Income Risk and Household Debt with Endogenous Collateral Constraints*

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Abstract

We present a heterogeneous-agent model with incomplete markets, in which household debt needs to be collateralized by durable holdings and the lowest attainable labor income flow. Labor income is risky and households decide how much non-durables to consume, on their position of secured debt and the durable stock. Consumers value durables not only as collateral for their debt but also derive utility from their durable stock. We show that an interest spread between the borrowing and lending rate implies local convexities in the policy functions for non-durable consumption and especially durable holdings which are important quantitatively. Moreover, an increase in income risk *decreases* average household debt because of the buffer-stock saving motive.

Keywords: household debt, durables, collateral constraint, income risk, incomplete markets, heterogeneous agents.

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1 Introduction

Household debt has increased substantially in developed countries during the last decades. This has been most dramatic in the US where household debt as a proportion of disposable income has been 46 percentage points higher in 2003 than in 1981; and consumer debt amounted to 67% of households' disposable income in 1981 (see, for example, Iacoviello, 2005). Household debt has increased also in many European countries although starting from lower levels (see ECRI, 2000). Thus, it is important to understand debt accumulation of households and its determinants.

About 75% of household debt are mortgages and other credit that is secured by collateral and cannot be defaulted upon. This motivates why we frame our analysis in a model in which all credit needs to be collateralized. Since the collateral consists of durables like housing or cars which generate utility, we set-up a model in which consumers derive utility from non-durable consumption and durable holdings.

The increase in household debt has been attributed to the contemporaneous increase in uninsurable income risk in the recent macro-literature (see, for example, Iacoviello, 2005). Hence, we assume that markets are incomplete and all debt needs to be secured so that uninsurable labor income risk influences the behavior of consumers in non-trivial ways.[1]

An important difference of our model compared with the previous literature is that we add an interest spread between the lending and borrowing rate in financial markets. This generates the empirically realistic finding that a mass of consumers holds no financial assets at all.[2] Such a spread has been analyzed by Carroll (2001, section 3) in a model without durables. As Carroll (2001) we find that there is only a small effect of the spread on non-durable consumption but the effect on the propensity to purchase durables is sizeable. This is because durables are an alternative vehicle to transfer resources intertemporally, especially if depreciation rates are low.

We calibrate our model to the US and show how the solution depends on the model's parameters in an intuitive way. In particular, we find that an increase in income risk reduces average household debt, also if we condition on those consumers who hold some debt. Thus, our model with buffer-stock savings in incomplete markets has different predictions than models which analyze approximations around the non-stochastic steady state (see, for example, Iacoviello, 2005). We argue that an increase in idiosyncratic income risk alone cannot explain the increase in household debt in the US and other developed countries.

The rest of this paper is structured as follows. In Section 2 we present, solve and calibrate the model. In Section 3 we discuss the model's implications for the relationship between income risk and household debt before we conclude in

[1] See Deaton (1991), Carroll (1997) and the general equilibrium analysis of Aiyagari (1994) for models of non-durable consumption; and Diaz and Luengo-Prado (2005) or Gruber and Martin (2003) for models with durables.

[2] We take these rates as given so that our analysis is partial equilibrium.

Section 4.

2 The model

Agents are risk-averse and have an infinite horizon. They derive utility from a durable good d and a non-durable good c. The instantaneous utility is given by $U(c, d) = u(c) + \phi w(d)$ where $u(.)$ and $w(.)$ are both strictly concave, and ϕ is the weight assigned to utility derived from the durable. We assume that $\lim_{d\to\infty} w'(d) = 0$ and that marginal utility $w'(d)$ is well defined at $d = 0$ so that our model is able to generate agents with no durable stock in at least some states of the world, as is realistic. A possible functional form is $w(d) = (d+\underline{d})^{\tau}$, with $\tau \leq 1$ and $\underline{d}> 0$. The asymmetry in the utility function with respect to non-durable and durable consumption is justified in the sense that durables are less essential than non-durable consumption such as food. Note that we implicitly assume that durables can be transformed into non-durable consumption with a linear technology so that the relative price is unity.

In specifying utility as above we have made a number of simplifying assumptions. We assume d to be a homogenous, divisible good. Moreover, utility is separable over time and at each point in time it is separable between durables and non-durables. Both assumptions are made for tractability given that it is more realistic to assume that durables are a bundle of characteristics and that utility derived from durables depends on non-durable consumption in non-trivial ways. Instead, as in much of the literature, we assume that the service flow derived from durables is proportional to the stock where we have normalized the factor of proportionality to 1 (see Waldman, 2003, for a critical review of these common assumptions).

We assume that markets are incomplete so that agents cannot fully diversify their risk. It is well known that in such an environment, it is necessary to assume that agents are impatient, $\beta < 1/(1 + r^a)$, where r^a is the lending rate which is taken as given in our partial equilibrium model. It follows from the results by Deaton and Laroque (1992) that agents hold a finite amount of financial assets a. Because of positive depreciation δ and $\lim_{d\to\infty} w'(d) = 0$, also the durable stock d is bounded from above. The collateral constraint and $d \geq 0$ then imply a compact state space so that standard dynamic programming techniques can be applied (see Araujo et al., 2002, for existence proofs in a general equilibrium context).

We assume that there are transaction costs in the financial market so that the lending rate r^a is smaller than the borrowing rate r^b: $r^a < r^b$. This assumption implies that some agents will hold no financial assets, $a = 0$. As we will see below this has interesting implications for the consumption propensities and the shape of the policy functions.

Timing. We specify our model in discrete time so that we have to make assumptions about the timing within a period. Figure 1 illustrates the time line. We assume that agents derive utility from the durable good d_t before the

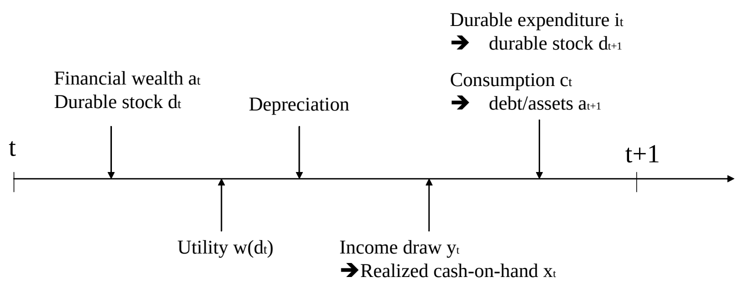

Figure 1: Timing in the model

durable depreciates at rate δ. Then the uncertain income y_t is drawn and the cash-on-hand available to the agent is

$$x_t \equiv (1 + r^j)a_t + y_t + (1 - \delta)d_t,\ j = a, b,$$

where r^b is interest rate on debt and r^a is the interest rate on financial assets a_t, with $r^b > r^a$.

The program. Rearranging the budget constraint,

$$c_t = (1 + r^j)a_t - a_{t+1} + y_t - (d_{t+1} - (1 - \delta)d_t) \ ,$$

we can write the value function as

$$V(x_t, d_t) = \max_{a_{t+1}, d_{t+1}} \left[u(\underbrace{x_t - a_{t+1} - d_{t+1}}_{c_t}) + \phi w(d_t) + \beta E_t V(x_{t+1}, d_{t+1}) \right]$$

We can further simplify the problem by noting that d_t is predetermined in period t and that the additive separable term $\phi w(d_t)$ does not affect the optimal choices of the consumer. Defining

$$\widetilde{V}\left(x_t\right) \equiv V\left(x_t, d_t\right) - \phi w\left(d_t\right)$$

the transformed maximization problem is

$$\widetilde{V}(x_t) = \max_{a_{t+1},d_{t+1}} \left[u(\underbrace{x_t - a_{t+1} - d_{t+1}}_{c_t}) + \beta\phi w(d_{t+1}) + \beta E_t \widetilde{V}(x_{t+1}) \right] \qquad (1)$$

under the constraints

$$\begin{aligned} a_{t+1} &= \begin{Bmatrix} (1+r_t^a)a_t + y_t - c_t - i_t \text{ if } a_t \geq 0 \\ (1+r_t^b)a_t + y_t - c_t - i_t \text{ if } a_t < 0 \end{Bmatrix} \\ d_{t+1} &= (1-\delta)d_t + i_t \\ \underbrace{\left(1+r_s^b\right)a_s + (1-\delta)d_s + \underline{y}}_{\underline{x_s} \equiv x_s(\underline{y})} &\geq 0,\ s > t \\ d_s &\geq 0,\ s \geq t\ . \end{aligned}$$

The first two constraints are the accumulation equations for the financial wealth a and the durable stock d. The third constraint is the collateral constraint. This constraint ensures that the lowest attainable cash-on-hand $\underline{x_s}$ guarantees full repayment (if income takes its smallest possible value $\underline{y}$). The assumption here is that the lender, who lends at the risk-free rate, knows the financial position (a,d) and the minimum of the support of the income distribution $\underline{y}$. The lender does not know individual income draws.

Problem (1) satisfies Blackwell's sufficient conditions (monotonicity and discounting) for a contraction mapping so that we can apply standard dynamic programming techniques to solve for the stationary equilibrium. Because of stationarity, we drop time indexes and use primes " $'$ " to denote a one-period lead (but for $u'(.)$ or $w'(.)$ which denote first derivatives of the instantaneous utility functions).

Equilibrium definition. A stationary equilibrium is given by the policy functions for non-durable consumption $c(x)$, durable investment $i(x)$, the accumulation equations $a'(x)$ and $d'(x)$, and the evolution of the state variable $x'(x)$ so that for given prices $\{r^a, r^b\}$

(i) the value function $\widetilde{V}(x|y)$ attains its maximal value.

(ii) the collateral constraint is not violated, i.e., $\underline{x} \geq 0$.

(iii) the durable stock is weakly positive, $d \geq 0$.

2.1 Euler equations and analytic results

For later reference, note that in the optimum

$$u'(c) = \beta(1+r^a)E_y u'(c')\ ,$$

if the agent holds positive financial assets a, and

$$u'(c) = \beta(1+r^b)\left(E_y u'(c') + E_y \kappa'\right)$$

if the agent holds debt and the collateral constraint is expected to bind so that $E_y \kappa' > 0$. Because of the interest spread $r^b > r^a$, both Euler equations can be slack. In this case the intertemporal rate of substitution of non-durable consumption is in-between the lending and borrowing rate:

$$1 + r^a < \frac{u'(c)}{E_y u'(c')} < 1 + r^b \ .$$

Then, agents hold zero financial assets, $a = 0$.

In the optimum, durable investment is chosen so that it satisfies the condition

$$u'(c) = \beta E_y \left(u'(c')(1-\delta) + \phi w'(d') + (1-\delta)\kappa' + \gamma' \right) \ ,$$

where $\gamma' \geq 0$ is the multiplier associated with the constraint $d' \geq 0$. As is intuitive, the agent aligns the marginal utility of foregone non-durable consumption today (resulting from durable investment) with the discounted marginal utility derived from the durable tomorrow and the additional marginal utility of non-durable consumption that is afforded by re-selling the durable good (taking into account its depreciation at rate δ).

Note that if the collateral constraint is expected to bind, $E_y \kappa' > 0$, present consumption is valued less and more resources are transferred to the future period. We can show the following

Remark 1: *If utility is separable in the durable d and non-durable consumption c, the instantaneous utility functions u(.) and w(.) are strictly concave, of the HARA family, and satisfy prudence so that $u'''(.) \geq 0$ and $w'''(.) \geq 0$, we can show:*

(i) If the constraints are not binding, $c(x)$, $d(x)$ are concave, $a(x)$ is convex and $\partial c(x)/\partial x > 0$, $\partial d(x)/\partial x > 0$. Moreover, $\partial a(x)/\partial x \geq 0$ if $\delta = 1$, and under additional restrictions on concavity also for $0 \leq \delta < 1$.

(ii) If the collateral constraint binds, $\partial a(x)/\partial x$ falls and can become negative.

(iii) If the Euler equations for non-durable consumption are slack, $c(x)$, $d(x)$ can be locally strictly convex and $a(x)$ can be locally strictly concave.

Proof: see the Appendix.

Remark 1(i) is an application of Theorem 1 in Carroll and Kimball (1996) to our model with durable and non-durable consumption. The concavity of the non-durable and durable consumption functions in models of incomplete markets is very intuitive. Precautionary motives imply that the consumption propensity falls as agents have more cash-on-hand.

The intuition for Remark 1(ii) is that the possibility of a binding collateral constraint increases the amount of financial wealth a for small values of x so that the slope is flatter. The optimality condition of borrowing agents

$$u'(c) = \beta(1 + r_t^b) \left(E_y u'(c') + E_y \kappa' \right)$$

illustrates that as $E_y\kappa'$ falls with more cash-on-hand x (the collateral constraint is expected to be less binding), $u'(c)$ decreases, ceteris paribus. The same holds for durable investment. The slope $\partial a(x)/\partial x$ can be negative if the propensity of non-durable and durable consumption is larger than 1 and the collateral constraint is relaxed as the durable stock increases.

The intuition for Remark 1(iii) is that the propensity to consume out of cash-on-hand has to increase if the Euler equations for non-durable consumption are slack since $a' = 0$ and $\partial a'/\partial x$ falls so that $\partial a'/\partial x = 0$. Hence, the consumption propensities increase since $\partial c/\partial x + \partial d/\partial x = 1$ if $a' = 0$. The consumption functions are no longer globally concave.

Moreover, the durable stock increases relative to non-durable consumption since the optimality conditions above (without multipliers for the constraints) imply

$$1 + r^a < \beta(1-\delta) + \beta\phi \frac{E_y w'(d')}{E_y u'(c')} < 1 + r^b \ .$$

The expected intra-temporal rate of substitution between durable and non-durable consumption tomorrow equals $\left[1 + r^a - \beta(1-\delta)\right]/(\beta\phi)$ if the agent lends and $\left[1 + r^b - \beta(1-\delta)\right]/(\beta\phi)$ if the agent borrows. Thus, as agents accumulate cash-on-hand in the region where $a' = 0$, $E_y w'(d')/E_y u'(c')$ falls until the intra-temporal rate of substitution equals $1 + r^a$.

The larger propensity for durable investment, for values of cash-on-hand x where $a' = 0$, is intuitive. As long as the depreciation rate is not too high, durables are an imperfect way to transfer resources intertemporally since the rate of transformation is optimally in-between the exogenous interest factors $1 + r^b$ and $1 + r^a$.

That financial market imperfections increase the propensity of durable and non-durable consumption is supported by empirical evidence (see, for example, Alessie et al., 1997, for estimates using the period of financial deregulation in the UK in the 1980s).[3]

2.2 Calibration and numerical results

Numerical algorithm. It is well known that problems like ours do not have a closed-form solution for optimal policies. Therefore, we pursue a numerical approach which relies on value function iteration. While this allows us to conveniently rely on the contraction properties of the Bellman operator, one of the main challenges for this technique is to find a way to get around the curse of dimensionality. This is where the formulation of the problem that reduces the number of state variables to the minimum pays off - by subsuming the portfolio positions and the income realization in the single variable cash-on-hand. Hence, the state variables are cash-on-hand and the state of uncertainty, which is modeled as a 2-state Markov chain. The range of cash-on-hand, x, is restricted to

[3] Bertola et al. (2005) provide alternative microfoundations to explain the higher propensity for durable purchases if there is an interest spread $r^b > r^a$ and agents can be liquidity constrained ($a = 0$). In this case, a monopolist dealer has an incentive to lower the credit price of a durable good to attract liquidity constrained customers.

an interval $[0, x_{\max}]$. We perform value function iteration on a grid over that interval. Our choice of $x_{\max}$ guarantees that, for every x and for every realization of uncertainty, the equilibrium policy will imply a value for x tomorrow that remains within that interval.[4] We use linear interpolation of the value function between these grid points.

A feature of our algorithm that greatly enhances the accuracy of our solutions is the fact that the maximizing choices for the policy (at each state and each iteration) are not selected from a discretized set of choices, but rather by solving these maximization problems continuously over portfolio choices. We rely on a numerical optimization routine[5], which can also handle the collateral constraint and sign restrictions, to perform this task and to obtain the implicit multipliers on the constraints. The policy functions over the range $[0, x_{\max}]$ are obtained from the optimal policy choices on the grid by interpolation, using cubic splines.

As has become standard in the literature (see, e.g., Judd, 1992, and Aruoba et al., 2006), we evaluate the accuracy of our solutions by the normalized Euler equation errors implied by the policy functions. These are smaller than $4 * 10^{-4}$ over the entire range where the Euler equations apply with equality, and in fact much smaller for most values that the state variables of our problem can assume.

Calibration. We normalize average labor income y to 1, and parametrize the utility functions

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \text{ and } w(d) = \frac{(d + \underline{d})^{1-\sigma} - 1}{1 - \sigma},$$

where, as mentioned above, $\underline{d} > 0$ allows the consumer to hold no durable stock. We set risk aversion for the non-durable and durable good $\sigma = 2$ which is well within the range of commonly used values, and assume $\underline{d} = 0.01$. It turns out that this parameter is rather unimportant and can be set to negligibly small values without changing the quantitative results much. This is because the region of d close to zero is not important in our simulations. We calibrate the size of the shocks and transition probabilities of our 2-state Markov chain as 0.4. This implies a coefficient of variation of 0.4 and a first-order autocorrelation of 0.86 which is within the range of reasonable values considered by Aiyagari (1994).

We calibrate our model to the US, following previous calibrations by Diaz and Luengo-Prado (2005) and Athreya (2004). Table 1 summarizes the calibration parameters. We calibrate the relative taste for the durable ϕ and the depreciation rate δ so that we match a ratio of the durable stock to disposable income of 1.6 and a ratio of non-durable consumption over durable investment

[4] In our algorithm, we choose the grid for cash-on-hand so that for an upper bound of cash-on-hand $\overline{x}$, the optimal policies imply that the maximal attainable cash-on-hand, $x'_{\max}$ (for the highest realization of income $y_{\max}$) is smaller than this upper bound: $x'_{\max} = (1 + r)\, a' + y_{\max} + (1 - \delta)\, d' < \overline{x}$. Using $\underline{x} = 0$ as a lower bound gives us a compact state space (this bound is implied by the collateral constraint).

[5] We are using the Matlab routine fmincon().

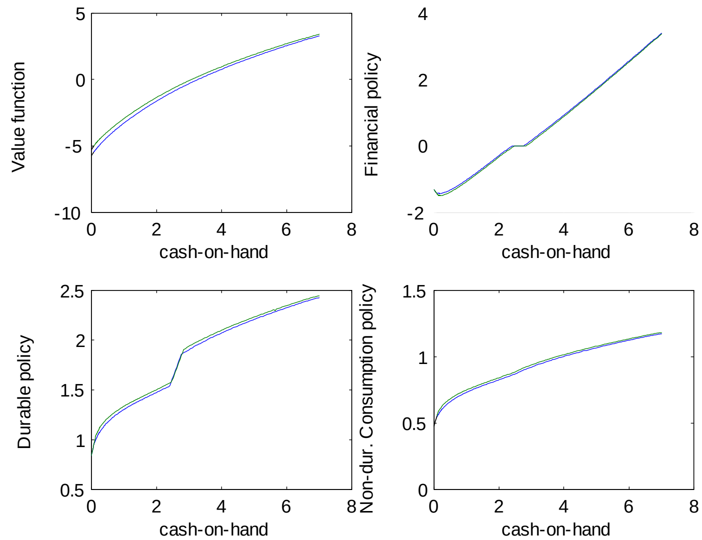

Figure 2: Value and policy functions in the good and bad state

slightly above 6 (see Diaz and Luengo-Prado, 2005, for the discussion of empirical estimates). This results in $\phi = 0.4$ and $\delta = 0.08$. The other parameters are rather standard and their sources are listed in Table 1.

The choice of the depreciation rate merits further discussion. We need a rather high depreciation rate so that a durable stock of 1.6, which is realistic empirically, is consistent with a ratio of non-durable consumption over durable investment of 6. Although a depreciation rate $\delta = 0.08$ is less realistic for housing, the rate is below commonly assumed values for other important durables like cars or computers. Thus, we view it as a reasonable approximation for the depreciation of a durable composite. We will also present results for a lower depreciation rate $\delta = 0.04$ which is closer to commonly used depreciation rates as used in Campbell and Hercowitz (2005).

Value function and policy functions. Figure 2 displays the solution for the value function and the policy functions in the bad and good income state. The value function is smooth and concave. Not surprisingly, the function shifts down in the bad state of the world. The policy functions have a slightly non-standard shape consistent with the results of Remark 1. Because of the interest spread $r^b > r^a$, $a = 0$ for an interval of cash-on-hand values. This local concavity of the financial policy implies local convexities in the policy functions for non-durable consumption and the durable stock. The local convexity is much more pronounced for the durable policy. This depends on whether the depreciation rate is low enough so that durables are a reasonably attractive vehicle to transfer

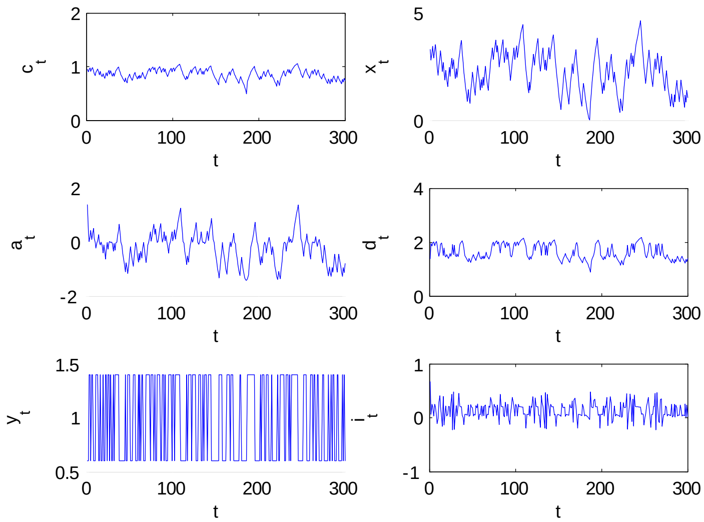

Figure 3: Time-series simulation of the economy without default

resources intertemporally.

Note that the constraint $d \geq 0$ is never binding whereas the collateral constraint is expected to bind for values of cash-on-hand close to zero. We now simulate our economy to find out more about the mean and distribution of the policy variables in the steady state.

Simulations. We simulate our economy for 10,000 periods. Figure 3 displays the results for an arbitrarily chosen subsample of 300 periods. If the exogenous income process y_t implies a long enough sequence of bad-state incomes, the agent accumulates financial debt as he borrows against the durable stock. If the bad shocks persist, the agent might not have the resources to keep the durable stock at his current level so that it depreciates. This tightens the collateral constraint and can sometimes imply that $x = 0$. The collateral constraint $x_t \geq 0$, however, is also important for behavior if $x > 0$, as long as the constraint is *expected* to bind.

Note that without income uncertainty, the impatient consumer would always be at his borrowing limit. Income uncertainty implies that the agent does not borrow as much and, if income is persistently good, he even accumulates some buffer-stock of assets, $a_t > 0$. Finally, we observe that durable investment is more volatile than consumption also because of the high propensity to invest if financial assets are zero. We return to this point below.

Table 2 displays the averages in the steady-state equilibrium for the main variables of interest. In column (1) we display the results for our benchmark economy. All values are expressed in average-income equivalents. On average, the consumer holds 2.3 of average income as cash-on-hand and borrows a sixth of average income with financial assets. The size of the durable stock of is 1.64 and the ratio of non-durable consumption over durable investment is 6.5 which is in line with empirical evidence for the US (see Diaz and Luengo-Prado, 2005).[6]

Given that the income shocks are purely idiosyncratic, the law of large numbers applies upon aggregation (see Uhlig, 1996) and the time-series distribution can be used as an approximation of the cross-sectional distribution in the steady state. Figure 4 displays such distributions for non-durable consumption c, durable holdings d, financial assets a, and cash-on-hand x. The density of cash-on-hand is bell-shaped and is truncated at $x = 0$, where collateral constraints bind. Thus, also the densities of c, d, and a have more mass at their lower bound of the support. Moreover, financial assets have a mass point at $a = 0$ when the (non-durable) consumption Euler equation is slack for both r^a and r^b. The frequency of agents with zero financial assets in Figure 4 is 11.7%. This is about the same order of magnitude as the 10% of US consumers between age[7] 25 and 50 which hold net non-housing wealth in the range from zero to two weeks' of their permanent income (see the discussion of these statistics based on the 1995 Survey of Consumer Finances in Carroll, 2001). The higher propensity to consume in the range where $a = 0$ implies that both the distribution for non-durable consumption and durable holdings are bimodal. Consistent with the much stronger change in the propensity to purchase durables observed in Figure 2, the bimodality is more pronounced for the distribution of durable holdings.

Changes in parameters. We now investigate how changes of the model's parameters alter the steady-state equilibrium. In Table 2, column (2), we compute the average equilibrium for lower risk-aversion, $\sigma = 1$. This reduces non-durable consumption and induces accumulation of cash-on-hand in terms of durables. Thus, the collateral constraint is laxer and consumers borrow more when bad income shocks occur.

In column (3) we investigate whether the parameter $\underline{d}$ is important in our benchmark equilibrium. We set $\underline{d} = 0$ and find no significant changes. As expected durable holdings increase slightly compared to non-durable consumption because the marginal utility derived from the durable is higher (for a given d). Thus, the ratio c/i falls. The larger durable stock also relaxes the collateral constraint. This allows agents to borrow more so that the average financial-asset position is lower. The increase in debt is not enough, however, to completely offset the increase in d so that cash-on-hand increases. The effect of increasing ϕ from 0.4 to 0.5 is qualitatively the same (see column (4)).

If agents are more impatient ($\beta = 0.9$), the consumers borrow more (see column (5)). At the same time the ratio c/i increases since non-durable con-

[6] Note that average disposable income $y + r^j a$ is nearly equal to average income since $r^j a \simeq 0$.

[7] Buffer-stock saving behavior should matter for consumers in this age range.

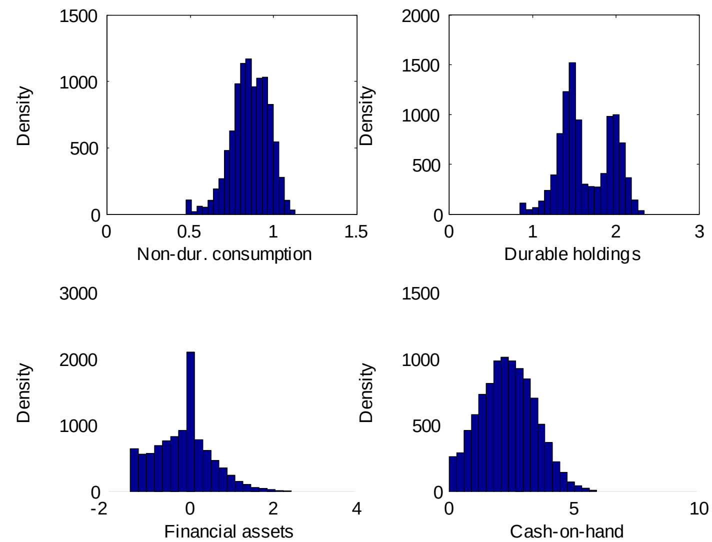

Figure 4: The steady-state distributions

sumption generates utility today whereas durable investment only generates utility tomorrow. Thus, the durable stock falls which also tightens the collateral constraint. Since consumers borrow more, the collateral constraint binds much more often.

When calibrating the model, we have mentioned that a depreciation rate $\delta = 0.08$ is rather high. In column (6) we lower the depreciation rate to $\delta = 0.04$. This increases the durable stock and non-durable consumption and lowers durable investment which is only a tenth of non-durable consumption. The larger cash-on-hand relaxes the collateral constraint and allows agents to borrow more in bad times so that the average financial asset position is lower.

If we lower the borrowing rate r^b to 0.02, not surprisingly agents borrow more (see column (7)). Cheaper borrowing also allows consumers to afford a larger durable stock. Total cash-on-hand decreases, however, because of more consumer debt. The fall in the borrowing rate also reduces the spread in the financial market so that agents hold zero financial assets less frequently and the kinks in the policy functions of durables and financial assets become less pronounced. This implies that the frequency of consumers with financial assets $a = 0$ is 2.3% which is similar to the empirically observed frequency of 2.5% for consumers holding precisely zero net non-housing worth in the 1995 Survey of Consumer Finances in the US (see Carroll, 2001). The lower frequency implies in our model that the distribution of durable holdings becomes less bimodal (the figures are not reported but are available upon request).

3 Income risk and household debt

We now apply our model to answer the question whether higher income risk is a good candidate for explaining the rise in household debt in the US in the last decades. We find that in our model the answer is no. The reason is that higher risk (in terms of shock size or persistence) increases the buffer-stock motive and thus *decreases* the debt holdings of agents. Instead, institutional financial market reforms that allow consumers to collateralize more of their debt are a more plausible explanation in our model.

The results are in Table 2, columns (8)-(10). In column (8) we increase the size of shocks from 0.4 to 0.5, which implies an increase of the standard deviation of log-income by 12%. This is much more than the increase of the cross-sectional variance of earnings in the US (15 basis points in the period between 1981 and 2003) to illustrate the point qualitatively. As can be seen in column (8), consumers hold more financial assets as buffer stock and also, conditional on holding debt, average debt increases from -0.35 to -0.22. The average durable stock increases slightly. The results are qualitatively the same if the shocks are more persistent (see column (9) where the transition probability falls from $p = 0.4$ to $p = 0.2$). Moreover, the equilibrium change is similar if we exogenously tighten the collateral constraint in column (10) where we no longer allow consumers to collateralize their durable stock. Thus, *relaxing* collateral constraints, does *increase* consumer debt. The bottom-line is that an increase in income risk does not increase consumer debt if the buffer-stock saving motive is strong. Instead, lower collateral requirements are a possible explanation for higher consumer debt (see Campbell and Hercowitz, 2005, for a discussion on how market innovations that followed the Monetary Control Act of 1980 and the Garn-St.Germain Act of 1982 relaxed collateral constraints on household debt in the US).

However, we cannot fully dismiss the hypothesis that more idiosyncratic income risk increased consumer debt for at least two reasons:

(i) In our partial-equilibrium model interest rates are exogenous. A general equilibrium effect as in Aiyagari (1994) would imply that interest rates have to fall until the asset market clears. This would reduce the buffer-stock saving motive. However, the results in Aiyagari suggest that it is unlikely that the general equilibrium effect outweighs the direct partial-equilibrium effect.

(ii) The access to borrowing and idiosyncratic risk maybe endogenously related. For example in Krueger and Perri (2005), limited enforcement of credit contracts implies that financial market development interacts with income volatility. If more volatile income makes the exclusion from credit markets in case of default more costly, this might foster financial market development. In this case, more volatile income will induce a higher buffer-stock but with respect to a laxer borrowing limit. Whether this implies more or less debt depends on which effect dominates quantitatively and is *a priori* unclear.

4 Conclusion

We set up and solve a heterogenous-agent model with incomplete markets in which households derive utility from non-durable consumption and durable holdings. We show that an interest spread between the borrowing and lending rate implies local convexities in the policy functions for non-durable consumption and especially durable holdings which are important quantitatively.

We apply our model to the question whether an increase in income risk can explain the increase in household debt observed in many developed countries in the past decades. Calibrating our model to the US economy, we find that an increase in income risk *reduces* average household debt, also if we condition on those consumers who hold some debt. Thus, our model with buffer-stock saving motives has different predictions than models which analyze approximations around a non-stochastic steady state (see, for example, Iacoviello, 2005). We argue that an increase in idiosyncratic income risk alone cannot explain the increase in household debt in the US and other developed countries.

In current research we extend our model to analyze interactions between aggregate and idiosyncratic risk and whether the observed *decrease* of aggregate risk in the US has facilitated the risk-sharing provided by financial intermediaries.

Appendix

Proof of Remark 1

The proof is based on results of Carroll and Kimball (1996).

Claim (i): If the constraints are not binding, $c(x)$, $d(x)$ *are concave and* $a(x)$ *is convex and* $\partial c(x)/\partial x > 0$, $\partial d(x)/\partial x > 0$, $\partial a(x)/\partial x \geq 0$.

Proof: We want to show that if $u(.)$ and $w(.)$ are HARA utility functions and $u'(.) > 0$, $u''(.) < 0$, $u'''(.) \geq 0$, and $w'(.) > 0$, $w''(.) < 0$, $w'''(.) \geq 0$, then $c(x)$, $d(x)$ are concave and $a(x)$ is convex and $\partial c(x)/\partial x > 0$, $\partial d(x)/\partial x > 0$, $\partial a(x)/\partial x \geq 0$.

Our problem is

$$\widetilde{V}_t\left(x_t\right) = \max_{a_{t+1},d_{t+1}} \left[u(\underbrace{x_t - a_{t+1} - d_{t+1}}_{c_t}) + \beta\phi w\left(d_{t+1}\right) + \beta E_t \widetilde{V}_{t+1}\left(x_{t+1}\right) \right]$$

where $x_t \equiv (1 + r^j)a_t + y_t + (1 - \delta)d_t$ so that the budget constraint

$$c_t = x_t - a_{t+1} - d_{t+1} \ .$$

To start we also assume a finite horizon so that we have the terminal condition

$$c_T = x_T.$$

We then proceed analogously as in Carroll and Kimball and prove Lemmas 1-3. For this we define as $\xi_t((1+r^j)a_{t+1}(x_t)+(1-\delta)d_{t+1}(x_t)) \equiv \beta E_t \widetilde{V}_{t+1}(x_{t+1})$, where

$$x_{t+1} \equiv (1+r^j)a_{t+1} + y_{t+1} + (1-\delta)d_{t+1}.$$

Note that $\xi_t(.)$ is written as a function of choice variables.

The first lemma shows that the property of prudence is conserved when aggregating across states of nature.

Lemma 1: If $\widetilde{V}'''_{t+1}\widetilde{V}'_{t+1}/\left[\widetilde{V}''_{t+1}\right]^2 \geq k$, *then* $\xi'''_t\xi'_t/\left[\xi''_t\right]^2 \geq k$.

Proof: see Carroll and Kimball, p. 985.

The second lemma shows that the property of prudence is conserved when aggregating intertemporally.

Lemma 2: If $\xi'''_t\xi'_t/\left[\xi''_t\right]^2 \geq k$ and $u'''u'/\left[u''\right]^2 \geq k$, $w'''w'/\left[w''\right]^2 = k$, then $\widetilde{V}'''_t\widetilde{V}'_t/\left[\widetilde{V}''_t\right]^2 \geq k$.

Proof: Following Carroll and Kimball, p. 985/986, we denote the marginal utility of non-durable consumption at the optimal consumption level with $z_t = u'(c^*_t(x_t)$. Neglecting the collateral constraint and interest spread, we know that in our problem the following equations hold in the optimum:

$$\begin{aligned}
z_t &= u'(c^*_t(x_t)) \ , \\
u'(c^*_t(x_t)) &= \widetilde{V}'_t(x_t) \ , \\
u'(c^*_t(x_t)) &= \beta(1+r^j)E_t\widetilde{V}'_{t+1}(x_{t+1}) = (1+r^j)\xi'_t \ , \\
u'(c^*_t(x_t)) &= \beta\phi w'(d_{t+1}) + (1-\delta)\xi'_t \ ,
\end{aligned}$$

where $\xi_t((1+r^j)a_{t+1}(x_t)+(1-\delta)d_{t+1}(x_t))$. We then define the functions $f_t(z_t)$, $g_t(z_t)$, $h_t(z_t)$, $l_t(z_t)$ as

$$f_t(z_t) = u'^{-1}(z_t) = c_t \ ,$$

$$h_t(z_t) = \widetilde{V}'^{-1}_t(z_t) = x_t \ ,$$

$$l_t(z_t) = w'^{-1}\left(\frac{z_t - (1-\delta)\xi'_t(.)}{\beta\phi}\right) = d_{t+1} \ ,$$

$$g_t(z_t) = \xi'^{-1}_t\left(\frac{z_t}{1+r^j}\right) - (1-\delta)l_t(z_t) = (1+r^j)a_{t+1} \ .$$

Noting from the last equation that

$$(1+r^j)a_{t+1} + (1-\delta)d_{t+1} = \xi'^{-1}_t\left(\frac{z_t}{1+r^j}\right) \ ,$$

we use this expression in as the argument of $\xi_t'(.)$ in the second equation which then simplifies to

$$l_t(z_t) = w'^{-1}\left(\frac{r^j+\delta}{\beta\phi\left(1+r^j\right)}z_t\right) = d_{t+1}$$

Dropping time indexes for functions f, g, l, h, we have

$$\begin{aligned} f'(z) &= \frac{1}{u''(c(z))}\ , \\ f'' &= -\frac{u'''(c)}{\left[u''(c)\right]^2}\underbrace{f'}_{\partial c/\partial z} = -\frac{u'''}{\left[u''\right]^3}\ , \end{aligned}$$

so that

$$-\frac{zf''}{f'} = \frac{u'''u'}{\left[u''\right]^2} \geq k\ .$$

Similarly,

$$-\frac{zh''}{h'} = \frac{\widetilde{V}_t'''\widetilde{V}_t'}{\left[\widetilde{V}_t''\right]^2}\ .$$

Furthermore,

$$\begin{aligned} l' &= \frac{r^j+\delta}{\beta\phi\left(1+r^j\right)w''}\ , \\ l'' &= -\frac{\left(r^j+\delta\right)w'''}{\beta\phi\left(1+r^j\right)\left[w''\right]^2}l'\ , \end{aligned}$$

so that

$$-\frac{zl''}{l'} = \frac{w'''w'}{\left[w''\right]^2} \geq k\ ,$$

where we use that

$$\frac{r^j+\delta}{\beta\phi\left(1+r^j\right)}z_t = w'(d_{t+1})\ .$$

Finally,

$$\begin{aligned} g' &= \frac{1}{(1+r^j)\xi''\left(\xi_t'^{-1}\left(\frac{z_t}{1+r^j}\right)\right)} - (1-\delta)\frac{r^j+\delta}{\beta\phi\left(1+r^j\right)w''}\ , \\ g'' &= -\frac{\xi'''}{(1+r^j)^2\left[\xi''\right]^3} + (1-\delta)\frac{\left(r^j+\delta\right)w'''}{\beta\phi\left(1+r^j\right)\left[w''\right]^2}l'\ . \end{aligned}$$

Thus,

$$-\frac{zg''}{g'} = \frac{\frac{\xi'''\xi'}{(1+r^j)\left[\xi''\right]^3} - (1-\delta)\frac{w'''w'}{\left[w''\right]^2}l'}{\frac{1}{(1+r^j)\xi''} - (1-\delta)l'}\ .$$

For $\delta = 1$, this simplifies to

$$-\frac{zg''}{g'} = \frac{\xi'''\xi'}{\left[\xi''\right]^2} \geq k \ ,$$

For $0 < \delta < 1$,

$$-\frac{zg''}{g'} = \frac{g'}{g' - (1-\delta)l'} \frac{\xi'''\xi'}{\left[\xi''\right]^2} - \frac{(1-\delta)l'}{g' - (1-\delta)l'} \frac{w'''w'}{[w'']^2} \ .$$

If we assume HARA utility so that $w'''w'/[w'']^2 = k$, then $\xi_t'''\xi_t'/\left[\xi_t''\right]^2 \geq k$ implies that

$$-\frac{zg''}{g'} \geq \frac{g'}{g' - (1-\delta)l'} k - \frac{(1-\delta)l'}{g' - (1-\delta)l'} k = k \ .$$

Now note that since

$$c_t = x_t - a_{t+1} - d_{t+1}$$

and

$$a_{t+1} = \frac{g}{(1+r^j)} - (1-\delta)l \ ,$$

we have

$$\begin{aligned} h &= f + \frac{g}{(1+r^j)} - (1-\delta)l + l \\ &= f + \frac{g}{(1+r^j)} + \delta l \ . \end{aligned}$$

That is, h is an additive function of f, g and l, so that

$$h' = f' + \frac{g'}{(1+r^j)} + \delta l'$$

and

$$h'' = f'' + \frac{g''}{(1+r^j)} + \delta l'' \ .$$

This implies that

$$\begin{aligned} -\frac{zh''}{h'} &= -z\frac{f'' + \frac{g''}{(1+r^j)} + \delta l''}{f' + \frac{g'}{(1+r^j)} + \delta l'} \\ &= \underbrace{\frac{f'}{f' + \frac{g'}{(1+r^j)} + \delta l'}}_{>0}\underbrace{\left(-\frac{zf''}{f'}\right)}_{\geq k} + \underbrace{\frac{\frac{g'}{(1+r^j)}}{f' + \frac{g'}{(1+r^j)} + \delta l'}}_{>0}\underbrace{\left(-\frac{zg''}{g'}\right)}_{\geq k} + \underbrace{\frac{\delta l'}{f' + \frac{g'}{(1+r^j)} + \delta l'}}_{>0}\underbrace{\left(-\frac{zl''}{l'}\right)}_{\geq k} \\ &\geq k \ , \end{aligned}$$

since this is a weighted average of expressions that are larger or equal than k.

As in Carroll and Kimball we move on to show Lemma 3, where we exploit again that HARA utility implies $w'''w'/[w'']^2 = k$ and $u'''u'/[u'']^2 = k$ with equality.

Lemma 3: If $\widetilde{V}_t'''\widetilde{V}_t'/\left[\widetilde{V}_t''\right]^2 \geq k$, $w'''w'/[w'']^2 = k$ and $u'''u'/[u'']^2 = k$, then the optimal consumption policy rules $c(x)$ and $d(x)$ are concave and liquid assets $a(x)$ are convex.

Proof: Note that

$$c_t(x) = f_t(h_t^{-1}(x)) \ .$$

Thus,

$$\frac{\partial c}{\partial x} = \frac{f'(h^{-1})}{h'(h^{-1})} = \frac{\widetilde{V}''}{u''} > 0$$

if $u'' < 0$, $\widetilde{V}'' < 0$ and

$$\begin{aligned}
\frac{\partial^2 c}{\partial x^2} &= \frac{\left(f''(h^{-1})/h'(h^{-1})\right)\left(h'(h^{-1})\right) - \left(f'(h^{-1})\right)\left(h''(h^{-1})/h'(h^{-1})\right)}{[h'(h^{-1})]^2} \\
&= \frac{f'(h^{-1})}{[h'(h^{-1})]^2}\left[\frac{f''(h^{-1})}{f'(h^{-1})} - \frac{h''(h^{-1})}{h'(h^{-1})}\right] \ .
\end{aligned}$$

Applying Lemma 2 we find

$$\frac{\partial^2 c}{\partial x^2} = \frac{f'(h^{-1})}{[h'(h^{-1})]^2}\frac{1}{z}\left[\underbrace{-\frac{zh''(h^{-1})}{h'(h^{-1})}}_{\geq k} - \underbrace{\frac{-zf''(h^{-1})}{f'(h^{-1})}}_{=k}\right] \ .$$

The sign of this derivative is smaller or equal than zero if $sgn(f'(h^{-1})) < 0$. Recalling that $f'(h^{-1}) = f'(z) = 1/u'' < 0$, this is the case for a strictly concave utility function. Analogous manipulations for $d_t(x) = l_t(h_t^{-1}(x))$ prove $\partial d(x)/\partial x > 0$ and $\partial^2 d(x)/(\partial x)^2 \leq 0$.

Since $a_{t+1}(x) = x_t - c_t(x) - d_{t+1}(x)$,

$$\frac{\partial a}{\partial x} = 1 - \frac{\partial c(x)}{\partial x} - \frac{\partial d(x)}{\partial x}$$

and

$$\frac{\partial^2 a}{\partial x^2} = -\frac{\partial^2 c(x)}{\partial x^2} - \frac{\partial^2 d(x)}{\partial x^2} \geq 0.$$

Thus, financial wealth increases or decreases with x, depending on whether the marginal propensity to consume $\partial c(x)/\partial x + \partial d(x)/\partial x \gtreqless 1$. The second derivative is certainly positive so that $a(x)$ is convex.

We now investigate the properties of the consumption propensities further. In particular, do we know whether $\partial c(x)/\partial x + \partial d(x)/\partial x > 1$?

Noting that

$$h' = f' + \frac{g'}{(1+r^j)} + \delta l'$$

we can write

$$\frac{\partial c}{\partial x} = \frac{f'(h^{-1})}{f'(h^{-1}) + \frac{g'(h^{-1})}{(1+r^j)} + \delta l'(h^{-1})}$$

and

$$\frac{\partial d}{\partial x} = \frac{l'(h^{-1})}{f'(h^{-1}) + \frac{g'(h^{-1})}{(1+r^j)} + \delta l'(h^{-1})} \ .$$

Thus,

$$\frac{\partial c}{\partial x} + \frac{\partial d}{\partial x} = \frac{f'(h^{-1}) + l'(h^{-1})}{f'(h^{-1}) + \frac{g'(h^{-1})}{(1+r^j)} + \delta l'(h^{-1})} < 1 \ ,$$

if $\delta = 1$ and $g'(h^{-1}) > 0$.

We now compute the derivative of $a(x) = g(h^{-1}(x))/\left(1+r^j\right)$:

$$\begin{aligned}
\frac{\partial a}{\partial x} &= \frac{1}{1+r^j} g'(h^{-1}(x))/h'(h^{-1}(x)) \\
&= \frac{\widetilde{V}_t''}{1+r^j}\left(\frac{1}{(1+r^j)\xi''} - (1-\delta)\frac{r^j+\delta}{\beta\phi\,(1+r^j)\,w''}\right) \ ,
\end{aligned}$$

which is certainly positive if $\delta = 1$ since $\widetilde{V}_t'' < 0, \xi'' < 0$. For $\delta < 1$, we need to impose an additional condition on the curvature

$$\begin{aligned}
\frac{1}{(1+r^j)\xi''} - (1-\delta)\frac{r^j+\delta}{\beta\phi\,(1+r^j)\,w''} &< 0 \text{ or} \\
\frac{\xi''}{\beta\phi w''} &< \frac{r^j+\delta}{1-\delta} \ .
\end{aligned}$$

In general the sign of $\partial a/\partial x$ depends on the relative curvature of the value function expected tomorrow, ξ_t'', and instantaneous utility derived from the durable, w''. Intuitively, a larger δ makes durables less useful to transfer utility and thus increase the marginal propensity of financial assets to transfer resources.

The lemmas derived above imply Theorem 1 as in Carroll and Kimball (1996). Note that the second-order derivatives for the policy functions hold with strict equality if $k > 0$ and there is some labor income uncertainty.

Carroll and Kimball show results for a finite horizon. In a finite horizon, we have that in the last period $V_T = u(c) + \phi w(d)$ so that prudence of $u(.)$ and $w(.)$ trivially also apply to V_T. Then one iterates forward using using Lemma

1 and 2. To extend these results to the infinite horizon one needs to apply the contraction property of V, for $T \to \infty$. Since cash on hand is finite, agents discount and V satisfies monotonicity, $\lim_{T\to\infty} V_t(x) = V(x)$ for all x (see Lucas and Stokey, 1989, ch. 3). Pointwise convergence implies that the properties of V_t are conserved as V_t converges towards V. ■

Claim (ii): If the collateral constraint binds, $\partial a(x)/\partial x$ falls and can become negative.

Proof: Intuitively, the value function will be more concave if the collateral constraint holds. The expression for the propensities derived above, then imply that $\partial c(x)/\partial x + \partial d(x)/\partial x$ increases if $\widetilde{V}''$ falls (i.e., increases in absolute value). This can imply $\partial a(x)/\partial x < 0$, which we now want to derive more formally. Adding the multiplier κ for the collateral constraint and γ for the constraint $d > 0$, the four equations used in Lemma 2 change to

$$\begin{aligned} z_t &= u'(c_t^*(x_t) \ , \\ u'(c_t^*(x_t) &= \widetilde{V}_t'(x_t) \ , \\ u'(c_t^*(x_t) &= (1+r^j)\left(\xi_t' + E_t\kappa\right) \ , \\ u'(c_t^*(x_t) &= \beta\phi w'(d_{t+1}) + (1-\delta)\left(\xi_t' + E_t\kappa\right) + E_t\gamma \ , \end{aligned}$$

so that

$$f_t(z_t) = u'^{-1}(z_t) = c_t \ ,$$

$$h_t(z_t) = \widetilde{V}_t'^{-1}(z_t) = x_t \ ,$$

$$l_t(z_t) = w'^{-1}\left(\frac{z_t - (1-\delta)\left(\xi_t'(.) + E_t\kappa\right) - E_t\gamma}{\beta\phi}\right) = d_{t+1} \ ,$$

$$g_t(z_t) = \xi_t'^{-1}\left(\frac{z_t}{1+r^j} - E_t\kappa\right) - (1-\delta)l_t(z_t) = (1+r^j)a_{t+1} \ .$$

Observing that

$$(1+r^j)a_{t+1} + d_{t+1} = \xi_t'^{-1}\left(\frac{z_t}{1+r^j} - E_t\kappa\right) \ ,$$

the third equation can be rewritten as

$$l_t(z_t) = w'^{-1}\left(\frac{\frac{r^j+\delta}{1+r^j}z_t - E_t\gamma}{\beta\phi}\right) = d_{t+1} \ .$$

Thus, a expectedly binding collateral constraint does not directly affect d_{t+1}. Instead if the constraint $d = 0$ is expected to bind this lowers $w'(d_{t+1})$ and thus induces a larger d_{t+1}, ceteris paribus.

More interestingly, let us investigate how the marginal propensity of $a(x)$ changes if the collateral constraint is binding (we neglect the constraint $d \geq 0$ for simplicity). Recall that $a(x) = g(h^{-1}(x))/\left(1+r^j\right)$:

$$
\begin{aligned}
\frac{\partial a}{\partial x} &= \frac{1}{1+r^j} g'(h^{-1}(x))/h'(h^{-1}(x)) \\
&= \frac{\widetilde{V}_t''}{1+r^j}\left(\frac{\frac{1}{1+r^j}-E_t\frac{\partial\kappa}{\partial z}}{\xi''}-(1-\delta)\frac{r^j+\delta}{\beta\phi\,(1+r^j)\,w''}\right) .
\end{aligned}
$$

Since a larger $z = u'(c^*(x))$ means a smaller c and x, $E_t\partial\kappa/\partial z > 0$, i.e. the collateral constraint is expected to become more binding for smaller x and thus larger z. Then, this derivative shows that the propensity $\partial a/\partial x$ falls if the collateral constraint is expected to bind. In particular, the propensity need no longer be positive. The intuition is that the possibility of a binding collateral constraint increases the amount of financial wealth for small values of x so that the slope is flatter. ■

Claim (iii): If the Euler equations for non-durable consumption are slack, $c(x)$, $d(x)$ can be locally strictly convex and $a(x)$ can be locally strictly concave.
Proof: We show that $c(x)$, $d(x)$ are locally strictly convex and $a(x)$ is locally strictly concave in the range where $a = 0$. In particular, $\partial c(x)/\partial x|_{a=0} > \partial c(x)/\partial x$ and $\partial d(x)/\partial x|_{a=0} > \partial d(x)/\partial x$ for given x, and $E_y w'(d')/E_y\mu'$ falls.

If $a_{t+1}(x) = 0$,

$$c_t = x_t - d_{t+1}$$

and thus

$$h = f + l \ .$$

Hence,

$$-\frac{zh''}{h'} = \underbrace{\frac{f'}{f'+l'}}_{>0}\underbrace{\left(-\frac{zf''}{f'}\right)}_{\geq k} + \underbrace{\frac{l'}{f'+l'}}_{>0}\underbrace{\left(-\frac{zl''}{l'}\right)}_{\geq k}$$

so that the curvature of $w(.)$ becomes much more important for the curvature of the value function. Also

$$\frac{\partial c}{\partial x} + \frac{\partial d}{\partial x} = \frac{f'(h^{-1}) + l'(h^{-1})}{f'(h^{-1}) + l'(h^{-1})} = 1 \ ,$$

so that the propensities increase since $\partial a(x)/\partial x > 0$ to the left of the range where $a(x) = 0$. The local increase of the propensities implies local convexity of the consumption functions. Moreover, $\partial a(x)/\partial x > 0$ is locally concave.

More formally, if $\partial a(x)/\partial x = 0$, the collateral constraint is certainly not binding and

$$\begin{aligned} z_t &= u'(c_t^*(x_t) \ , \\ u'(c_t^*(x_t) &= \widetilde{V}_t'(x_t) \ , \\ (1+r^a)\xi_t' &< u'(c_t^*(x_t) < (1+r^b)\xi_t' \\ u'(c_t^*(x_t) &= \beta\phi w'(d_{t+1}) + (1-\delta)\xi_t' \ , \end{aligned}$$

so that

$$f_t(z_t) = u'^{-1}(z_t) = c_t \ ,$$

$$h_t(z_t) = \widetilde{V}_t'^{-1}(z_t) = x_t \ ,$$

$$l_t(z_t) = w'^{-1}\left(\frac{z_t - (1-\delta)\xi_t'(.))}{\beta\phi}\right) = d_{t+1} \ ,$$

$$g_t(z_t) = \xi_t'^{-1}(z_t + \lambda^b) - (1-\delta)l_t(z_t) = (1+r^b)a_{t+1}$$

or

$$g_t(z_t) = \xi_t'^{-1}(z_t - \lambda^a) - (1-\delta)l_t(z_t) = (1+r^a)a_{t+1}$$

with $\lambda^a > 0$ and $\lambda^b > 0$.

This implies

$$\begin{aligned} \frac{\partial a}{\partial x} &= \frac{1}{1+r^b} g'(h^{-1}(x))/h'(h^{-1}(x)) \\ &= \frac{\widetilde{V}_t''}{1+r^b}\left(\frac{1+\frac{\partial\lambda^b}{\partial z}}{(1+r^b)\xi''} - (1-\delta)\frac{r^b+\delta}{\beta\phi\,(1+r^b)\,w''}\right) \ . \end{aligned}$$

For the range $a_{t+1}(x) = 0$, $\partial\lambda^b/\partial z < 0$ so that $\partial a/\partial x = 0$ (Note that $\partial\lambda^b/\partial x > 0$.). Similarly, for the lending Euler-equation,

$$\frac{\partial a}{\partial x} = \frac{\widetilde{V}_t''}{1+r^b}\left(\frac{1-\frac{\partial\lambda^a}{\partial z}}{(1+r^b)\xi''} - (1-\delta)\frac{r^b+\delta}{\beta\phi\,(1+r^b)\,w''}\right) \ ,$$

with $\partial\lambda^a/\partial z > 0$ (Note that $\partial\lambda^a/\partial x < 0$.). ■

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Parameters Values	*Sources / Targets*
lending rate: $r^a = 0.01$	Mehra and Prescott (1985)
borrowing rate: $r^b = 0.044$	Athreya (2004)
discount factor: $\beta = 0.96$	Aiyagari (1994)
risk aversion: $\sigma = 2$	for example, Aiyagari (1994)
transition probability $p = 0.4$	$\rightarrow$ coefficient of variation of 0.4, e.g. Aiyagari (1994)
size of the shock 0.4	$\rightarrow$ 1st order autocorrelation 0.86, e.g. Aiyagari (1994)
minimum durable: $\underline{d}$= 0.01	-
depreciation rate: $\delta = 0.08$	$\rightarrow$ ratio $c/i \in 6 - 6.5$, Diaz and Luengo-Prado (2005)
weight of durable utility: $\phi = 0.4$	$\rightarrow$ durable stock $d \in 1.4 - 1.6$, DLP (2005)

Table 1: Parameter values for the calibration.

Variables	*Benchmark*	$\sigma = 1$	$\underline{d} = 0$	$\phi = 0.5$	$\beta = 0.9$
	(1)	**(2)**	**(3)**	**(4)**	**(5)**
cash-on-hand x	2.330	2.794	2.335	2.417	1.207
financial assets a	-0.170	-0.585	-0.172	-0.227	-1.085
durable stock d	1.641	2.610	1.649	1.799	1.453
durabl. inv. i	0.131	0.208	0.132	0.144	0.116
non-d. cons. c	0.859	0.769	0.858	0.845	0.840
ratio c/i	6.541	3.682	6.501	5.869	7.228
	$\delta = 0.04$	$r^b = 0.02$	*shock size* 0.5	$p = 0.2$	*no collat.* d
	(6)	**(7)**	**(8)**	**(9)**	**(10)**
cash-on-hand x	2.684	1.789	2.799	2.886	2.671
financial assets a	-0.358	-0.755	0.239	0.295	0.132
durable stock d	2.142	1.691	1.696	1.717	1.673
durabl. inv. i	0.086	0.135	0.136	0.137	0.134
non-d. cons. c	0.900	0.853	0.864	0.873	0.866
ratio c/i	10.499	6.304	6.366	6.359	6.476

Table 2: Means in the steady-state equilibrium