government's objective is to maximize the welfare of a representative household, government expenditures increase the utility of the representative household, only distortionary labor income taxes are available, and the cycle is driven by exogenous technology shocks. We also assume that a complete set of one-period Arrow securities are available in each period and state, but securities with longer maturities are not.

First, following Lucas and Stokey (1983), Zhu (1992), and Chari, Christiano and Kehoe (1994), we assume that the government can commit, and characterize the optimal competitive equilibrium or Ramsey equilibrium. We follow a suggestion by Kydland and

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Prescott (1980) and Chang (1998), we introduce a pseudo state variable (the promised value of government liabilities), and we divide the Ramsey problem into a first-period problem and a continuation problem. In the continuation problem, we characterize the optimal competitive equilibrium among the subset of equilibria which depend recursively on the natural state variables (the technology shock and government liabilities) as well as the pseudo state variable. Given the solution of the continuation problem, the first-period problem is a static problem of determining optimally the first-period allocation as well as the next-period values of the state.

Our main result for the commitment case is that, in the benchmark case that the utility function is separable between leisure and the composite consumption good, and constant elasticity of substitution between private and public consumption, taxes, government expenditures and the primary surplus should all be positive constant fractions of production. If the technology shock is positively serially correlated, both government liabilities and the fiscal surplus should be pro-cyclical. The intuition is the following. Since households and the government obtain utility from the expected present discounted value of public consumption, the government should purchase goods in periods and states where the technology shock and the goods supply are high and the intertemporal price of goods is low, so government expenditures should be pro-cyclical. Also, since the tax distortion increases more than proportionally with the tax rate, to smooth the tax distortion across periods and states the government should smooth the tax rate across periods and states, leading to pro-cyclical taxes. In the benchmark case, both taxes and government expenditures are constant positive fractions of production, so the primary surplus is a constant fraction of production. Since the present value of primary surpluses is equal to the positive initial government liabilities, the primary surplus is a positive constant fraction of production. If the technology shock is positively serially correlated, so is production. Then, the present value of primary surpluses is pro-cyclical, and so are government liabilities. We finally show that pro-cyclical government liabilities imply pro-cyclical fiscal surpluses.

We then relax the commitment assumption, and we show numerically that, for a realistic value of the preferences discount factor, the Ramsey equilibrium is sustainable. As a first step, following Kydland and Prescott (1977), Klein and Rios-Rull (2003) and Klein, Krusell and Rios-Rull (2004), we define Markov perfect equilibria, and we compare the outcome of a Markov perfect equilibrium with that one of the Ramsey equilibrium. Then, adapting and modifying tools developed in Stokey (1989), Chari and Kehoe (1990), Stokey (1991), Chang (1998) and Phelan and Stacchetti (2001), we define recursive sustainable equilibria, and we show numerically that the continuation of the Ramsey equilibrium can be sustained as a recursive sustainable equilibrium by the threat to revert to a Markov perfect equilibrium. Since the continuation of the Ramsey equilibrium is sustainable, it follows immediately that there is a sustainable equilibrium with the same outcome and value as the Ramsey equilibrium.

The focus of this study is the optimal setting of taxes, government expenditures, fiscal and primary surpluses and government liabilities over the business cycle, assuming that the cycle is driven by technology shocks. Two features of the model which are necessary for the analysis are that government expenditures are endogenous, and the government period budget constraint does not necessarily balance. Most previous studies model government expenditures exogenously, and answer a public finance question. Among them, Lucas and

Stokey (1983) characterize the optimal labor income tax policy with commitment in a model subject to government expenditures shocks. In addition, they show that the optimal policy with commitment is time-consistent if a complete set of Arrow-Debreu securities for all future periods and states is available in each period and state. Chamley (1986) introduces capital and characterizes the optimal labor and capital income tax policy with commitment. Zhu (1992) Chari, Christiano and Kehoe (1994), and Stockman (1998) characterize the optimal tax policy with commitment in a model subject to government expenditures shocks and technology shocks, while Klein and Rios-Rull (2003) characterize the Markov equilibrium. Other studies let government expenditures be determined endogenously, but add the strong assumption that the government budget constraints balance in all periods and states. Kydland and Prescott (1980) focus on the optimal recursive competitive equilibrium, Phelan and Stacchetti (2001) on sequential equilibria, while Klein, Krusell and Rios-Rull (2004) on the Markov equilibrium.

In what follows, section 2 describes the model and defines the competitive equilibrium. Section 3 studies the Pareto optimum, which characterizes the optimal competitive equilibrium in the case that lump-sum taxes are available. Section 4 assumes that only distortionary labor income taxes are available and studies the Ramsey equilibrium. Section 5 considers Markov perfect equilibria, and section 6 shows numerically that the Ramsey equilibrium is sustainable. Section 7 concludes.

2 Model

Let the state of the economy be described by the first-order Markov process $\{s_t\}_{t=0}^{\infty}$, $s_t \in S$, S finite, with transition probabilities $\pi(s_{t+1}|s_t)$. The initial state s_0 is given. Let $s^t \equiv \{s_j\}_{j=0}^{t}$ be the history of the state up to period t, and let $\pi_t(s^t|s_0)$ be the probability of s^t conditional on s_0. Let $\xi(s_t)$ be the technology shock in period t.

In each period t and history s^t, households are endowed with $\overline{n} > 0$ hours, and they choose to work $n_t(s^t) \in [0, \overline{n}]$ hours. Each hour of work produces $\xi(s_t)$ units of a non-storable consumption good, so aggregate production is $y_t(s^t) \equiv \xi(s_t)n_t(s^t)$. Production can be used for private consumption $c_t(s^t) \geq 0$ or public consumption $g_t(s^t) \geq 0$. The feasibility constraints are then

$$c_t(s^t) + g_t(s^t) \leq \xi(s_t)n_t(s^t), \text{ all } t,\ s^t \tag{1}$$

The households' and government preferences are described by

$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t u(c_t(s^t), n_t(s^t), g_t(s^t))\pi_t(s^t|s_0)$$

where $\beta \in (0, 1)$ is the preferences discount factor, and $u(c, n, g)$ is twice continuously differentiable, strictly increasing in its first and third arguments c and g, strictly decreasing in its second argument n, strictly concave, and satisfies the Inada conditions $\lim_{c\to 0} u_c(c, n, g) = \infty$ for all n, g, $\lim_{n\to\overline{n}} u_n(c, n, g) = -\infty$ for all c, g, and $\lim_{g\to 0} u_g(c, n, g) = \infty$ for all c, n. Alternative, in place of the second condition, one could assume that $\lim_{n\to+\infty} u_n(c, n, g) = -\infty$ for all c, g, and that $\overline{n}$ is large enough so that $n < \overline{n}$ is optimal in all the following optimization problems.

A complete set of one-period Arrow securities is available in each period and history, but securities with longer maturities are not. Let $q_t(s^t, s_{t+1}) > 0$ be the price of consumption goods in period $t+1$ and history $\{s^t, s_{t+1}\}$ in terms of consumption goods in period t and history s^t. Let $b_t(s^t)$ be the households' real assets equal to the government real liabilities in period t and history s^t, and let $b_0(s_0) \geq 0$ be given. Let $\tau_t(s^t) < 1$ be the labor income tax rate in period t and history s^t. The primary surplus

$$\delta_t^p(s^t) \equiv \tau_t(s^t) y_t(s^t) - g_t(s^t)$$

is the difference between taxes and government expenditures, while the fiscal surplus

$$\delta_t^f(s^t) \equiv \delta_t^p(s^t) - \left(1 - \sum_{s_{t+1} \in S} q_t(s^{t+1})\right) b_t(s^t)$$

is the primary surplus minus the interests on government liabilities. The real interest rate $r_t(s^t)$ is defined by

$$\frac{1}{1 + r_t(s^t)} \equiv \sum_{s_{t+1} \in S} q_t(s^{t+1})$$

A competitive equilibrium is an allocation $\{c_t(s^t) \geq 0,\ n_t(s^t) \in [0, \overline{n}],\ g_t(s^t) \geq 0,\ b_{t+1}(s^{t+1})\}_{t=0}^{\infty}$, and a tax and price system $\{\tau_t(s^t) < 1,\ q_t(s^{t+1}) > 0\}_{t=0}^{\infty}$ such that:

- Given $\{g_t(s^t),\ \tau_t(s^t),\ q_t(s^{t+1})\}_{t=0}^{\infty}$, $\{c_t(s^t),\ n_t(s^t),\ b_{t+1}(s^{t+1})\}_{t=0}^{\infty}$ solves the representative household's problem:

$$\max_{\{c_t(s^t) \geq 0, n_t(s^t) \in [0, \overline{n}], b_{t+1}(s^{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t), n_t(s^t), g_t(s^t)) \pi_t(s^t | s_0)$$

subject to: $c_t(s^t) - [1 - \tau_t(s^t)] \xi(s_t) n_t(s^t) + \sum_{s_{t+1} \in S} q_t(s^{t+1}) b_{t+1}(s^{t+1}) = b_t(s_t)$, all t, s^t

$$\lim_{t \to \infty} \sum_{s^t} \left(\prod_{j=1}^{t} q_{j-1}(s^j) \right) b_t(s^t) = 0$$

- The government budget constraints are satisfied

$$b_t(s_t) = \tau_t(s^t) \xi(s_t) n_t(s^t) - g_t(s^t) + \sum_{s_{t+1} \in S} q_t(s^{t+1}) b_{t+1}(s^{t+1}), \text{ all } t,\ s^t$$

- The market for consumption goods is in equilibrium:

$$c_t(s^t) + g_t(s^t) = \xi(s_t) n_t(s^t), \text{ all } t,\ s^t$$

By Walras' Law, the household's budget constraints and the consumption goods market equilibrium conditions imply that government budget constraints are satisfied. From the definition of fiscal surplus and the government budget constraints,

$$\delta_t^f(s^t) = \sum_{s_{t+1} \in S} q_t(s^{t+1}) \left(b_t(s^t) - b_{t+1}(s^{t+1}) \right)$$

From the necessary conditions of the household's problem,

$$\begin{aligned}
q_t(s^{t+1}) &= \frac{\beta^{t+1}u_c(c_{t+1}(s^{t+1}), n_{t+1}(s^{t+1}), g_{t+1}(s^{t+1}))\pi_{t+1}(s^{t+1}|s_0)}{\beta^t u_c(c_t(s^t), n_t(s^t), g_t(s^t))\pi_t(s^t|s_0)} \\
&= \frac{\beta u_c(c_{t+1}(s^{t+1}), n_{t+1}(s^{t+1}), g_{t+1}(s^{t+1}))\pi(s_{t+1}|s_t)}{u_c(c_t(s^t), n_t(s^t), g_t(s^t))}, \text{ all } t,\ s^t \\
[1-\tau_t(s^t)]\xi(s_t) &= \frac{-u_n(c_t(s^t), n_t(s^t), g_t(s^t))}{u_c(c_t(s^t), n_t(s^t), g_t(s^t))}, \text{ all } t,\ s^t
\end{aligned}$$

These two conditions, evaluated in equilibrium, express the tax and price system as a function of the allocation, and they define the tax and price system associated with a given allocation. We will refer to them as the equilibrium conditions for the tax and price system.

Substituting the previous expressions for tax rates and prices into the household's budget constraints, we obtain the implementability constraints

$$\begin{aligned}
&u_c(c_t(s^t), n_t(s^t), g_t(s^t))[c_t(s^t) - b_t(s^t)] + u_n(c_t(s^t), n_t(s^t), g_t(s^t))n_t(s^t) \\
&+ \sum_{s_{t+1}\in S} \beta u_c(c_{t+1}(s^{t+1}), n_{t+1}(s^{t+1}), g_{t+1}(s^{t+1}))b_{t+1}(s^{t+1})\pi(s_{t+1}|s_t) = 0, \text{ all } t,\ s^t \\
&\lim_{t\to\infty} \sum_{s^t} \beta^t u_c(c_t(s^t), n_t(s^t), g_t(s^t))b_t(s^t)\pi_t(s^t|s_0) = 0
\end{aligned} \tag{2}$$

Definition 1 (Implementable allocations.) *An allocation $\{c_t(s^t) \geq 0,\ n_t(s^t) \in [0, \overline{n}],$ $g_t(s^t) \geq 0,\ b_{t+1}(s^{t+1})\}_{t=0}^{\infty}$ is implementable if it satisfies the feasibility constraints 1 with equality and the implementability constraints 2.*

One can show that competitive equilibria are implementable allocations together with their associated tax and price systems $\{\tau_t(s^t) < 1,\ q_t(s^{t+1}) > 0\}_{t=0}^{\infty}$.

Without loss of generality, we focus on implementable allocations for which $b_t(s^t)$ is the following function of current and future consumption and labor,

$$\begin{aligned}
&u_c(c_t(s^t), n_t(s^t), g_t(s^t))b_t(s^t) \\
&= \sum_{j=t}^{\infty} \sum_{s^j} \beta^{j-t}\{u_c(c_j(s^j), n_j(s^j), g_j(s^j))c_j(s^j) + u_n(c_j(s^j), n_j(s^j), g_j(s^j))n_j(s^j)\}\pi_{t,j}(s^j|s^t)
\end{aligned}$$

where $\pi_{t,j}(s^j|s^t)$ is the probability of s^j conditional on s^t.

3 Pareto optimum

We begin considering the Pareto optimum, which characterizes the optimal competitive equilibrium in the case that lump-sum taxes are available. A Pareto optimum is a contingent sequence $\{c_t(s^t),\ n_t(s^t),\ g_t(s^t)\}_{t=0}^{\infty}$ which solves the following Pareto problem:

$$\max_{\{c_t(s^t)\geq 0, n_t(s^t)\in[0,\overline{n}], g_t(s^t)\geq 0\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t(s^t), n_t(s^t), g_t(s^t))\pi_t(s^t|s_0)$$

$$\text{subject to: } c_t(s^t) + g_t(s^t) \leq \xi_t(s^t)n_t(s^t), \text{ all } t,\ s^t$$

The Pareto optimum solves a sequence of static problems. For comparison with the following sections, we formulate the problem recursively. A recursive Pareto optimum is a set of policy functions $c(s)$, $n(s)$ and $g(s)$ solving the following static optimization problem:

$$\text{For all } s: \quad \max_{\{c\geq 0, n\in[0,\overline{n}], g\geq 0\}} u(c,n,g) \text{ subject to: } c+g\leq \xi(s)n$$

Since u is continuous and strictly concave, and the constrained region is compact and convex, a solution exists and is unique. Since u satisfies the Inada conditions, $c > 0$, $n \in (0, \overline{n})$ and $g > 0$. Since u is strictly increasing in c and g and strictly decreasing in n, the feasibility constraint holds with equality.

Notice that the solution is a time-invariant function of technology $\{c(\xi(s)),\ n(\xi(s)),\ g(\xi(s)),\ \mu(\xi(s))\}$. Moreover, the previous problem consists in maximizing u subject to a budget constraint where the price of labor in terms of both private and public consumption is $\xi(s)$, and the income is 0. Then, the effect of an increase in the technology shock on consumption and labor is the sum of an income effect and a substitution effect. If private consumption is a normal good, both effects work in the same direction, and private consumption increases, $(dc/d\xi > 0)$. The same holds in the case of public consumption $(dg/d\xi > 0)$. As a consequence, if both private and public consumption are normal goods, aggregate production, which is equal to the sum of private and public consumption, increases with the technology shock $(dy/d\xi > 0)$. Both private and public consumption are, then, pro-cyclical. With regard to leisure, however, if leisure is a normal good, the income effect increases leisure, while the substitution effect decreases it.

The Lagrangian is

$$\mathcal{L} = u(c,n,g) + \mu[\xi(s)n - c - g]$$

where $\mu \geq 0$ is the Kuhn-Tucker multiplier associated with the feasibility constraint. The necessary and sufficient conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c} &= u_c(c,n,g) - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial n} &= u_n(c,n,g) + \mu\xi(s) = 0 \\
\frac{\partial \mathcal{L}}{\partial g} &= u_g(c,n,g) - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu} &= \xi(s)n - c - g = 0
\end{aligned}$$

which form a system of four equations in the four unknown $\{c,\ n,\ g,\ \mu\}$. For comparison with later sections, notice that $u_c = u_g$, $-u_n = \xi u_c$ and $-u_n = \xi u_g$ in the Pareto optimum.

The result we are mostly interested is that government expenditures should be set pro-cyclically. The intuition is that, as long as there is some substitutability between public consumption in different periods and histories public consumption should be higher in periods and histories where the supply of consumption goods is higher and the intertemporal price of consumption goods is lower.

4 Ramsey equilibrium

We now turn to the analysis of the optimal competitive equilibrium, or Ramsey equilibrium. Following Lucas and Stokey (1983), the allocation of the Ramsey equilibrium solves the following Ramsey problem:

$$\max_{\{c_t(s^t)\geq 0, n_t(s^t)\in[0,\overline{n}], g_t(s^t)\geq 0, b_{t+1}(s^{t+1})\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t}\beta^t u(c_t(s^t), n_t(s^t), g_t(s^t))\pi_t(s^t|s_0)$$

subject to:

$$u_c(c_t(s^t), n_t(s^t), g_t(s^t))[c_t(s^t) - b_t(s^t)] + u_n(c_t(s^t), n_t(s^t), g_t(s^t))n_t(s^t)$$
$$+ \sum_{s_{t+1}\in S}\beta u_c(c_{t+1}(s^{t+1}), n_{t+1}(s^{t+1}), g_{t+1}(s^{t+1}))b_{t+1}(s^{t+1})\pi(s_{t+1}|s_t) = 0, \text{ all } t,\ s^t$$

$$\lim_{t\to\infty}\sum_{s^t}\beta^t u_c(c_t(s^t), n_t(s^t), g_t(s^t))b_t(s^t)\pi_t(s^t|s_0) = 0$$

$$c_t(s^t) + g_t(s^t) \leq \xi_t(s^t)n_t(s^t), \text{ all } t,\ s^t$$

Notice that we write the feasibility constraints with inequality instead of equality. This allows to determine the sign of the multipliers without affecting the solution, since the feasibility constraints are binding at the optimum.

We would like to express the conditions defining implementable allocations (except the limit condition) recursively. We follow a suggestion by Kydland and Prescott (1980) and Chang (1998), and define the pseudo state variable (the promised value of government liabilities):

$$\theta_t(s^t) \equiv u_c(c_t(s^t), n_t(s^t), g_t(s^t))b_t(s^t), \text{ all } t,\ s^t$$

After substituting the expression for $\theta_t(s^t)$ in the conditions defining implementable allocations, we restrict attention to the implementable allocations which, in the first period, depend on the natural state variables s_t and $b_t(s^t)$, while, in the following periods, depend recursively on the natural state variables s_t and $b_t(s^t)$ as well as the pseudo state variable $\theta_t(s^t)$. Incidentally, we notice that it would be equivalent (and not more restrictive) to restrict attention to the implementable allocations which, in *all* periods, depend recursively on the natural state variables s_t and $b_t(s^t)$ as well as the pseudo state variable $\theta_t(s^t)$ for any arbitrary initial value of the pseudo state variable $\theta_0(s_0)$.

However, for this model, it is possible and convenient to further restrict attention to the following recursive implementable allocations which, in the periods after the first, do not depend on $b_t(s^t)$:

Definition 2 (Recursive implementable allocations.) *A recursive implementable allocation is a set of first-period functions and continuation functions.*

The first-period functions are $\{c_0(b_0, s_0) \geq 0,\ n_0(b_0, s_0) \in [0, \overline{n}],\ g_0(b_0, s_0) \geq 0,$ *and* $\theta_1(b_0, s_0, s_1)\}$ *satisfying*

$$u_c(c_0, n_0, g_0)b_0(s_0) = u_c(c_0, n_0, g_0)c_0 + u_n(c_0, n_0, g_0)n_0 + \sum_{s_1\in S}\beta\theta_1(s_1)\pi(s_1|s_0)$$

$$c_0 + g_0 = \xi(s_0)n_0$$

The continuation functions are allocation functions $\{c(\theta,s) \geq 0,\ n(\theta,s) \in [0,\overline{n}],\ g(\theta,s) \geq 0,\ b(\theta,s)\}$ *and a law of motion for the pseudo state variable* $\theta'(\theta,s,s')$ *satisfying:*

$$\theta = u_c(c,n,g)c + u_n(c,n,g)n + \sum_{s' \in S} \beta \theta'(s')\pi(s'|s)$$

$$\lim_{t\to\infty} \sum_{s^t} \beta^t \theta_t(s^t)\pi_t(s^t|s_0) = 0$$

$$c + g = \xi(s)n$$

$$\theta = u_c(c,n,g)b$$

where $\theta_1(s^1) \equiv \theta_1(b_0,s_0,s_1)$*, and* $\theta_t(s^t)$ *is obtained starting from* $(\theta_1(s^1),s_1)$ *and iterating with the law of motion* $\theta'(\theta,s,s')$*.*

Notice that the first-period functions depend on b and s, while the continuation functions depend on θ and s. Our strategy is to focus on recursive implementable allocations ignoring the limit condition in the definition, characterize the recursive solution, and check that the limit condition is satisfied.

The Ramsey problem, then, can be divided into the following first-period problem and continuation problem.

The continuation problem consists in finding a value function $w(\theta,s)$, and policy functions $\{c(\theta,s) \geq 0,\ n(\theta,s) \in [0,\overline{n}],\ g(\theta,s) \geq 0,\ \theta'(\theta,s,s')\}$ solving the following Bellman equation:

$$\text{For all } \theta, s\text{: } w(\theta,s) = \max_{\{c\geq 0, n\in[0,\overline{n}], g\geq 0, \theta'(s')\}} \left\{ u(c,n,g) + \sum_{s'\in S} \beta w(\theta'(s'),s')\pi(s'|s) \right\}$$

$$\text{subject to: } \theta = u_c(c,n,g)c + u_n(c,n,g)n + \sum_{s'\in S} \beta\theta'(s')\pi(s'|s)$$

$$c + g \leq \xi(s)n$$

Once the continuation problem has been solved, government liabilities $b(\theta,s)$ are determined by

$$b(\theta,s) \equiv u_c(c(\theta,s),n(\theta,s),g(\theta,s))/\theta$$

In all the numerical examples considered below, for fixed s, the function $b(\theta,s)$ is strictly increasing for values of θ smaller than a threshold (the values corresponding to the good part of the Laffer curve), and it is strictly decreasing for higher values. Then, we focus on the values of θ smaller than the threshold, we invert $b(\theta,s)$, and we express θ as a function of b and s — Let $\theta(b,s)$ denote the function. Then, we determine the continuation value of the Ramsey equilibrium as a function of b and s by

$$\text{For all } b, s\text{: } v^C(b,s) \equiv w(\theta(b,s),s)$$

Given the value function $w(\theta, s)$, the first-period Ramsey problem is

$$v^R(b_0(s_0), s_0) \max_{\{c_0 \geq 0, n_0 \in [0,\overline{n}], g_0 \geq 0, \theta_1(s_1)\}} \left\{ u(c_0, n_0, g_0) + \sum_{s_1 \in S} \beta w(\theta_1(s_1), s') \pi(s_1|s_0) \right\}$$

$$\text{subject to: } u_c(c_0, n_0, g_0) b_0(s_0) = u_c(c_0, n_0, g_0) c_0 + u_n(c_0, n_0, g_0) n_0 + \sum_{s_1 \in S} \beta \theta_1(s_1) \pi(s_1|s_0)$$

$$c_0 + g_0 \leq \xi(s_0) n_0$$

Notice that, in general, $v^R(b, s) > v^C(b, s)$, so the continuation of the Ramsey equilibrium is not a Ramsey equilibrium for the continuation economy.

4.1 Results

The Lagrangian for the continuation problem is

$$\begin{aligned}
\mathcal{L} &= u(c, n, g) + \sum_{s' \in S} \beta w(\theta'(s'), s') \pi(s'|s) \\
&+ \lambda \left[u_c(c, n, g) c + u_n(c, n, g) n + \sum_{s' \in S} \beta \theta'(s') \pi(s'|s) - \theta \right] + \mu[\xi(s) n - c - g]
\end{aligned}$$

where λ is the Lagrange multiplier associated with the implementability constraint, and $\mu \geq 0$ is the Kuhn-Tucker multiplier associated with the feasibility constraints.

Assuming that the solution satisfies $c > 0$, $n \in (0, \overline{n})$ and $g > 0$, and that it satisfies the feasibility constraints with equality, the necessary conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c} &= (1 + \lambda) u_c(c, n, g) + \lambda u_{cc}(c, n, g) c + \lambda u_{nc}(c, n, g) n - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial n} &= (1 + \lambda) u_n(c, n, g) + \lambda u_{cn}(c, n, g) c + \lambda u_{nn}(c, n, g) n + \mu \xi(s) = 0 \\
\frac{\partial \mathcal{L}}{\partial g} &= u_g(c, n, g) + \lambda u_{cg}(c, n, g) c + \lambda u_{ng}(c, n, g) n - \mu = 0 \\
\frac{\partial \mathcal{L}}{\partial \theta'(s')} &= \beta w_\theta(\theta'(s'), s') \pi(s'|s) + \beta \lambda \pi(s'|s) = 0, \text{ all } s' \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= u_c(c, n, g) c + u_n(c, n, g) n + \sum_{s' \in S} \beta \theta'(s') \pi(s'|s) - \theta = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu} &= \xi(s) n - c - g = 0 \\
w_\theta(\theta, s) &= \frac{\partial \mathcal{L}}{\partial \theta} = -\lambda
\end{aligned}$$

The Lagrangian for the first-period problem is

$$\begin{aligned}\mathcal{L}_0 = u(c_0, n_0, g_0) &+ \sum_{s_1 \in S} \beta w(\theta_1(s_1), s')\pi(s_1|s_0) \\ &+ \lambda_0 \left[u_c(c_0, n_0, g_0)[c_0 - b_0(s_0)] + u_n(c_0, n_0, g_0)n_0 + \sum_{s_1 \in S} \beta \theta_1(s_1)\pi(s_1|s_0) \right] \\ &+ \mu_0[\xi(s_0)n_0 - c_0 - g_0]\end{aligned}$$

where λ_0 is the Lagrange multiplier associated with the implementability constraint, and $\mu_0 \geq 0$ is the Kuhn-Tucker multiplier associated with the feasibility constraints.

Assuming that the solution satisfies $c_0 > 0$, $n_0 \in (0, \overline{n})$ and $g_0 > 0$, and that it satisfies the feasibility constraints with equality, the necessary conditions are

$$\begin{aligned}\frac{\partial \mathcal{L}_0}{\partial c_0} &= (1 + \lambda_0)u_c(c_0, n_0, g_0) + \lambda_0 u_{cc}(c_0, n_0, g_0)[c_0 - b_0(s_0)] + \lambda_0 u_{nc}(c_0, n_0, g_0)n_0 - \mu_0 = 0 \\ \frac{\partial \mathcal{L}_0}{\partial n_0} &= (1 + \lambda_0)u_n(c_0, n_0, g_0) + \lambda_0 u_{cn}(c_0, n_0, g_0)[c_0 - b_0(s_0)] + \lambda_0 u_{nn}(c_0, n_0, g_0)n_0 + \mu_0 \xi(s_0) = 0 \\ \frac{\partial \mathcal{L}_0}{\partial g_0} &= u_g(c_0, n_0, g_0) + \lambda_0 u_{cg}(c_0, n_0, g_0)[c_0 - b_0(s_0)] + \lambda_0 u_{ng}(c_0, n_0, g_0)n_0 - \mu_0 = 0 \\ \frac{\partial \mathcal{L}_{\prime}}{\partial \theta_1(s_1)} &= \beta w_\theta(\theta_1(s_1), s_1)\pi(s_1|s_0) + \beta \lambda_0 \pi(s_1|s_0) = 0, \text{ all } s_1 \\ \frac{\partial \mathcal{L}_0}{\partial \lambda_0} &= u_c(c_0, n_0, g_0)[c_0 - b_0(s_0)] + u_n(c_0, n_0, g_0)n_0 + \sum_{s_1 \in S} \beta \theta_1(s_1)\pi(s_1|s_0) = 0 \\ \frac{\partial \mathcal{L}_0}{\partial \mu_0} &= \xi(s_0)n_0 - c_0 - g_0 = 0 \\ v_b^R(b_0(s_0), s_0) = \frac{\partial \mathcal{L}}{\partial b_0(s_0)} &= -\lambda_0 u_c(c_0, n_0, g_0)\end{aligned}$$

From the necessary conditions of the first-period problem,

$$-\lambda_0(b_0(s_0), s_0) = w_\theta(\theta_1(s_1), s_1), \text{ all } s_1$$

so $w_\theta(\theta_1(s_1), s_1)$ is constant and equal to λ_0 for all s_1. From the necessary conditions of the continuation problem,

$$w_\theta(\theta, s) = -\lambda(\theta, s) = w_\theta(\theta'(\theta, s, s'), s'), \text{ all } s'$$

It follows that $w_\theta(\theta, s)$ is constant along the equilibrium path, i.e. for all the values of (θ, s) which are reached starting from $(\theta_1(s_1), s_1)$, all s_1, and iterating on the law of motion $\theta'(\theta, s, s')$ for all s'. Also, $\lambda(\theta, s)$ is constant and equal to λ_0 along the equilibrium path as well. Then, recalling that $b_0(s_0) \geq 0$ and $g_t(s^t) > 0$ all t and s^t, one can show that $\lambda > 0$ following the same argument as in Lucas and Stokey (1983).

Also, along the equilibrium path, $\theta'(\theta, s, s')$ is not a function of θ or s. It follows that, along the equilibrium path and after the first period, it is possible to express the current value of θ as a function of the current value of s — Let $\theta(s)$ denote the function. The limit condition then becomes

$$\lim_{t\to\infty}\sum_{s^t}\beta^t u_c(c_t(s^t),n_t(s^t),g_t(s^t))b_t(s^t)\pi_t(s^t|s_0)=0$$
$$\lim_{t\to\infty}\sum_{s^t}\beta^t\theta_t(s^t)\pi_t(s^t|s_0)=0$$
$$\lim_{t\to\infty}\sum_{s^t}\beta^t\theta(s_t)\pi_t(s^t|s_0)=0$$

and is satisfied since $\theta(s_t)$ takes a finite set of values and is therefore bounded, while β^t converges to zero.

Comparing the necessary conditions of the first-period and continuation problems, and recalling that $\lambda=\lambda_0$, it follows that period 0 is special whenever $b_0(s_0)>0$. The reason is that the real government liabilities in period 0 are given and are financed through distortionary taxes in other periods and histories. To minimize the tax distortions, the benevolent government has an incentive to decrease the price of consumption goods in period 0 relative to other periods and histories.

In what follows, we restrict attention to the equilibrium path and to periods following the first one. For convenience, we rewrite the following necessary conditions of the continuation problem:

$$\begin{aligned}(1+\lambda)u_c(c,n,g)+\lambda u_{cc}(c,n,g)c+\lambda u_{nc}(c,n,g)n-\mu&=0\\(1+\lambda)u_n(c,n,g)+\lambda u_{cn}(c,n,g)c+\lambda u_{nn}(c,n,g)n+\mu\xi(s)&=0\\u_g(c,n,g)+\lambda u_{cg}(c,n,g)c+\lambda u_{ng}(c,n,g)n-\mu&=0\\\xi(s)n-c-g&=0\end{aligned}\tag{3}$$

Given the constant λ, the previous conditions form a system of four equations in the four unknown c, n, g and μ.

Assuming that the solution exists and is unique, it is a time-invariant function of technology $\{c(\xi(s)),\ n(\xi(s)),\ g(\xi(s)),\ \mu(\xi(s))\}$, i.e. it does not depend on θ and it depends on s only through the technology shock $\xi(s)$. It follows that production y the tax rate τ and the primary surplus δ^p are time-invariant functions of technology as well. Prices q depend on the current as well as the next period technology. Also, recalling that $b_t(s^t)$ is a function of current and future consumption and labor, government liabilities $b_t(s^t)$ are time-invariant functions of the current state s_t, and so is the fiscal surplus δ^f. If $\xi(s)$ is a one-to-one function, both government liabilities and the fiscal surplus are time-invariant functions of technology. We emphasize that, in the periods following the first one, all the variables of interest in this economy are time-invariant functions of the current state s, so they are all perfectly correlated with each other.

It is instructive to consider the case that u is separable in its three arguments, and compare some properties of the Ramsey equilibrium and the Pareto optimum. In the Ramsey equilibrium, $-(1+\lambda)u_n-\lambda u_{nn}n=\xi u_g$, so $-u_n<\xi u_g$, while $-u_n=\xi u_g$ in the Pareto optimum. Also, $-(1+\lambda)u_n-\lambda u_{nn}n=(1+\lambda)\xi u_c+\lambda\xi u_{cc}c$, so $-u_n<\xi u_c$, while $-u_n=\xi u_c$ in the Pareto optimum. Then, recalling that $(1-\tau)\xi u_c=-u_n$, the tax rate τ is strictly positive. Intuitively, the distortionary labor income tax tends to discourage labor, production, private consumption and public consumption relative to the Pareto optimum. Finally, $(1+\lambda)u_c + \lambda u_{cc}c = u_g$, so the sign of $u_g - u_c$ is the same as the sign of $u_c + u_{cc}c$, while $u_c = u_g$ in the Pareto optimum.

4.2 Numerical solution

The results so far suggest the following efficient strategy to solve the Ramsey problem numerically.

First, consider the continuation problem. Using the necessary conditions 3, c, n, g and μ are expressed as functions of λ and s — Let $c(\lambda, s)$, $n(\lambda, s)$, $g(\lambda, s)$ and $\mu(\lambda, s)$ denote the four functions. Then, recalling that λ is constant along the equilibrium path, the value ω associated with λ and s is determined by the functional equation

$$\text{For all } \lambda, s\text{: } \omega(\lambda, s) = u(c(\lambda, s), n(\lambda, s), g(\lambda, s)) + \sum_{s' \in S} \beta\omega(\lambda, s')\pi(s'|s)$$

Notice that, for fixed λ, the previous is a simple linear system of equations in the unknowns $\omega(\lambda, s)$ for all $s \in S$. Let $\underline{\omega}(\lambda)$ be the column vector of the unknowns $\omega(\lambda, s)$, let $\underline{u}(\lambda)$ be the column vector of the constants $u(c(\lambda, s), n(\lambda, s), g(\lambda, s))$, and let P be the transition matrix of the state. Then,

$$\begin{aligned} \underline{\omega}(\lambda) - \beta P\underline{\omega}(\lambda) &= \underline{u}(\lambda) \\ \underline{\omega}(\lambda) &= (I - \beta P)^{-1}\underline{u}(\lambda) \end{aligned}$$

where I is the identity matrix.

The next step is to determine the function $\lambda(\theta, s)$. However, it is convenient to first determine θ as a function of λ and s. Recall that, along the equilibrium path and after the first period, the current value of θ is a function of the current value of s. Then, the function $\theta(\lambda, s)$ is determined by the functional equation

$$\begin{aligned} \text{For all } \lambda, s\text{: } & u_c(c(\lambda, s), n(\lambda, s), g(\lambda, s))c(\lambda, s) + u_n(c(\lambda, s), n(\lambda, s), g(\lambda, s))n(\lambda, s) \\ & + \sum_{s' \in S} \beta\theta(\lambda, s')\pi(s'|s) - \theta(\lambda, s) = 0 \end{aligned}$$

Here again, for fixed λ, the previous is a simple linear system of equations in the unknowns $\theta(\lambda, s)$ for all $s \in S$. Let $\underline{\theta}(\lambda)$ be the column vector of the unknowns $\theta(\lambda, s)$, let $\underline{u_c}(\lambda)\underline{c}(\lambda) + \underline{u_n}(\lambda)\underline{n}(\lambda)$ be the column vector of the constants, and let P be the transition matrix of the state. Then,

$$\begin{aligned} \underline{\theta}(\lambda) - \beta P\underline{\theta}(\lambda) &= \underline{u_c}(\lambda)\underline{c}(\lambda) + \underline{u_n}(\lambda)\underline{n}(\lambda) \\ \underline{\theta}(\lambda) &= (I - \beta P)^{-1}[\underline{u_c}(\lambda)\underline{c}(\lambda) + \underline{u_n}(\lambda)\underline{n}(\lambda)] \end{aligned}$$

where I is the identity matrix.

Notice that, for fixed s, $\theta(\lambda, s)$ may have the properties of a Laffer curve, i.e. it may be inverted-U shaped, so the correspondence $\lambda(\theta, s)$ may have two values for small values of θ, and no values for large values of θ. This is one reason why it is convenient to first characterize $\theta(\lambda, s)$ rather than $\lambda(\theta, s)$. In the case that $\theta(\lambda, s)$ has the properties of a Laffer curve, we focus on the good part of the Laffer curve, i.e. the part of the curve corresponding to small values of λ until the maximum value of θ is reached. On that part, we invert the function $\theta(\lambda, s)$, and obtain $\lambda(\theta, s)$. Alternatively, for fixed s, $\theta(\lambda, s)$ may be a strictly increasing function for all $\lambda > 0$, as it occurs in the numerical examples considered below. In this case, we invert the function $\theta(\lambda, s)$ for all $\lambda > 0$. Once $\lambda(\theta, s)$ has been obtained, the value function is determined by $w(\theta, s) = \omega(\lambda(\theta, s), s)$, and similarly the associated policy functions.

Government liabilities b as a function of λ and s are determined by

$$b = \theta(\lambda, s)/u_c(c(\lambda, s), n(\lambda, s), g(\lambda, s))$$

For fixed s, $b(\lambda, s)$ may have the properties of a Laffer curve, as it occurs in the numerical example considered later in this section. Then, we focus on the good part of the Laffer curve, invert the function $b(\lambda, s)$ on that part, obtain $\lambda(b, s)$, and determine the value function by $v^C(b, s) = \omega(\lambda(b, s), s)$.

Once $w(\theta, s)$ has been obtained, determining the value function $v^R(b, s)$ and the associated policy functions is a simple static optimization problem. We substitute the two constraints into the objective function and use a grid search method. Alternatively, one can use the necessary conditions to determine c, g, n, μ and b as functions of (λ, s). We find that this second method works only for positive values of b.

4.3 Benchmark utility function

We now characterize the Ramsey equilibrium path in the periods following the first one in the case that the utility function is

$$u(c, n, g) \equiv \begin{cases} A\frac{f(c,g)^{1-\sigma}-1}{1-\sigma} - \Phi\frac{n^{1+\varphi}}{1+\varphi} & \sigma > 0, \sigma \neq 1 \\ A\log(f(c, g)) - \Phi\frac{n^{1+\varphi}}{1+\varphi} & \sigma = 1 \end{cases}$$

$$f(c, g) \equiv \begin{cases} \left(\alpha c^{\frac{\epsilon-1}{\epsilon}} + (1-\alpha)g^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}} & \epsilon > 0, \epsilon \neq 1 \\ c^{\alpha}g^{1-\alpha} & \epsilon = 1 \end{cases}$$

where $A > 0$, $\alpha \in (0, 1)$, $\Phi > 0$ and $\varphi > 0$. The utility function is separable between the composite consumption good and labor. The elasticity of substitution between private and public consumption is constant and equal to ϵ. Although $\lim_{n \to \overline{n}} u_n(c, n, g) = -\Phi\overline{n}^{\varphi} > -\infty$ we assume that $\overline{n}$ is large enough so that the solution is still described by the necessary conditions 3.

Let us start considering the simpler case $\epsilon = 1$. In this case, the necessary conditions 3 become

$$\begin{aligned} [1 + \lambda\alpha(1-\sigma)]u_c - \mu &= 0 \\ (1 + \lambda + \lambda\varphi)u_n + \mu\xi &= 0 \\ [1 + \lambda\alpha(1-\sigma)]u_g - \mu &= 0 \\ \xi(s)n - c - g &= 0 \end{aligned}$$

where we have used $u_{cc}c = [\alpha(1-\sigma)-1]u_c$, $u_{gc}c = \alpha(1-\sigma)u_g$, and $u_{nn}n = \varphi u_n$. It is easy to show that λ satisfies $1+\lambda\alpha(1-\sigma) > 0$.

Notice that $u_c = u_g$ like in the Pareto optimum. However, $-(1+\lambda+\lambda\varphi)u_n = \xi[1+\lambda\alpha(1-\sigma)]u_c$, so $-u_n < \xi u_c = \xi u_g$, while $-u_n = \xi u_c = \xi u_g$ in the the Pareto optimum.

Using the expression for the tax rate,

$$1-\tau = \frac{-u_n}{\xi u_c} = \frac{1+\lambda\alpha(1-\sigma)}{1+\lambda+\lambda\varphi} < 1$$

so the tax rates are strictly positive and constant along the equilibrium path. Taxes $\tau\xi n$ are then a strictly positive constant fraction of production. Moreover, from $u_c = u_g$, it follows that private consumption c and public consumption g are respectively strictly positive constant fractions α and $1-\alpha$ of production. The primary surplus δ^p is then a constant fraction $\tau-(1-\alpha)$ of production.

From $c = \alpha\xi n$ and $g = (1-\alpha)\xi n$, it follows that u_c is equal to $B(\xi n)^{-\sigma}$, where $B \equiv A\left(\alpha^\alpha(1-\alpha)^{1-\alpha}\right)^{1-\sigma}$. Then,

$$\begin{aligned}(1+\lambda+\lambda\varphi)(-u_n) &= [1+\lambda\alpha(1-\sigma)]u_c\xi \\ (1+\lambda+\lambda\varphi)\Phi n^\varphi &= [1+\lambda\alpha(1-\sigma)]B(\xi n)^{-\sigma}\xi \\ (1+\lambda+\lambda\varphi)\Phi n^{\varphi+\sigma} &= [1+\lambda\alpha(1-\sigma)]B\xi^{1-\sigma}\end{aligned}$$

so labor n is a strictly increasing (decreasing) function of technology ξ if and only if $1-\sigma > 0$ ($1-\sigma < 0$). Also,

$$\begin{aligned}(1+\lambda+\lambda\varphi)\Phi n^\varphi &= [1+\lambda\alpha(1-\sigma)]B(\xi n)^{-\sigma}\xi \\ (1+\lambda+\lambda\varphi)\Phi(\xi n)^{\varphi+\sigma} &= [1+\lambda\alpha(1-\sigma)]B\xi^{1+\varphi}\end{aligned}$$

so production ξn is a strictly increasing function of technology ξ.

The following conclusions depend on the assumption that $b_0(s_0) \geq 0$. With analogous arguments, one can show that they are reversed in the case that $b_0(s_0) < 0$. First notice that, in the case that $b_0(s^0) = 0$, period 0 should be treated as all other periods and histories, so the primary surplus should be equal to zero in all periods and histories, implying that $b_t(s^t) = 0$ all t and s^t. Suppose that $b_0(s_0) > 0$ implies $b_1(s^1) > 0$ for at least one history s^1. For instance, this is the case in the numerical examples considered below. Then, the following holds. Since $b_1(s^1)$ is equal to the present discounted value of current and future primary surpluses, and since the primary surplus is a constant fraction of production, the primary surplus is strictly positive in all periods and histories. Since government liabilities $b_t(s^t)$ are equal to the present discounted value of current and future primary surpluses, they are strictly positive in all periods and histories. Also, since production is strictly increasing in technology, the primary surplus is strictly increasing in technology.

Assume that the transition probability is strictly monotone (page 220 of Stokey and Lucas with Prescott (1989)), so the expected future value of a strictly increasing function of the state is strictly increasing in the current state. Also, assume that technology $\xi(s)$ is a strictly increasing function of the state, so the expected future value of a strictly increasing function of technology is strictly increasing in the current technology. (For simplicity, one could directly assume that $\xi(s) = s$.) These assumptions imply that technology is positively serially correlated. Then, the expected value of the primary surplus in any future period and history is strictly increasing in current technology. Since government liabilities $b_t(s^t)$ are equal to the present discounted value of current and future primary surpluses, government liabilities are strictly increasing in technology and are pro-cyclical. Then, the fiscal surplus tends to be positive (negative) in periods of high (low) production, and tends to be pro-cyclical. For instance, in the case that there are only two states $s_1 < s_2$, so $\xi(s_1) < \xi(s_2)$, government liabilities take a strictly smaller value when the state is s_1 than when it is s_2. Then, from the previously derived expression

$$\delta_t^f(s^t) = \sum_{s_{t+1} \in S} q_t(s^{t+1}) \left(b_t(s^t) - b_{t+1}(s^{t+1}) \right)$$

it follows that δ^f is strictly negative (positive) when the state is s_1 (s_2). In this case, then, the fiscal surplus is strictly increasing in technology and pro-cyclical.

Most of the previous conclusions hold in the general case of constant elasticity of substitution ϵ. First, using the necessary conditions, one guesses and verifies that the private consumption c and public consumption g are strictly positive constant fractions of production. Then, one shows that $u_{cc}c$ and u_c are proportional, and that $u_{gc}g$ and u_g are proportional. Also, u_c is proportional to a power of production ξn. The arguments to show the other conclusions parallel the case of unitary elasticity. Production is strictly increasing in technology. Taxes and the primary surplus are strictly positive constant fractions of production. Government liabilities are pro-cyclical, and the fiscal surplus tends to be pro-cyclical.

4.4 Numerical example

The following numerical example document the previous conclusions. It has been solved using the numerical strategy described above. The utility function is

$$u(c, n, g) \equiv A[\alpha \log(c) + (1 - \alpha) \log(g)] - \Phi \frac{n^{1+\varphi}}{1+\varphi}$$

with $A = 1$, $\alpha = 0.75$, $\Phi = 1$ and $\varphi = 1$. Total available hours are $\overline{n} = 4$. The preferences discount factor is $\beta = 0.99$. The state space is $S = \{s_1, s_2\}$, and the transition matrix is $[0.95, 0.05; 0.05, 0.95]$. The technology shock is $\xi(s_1) = 0.9$ and $\xi(s_2) = 1.1$. The parameter values are chosen so that, approximately, in the steady state, the ratio of public consumption to private consumption is $1/3$, the ratio of labor to total available hours is $1/4$, the labor supply elasticity is 1, the real interest rate is 4%, and the first-order serial correlation and standard deviation of the technology shock are respectively 0.9 and 10%. We chose an unrealistically high value for the standard deviation in order to plot clearer figures. The utility is separable in its three arguments, and the elasticity of substitution between private and public consumption is constant and equal to one.

Let us first focus on the solution of the continuation problem. Figures 1 and 2 plot several functions of λ for each level of s. Recall that λ is constant along the equilibrium path. Then, all the variables of interest are perfectly correlated with each other, and, for each fixed λ, the sign of the correlation can be easily inferred from the figures. Notice that we also consider small values of λ corresponding to negative values of initial government liabilities $b_0(s_0)$.

For this economy, $(1 + \lambda + \lambda\varphi)\Phi n^{\varphi+1} = 1$, so labor n does not depend on s. Aggregate production ξn is the product of technology and labor. Private and public consumption are respectively constant fractions α and $1 - \alpha$ of aggregate production. The multiplier μ is equal to the inverse of aggregate production.

The tax rate τ is constant and equal to $(\lambda + \lambda\varphi)/(1 + \lambda + \lambda\varphi)$. The ratio of the primary surplus δ^p to aggregate production is then constant and equal to $\tau - (1 - \alpha)$. Notice that there is a threshold value λ^* (equal to 1/6 in this numerical example) such that $\delta^p/\xi n$ is a strictly negative constant for $\lambda < \lambda^*$, and a strictly positive constant for $\lambda > \lambda^*$. Then, for $\lambda < \lambda^*$, the primary surplus is negative in all periods and counter-cyclical, while for $\lambda > \lambda^*$ it is positive in all periods and pro-cyclical. The same applies to debt b. The fiscal surplus δ^f is counter-cyclical for $\lambda < \lambda^*$, and pro-cyclical for $\lambda > \lambda^*$, and takes both positive and negative values along all equilibrium paths.

One can show that, since utility is logarithmic, the value of debt $\theta(\lambda, s)$ does not depend on s, and is strictly increasing for all $\lambda > 0$, so it does not have the properties of a Laffer curve. By considering a larger domain for λ, the last two panels show that both the primary surplus δ^p and the debt b have the properties of a Laffer curve. By focusing on values of λ lower than the value that maximizes the primary surplus (equal to 1.5 in this numerical example) we restrict attention to the good part of the Laffer curve.

Let us now turn to the first-period problem, and compare its optimal policies with the ones of the continuation problem. Figures 3 and 4 plot the relevant variables as functions of debt d for each level of s, for the continuation problem. Figures 5 and 6 plot the relevant variables as functions of debt d for each level of s, for the first-period problem.

There are important qualitative differences. In the continuation problem, the higher government liabilities, the higher taxes and the primary surplus, in accordance with the general principle that the tax distortion should be smoothed across periods and states. Then, the higher the tax rate, the lower labor, production and consumption.

The first-period problem, however, is a one-period problem taking current government liabilities as given. In the case that government liabilities are strictly positive, since current government liabilities must be financed by (current and) future surpluses, the higher real government liabilities b, the stronger the governments incentive to decrease the intertemporal price of current consumption goods in terms of future consumption goods. Hence, the higher government liabilities, the stronger the government's incentive to decrease the tax rate, to increase labor, production and consumption, and to decrease the marginal utility of current consumption. In the numerical example, the incentive outweighs the general principle that the tax distortion should be smoothed across periods and states.

This, however, need not be true in general. The lower the preferences discount factor β, the more important becomes the general principle of tax smoothing relative to the government's incentive to decrease the intertemporal price of current consumption goods in terms of future consumption goods. For instance, figures 7 and 8 plot the relevant variables as functions of debt d for each level of s, for the first-period problem, for the case that $\beta = 0.5$. In this case, the policies of the first-period problem are qualitative similar to those of the continuation problem. Specifically, the higher government liabilities, the higher the tax rate.

5 Markov perfect equilibria

We now turn to the study of the optimal policy without commitment. In each period t, first the state variable s_t, the technology shock $\xi(s_t)$ and the government real liabilities $b_t(s^t)$ are realized, then the government chooses the tax rate τ_t and public consumption g_t, and finally the households choose private consumption c_t and labor n_t. In this section, we characterize Markov perfect equilibria, studied by Klein and Rios-Rull (2003) and Klein, Krusell and Rios-Rull (2004), and show that their values are lower than the Ramsey equilibrium. In the next section, we determine whether the value of the Ramsey equilibrium can be sustained by the threat to revert to a Markov perfect equilibrium.

In the words of Kydland and Prescott (1977), the policy associated with a Markov perfect equilibrium is the optimal policy given the past choices of households and the government, and given that the future policy is chosen in the same way. In the next section, we will focus on sustainable equilibria. Markov perfect equilibria are sustainable equilibria which depend recursively on the natural state variables only, namely s_t and $b_t(s^t)$. Hence, they are self-sustainable in the sense that they can be sustained without any threat to revert to other sustainable equilibria. They are also the limit as the time horizon goes to infinity of the sustainable equilibria of the finite-horizon economy.

A Markov perfect equilibrium is a set of a value function $v(b,s)$, associated policy functions $c(b,s)$, $n(b,s)$, $g(b,s)$, $b'(b,s,s')$, and a *future* policy function $\theta(b,s)$ to be adopted in all future periods, which satisfy

- Given the future policy function $\theta(b,s)$, the value function $v(b,s)$ and its associated policy functions solve the Bellman equation:

$$\begin{aligned}
\text{For all } b,s\text{: } v^M(b,s) &= \max_{\{c\geq 0, n\in[0,\overline{n}], g\geq 0, b'(s')\}} \left\{ u(c,n,g) + \sum_{s'\in S} \beta v^M(b'(s'),s')\pi(s'|s) \right\} \\
\text{subject to: } u_c(c,n,g)b &= u_c(c,n,g)c + u_n(c,n,g)n + \sum_{s'\in S} \beta\theta(b'(s'),s')\pi(s'|s) \\
c + g &\leq \xi(s)n
\end{aligned}$$

- The future and the current policy functions are the same:

$$\text{For all } b,s\text{: } \theta(b,s) = u_c(c(b,s),n(b,s),g(b,s))b$$

In the Ramsey equilibrium, the government chooses optimally a unique set of policy functions to be adopted both in the current period and in all future periods. In particular, the government chooses the future policy functions taking into full account how the current households' choices depend on the future policy functions. In a Markov perfect equilibrium, however, the government chooses optimally a set of policy functions to be adopted only in the current period, for a given set of policy functions to be adopted in all future periods. Only after the current period optimization problem has been solved, the requirement that current and future policy functions be the same is added. It might help the intuition considering the government in the current period as a different agent from the government in all future periods, so it cannot choose the policy functions to be adopted in all future periods. The future policy functions will be chosen by the future government taking as given the households' choices in the current period, and ignoring their dependence on the future policy functions. The value of a Markov perfect equilibrium is, then, lower, in general, than the value of the Ramsey equilibrium.

5.1 Results

The Lagrangian is

$$\begin{aligned}\mathcal{L} &= u(c,n,g) + \sum_{s'\in S} \beta v^M(b'(s'),s')\pi(s'|s) \\ &+ \lambda\left[u_c(c,n,g)c + u_n(c,n,g)n + \sum_{s'\in S}\beta\theta(b'(s'),s')\pi(s'|s) - u_c(c,n,g)b\right] \\ &+ \mu[\xi(s)n - c - g]\end{aligned}$$

where λ is the Lagrange multiplier associated with the implementability constraint, and $\mu \geq 0$ is the Kuhn-Tucker multiplier associated with the feasibility constraints.

Assuming that the solution satisfies $c > 0$, $n \in (0, \overline{n})$ and $g > 0$, and that it satisfies the feasibility constraints with equality, the necessary conditions are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial c} &= (1+\lambda)u_c(c,n,g) + \lambda u_{cc}(c,n,g)(c-b) + \lambda u_{nc}(c,n,g)n - \mu = 0 \\ \frac{\partial\mathcal{L}}{\partial n} &= (1+\lambda)u_n(c,n,g) + \lambda u_{cn}(c,n,g)(c-b) + \lambda u_{nn}(c,n,g)n + \mu\xi(s) = 0 \\ \frac{\partial\mathcal{L}}{\partial g} &= u_g(c,n,g) + \lambda u_{cg}(c,n,g)(c-b) + \lambda u_{ng}(c,n,g)n - \mu = 0 \\ \frac{\partial\mathcal{L}}{\partial b'(s')} &= \beta v_b^M(b'(s'),s')\pi(s'|s) + \beta\lambda\theta_b(b'(s'),s')\pi(s'|s) = 0 \text{ all } s' \\ \frac{\partial\mathcal{L}}{\partial\lambda} &= u_c(c,n,g)c + u_n(c,n,g)n + \sum_{s'\in S}\beta\theta(b'(s'),s')\pi(s'|s) - u_c(c,n,g)b = 0 \\ \frac{\partial\mathcal{L}}{\partial\mu} &= \xi(s)n - c - g = 0 \\ v_b^M(b,s) = \frac{\partial\mathcal{L}}{\partial b} &= -\lambda u_c(c,n,g)\end{aligned}$$

Also, from the Markov perfect equilibrium definition,

$$\begin{aligned}\text{For all } b,s\text{: } \theta(b,s) &= u_c(c(b,s),n(b,s),g(b,s))b \\ \theta_b(b,s) &= u_c(c(b,s),n(b,s),g(b,s)) + \frac{du_c(c(b,s),n(b,s),g(b,s))}{db}b\end{aligned}$$

Relative to the Ramsey equilibrium, it is harder to characterize how variables co-vary with the technology shock. First, λ is not constant any more but depends on b and s. Also, government liabilities b appear in the necessary conditions. Next-period government liabilities $b'(b,s,s')$ depend on the next-period state s' as well, i.e. government liabilities are state-contingent and vary with the technology shock. Hence, as s and $\xi(s)$ vary, b varies as well, and the full effect of s and $\xi(s)$ on any variable is the sum of its direct effect and its indirect effect through b.

The following observations, however, may be helpful. On one hand, the higher real government liabilities b, the higher the tax rate, the lower labor, production and consumption, the higher the marginal utility of consumption u_c. On the other hand, since current government liabilities b must be financed by (current and) future surpluses, the higher real government liabilities b, the stronger the governments incentive to increase labor and consumption and decrease the marginal utility of consumption u_c, in order to decrease the intertemporal price of current consumption goods. This incentive is stronger when government liabilities are higher, and disappears when they are equal to zero. Hence, although the sign of the term $du_c = db$ in the last equation is uncertain, it is positive for values of b sufficiently close to zero.

Then, at least for small values of b, from the previous conditions,

$$\begin{aligned}
\frac{-v_b^M(b,s)}{u_c(c,n,g)} = \lambda(b,s) &= \frac{-v_b^M(b'(s'),s')}{\theta_b(b'(s'),s')} \text{ all } s' \\
&= \frac{-v_b^M(b'(s),s)}{\theta_b(b'(s),s)} \\
&= \frac{-v_b^M(b'(s),s)}{u_c(b'(s),s)} \frac{u_c(b'(s),s)}{\theta_b(b'(s),s)} < \frac{-v_b^M(b'(s),s)}{u_c(b'(s),s)} = \lambda(b'(s),s)
\end{aligned}$$

which shows that $\lambda(b,s)$, a measure of tax distortion, increases over time for any fixed s. Since $\lambda(b,s)$ is strictly increasing in b on the good part of the Laffer curve, government liabilities increase over time.

Recall that government liabilities b are stationary in the Ramsey equilibrium. Relative to the Ramsey policy, Markov perfect policies take b as given and decrease u_c in order to decrease the intertemporal value of b. This is implemented through a lower tax rate leading to higher labor, production and consumption and a lower marginal utility of consumption u_c relative to the Ramsey policy. As a result of the lower tax rate, future government liabilities b are higher relative to the Ramsey policy, and therefore government liabilities increase over time. Notice that the difference between the Ramsey policy and Markov perfect policies is that the Ramsey policy commits to a fixed current u_c, while Markov perfect policies manipulate it. In fact, if u_c were held fixed in the expression $\theta = u_c b$, then $\theta_b = u_c$, so $\lambda(b,s) = \lambda(b'(s),s)$, and the Markov perfect policy would be the same as the Ramsey policy.

A Markov perfect equilibrium can be computed with the following two methods. Given an initial function $\theta(b,s)$, one can determine the other functions by the standard value function iterations method. Then, one can use the just obtained functions to update the function $\theta(b,s)$, and iterate until convergence. Alternatively, given initial functions $\theta(b,s)$ and $v^M(b,s)$, obtain the other functions solving the simple static optimization problem in the definition of Markov perfect equilibrium. Then, using the just obtained functions, update both functions $\theta(b,s)$ and $v^M(b,s)$ simultaneously, and iterate until convergence. This second method corresponds to computing the Markov perfect equilibrium of the infinite horizon economy as the limit as the time horizon goes to infinity of the Markov perfect equilibrium of the finite horizon economy.

We have used the second method to compute a Markov perfect equilibrium for the same numerical example considered at the end of section 4. However, we have restricted the choice of consumption c to be below 0.75, which is the maximum value that it takes in the continuation of the Ramsey equilibrium. This amounts to constraining the government not to depart too far from the optimal competitive equilibrium. In particular, the government never chooses a negative tax rate. We conjecture that the constraint on consumption gives discipline to the government, and allows it to reach a strictly higher value $v^M(b,s)$. Without the constraint, indeed, we find that $v^M(b,s) = -\infty$ for all positive values of b. However, we do not emphasize this finding because we are not confident enough that the numerical method without the constraint delivers correct results. Figures 9 and 10 plot the relevant variables as functions of debt d for each level of s. Contrary to our previous argument, in this numerical example, consumption and the marginal utility of consumption do not change with b.

6 Ramsey as sustainable equilibrium

In this section, we show that, for a realistic value of the preferences discount factor β, the value of the Ramsey equilibrium can be sustained by a sustainable equilibrium, as defined by Chary and Kehoe (1990).

Histories are re-defined as including the government's choices of tax rates and public consumption. A sustainable equilibrium is a sequence of functions of the histories which is a competitive equilibrium and satisfies the requirement that the government is optimizing at each period and history. Chang (1998) and Phelan and Stacchetti (2001) show that, after adding the continuation value of sustainable equilibria as a state variable, the set of values of all sustainable equilibria can be characterized recursively. Also, the value of any sustainable equilibrium can be obtained with a recursive sustainable equilibrium.

Here, we adapt their arguments to our model, modifying them. A recursive sustainable equilibrium is a value correspondence $V(b,s)$, policy functions $c(v,b,s)$, $n(v,b,s)$, $g(v,b,s)$, $b'(v,b,s,s')$, a law of motion $v'(v,b,s,c,n,g,b'(s'),s') : (b',s') \to V(b',s')$ for the continuation value, and a *future* policy function $\theta(v,b,s)$, satisfying:

- Given $v'(v,b,s,c,n,g,b'(s'),s')$ and $\theta(v,b,s)$, the value correspondence $V(b,s)$ and the policy functions solve the Bellman equation:

  For all b, s, all $v \in V(b,s)$:

  $$v = \max_{\{c\geq 0, n\in[0,\overline{n}], g\geq 0, b'(s')\}} \left\{ u(c,n,g) + \sum_{s'\in S} \beta v'(c,n,g,b'(s'),s')\pi(s'|s) \right\}$$

  subject to:

  $$u_c(c,n,g)b = u_c(c,n,g)c + u_n(c,n,g)n + \sum_{s'\in S} \beta\theta(v'(c,n,g,b'(s'),s'),b'(s'),s')\pi(s'|s)$$

  $$c + g \leq \xi(s)n$$

- The future and the current policy functions are the same:

  For all b, s, for all $v \in V(b,s)$: $\theta(v,b,s) = u_c(c(v,b,s), n(v,b,s), g(v,b,s))b$

First, notice that, if the value correspondence $V(b,s)$ is a value function, so the law of motion for the continuation value can depend only on $(b'(s'), s')$, then the recursive sustainable equilibrium is a self-sustainable equilibrium.

In the general case that $V(b,s)$ is not a function, the law of motion for the continuation value plays a crucial role. The future continuation value $v'(c,n,g,b'(s'),s')$ as a function of the last two arguments is constrained to be in the value correspondence $V(b'(s'),s')$ but can vary with the current government choices $c, n, g, b'(s')$. The current government takes as given the law of motion for the continuation value, and correctly believes that the future continuation value depends on its current choices. This may give an extra-incentive to the current government to make a choice closer to the Ramsey choice.

To determine whether the value $v^R(b,s)$ of the Ramsey equilibrium can be sustained by a sustainable equilibrium, we first determine whether the continuation value $v^C(b,s)$ of the Ramsey equilibrium can be sustained by the following recursive sustainable equilibrium. Let $V(b,s) = \{v^M(b,s), v^C(b,s)\}$, so the value correspondence is made of only two value functions, the one of a Markov perfect equilibrium and the one of the continuation value of the Ramsey equilibrium. If $v = v^M(b,s)$, the law of motion for the continuation value is equal to $v^M(b'(s'),s')$ no matter what is the current government's choice. If $v = v^C(b,s)$, however, the law of motion for the continuation value is equal to $v^C(b'(s'),s')$ if the current government's choice is the Ramsey choice, and it is equal to $v^M(b'(s'),s')$ otherwise. Let the policy functions be the ones associated with the continuation of the Ramsey equilibrium and the Markov perfect equilibrium. The previous is a recursive sustainable equilibrium in the case that $v^C(b,s) \geq v^M(b,s)$ for all (b,s).

In the case that $v^C(b,s) \geq v^M(b,s)$ for all (b,s), we only need to establish that, in the first period, $v^R(b,s) \geq v^M(b,s)$ for all (b,s), which is immediately true since the Ramsey equilibrium is the optimal competitive equilibrium. It follows that the value of the Ramsey equilibrium is higher than the value of the Markov perfect equilibrium at each period and history, and the value $v^R(b,s)$ of the Ramsey equilibrium can be sustained by the threat to revert to a Markov perfect equilibrium.

Figure 11 plots the value functions $v^R(b,s)$, $v^C(b,s)$ and $v^M(b,s)$ for the numerical example described at the end of section 4. All the value functions are decreasing in debt b and increasing in the state s. Notice that $v^R(b,s) > v^C(b,s) > v^M(b,s)$ all (b,s). Since the condition $v^C(b,s) \geq v^M(b,s)$ holds for all (b,s), the continuation of the Ramsey equilibrium can be sustained as a recursive sustainable equilibrium. Then, the Ramsey equilibrium can be sustained as well.

We emphasize that the condition $v^C(b,s) \geq v^M(b,s)$ for all (b,s) needs not be true in general, and it ceases to hold when the preferences discount factor β decreases below a threshold value. For the numerical example considered, the threshold value is below $\beta = 0.8$.

Also, notice that the value of the Ramsey equilibrium cannot be sustained by a recursive sustainable equilibrium as defined in this paper. To sustain it as a recursive sustainable equilibrium as in Chang (1998) and Phelan and Stacchetti (2001), one should include in the state variables the promised value of government liabilities, or another variable that equivalently keeps track of the competitive equilibrium conditions.

7 Conclusion

Our main result under commitment is that, in a benchmark case, taxes, government expenditures and the primary surplus should all be constant positive fractions of production, and both government liabilities and the fiscal surplus should be pro-cyclical. In addition, we have shown that, for a realistic value of the preferences discount factor, there is a sustainable equilibrium with the same outcome and value as the Ramsey equilibrium.

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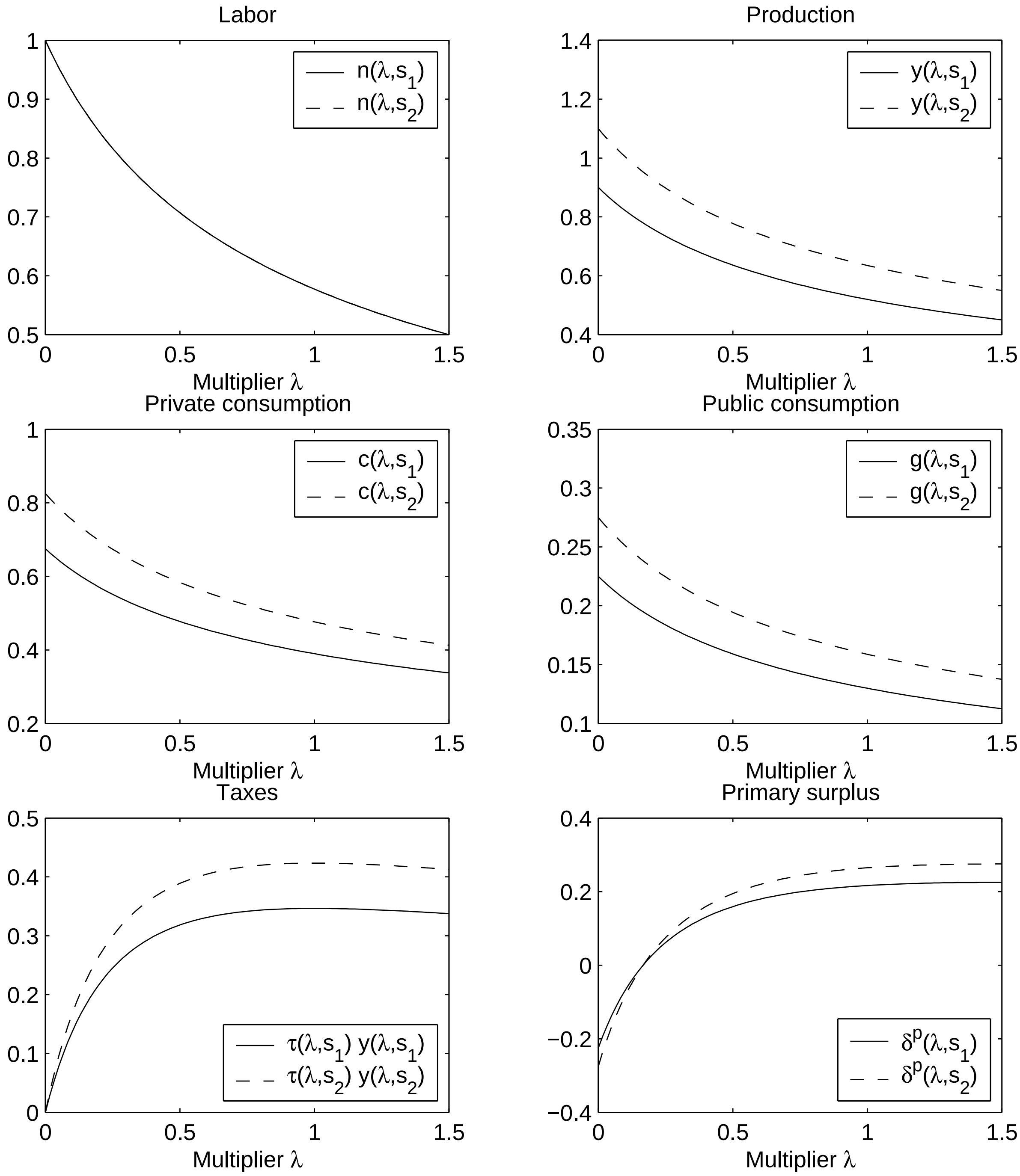

Figure 1: Continuation of the Ramsey equilibrium. The functional form of the utility function implies that labor does not depend on s. Production is the product of the technology shock with labor. All the other variables are constant fractions of production. The primary surplus δ^p is negative and counter-cyclical for small values of λ, positive and pro-cyclical for large values of λ.

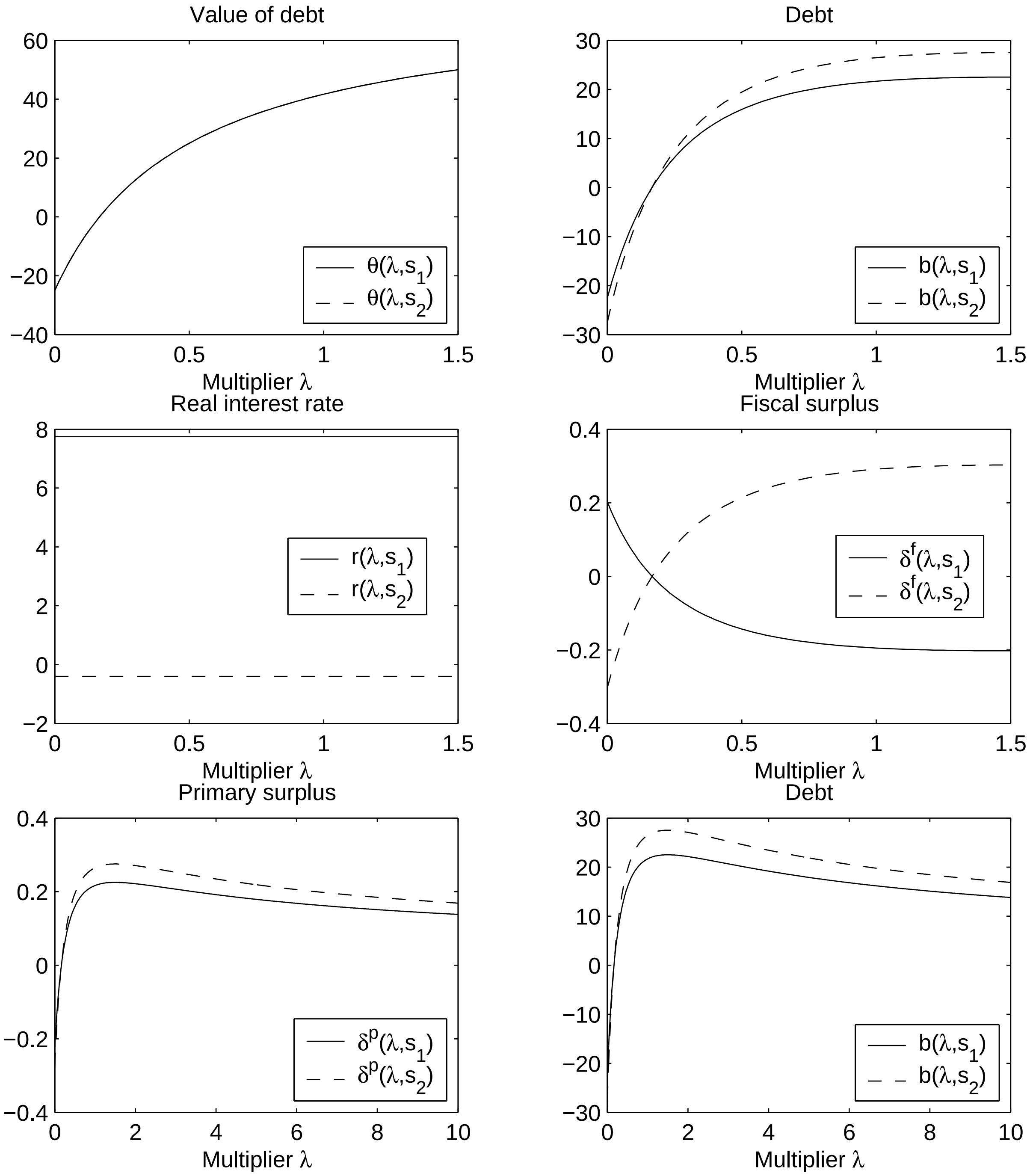

Figure 2: Continuation of the Ramsey equilibrium. The functional form of the utility function implies that the value of debt θ does not depend on s. The debt b is negative and counter-cyclical for small values of λ, positive and pro-cyclical for large values of λ. The fiscal surplus δ^f is counter-cyclical for small values of λ, and pro-cyclical for large values of λ. Both the primary surplus δ^p and the debt b have the properties of a Laffer curve.

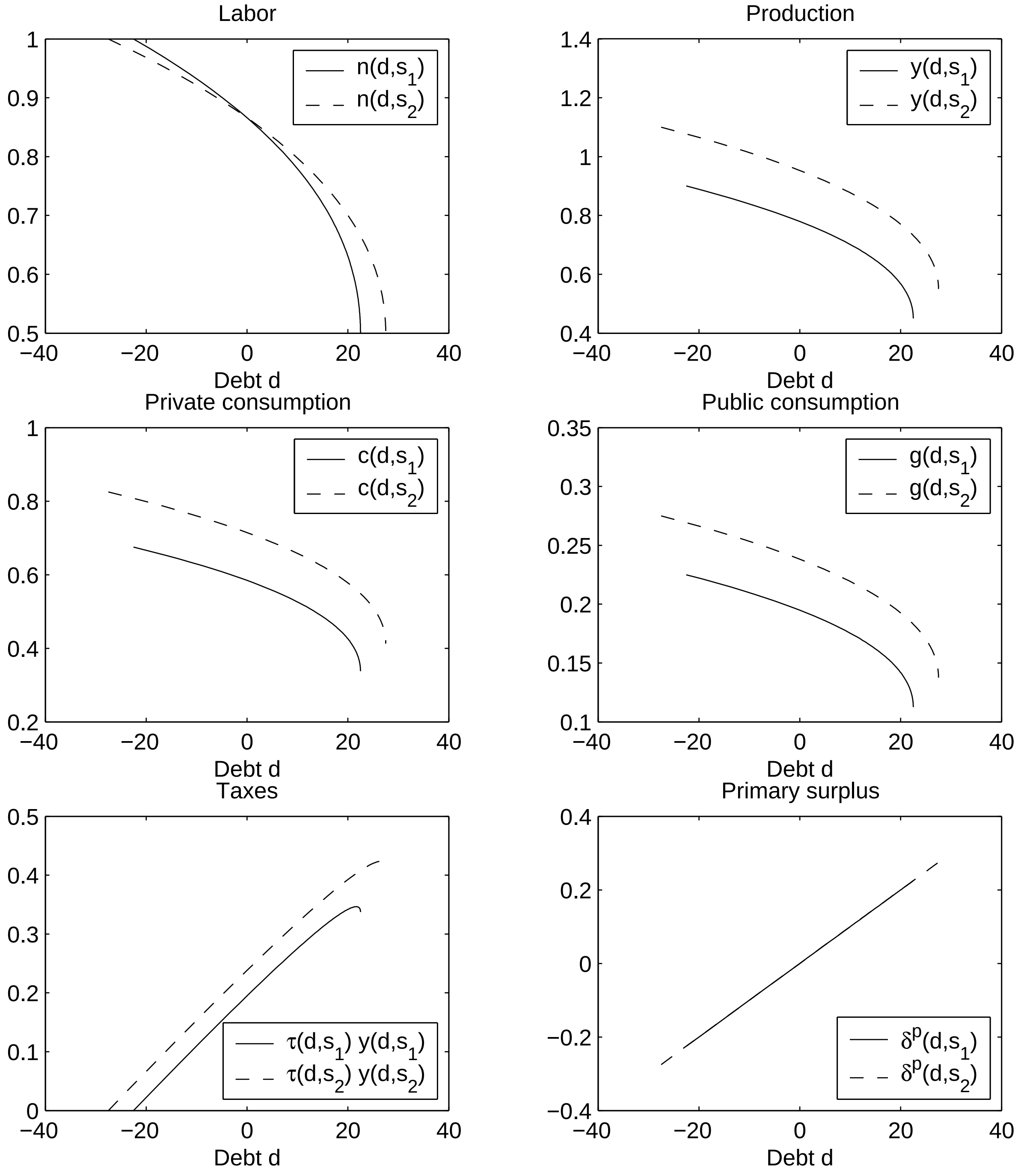

Figure 3: Continuation of the Ramsey equilibrium.

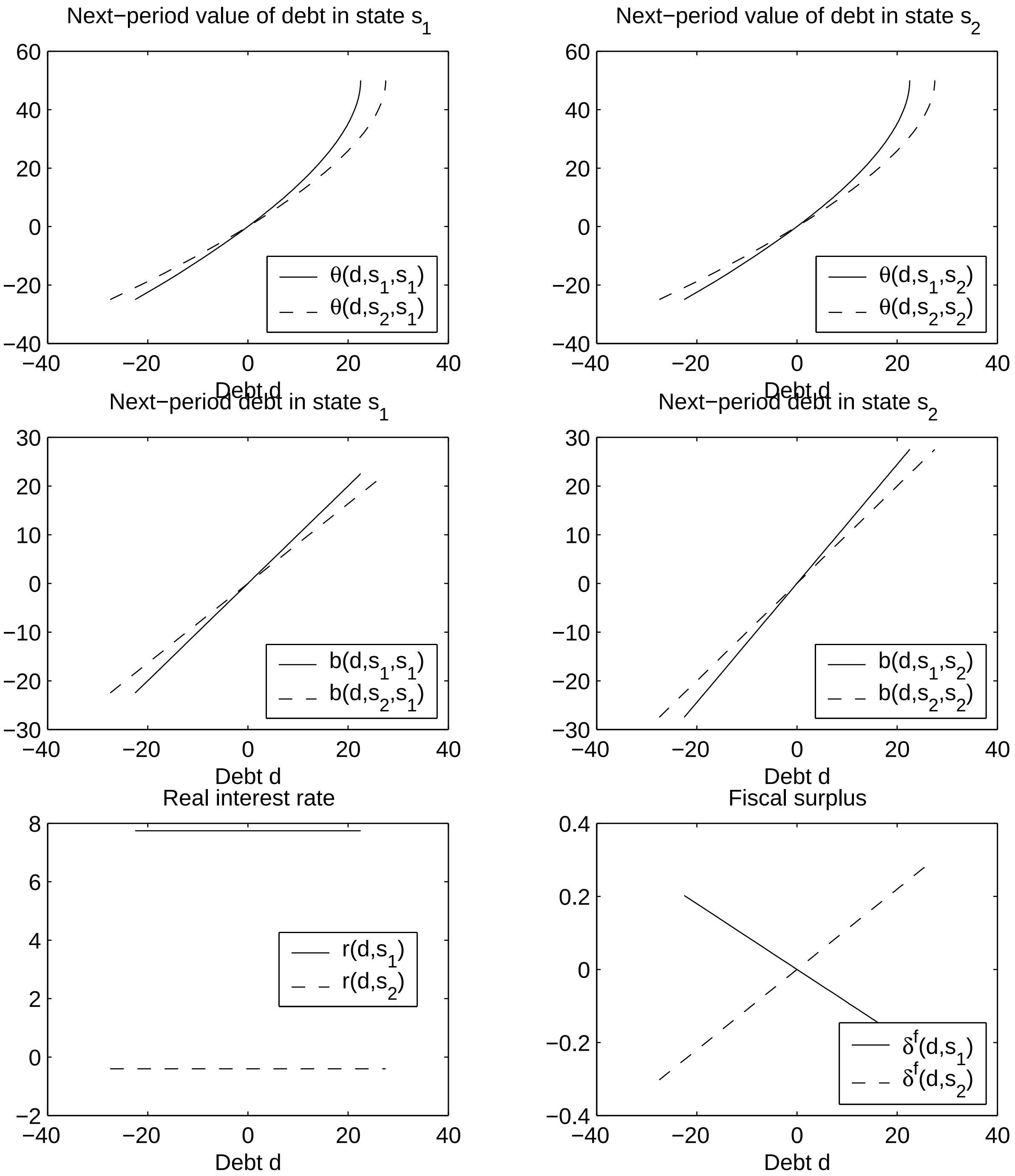

Figure 4: Continuation of the Ramsey equilibrium.

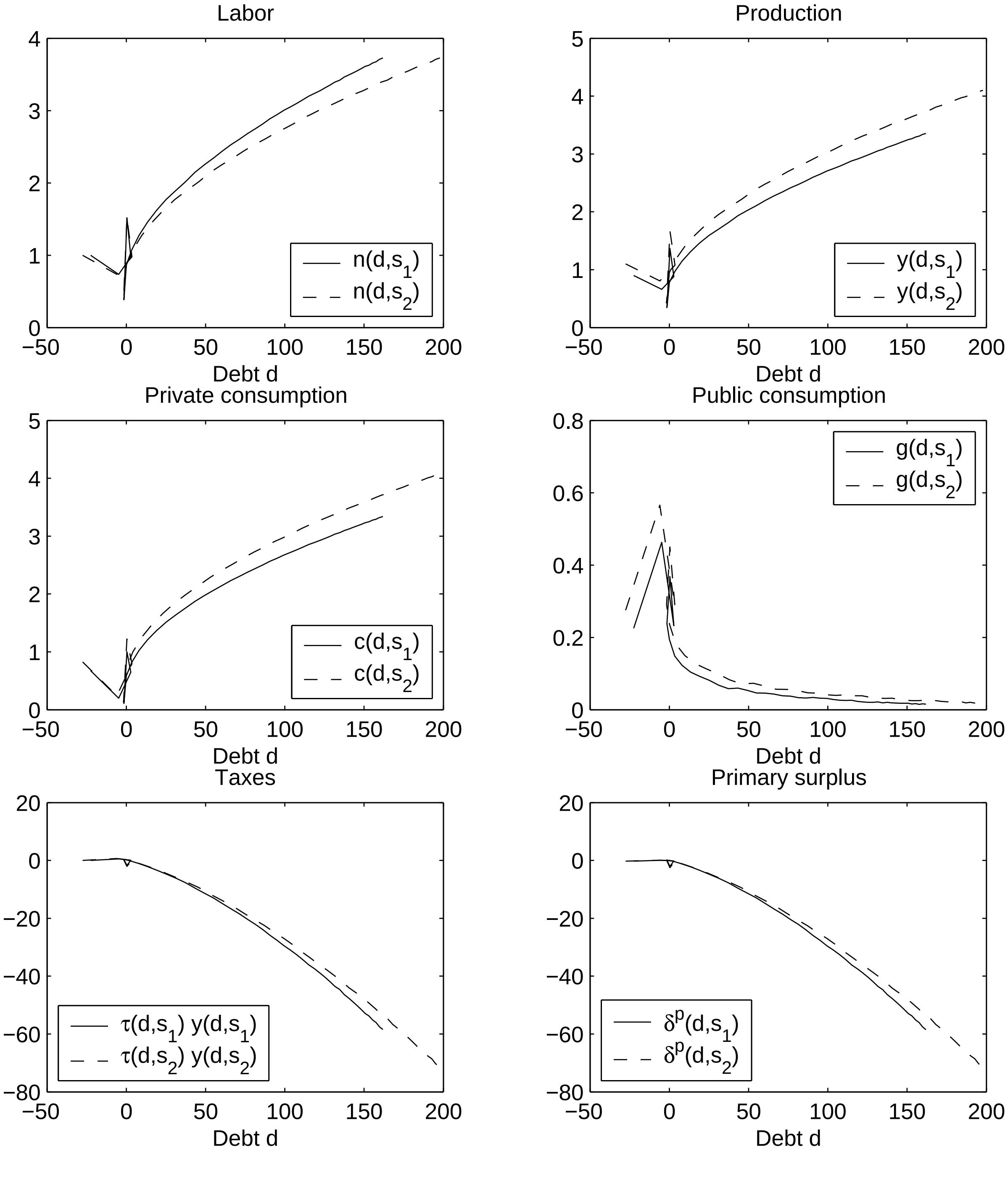

Figure 5: Ramsey equilibrium.

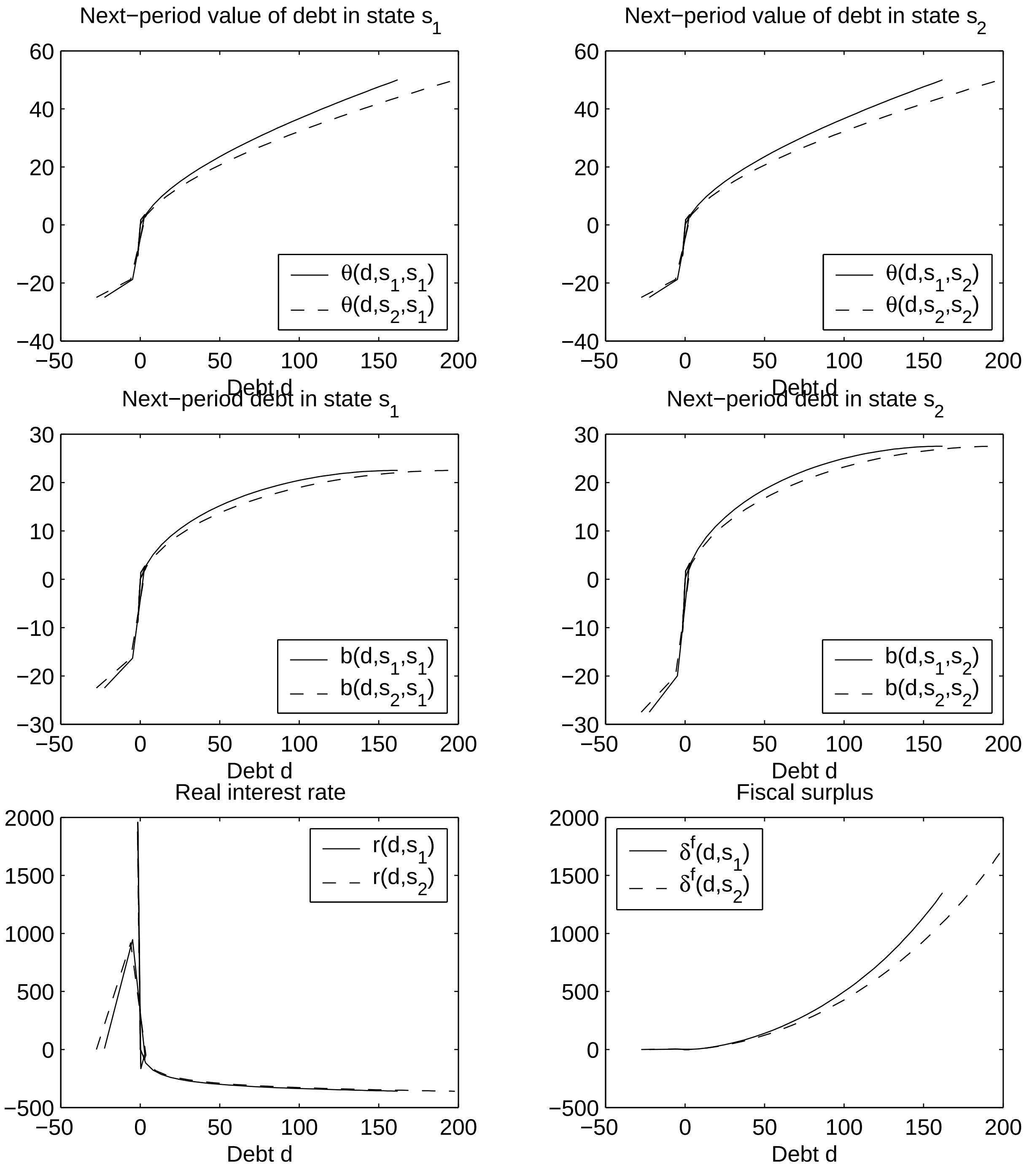

Figure 6: Ramsey equilibrium.

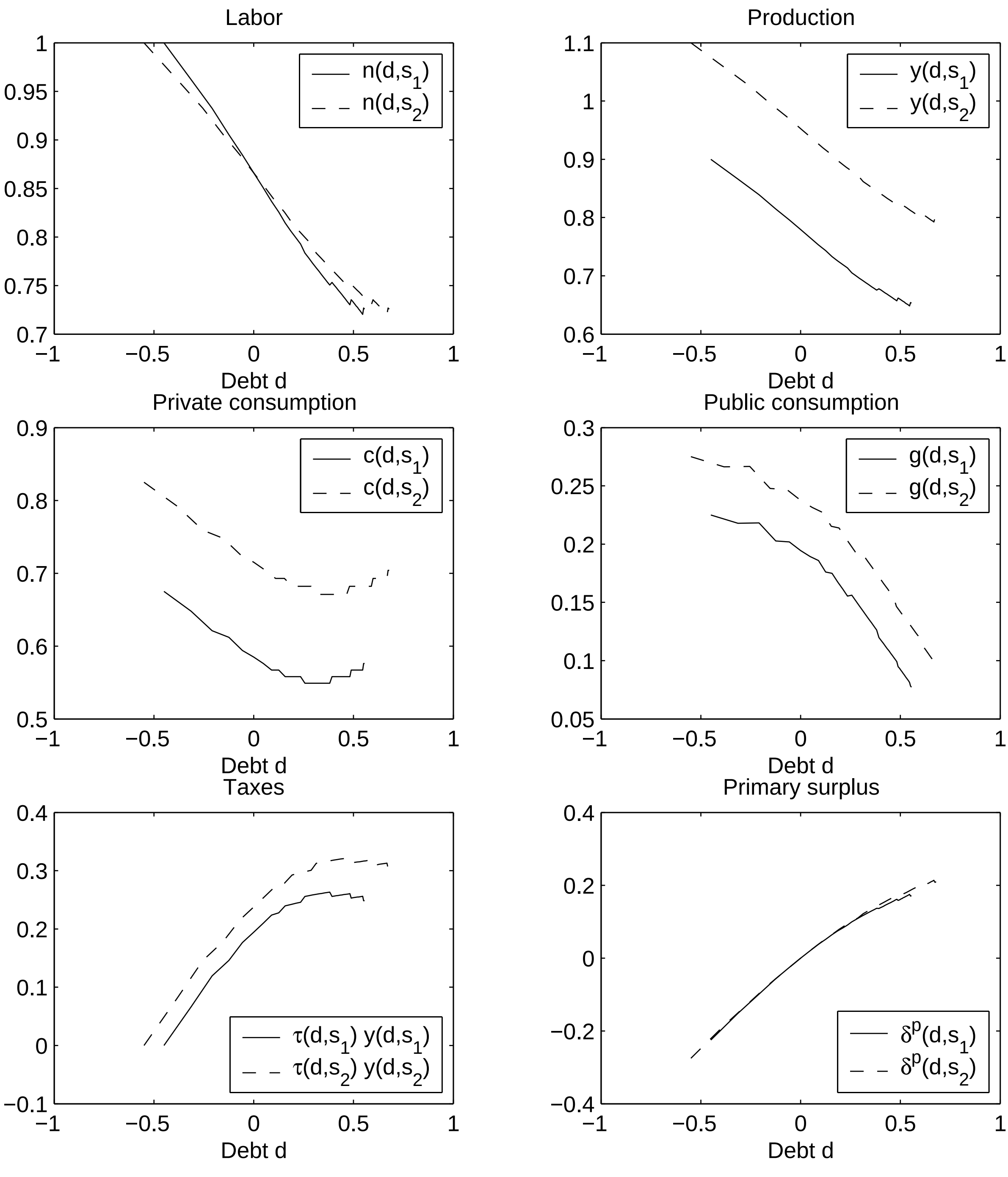

Figure 7: Ramsey equilibrium. $\beta = 0.5$.

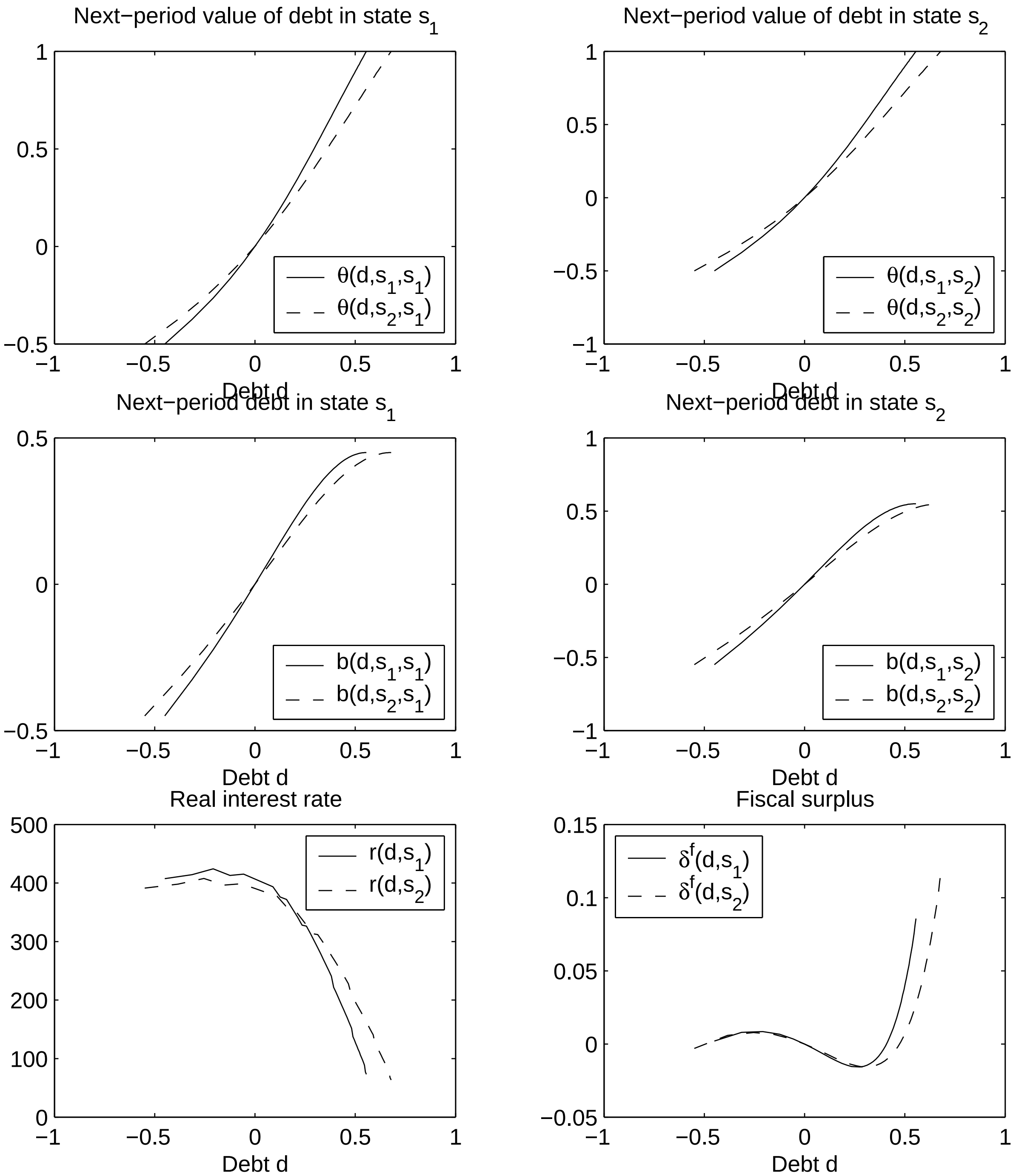

Figure 8: Ramsey equilibrium. $\beta = 0.5$.

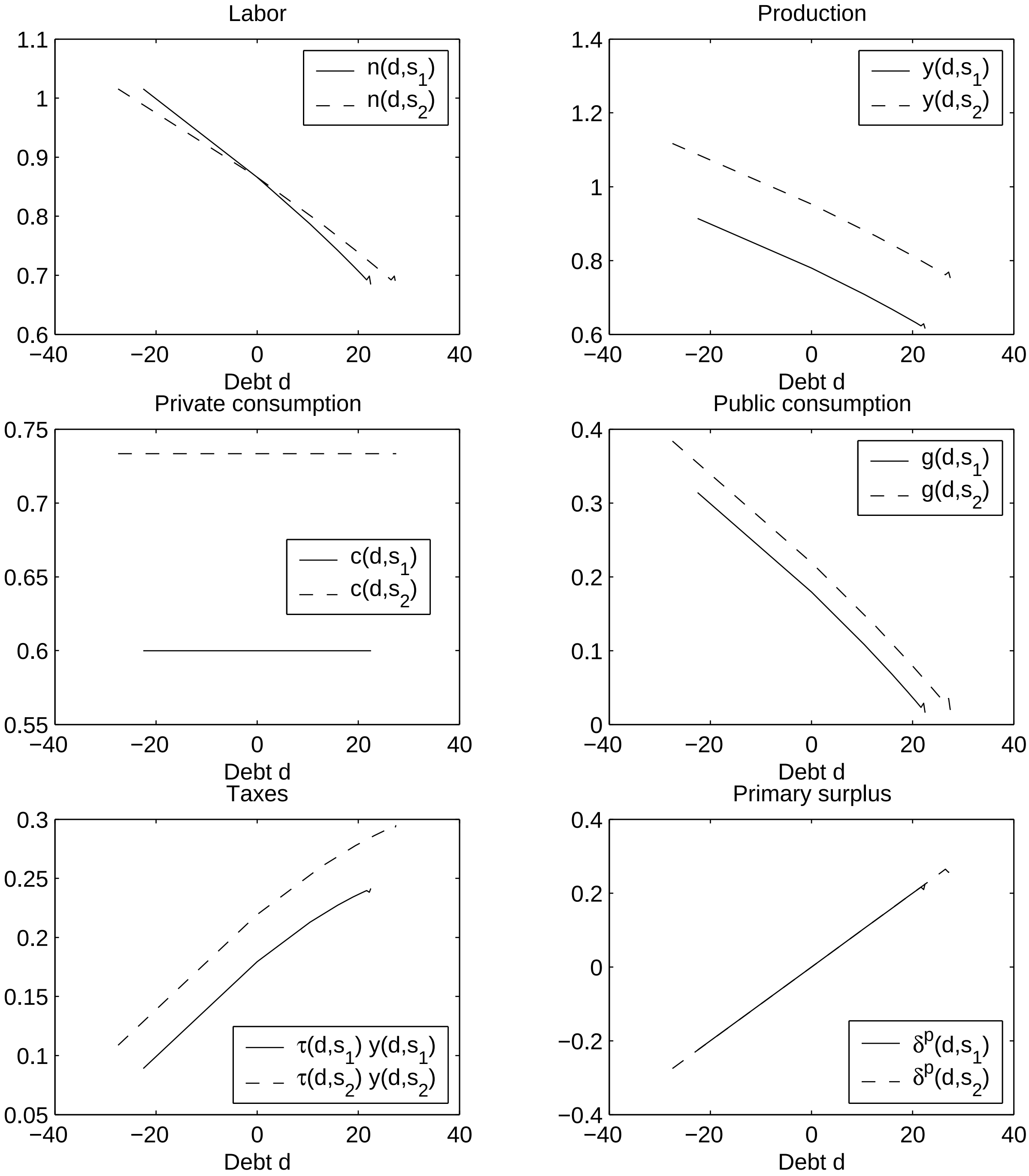

Figure 9: Markov perfect equilibrium.

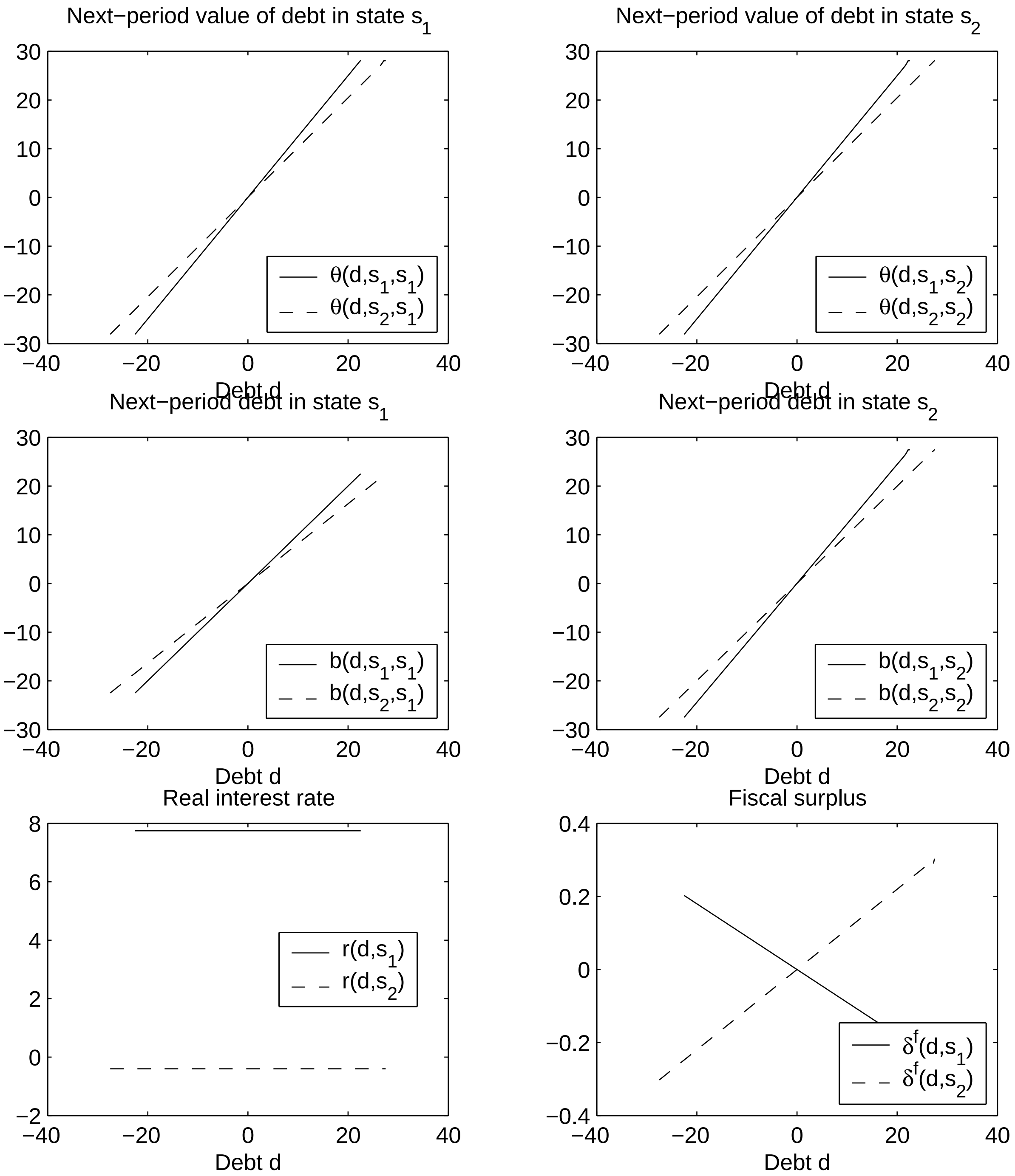

Figure 10: Markov perfect equilibrium.

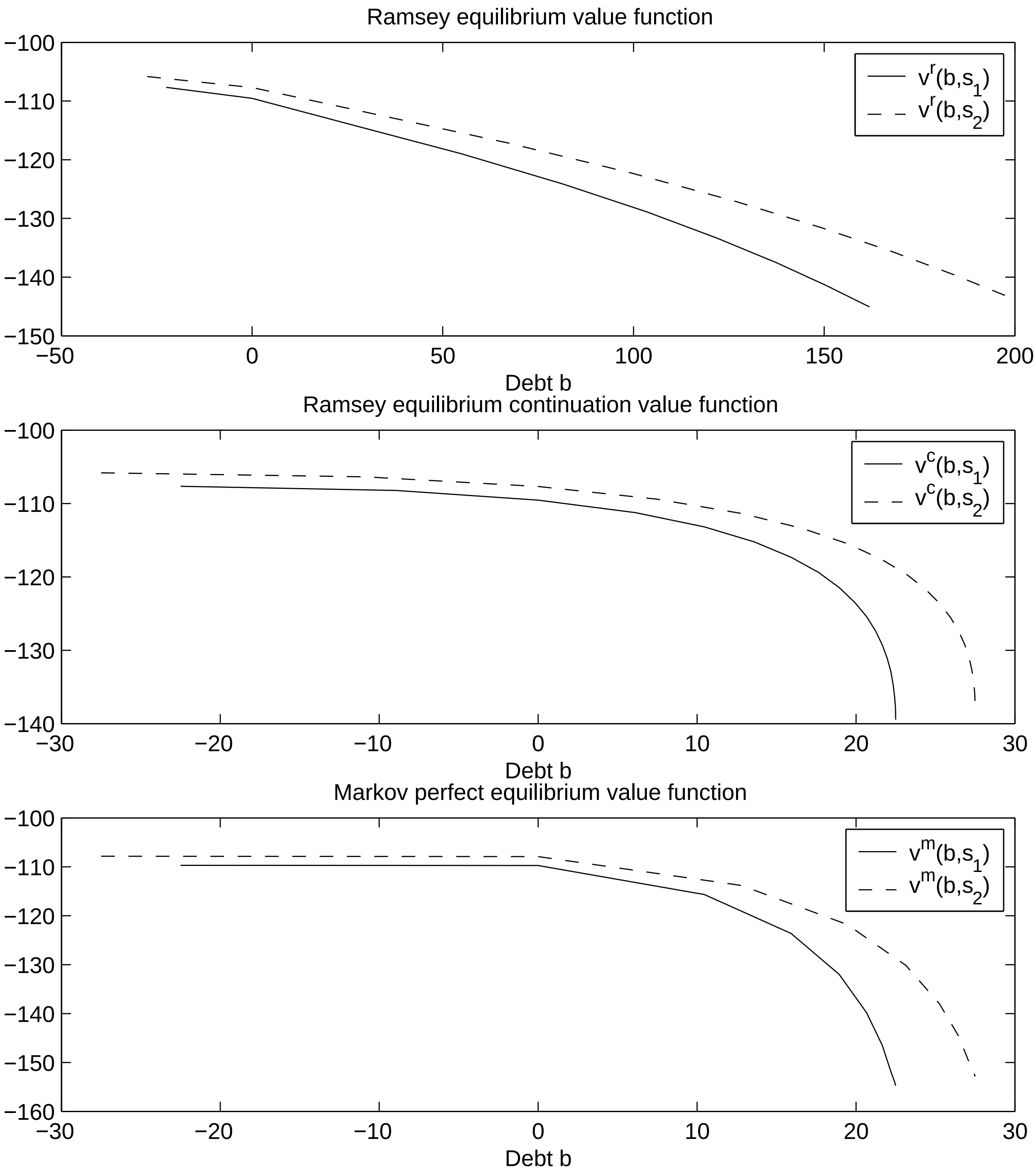

Figure 11: Value functions of b and s. Notice that $v^R(b,s) > v^C(b,s) > v^M(b,s)$ all (b,s).