The Eeckhout Condition and the Subgame Perfect Implementation of Stable Matching*

Sang-Chul Suh†
University of Windsor

Quan Wen‡
Vanderbilt University

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Abstract

We investigate an extensive form sequential matching game of perfect information. We show that the subgame perfect equilibrium of the sequential matching game leads to the unique stable matching when the Eeckhout Condition (2000) for existence of a unique stable matching holds, regardless of the sequence of agents. This result does not extend to preferences that violate the Eeckhout Condition, even if there is a unique stable matching.

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†Department of Economics, University of Windsor, Windsor, Ontario, N9B 3P4, Canada. E-mail: scsuh@uwindsor.ca

‡Department of Economics, Vanderbilt University, VU Station B #351819, 2301 Vanderbilt Place, Nashville, TN 37235-1819, U.S.A. E-mail: quan.wen@vanderbilt.edu

1 Introduction

Gale and Shapley (1962) introduced a marriage problem which is described by a group of men and women, and their preferences over their potential mates.[1] They proved the existence of the stable matching by providing an algorithm which leads to a stable matching. We consider the issue of the implementation of stable matchings in relation to the following extensive form sequential matching game.[2] Men and women move sequentially according to a certain order. A typical agent can choose one of three possible actions at his or her move. The agent can either accept one among those who proposed to this agent at previous stages, or propose to a possible mate who moves at a later stage, or choose to be single. An agent has perfect information about the game structure, the others' preferences, and the actions of the agents who move in the previous stages. In addition, we concentrate on the marriage problems in which the preferences are strict so that each corresponding sequential matching game has a unique subgame perfect equilibrium(SPE). One may wonder whether the SPE outcome of a sequential matching game is a stable matching of the corresponding marriage problem. Eeckhout (2000) provided a sufficient condition which identifies a set of preferences guaranteeing a unique stable matching. Within such a domain of preference profiles, we show that the SPE of every sequential matching game, regardless of the order of moves, coincides with the unique stable matching. This result, however, does not extend to preference profiles that violate the Eeckhout Condition, even if there is a unique stable matching. We provide a counter example where there is a unique stable matching but the SPE outcome is not the stable matching.

2 An Extensive Form Matching Game

We first introduce some definitions and notations that are necessary to describe a marriage problem. For a description of an extensive form game, we refer readers to textbooks, for

[1]For an introduction of the marriage problem, refer to Roth and Sotomayor (1990).

[2]The issue of the implementation of stable matchings has been investigated in Nash (undominated or strong Nash) equilibrium. Refer to Alcalde (1996), Kara and Sönmez (1996), Ma (1995), Shin and Suh (1996), and Sönmez (1997).

example, Osborne and Rubinstein (1994).

There are two non-empty and finite sets of agents, $M = \{m_1, \ldots, m_n\}$ for "men" and $W = \{w_1, \ldots, w_n\}$ for "women," where $M \cap W = \phi$ and $\| M \| = \| W \| = n$. Each agent $i \in M \cup W$ has a complete, irreflexive, and transitive strict preference relation P_i defined over $W \cup \phi$ for $i \in M$ and over $M \cup \phi$ for $i \in W$. We read aP_ib as "a is strictly preferred to b by agent i." A *matching* is a function $\mu : M \cup W \to M \cup W$ such that

$\mu(m) \in W \cup \phi$ for all $m \in M$, $\mu(w) \in M \cup \phi$ for all $w \in W$, and

$\mu(m) = w$ if and only if $\mu(w) = m$ for all $m \in M$ and $w \in W$.

Given a preference profile $P = (P_i)_{i \in M \cup W}$, a matching μ is *individually rational* if for all $i \in M \cup W$, $\mu(i) P_i \phi$. Given a matching μ and P, a pair $(m, w) \in M \times W$ *blocks* the matching μ if $wP_m\mu(m)$ and $mP_w\mu(w)$. A matching is *stable* if it is individually rational and does not allow any blocking pair.

We now introduce an extensive form sequential matching game. Let all agents be ordered as $1, 2, \ldots, \| M \cup W \|$ according to the order of their moves. Different orders of moves lead to different games, hence possibly different SPE outcomes.

Stage 1. Agent 1 may either propose to a possible mate or choose to be single.

Stage 2. Agent 2 observes agent 1's move. If agent 1 proposed to agent 2, then agent 2 may either accept agent 1's proposal, or (reject agent 1's proposal and) propose to a potential mate in $M \cup W \setminus \{1\}$, or choose to be single. If agent 1 did not propose to agent 2, then agent 2 may only either propose to a potential mate in $M \cup W \setminus \{1\}$, or choose to be single.

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Stage i. After observing agents' moves in all previous stages, agent i may either accept only one of the proposals directed to him/her (and reject the others if there are any), or (reject all proposals to him/her and) propose to a potential mate in $M \cup W \setminus \{1, \ldots, i\}$, or choose to be single. Note that if agent i has no one to accept and no one to propose to, then being single is the only choice left.

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Stage $\| M \cup W \|$. After observing all the agents' moves, agent $\| M \cup W \|$ may either accept one of the proposals directed to him/her (and reject the others if there are any), or (reject all the previous proposals to him/her and) choose to be single.

All the agents who either choose to be single or are rejected will have no mates, and all the other agents will be matched to agents they proposed to or were accepted by. It is easy to see that such a sequential matching game is a well defined finite game with perfect information, and hence has a unique SPE under strict preferences.

3 The Main Result

The following condition on agents preferences, introduced by Eeckhout (2000), is sufficient for the existence of a unique stable matching:[3]

The Eeckhout Condition: *It is possible to rename (rearrange) agents so that*

i) For any agent $m_i \in M$, $w_i P_{m_i} w_j$ *for all* $j > i$*;*

ii) For any agent $w_i \in W$, $m_i P_{w_i} m_j$ *for all* $j > i$*.*

Note that agent m_i may or may not prefer w_i to w_k for $k < i$. Under the Eeckhout Condition, μ is the unique stable matching if and only if $w_i = \mu(m_i)$ for all i. Our main result is:

Theorem 1 *Suppose that agents' preferences satisfy the Eeckhout Condition. Then regardless of the order of agents' moves, the subgame perfect equilibrium of the sequential matching game leads to the unique stable matching.*

Proof We will prove the theorem by using an inductive argument. First, since m_1 prefers w_1 to any other woman under the Eeckhout Condition, it is a strictly dominant strategy for

[3]See Clark (2005) for another sufficient condition, called the No Crossing Condition, which is interesting but stronger than the condition by Eeckhout (2000).

m_1 to propose to w_1 if he moves before w_1 or to accept w_1 if w_1 proposes to him. Similarly, it is a strictly dominant strategy for w_1 to propose to m_1 if she moves before m_1 or to accept m_1 if m_1 proposes to her. Their strategies off the equilibrium path, such as w_1's strategy after m_1 did not propose to w_1, are irrelevant to the equilibrium outcome. Consequently, m_1 and w_1 will be matched in the SPE, regardless of the order of their moves in the sequential matching game.

For any $k \leq n$, assume that, for all $i < k$, (1) no $m \in \{m_1, \cdots, m_{i-1}\}$ proposes to w_i and no $w \in \{w_1, \cdots, w_{i-1}\}$ proposes to m_i, (2) m_i either proposes to w_i if he moves before w_i or accepts w_i if w_i proposes to m_i, and (3) w_i either proposes to m_i if w_i moves before m_i or accepts m_i if m_i proposes to w_i. Accordingly, m_i and w_i will be matched for all $i < k$, regardless of the order of their moves.

It is impossible for m_k to be matched with any $w \in \{w_1, \cdots, w_{k-1}\}$ and w_k to be matched with any $m \in \{m_1, \cdots, m_{k-1}\}$. Under the Eeckhout Condition, m_k prefers w_k to any $w \in \{w_{k+1}, \cdots, w_n\}$. Therefore, m_k will either propose to w_k if no $m \in \{m_1, \cdots, m_{k-1}\}$ proposes to w_k or accept w_k if w_k proposes to him. Similarly, w_k will either propose to m_k if no $w \in \{w_1, \cdots, w_{k-1}\}$ proposes to m_k or accept m_k if m_k proposes to her. Accordingly, m_k and w_k will be matched in the equilibrium, regardless their places to move.

By induction, the resulting SPE outcome coincides with the unique stable matching with m_i and w_i matched for all $i \in \{1, \cdots, n\}$. **Q.E.D.**

4 A Counter Example

Can Theorem 1 be extended to preference profiles that violate the Eeckhout Condition? We provide a counter example with three men and three women to demonstrate that the answer to this question is negative, even if there is a unique stable match. We show that the SPE outcome of a sequential matching game in which all the men move before all the women is not the stable matching, but the SPE outcome of a sequential matching game in which all the women move before all the men is the stable matching.[4]

There are three men and three women, $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$, with the following preferences:

P_{m_1}	P_{m_2}	P_{m_3}	P_{w_1}	P_{w_2}	P_{w_3}
w_2	w_2	w_1	m_2	m_3	m_1
w_3	w_1	w_2	m_3	m_1	m_2
w_1	w_3	w_3	m_1	m_2	m_3

which violate the Eeckhout Condition. Nevertheless, this marriage problem has a unique stable matching:

$$\mu^* = [(m_1, w_3), (m_2, w_1), (m_3, w_2)].$$

Consider the following order of moves: $(m_1, m_2, m_3, w_1, w_2, w_3)$, i.e., m_1 moves first, m_2 moves next, etc. Let a_i be the action taken by agent i. The following strategy $\sigma = (\sigma_{m_1}, \sigma_{m_2}, \sigma_{m_3}, \sigma_{w_1}, \sigma_{w_2}, \sigma_{w_3})$ is the unique SPE:

$$\begin{aligned}
\sigma_{m_1} &= w_3; \\
\sigma_{m_2} &= \begin{cases} w_2 & \text{if } a_{m_1} \neq w_2 \\ w_1 & \text{if } a_{m_1} = w_2; \end{cases} \\
\sigma_{m_3} &= \begin{cases} w_1 & \text{if } a_{m_2} = w_2 \\ w_2 & \text{if } a_{m_2} \neq w_2; \end{cases} \\
\sigma_{w_1} &= \begin{cases} m_2 & \text{if } a_{m_2} = w_1 \\ m_3 & \text{if } a_{m_2} \neq w_1, \text{ and } a_{m_3} = w_1 \\ m_1 & \text{if } a_{m_2} \neq w_1, \ a_{m_3} \neq w_1 \text{ and } a_{m_1} = w_1 \\ w_1 & \text{otherwise;} \end{cases} \\
\sigma_{w_2} &= \begin{cases} m_3 & \text{if } a_{m_3} = w_2 \\ m_1 & \text{if } a_{m_3} \neq w_2, \text{ and } a_{m_1} = w_2 \\ m_2 & \text{if } a_{m_3} \neq w_2, \text{ and } a_{m_1} \neq w_2, \text{ and } a_{m_2} = w_2 \\ w_2 & \text{otherwise;} \end{cases} \\
\sigma_{w_3} &= \begin{cases} m_1 & \text{if } a_{m_1} = w_3 \\ m_2 & \text{if } a_{m_1} \neq w_3, \text{ and } a_{m_2} = w_3 \\ m_3 & \text{if } a_{m_1} \neq w_3, \text{ and } a_{m_2} \neq w_3, \text{ and } a_{m_3} = w_3 \\ w_3 & \text{otherwise.} \end{cases}
\end{aligned}$$

It is easy to see that the strategies σ_{w_1}, σ_{w_2}, and σ_{w_3} are sequentially rational. Since all the men move prior to all the women, no woman has any chance to propose. Hence the

[4]Eeckhout (2000) claims that the Eeckhout Condition is also necessary for the uniqueness of a stable matching when there are three men and three women. Although our example demonstrates that this is not the case, this discrepancy does not change our results

choices available to a woman are: accepting a proposal, if there is any, or rejecting all other proposals to her. Therefore, for a woman, her best response is to accept a proposal from the man whom she prefers most among those who proposed to her. The strategies σ_{w_1}, σ_{w_2} and σ_{w_3} describe such choices.

Now consider σ_{m_3}. Given w_1's preferences, if m_2 did not propose to w_1, then m_3's proposal to w_1 would be accepted by w_1. But if m_2 proposes to w_1, m_3 has no chance of being matched to w_1. Hence, his best strategy is: propose to w_1 if m_2 did not propose to her and propose to w_2 if m_2 proposed to w_1.

Consider σ_{m_2}. If m_1 did not propose to w_2, then m_2's proposal to w_2 would be accepted by w_2. Suppose m_1 proposed to w_2. Since w_2 prefers m_1 to m_2, m_2's proposal to w_2 would be rejected by w_2. Hence, his best strategy is: propose to w_2 if m_1 did not propose to w_2 and propose to w_1 if m_1 proposed to w_2.

Consider σ_{m_1}. Two possible best strategies are proposing to w_2 and w_3. Suppose he proposed to w_2. Then m_2 will propose to w_1 and m_3 will propose to w_2. The resulting matching is $[(m_1, \phi), (m_2, w_1), (m_3, w_2), (\phi, w_3)]$. Suppose he proposed to w_3. Then m_2 will propose to w_2 and m_3 will propose to w_1. The resulting matching is $[(m_1, w_3), (m_2, w_2), (m_3, w_1)]$. Therefore, proposing to w_3 is his best strategy. Using the backward induction argument, we can show that σ is the SPE and the resulting matching is

$$\mu_\sigma = [(m_1, w_3), (m_2, w_2), (m_3, w_1)].$$

μ_σ is not a stable matching since it is blocked by (m_1, w_2).

Now consider the following order of moves: $(w_1, w_2, w_3, m_1, m_2, m_3)$. For this order of moves where all the women move prior to all the men, one can show that the SPE outcome of this sequential matching game is the unique stable matching μ^*.

This example demonstrates that different orders of moves lead to different SPE outcomes. Under a list of preference profiles which does not satisfy the Eeckhout Condition, a SPE outcome may not even be a stable matching. This raises an open question whether or not the Eeckhout Condition is also necessary for the SPE outcome of every sequential matching game to be a stable matching of the corresponding matching problem. However, this does not exclude the possibility that, without the Eeckhout Condition, a stable matching can be a SPE outcome of a sequential game of a certain specific order of moves.

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