

On the Provision of Public Goods in Dynamic Contracts: Lack of Commitment

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Abstract

We study a model of efficient risk sharing between two agents, A and B, who enjoy a non-durable common good. Only agent B can provide the common good whereas agent A can merely contribute indirectly by making transfers to the provider, agent B. We consider self-enforcing equilibria in the absence of commitment. We characterize the Pareto frontier of the subgame perfect equilibrium payoffs. The main results are: First, the consumption of the public good is significantly more stable than are the private consumptions. Second, in the absence of aggregate uncertainty, agents' consumptions are invariant to distribution of income in most cases. In the remaining cases, private consumptions and continuation values covary positively with respective incomes. Third, if some first best allocation is sustainable, the long-term equilibrium converges to the first best allocation. Otherwise, agents' utilities oscillate over a finite set of values. We find that an increase in the provider's deviation lifetime utility shifts the frontier of the set of subgame perfect equilibrium payoffs to exclude the lowest values of the provider (hence the highest values of the other). A decrease in the provider's deviation lifetime utility shifts the frontier of the set to include lower values for the provider (hence higher values for the other).

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1 Introduction

The question of risk-sharing in the absence of commitment, and the efficiency of self-sustainable contracts, has been widely studied in the dynamic contracting literature. But what happens when parties not only have mutual insurance motives, but also enjoy the consumption of a shared common good? Which consumption allocations can be supported? What are the short term and long term implications of the optimal dynamic contract?

In this paper, we address the problem of two infinitely lived agents A and B , who receive a random income each period, and derive utility from the consumption of a private good and a public good. In the traditional model where agents care only about the private good and information is symmetric, agents devise incentive compatible contracts to insure each other against their income fluctuations. In the present case, the question of the allocation of total income between private good and public good is added to the decision making process, so that monetary transfers between agents serve two purposes now. Moreover, we assume that only agent B can provide the public good, which creates an asymmetry in the game.

Examples of such a situation are numerous. In the context of family economics where children are considered to be public goods, the interaction in a divorced or separated couple can be illustrated in a repeated game where only the custodial parent can spend on the child. The noncustodial parent can only make monetary transfers (or child support payments), to the custodial parent, who then allocates her post-transfer income between her private consumption and the child's consumption. Depending on the income levels each period, the transfers can go either way, serving as both risk-sharing and public good provision tools. Assuming symmetric information, it is a double sided lack of commitment problem, where parents voluntarily agree on a set of transfer payments and child expenditure, but where both can renege on their part of the contract at any time.

One can also adapt the present scenario to think of a self-interested institution whose prerogatives include tax collection from individuals, and who in return supplies a public good as well as insurance against low income shocks through a redistributive scheme.¹

Yet another example would be that of sovereign debt, where the government of the borrowing country can spend the debt in two ways; either by distributing it directly to the people to spend on consumption goods, or by investing it in capital and infrastructure. Assuming that the first way is more effective in ensuring reelection (one can look at it as a bribe to buy individuals' votes), it is comparable to the private spending above, while investing in capital and infrastructure serves better as a guarantee to repay the debt, and thus is comparable to the public good spending.

The timing of the game makes it all the more interesting, since it highlights the asymmetry in agents' incentives and payoffs. At the realization of the income shock, agents simultaneously transfer a nonnegative amount of income to each other. Agent A then consumes his post-transfer income, while agent B decides on how to split her income between her private consumption, and the public good consumption. Hence, while the optimal contract needs to ensure that agents A and B comply with the dictated transfer, it should also guarantee in a second term that agent B complies with the dictated income allocation.

This is the first paper which attempts two tasks: First, to characterize the properties of the

¹We would like to thank Árpád Ábrahám for pointing this out.

optimal dynamic contract between two agents who enjoy a common good when there is lack of commitment. Second, to investigate the effects of the presence of a public good on insurance, on incentives and on welfare.

In this sense, we build on the contracting literature by using the dynamic programming methods of Thomas and Worrall (1990) in order to characterize the Pareto frontier of subgame perfect equilibrium payoffs by using the continuation utility as a state variable. We also draw on Kocherlakota (1996) to model the efficient allocation of consumption in economies with risk averse agents and double-sided lack of commitment. There are numerous papers that follow his work, Alvarez and Jermann (2000, 2001), Kehoe and Levine (2001), Kehoe and Perri (2002) are prime examples. However, this strand of the literature has not considered the presence of public goods in models of insurance in the absence of commitment.

We look for the Pareto optimal subgame perfect equilibria (SPE) of the repeated game, and show that the worst SPE is the Stackelberg equilibrium of the stage game.

The paper is organized as follows. We start by setting up a general model where the agents play a repeated game. In the absence of outside enforcement, the threat of punishment is the Nash-Stackelberg equilibrium outcome. In that case, agent B, the provider, assimilates agent A's transfer with her other sources of income and decides on the provision of the public good according to her own preferences. Agent A takes agent B's spending rule as given and chooses his transfer optimally.

For the characterization of the optimal contract, we focus on the simplified case where the agents have identical preferences and the income shocks are iid. The restriction on the provision of the public good adds a new layer of incentive constraint: The provider is not only required to insure the other agent, but also to provide the "adequate" amount of public good. We find that in the optimal contract, the public good's consumption is significantly more stable across states and across Pareto weights. In the absence of aggregate uncertainty, parents are able to insure each other across the states in many cases, though the allocation of private consumption depends on the relative Pareto weight. In the remaining cases, agents' equilibrium values will covary positively with their income.

We conduct computational experiments to study the effects of heterogeneous preferences and child support policies. If the agent B's deviation lifetime utility increases, the set of subgame perfect equilibria shrink to exclude the lowest values of agent B (thus the highest values of agent A). If her deviation utility decreases, the set will expand to include lower values for agent B (thus higher values for agent A). Analogous argument will apply for agent A's values.

2 The Model

We study a repeated game between two agents: A and B , who each enjoy the consumption of a private good and a public good. Agent B is called the provider, so she is the only one capable of spending directly on the public good, whereas agent A is merely a contributor who can make monetary transfers to agent B .

Time is discrete and agents discount future utility at rate β . The time horizon is infinite. The state of the world in period τ is stochastic and is determined by the realization of a discrete i.i.d. random variable θ , with support equal to $\{1, \dots, S\}$. The probability that θ takes on the value s is denoted by π_s , where $\pi_s > 0$ for all s . Agents A and B 's incomes are denoted by Y_τ^a and Y_τ^b respectively, and are determined by the realization of θ in every period. We assume that agents cannot save, but they can transfer income to each other. Letting a_τ and b_τ be agents A and B 's consumptions of the private good respectively at time τ , and c_τ be that of the public good. Agent A 's lifetime utility at τ can be written as

$$E_\tau \sum_{r=0}^{\infty} \beta^r u(a_{\tau+r}, c_{\tau+r})$$

and agent B 's utility as

$$E_\tau \sum_{r=0}^{\infty} \beta^r z(b_{\tau+r}, c_{\tau+r}).$$

where $u(\cdot, \cdot)$ and $z(\cdot, \cdot)$ are increasing, strictly concave and continuously differentiable in both arguments.

2.1 First Best Allocation

We write the planner's problem where the agents' incomes are pooled in one resource constraint, and solve for the first best sequence of consumptions. Letting μ be the relative weight on agent B 's utility, the planner's problem at τ is

$$\begin{aligned} \max_{\{a_{\tau+r}, b_{\tau+r}, c_{\tau+r}\}_{r=0}^{\infty}} \quad & E_\tau \sum_{r=0}^{\infty} \beta^r [u(a_{\tau+r}, c_{\tau+r}) + \mu z(b_{\tau+r}, c_{\tau+r})] \\ \text{s.t.} \quad & a_{\tau+r} + b_{\tau+r} + c_{\tau+r} = Y_{\tau+r}^a + Y_{\tau+r}^b \quad \text{for all } r. \end{aligned}$$

so the objective function is also increasing and strictly concave in the public good consumption.

The first order conditions imply the following relation:

$$\frac{u_a(a_{\tau+r}, c_{\tau+r})}{z_b(b_{\tau+r}, c_{\tau+r})} = \frac{u_c(a_{\tau+r}, c_{\tau+r})}{z_b(b_{\tau+r}, c_{\tau+r}) - z_c(b_{\tau+r}, c_{\tau+r})} = \mu$$

This is a standard result which simply dictates that in the first best, the ratio of marginal utilities of both the private and public goods is constant across states and dates, and is equal to the relative welfare weight.

Rewrite the first order conditions.

$$\begin{aligned} u_a(a_s, c_s) - u_c(a_s, c_s) &= \mu z_c(b_s, c_s) \\ z_b(b_s, c_s) - z_c(b_s, c_s) &= \frac{1}{\mu} u_c(a_s, c_s) \end{aligned}$$

Consider a hypothetical case where each agent could decide on how much to allocate to the public good out of their income, without taking into account the other agent's action. In this case, agent B 's decision would be given by $z_b(b_s, c_s) = z_c(b_s, c_s)$. As for agent A , assuming that he had the power to spend on the public good, his policy function would set $u_a(a_s, c_s) = u_c(a_s, c_s)$. We call these conditions the intra-temporal optimality conditions and the consumptions that satisfy them individually optimal. The first order conditions imply that the intra-temporal optimality conditions of the agents are never satisfied. This is a standard result in settings with public goods and it holds even in the first best. The social planner will internalize the effects of public good consumption decisions on both agents' utilities. Note that as μ , the relative Pareto weight of agent B increases, her wedge decreases while that of agent A increases, and vice versa for agent A .

2.2 Subgame Perfect Equilibria

The interaction between agents in our environment involves a two-part decision making process in each period. At the beginning of period τ , both agents observe the realization of θ . Each agent simultaneously makes a nonnegative transfer: $t_\tau^a \in [0, Y_\tau^a]$ from agent A to agent B and $t_\tau^b \in [0, Y_\tau^b]$ from agent B to agent A . Agent A then consumes $a_\tau = Y_\tau^a - t_\tau^a + t_\tau^b$, while agent B decides how to split her disposable income between the public good, c_τ , and the private good, $b_\tau = Y_\tau^b + t_\tau^a - t_\tau^b - c_\tau$.

We define an allocation $\{t_\tau^a, t_\tau^b, c_\tau\}_{\tau=1}^\infty$ to be a vector of state-dependent transfers and public good consumptions. A period τ history in this game consists of a sequence of realizations for θ, t^a, t^b and c :

$$h^\tau = (\theta_1, t_1^a, t_1^b, c_1, \theta_2, t_2^a, t_2^b, c_2, \dots, \theta_{\tau-1}, t_{\tau-1}^a, t_{\tau-1}^b, c_{\tau-1}, \theta_\tau)$$

A strategy for agent A at τ is a mapping from possible histories at τ into a transfer. Agent B 's strategy is composed of two decisions, one for the transfer in the first part of the game and one for the public good consumption in the second part. That is, her choice of c depends on her

observation of θ as well as t^a and t^b . Hence, we define two separate strategies for agent B : the first as a mapping from the possible histories into a transfer amount and the second as a mapping from the possible histories and current transfer amounts into a public good consumption.

A subgame-perfect equilibrium specifies:

- i. A strategy for agent A such that his transfer after any history is optimal given agent B 's transfer and consumption strategies;
- ii. A transfer strategy for agent B such that her transfer after any history is optimal, given agent A 's strategy and her consumption strategy;
- iii. A consumption strategy for agent B given the observed history, current period transfers, and agent A 's and own transfer strategies.

Our aim is to characterize the Pareto frontier of the set of subgame-perfect equilibrium payoffs. To do that, we first identify the lowest subgame-perfect equilibrium payoff.

The presence of a public good prevents the agents from going into autarky in the standard sense of the term. Since agent A cares about the public good, he will find it optimal to transfer some of his income to agent B to be spent toward its provision (conditional on his income relative to that of agent B being higher than some threshold). Autarky in our case corresponds to agents playing a Stackelberg game every period. For a history h^τ and for current period transfers t_τ^a and t_τ^b , agent B 's optimal public good consumption strategy \hat{c}_τ must satisfy

$$\hat{c}_\tau = \arg \max_C z(Y_\tau^b + t_\tau^a - t_\tau^b - C, C).$$

Agent B 's optimal decision of public good provision in autarky is defined by the following first order condition

$$z_b(Y_\tau^b + t_\tau^a - t_\tau^b - \hat{c}_\tau, \hat{c}_\tau) = z_c(Y_\tau^b + t_\tau^a - t_\tau^b - \hat{c}_\tau, \hat{c}_\tau) \quad (1)$$

which defines \hat{c}_τ as a function that maps agent B 's disposable income, her own income and the transfer she receives from agent A net of her transfer to agent A , into public good consumptions. It is a straightforward division rule which depends only on the level of post-transfer income. We call this the Stackelberg rule.

Next, the optimal transfer strategy of agent B is to transfer zero to agent A , regardless of agent A 's strategy. Denote her optimal transfer policy as $\hat{t}_\tau^b = 0$. She then allocates the available income $Y_\tau^b + t_\tau^a$ between her private consumption and the public good expenditure according to the Stackelberg rule.

Finally, given a transfer from agent B and a public good consumption strategy, agent A picks the transfer that solves his problem

$$\begin{aligned} \hat{t}_\tau^a &= \arg \max_{T_\tau} && u(Y_\tau^a - T_\tau + t_\tau^b, \hat{c}_\tau) \\ \text{s.t.} &&& z_b(Y_\tau^b + T_\tau - \hat{t}_\tau^b - \hat{c}_\tau, \hat{c}_\tau) = z_c(Y_\tau^b + T_\tau - \hat{t}_\tau^b - \hat{c}_\tau, \hat{c}_\tau) \end{aligned} \quad (2)$$

where \hat{c}_τ denotes the level of public good chosen by agent B according to the Stackelberg rule, and agent B 's transfer to agent A is zero.

To summarize, in equilibrium, agent B transfers zero to agent A . Agent A decides on his transfer by using (2). Agent B takes the transfer and allocates her income according to the Stackelberg rule (1). We denote the corresponding transfers and consumption values by $\{t_\tau^{ast}, t_\tau^{bst}, c_\tau^{st}\}_{\tau=0}^\infty$.

This is the worst equilibrium. Agent B cannot credibly decrease the public good consumption to affect agent A 's payoff without reducing her own payoff as well. Agent A cannot cut his transfer without consequently decreasing the public good consumption. The resulting lifetime utilities for agent A and agent B are respectively,

$$V_{stack} = E_\tau \sum_{r=1}^{\infty} \beta^{r-1} u(Y_{\tau+r}^a - t_{\tau+r}^{ast}, c_{\tau+r}^{st}) \quad (3a)$$

$$W_{stack} = E_\tau \sum_{r=1}^{\infty} \beta^{r-1} z(Y_{\tau+r}^b + t_{\tau+r}^{ast} - c_{\tau+r}^{st}, c_{\tau+r}^{st}). \quad (3b)$$

Proposition 1 *An allocation $\{t_j^a, t_j^b, c_j\}_{j=\tau}^\infty$ is subgame perfect if and only if it satisfies*

$$u(Y_\tau^a - t_\tau^a + t_\tau^b, c_\tau) + E_\tau \sum_{r=1}^{\infty} \beta^r u(Y_{\tau+r}^a - t_{\tau+r}^a + t_{\tau+r}^b, c_{\tau+r}) \geq u(Y_\tau^a - \hat{t}_\tau^a + t_\tau^b, \hat{c}_\tau) + \beta V_{stack} \quad (4a)$$

$$z(Y_\tau^b + t_\tau^a - t_\tau^b - c_\tau, c_\tau) + E_\tau \sum_{r=1}^{\infty} \beta^r z(Y_{\tau+r}^b + t_{\tau+r}^a - t_{\tau+r}^b - c_{\tau+r}, c_{\tau+r}) \geq z(Y_\tau^b + t_\tau^a - c_\tau^*, c_\tau^*) + \beta W_{stack} \quad (4b)$$

for all dates and states, where \hat{c}_τ , \hat{t}_τ^a and c_τ^* are the optimal deviation transfer and consumptions defined as

$$\begin{aligned} \hat{t}_\tau^a &= \arg \max_T u(Y_\tau^a - T + t_\tau^b, \hat{c}_\tau) \\ \hat{c}_\tau &= \arg \max_C z(Y_\tau^b + \hat{t}_\tau^a - t_\tau^b - C, C) \\ c_\tau^* &= \arg \max_C z(Y_\tau^b + t_\tau^a - C, C). \end{aligned}$$

Proof. In Appendix A. ■

The two inequalities above simultaneously determine the subgame perfect allocations $\{t_j^a, t_j^b, c_j\}_{j=\tau}^\infty$.

It is useful to remember that the Stackelberg game per se starts only the period after the first deviation. But because of the timing of the stage game, a deviation by agent A sets off a Stackelberg reaction on behalf of agent B in the second part of the present period, whereas a deviation by agent B only has repercussions on agent A 's actions starting the following period.

Thus, the right hand side of the first inequality (4a) is agent A 's payoff if he deviates by transferring the amount \hat{t}_τ^a (defined as his best response to the transfer t_τ^b and the consequent consumption \hat{c}_τ), inducing agent B to split her income according to the Stackelberg rule this period, and reverting to the Stackelberg equilibrium forever.

The right hand side of the second inequality (4b) is agent B 's payoff if she deviates by transferring nothing to agent A , splitting her disposable income according to the Stackelberg rule, and reverting to the Stackelberg equilibrium forever.

Consider an allocation $\{t_j^a, t_j^b, c_j\}_{j=\tau}^\infty$ satisfying the conditions above, and let the agents follow a strategy whereby they transfer and consume the amounts dictated by the allocation as long as both have done so in the past, otherwise, they revert to the Stackelberg equilibrium. We say that these strategies form a contract.

Existence of subgame perfect allocations that are not Stackelberg allocations is not straightforward to see. One could imagine a contract which dictates in a certain state a substantial transfer from agent A to agent B (say when agent A 's income is high and agent B 's is low), and a high public good expenditure to match that. If agents care a lot about their private consumption, and not enough about the future, both of the above incentive constraints could bind or even be violated.

In Appendix C, we provide proofs and conditions for the existence of subgame perfect allocations that are not Stackelberg allocations, for the different utility specifications we study. In general, these conditions depend on the preferences of the agents, the discount factor and the distribution of the income shock. For the general case, we assume that these parameters are such that such equilibria exist.

2.3 Recursive Formulation

Definition *A subgame perfect allocation is efficient if and only if there is no other subgame perfect allocation that Pareto dominates it, and an optimal contract is one which implements such an allocation.*

Let V_{\max} be the maximal payoff agent A can obtain in a subgame-perfect equilibrium, and W_{\max} be that attainable by agent B . Define the function $V : [W_{\text{stack}}, W_{\max}] \longrightarrow [V_{\text{stack}}, V_{\max}]$

to be the following.

$$\begin{aligned}
V(W) &= \max_{\{t_\tau^a, t_\tau^b, c_\tau\}_{\tau=1}^\infty} E_0 \sum_{\tau=1}^\infty \beta^{\tau-1} u(Y_\tau^a - t_\tau^a + t_\tau^b, c_\tau) \\
\text{s.t.} & \quad \{t_\tau^a, t_\tau^b, c_\tau\}_{\tau=1}^\infty \text{ is a subgame perfect allocation.} \\
& \quad E_0 \sum_{\tau=1}^\infty \beta^{\tau-1} z(Y_\tau^b + t_\tau^a - t_\tau^b - c_\tau, c_\tau) \geq W
\end{aligned}$$

One can think of agent A choosing the allocation $\{t_\tau^a, t_\tau^b, c_\tau\}_{\tau=1}^\infty$ to maximize his utility, while providing agent B with an ex-ante promised lifetime utility W , and respecting the incentive constraints for every possible history. The function V is the Pareto frontier of subgame perfect equilibrium payoffs.

Following Thomas and Worrall (1990) and Kocherlakota (1996), we can characterize the frontier V by the recursive program

$$\begin{aligned}
V(W) &= \max_{\{t_s^a, t_s^b, c_s, W_s\}_{s=1}^S} \sum_{s=1}^S \pi_s \{u(Y_s^a - t_s^a + t_s^b, c_s) + \beta V(W_s)\} \\
\text{s.t.} & \quad \sum_{s=1}^S \pi_s \{z(Y_s^b + t_s^a - t_s^b - c_s, c_s) + \beta W_s\} \geq W \\
& \quad z(Y_s^b + t_s^a - t_s^b - c_s, c_s) + \beta W_s \geq z(Y_s^b + t_s^a - c_s^*, c_s^*) + \beta W_{stack} \quad \text{for all } s \\
& \quad u(Y_s^a - t_s^a + t_s^b, c_s) + \beta V(W_s) \geq u(Y_s^a - \hat{t}_s^a + t_s^b, \hat{c}_s) + \beta V_{stack} \quad \text{for all } s \\
& \quad t_s^a \geq 0 \quad \text{for all } s \\
& \quad t_s^b \geq 0 \quad \text{for all } s \\
& \quad W_s \in [W_{stack}, W_{\max}] \quad \text{for all } s.
\end{aligned}$$

Taking agent B 's promised lifetime utility as a state variable, agent A chooses state contingent transfer amounts, public good expenditures and continuation values for agent B . This immediately specifies his own value from the contract, as his own continuation utility has to be on the Pareto frontier.

The second and third constraints are agent B 's and agent A 's incentive constraints. They ensure that the contract is self-enforceable. The non-negativity constraints on the transfers follow. Finally, the last constraint puts bounds on agent B 's utility from the contract. Agent B cannot be kept in the contract if her utility from it is less than her Stackelberg utility. On the other hand, her maximum value is limited by agent A 's minimum utility, V_{stack} , and is determined endogenously.

It is worthwhile to analyze briefly the deviation utilities of the agents. When agent B deviates, she takes agent A 's transfer that is dictated by the contract, makes no transfer to him, and

allocates her disposable income $Y_s^b + t_s^a$ according to the Stackelberg rule, equation 1). When agent A deviates, he takes agent B 's transfer in the contract, and decides on how much to transfer to agent B according to his autarky rule, equation 2). Agent B responds to this by using equation (1) to determine the public good consumption, in this case, given by \hat{c}_s .

The restriction on the provision of the public good, i.e. agent B being the sole provider, adds another layer of incentive constraints to the general insurance problem. Agent A 's transfers, and subsequently the public good consumption levels, are limited above by agent B 's consumption levels and continuation values. For example, even if agent A is hit by a very high income shock in one state and would want to split that surplus between his private consumption and the public good consumption, he is limited in doing so because any large transfer to agent B that is not matched by a substantial private consumption or continuation utility for her, would lead her to deviate.

The next proposition places a constraint on the direction of the transfers in the contract.

Proposition 2 *In the optimal contract, in any state s , either $t_s^a \geq 0$ and $t_s^b = 0$ or $t_s^a = 0$ and $t_s^b \geq 0$.*

Proof. In Appendix A. ■

Intuitively, consider two contracts, the first dictating that both agents transfer positive amounts to each other at the same time, $t_s^a > 0$ and $t_s^b > 0$, the second dictating that the difference $|t_s^a - t_s^b|$ be transferred from the corresponding agent. Remember that the agent who deviates keeps the transfer sent by the other agent. In the first contract, the agent who deviates will have a higher income at his or her disposal in the period of deviation, since he or she will not make the transfer dictated, but keep the transfer of the other agent. In the second contract, if the agent who deviates is not the transferrer, he or she will only be able to keep the difference of the original transfers. If the agent who deviates is the transferrer, he or she will not receive any transfers from the other agent. Now, all the allocations that are available under the first contract are also available in the second one. However, it is clear that under the second contract the deviation utilities are weakly lower, making deviating less likely. Hence, the second contract supports a wider range of subgame perfect equilibria.

Lemma 3 *Agent A 's incentive constraint does not bind in a state where he is a net receiver.*

Proof. In Appendix A. ■

Intuitively, this is because when agent A is a net receiver, he can only deviate by transferring more than the contract dictated amount, which would increase agent B 's utility. This could not be a profitable deviation, since the contract would not have been Pareto optimal.

Proposition 4 *In the states where agent A is the net transferrer in autarky and in the contract, the equilibrium transfer, public good consumption and continuation values are invariant to redistribution of income (as long as agent A is still the transferrer after the redistribution). For the states where agent B is the net transferrer, this will not generally hold.*

Proof. In Appendix A. ■

To see why this proposition holds, note that the Stackelberg equilibrium displays invariance to redistribution of income so long as the redistribution preserves a positive transfer from agent A to agent B . Now, suppose that in the contract agent A is the transferrer in a set of states for a given promised utility to agent B , and there is no aggregate uncertainty. His deviation utility is the same in all these states, equal to the Stackelberg utility.

Optimality then dictates that his post-transfer income be the same in all the states within that set, since he would not choose a positive transfer that leaves him poorer in one state than another. It follows that the consumption allocation and the continuation utilities for states in this set will be invariant to the distribution of income.

This result will not generally hold for agent B . If agent B is the transferrer in the contract for a given set of states, her deviation transfer is zero for these states, which means her deviation utility will be state dependent, as will be the compensation necessary to keep her in the contract. As a result, the distribution of income will matter for the states in which agent B is the transferrer.

This result has an interesting implication for the contract. In particular, it means that when agent A is the net transferrer, a redistribution in income (such that agent A is still the transferrer) does not alter the incentives and actions of the agents in any way. Their decisions depend only on the final income at their disposal after the transfer. This feature of the model suggests that in many cases, what matters for the public good consumption is not the distribution of income between the two agents, but rather, their aggregate income.

Standard results of two-sided lack of commitment models are that if one agent's incentive constraint binds in a state, he/she is compensated by higher consumptions and the continuation values. In the presence of a public good, it is not trivial to see how the agents are compensated when their incentive constraints bind. We heuristically address the following questions: Is it better to provide higher consumption of the private good, or of the public good? Is there an optimal combination of the two that should be offered? Below, we present some answers using the first order conditions of the problem without assuming a functional form for the agents. We conduct the following exercise assuming that the first order conditions are necessary and sufficient, which they are for the utility functions used in this paper.² Let μ be the multiplier associated with the promise keeping constraint, and $\lambda_s^a, \lambda_s^b, \phi_s^a$ and ϕ_s^b the multipliers associated with incentive constraints for agent A and agent B , and the nonnegativity constraints, for each

²See Appendix B for a more rigorous treatment of the issue for the utility functions we use later in the paper.

state s , respectively. The first order conditions and the envelope condition imply the following

$$u_a(a_s, c_s) - u_c(a_s, c_s) = \frac{\mu + \lambda_s^b}{1 + \lambda_s^a} \left(z_c(b_s, c_s) - \lambda_s^b z_b(\hat{b}_s, \hat{c}_s) + \phi_s^a \right) \quad (5a)$$

$$u_a(a_s, c_s) - u_c(a_s, c_s) = \frac{\mu + \lambda_s^b}{1 + \lambda_s^a} \left(z_c(b_s, c_s) - \phi_s^b \right) \quad (5b)$$

$$z_b(b_s, c_s) - z_c(b_s, c_s) = \frac{1 + \lambda_s^a}{\mu + \lambda_s^b} u_c(a_s, c_s) \quad (5c)$$

$$(1 + \lambda_s^a) V'(W_s) = V'(W) - \lambda_s^b$$

where \hat{b}_s and \hat{c}_s refer to the deviation consumption levels of agent B in state s .

Assume that we are at a given promised utility W to agent B . Consider states in which agent A 's incentive constraint binds, i.e. $\lambda_s^a > 0$. He is compensated in two ways: first through consumption levels that make the wedge between his intra-temporal optimal consumption and his actual consumption smaller (while making that of agent B wider), and second, by providing him with a higher continuation utility.

In states where agent B 's incentive constraint binds, i.e. λ_s^b , the identity of the transferrer matters. When agent B is transferring, she is compensated by consumption levels that decrease the gap between her actual consumption and her intra-temporal optimal consumption (and increase that gap for agent A), and her continuation utility increases while that of agent A decreases.

In the states where agent B 's incentive constraint binds and she is a receiver, her wedge is smaller and her continuation utility is larger, as above. However, the magnitude of agent A 's wedge and its sign depend on how his transfer affects agent B 's incentive to deviate. The right-hand side equation (5a) captures this. The multiplier on agent B 's incentive constraint λ^b both increases and decreases the wedge of agent A . Agent A compensates agent B by transferring a larger amount to her and/or letting her consume more of the private good and less of the public good. Making a larger transfer to agent B would affect her incentives adversely. So, agent A might decrease his transfer so much that agent B has very little to spend on the provision of the public good.

It is necessary to assign functional forms to further characterize the optimal contract. We assume that the utility functions of the agents are of the Cobb-Douglas form for analytical tractability. Cobb-Douglas constitutes a widely used benchmark.

3 Cobb Douglas Utility Functions with iid Income Shocks

In the case where only one agent can spend on the public good, mutual insurance becomes more valuable. If agent B is facing high income risk, agent A would have an incentive to insure her given that the public good consumption will also fluctuate with her income. As for agent B , she would only need to insure agent A to keep him in the contract, which implies that she may make transfers when his income is relatively low. Hence, transfers might go both ways.

We characterize the short term and long term equilibria of the model for the special case where the agents' preferences are identical and there is no aggregate uncertainty. Later on, we compute the model to allow for agents who have different preferences over the public good. Formally, the utility functions of agent A and agent B are given by

$$\begin{aligned} u(a_s, c_s) &= a_s^{1-\alpha} c_s^\alpha \\ z(b_s, c_s) &= b_s^{1-\alpha} c_s^\alpha \end{aligned}$$

and the aggregate income $Y_s^b + Y_s^a$ in any state s is equal to Y .

3.1 Optimal deviation strategies

We start by finding the optimal deviation strategies of the agents, as well as their payoffs from the Stackelberg equilibrium.

We know that agent B 's optimal deviation transfer is zero, since she does not care about agent A 's consumption. Given a non-negative transfer t_s^a from agent A , agent B allocates her income optimally between her own consumption and the public good's according to the following policy function:

$$c_s^* = \alpha (Y_s^b + t_s^a).$$

Her deviation utility will be

$$z(Y_s^b + t_s^a - c_s^*, c_s^*) = G(Y_s^b + t_s^a)$$

where $G = (1 - \alpha)^{1-\alpha} \alpha^\alpha$. We already saw that agent A would never deviate when $t_s^b > 0$. When agent A deviates in his transfer, he anticipates agent B 's response to be the Stackelberg consumption rule. So given agent B 's optimal consumption response, and the nonnegativity constraint on \hat{t}_s^a , the decision rules of agent A and agent B in equilibrium are as follows

$$\begin{aligned} \hat{t}_s^a &= \max \{0, \alpha Y_s^a - (1 - \alpha) Y_s^b\} \\ \hat{c}_s &= \max \{\alpha Y_s^b, \alpha^2 Y\} \end{aligned}$$

agent A 's deviation utility will be

$$u(Y_s^a - \hat{t}_s^a, \hat{c}_s) = \begin{cases} \alpha^\alpha GY & \text{if } \hat{t}_s^a = \alpha Y_s^a - (1 - \alpha)Y_s^b \\ \alpha^\alpha (Y_s^a)^{1-\alpha} (Y_s^b)^\alpha & \text{if } \hat{t}_s^a = 0 \end{cases} \quad (6)$$

After the first deviation by any of agent A or agent B , the agents play the Stackelberg strategies forever. For states where the Stackelberg transfer is positive, the resulting Stackelberg consumption levels are all a constant fraction of total income, and depend only on the preference parameter α . In other words, the Stackelberg consumption allocations will be invariant to distribution of income. The Stackelberg continuation values can be written as in equations (3a) and (3b).

3.2 Optimal Subgame Perfect Equilibria

Let the function $V : [W_{stack}, W_{max}] \rightarrow [V_{stack}, V_{max}]$ be the Pareto frontier of the set of all the subgame perfect equilibrium values. It can be characterized by the following recursive program

$$\begin{aligned} V(W) = & \max_{\{t_s^a, t_s^b, c_s, W_s\}_{s \in S}} \sum_{s=1}^S \pi_s \{(Y_s^a - t_s^a + t_s^b)^{1-\alpha} c_s^\alpha + \beta V(W_s)\} \\ \text{s.t. } & \sum_{s=1}^S \pi_s \{(Y_s^b + t_s^a - t_s^b - c_s)^{1-\alpha} c_s^\alpha + \beta W_s\} \geq W \\ & (Y_s^b + t_s^a - t_s^b - c_s)^{1-\alpha} c_s^\alpha + \beta W_s \geq G(Y_s^b + t_s^a) + \beta W_{stack} \quad \text{for all } s \\ & (Y_s^a - t_s^a + t_s^b)^{1-\alpha} c_s^\alpha + \beta V(W_s) \geq u(Y_s^a - \hat{t}_s^a, \hat{c}_s) + \beta V_{stack} \quad \text{for all } s \\ & t_s^a \geq 0 \quad \text{for all } s \\ & t_s^b \geq 0 \quad \text{for all } s \\ & W_s \in [W_{stack}, W_{max}] \quad \text{for all } s \end{aligned}$$

What are the basic properties of this dynamic problem? Lemma 3 shows that agent A 's incentive constraint is trivially satisfied when he is the receiver. Hence we do not need to worry about the choice variable in the left hand side of agent A 's incentive constraint. Moreover, it is straightforward to show that for CES utility functions, the value functions are linearly homogenous in income. Hence, agent B 's deviation utility is linear in t_s^a , allowing for a convex constraint set. So the frontier is concave. As for differentiability, we refer to Koepl (2003) who finds sufficient conditions for the differentiability of the efficient frontier in risk sharing problems with lack of commitment. His proof is adapted to Kocherlakota-type setting, but extends immediately to our model.³

³His conditions are as follows. Let S_1 be the set of states where agent one's incentive constraint binds, and S_2 be the set of states where agent two's incentive constraint binds. If $S_1 \sqcup S_2 \neq S$ at W_0 , then V is differentiable at W_0 . If there exists an incentive compatible first best allocation, then V is differentiable everywhere.

Given that the problem is convex, the first order conditions are necessary and sufficient. Let μ be the multiplier associated with the promise keeping constraint, and $\lambda_s^a, \lambda_s^b, \phi_s^a$ and ϕ_s^b the multipliers associated with state by state incentive constraints for agent A and agent B , and the nonnegativity constraints, respectively. Assuming that we are at an interior solution for W_s , the first order conditions for each state s are below.

$$\begin{aligned}
(1 + \lambda_s^a)(1 - \alpha)(Y_s^a - t_s^a + t_s^b)^{-\alpha} c_s^\alpha &= (\mu + \lambda_s^b)(1 - \alpha)(Y_s^b + t_s^a - t_s^b - c_s)^{-\alpha} c_s^\alpha - \lambda_s^b G + \phi_s^a \\
(1 + \lambda_s^a)(1 - \alpha)(Y_s^a - t_s^a + t_s^b)^{-\alpha} c_s^\alpha &= (\mu + \lambda_s^b)(1 - \alpha)(Y_s^b + t_s^a - t_s^b - c_s)^{-\alpha} c_s^\alpha - \phi_s^b \\
(1 + \lambda_s^a)\alpha(Y_s^a - t_s^a + t_s^b)^{1-\alpha} &= (\mu + \lambda_s^b)\left((1 - \alpha)(Y_s^b + t_s^a - t_s^b - c_s)^{-\alpha} c_s^\alpha - \alpha(Y_s^b + t_s^a - t_s^b - c_s)^{1-\alpha} c_s^{\alpha-1}\right) \\
V'(W_s)(1 + \lambda_s^a) &= -\mu - \lambda_s^b
\end{aligned}$$

The envelope condition gives:

$$V'(W) = -\mu$$

Our aim is to characterize the frontier. We prove in Appendix C that both of the incentive constraints cannot simultaneously bind in a given state s . So, for any given promised value W to agent B , we can divide the states into four sets.

- S_0^W where no incentive constraint binds.
- S_a^W where agent A 's incentive constraint binds.
- S_{b1}^W where agent B 's incentive constraint binds and $t_s^b > 0$.
- S_{b2}^W where agent B 's incentive constraint binds and $t_s^a > 0$.

When drawing the figure for the Pareto frontier, the origin depicts the Stackelberg values for both agents.⁴ We let the values on the X-axis denote agent B 's lifetime utility, while those on the Y-axis denote agent A 's lifetime utility.

In the following sections, we look at how each of the four sets of states evolves as we increase or decrease the value W . Generally, the incentive constraint of an agent is more likely to bind for lower values of his or her lifetime utility. Hence, in the left region of the frontier, agent B 's incentive constraint will bind, in the right, agent A 's will bind. Consequently, as the value of one agent increases (and the other agent's decreases), the number of states for which none of the incentive constraints bind first increases, then decreases.

More precisely, we find that the set of states for which agent A 's incentive constraint binds will be non-empty for lower values of agent A . Also, his incentive constraint will bind first in

⁴See Appendix.

states where his income is relatively high, then as his value decreases, for the lower relative income states as well.

The sets of states for which agent B 's incentive constraint binds will be non-empty for lower values of agent B . The set where agent B is the transferrer expands as we move towards lower values for her. Also, her incentive constraint will bind first in states where her income is relatively high, then for the lower income states, analogously to agent A 's case.

For the set of states where agent A is the transferrer, we also show that if agent B 's incentive constraint binds in one state where agent A transfers, it will bind in all states where agent A transfers. Since this is more likely as agent B 's value increases, the set where agent A is the transferrer will grow as we move towards the range of values where the constrained efficient equilibrium is sustainable in the contract.

First, we know that if for a given promised utility, the incentive constraints do not bind in a set of the states, the agents will be able to achieve the constrained efficient allocation in this set of states in equilibrium. So we start by characterizing the consumptions and the continuation values for the states in the set S_0^W , which will be used as the benchmark. As for the specifics of how this set changes with income and with values of the agents, it is useful to think of it as the complement of the union of the above other sets.⁵

3.2.1 States where no incentive constraint binds

Let us first establish the first best and the constrained efficient levels of consumption. Letting $M_0 = \mu^{\frac{1}{\alpha}}$, the first order conditions and the envelope condition imply the following.

$$\begin{aligned} c_s &= \alpha Y & (7) \\ b_s &= \frac{M_0}{1 + M_0} (1 - \alpha) Y \\ a_s &= \frac{1}{1 + M_0} (1 - \alpha) Y. \end{aligned}$$

As expected, we get the perfect insurance outcome where agent A and agent B 's consumptions are constant across states, and depend only on agent B 's promised utility. More significantly, though, the public good consumption is a constant fraction of total income across all states and promised utilities. Each agent, if he or she were to have complete control over the allocation of the total income, would choose $c = \alpha Y$ and consume the rest his/herself as the intra-temporal optimality conditions dictate. Even though the agents cannot implement this particular allocation, the public good consumption will reflect the agreement over how much public good should

⁵Note that for a given W , $S_0^W \cup S_a^W \cup S_{b_1}^W \cup S_{b_2}^W = S$. To see how S_0^W moves over the W -axis, we ask the reader to think of this set as the complement of the union of all other sets.

be provided. The agents argue only over how to allocate the residual income once the public good consumption is set. Given that the public good consumption is constant across relative Pareto weights, the values of the agents vary only with the changes in their private consumptions. As agent B 's value W increases, agent B 's share of consumption increases at the expense of agent A 's.

We can go back and retrieve the transfers for a specific value W of agent B to help us determine the identity of the transferrer:

$$t_s^a - t_s^b = \frac{\alpha + M_0}{1 + M_0} Y_s^a - \frac{1 - \alpha}{1 + M_0} Y_s^b$$

If this difference is positive, agent A is making the transfer, otherwise, agent B is. Notice that for the same state s , the difference in transfers $t_s^a - t_s^b$ is increasing in W . That is, the higher agent B 's promised value, the larger is the net transfer from agent A to agent B (by the concavity of the Pareto frontier). This difference is increasing in agent A 's income, and decreasing in agent B 's.

The envelope condition finally tells us that $V'(W_s) = V'(W)$, so the continuation values stay the same.

We refer to these values as the constrained efficient consumption and transfer values throughout the paper and we denote them with the FB superscript.

3.2.2 States where agent A 's incentive constraint binds

This set of states is characterized by the following consumption levels and continuation utilities.

$$\begin{aligned} c_s &= \alpha Y \\ m_s &= \frac{M_s^a}{1 + M_s^a} (1 - \alpha) Y \\ f_s &= \frac{1}{1 + M_s^a} (1 - \alpha) Y \\ t_s^a &= \frac{\alpha + M_s^a}{1 + M_s^a} Y_s^a - \frac{1 - \alpha}{1 + M_s^a} Y_s^b \\ V'(W_s) &= -\frac{\mu}{1 + \lambda_s^a} \end{aligned}$$

where $M_s^a = \left(\frac{\mu}{1 + \lambda_s^a} \right)^{\frac{1}{\alpha}}$.

The notable result is that consumption of the public good is equal to the first best value, so again is independent of state and promised utility. There is a gain from keeping the consumption of the public good constant because it is difficult to use it as a reward or punishment tool. One cannot reward one agent by increasing c_s without rewarding the other agent as well.

In order to stay in the contract, agent A would like to consume more, and in a proportion that is closer to what is dictated by his intra-temporal optimality condition. The most efficient way is to achieve that is by keeping c_s constant, and diverting resources away from agent B to the agent A. Since $M_s^a < M_0$, compared to the constrained efficient levels, agent B's consumption must be lower and agent A's consumption higher. From the concavity of the frontier V , we can also deduce that $W_s \leq W$. Agent A's continuation value increases while that of agent B decreases.

Notice for a given state s , the difference in transfers $t_s^a - t_s^b$ is increasing in W . That is, the higher agent B's promised value, the larger is the net transfer from agent A to agent B (by the concavity of the Pareto frontier). This difference is increasing in the agent A's income, and decreasing in agent B's. Since agent A's incentive constraint can only bind if he's a net transferrer, this is more likely to happen when the agent B's value is high, and when agent A's income is high.

For the rest of the analysis of S_a^W , it is useful to consider two different types of income processes. First, income processes which imply that agent A is the transferrer in all states in autarky. Second, income processes which imply that agent A is the transferrer in only some states in autarky, if any.

Consider the first group of income processes. Agent A being the transferrer in all states in autarky implies that his income is high enough relative to that of agent B's that even in autarky, he chooses to transfer to agent B in all of the states. That is, he suffers more from the volatility of the public good consumption than of his private consumption. His gain from the contract is generated only by securing a steady consumption of the public good through insuring agent B.

It can easily be shown that if agent A is the transferrer in a given state in autarky, he is the transferrer in that state in the optimal contract as well. This implies that under the specified income process, agent A is the transferrer in all the states in the contract. Proposition 4 then shows that in the absence of aggregate uncertainty, the consumption levels and the continuation values will be constant across states for a given promised utility to agent B. Thus, if agent A's incentive constraint binds in state s for a given W , it will bind in all states for W . If agent A's incentive constraint binds for all states, it means that in every state, he's indifferent between staying in the contract or reverting to the Stackelberg equilibrium forever. His value in the contract then must be equal to his Stackelberg utility, which means agents are at the point $((W_{\max}, V(W_{\max}))$ of the frontier.

In other words, if the income process is such that agent A is the transferrer in all of the states in autarky, the incentive constraint of agent A binds in all states at his lowest sustainable value, V_{stack} only. Proposition 5 summarizes these results.

Proposition 5 *If $t_s^{st} > 0$ for all s , then $S_a^W = \{1, 2, \dots, S\}$ for $W = W_{\max}$ and $S_a^W = \emptyset$ for $W \neq W_{stack}$.*

Proof. In Appendix A. ■

Now, consider the income processes for which agent A is the transferrer only in some states in autarky. If agent A 's income is relatively low in some states such that he transfers nothing in autarky for these states, he needs to be insured by agent B in the contract against low income shocks. This can be done in two ways, by a direct transfer from agent B to agent A and/or by compensation through the public good consumption. Given Lemma 3, we can ignore the states in which agent B is a transferrer since they will never be in S_a^W .

As the value of agent B increases, agent A will transfer more and more to her, while letting her allocate her income more and more independently. Moreover, as the promised utility to agent B increases, agent A transfers in his lower income states as well. S_a^W will expand from higher income levels of agent A to lower ones as we move right along the W -axis. See Figure S_a^W in the appendix.

Consider a state $j \in S_a^W$. If the incentive constraint of agent A binds in j , it binds in that state for all higher promised utilities to agent B . So, there is a threshold W for each state j where agent A 's incentive constraint starts binding. The consumptions and continuation values in state j are equated across promised utilities to agent B . This means that in Figure S_a^W we will observe consumption levels and continuation values in state j to be constant for all values between W and W_{max} . An immediate implication is that, regardless of the promised utility to agent B , once agent A 's incentive constraint binds, he immediately jumps to a continuation utility for which his incentive constraint does not bind at any state. Since his continuation utility is always greater than his ex-ante value, he cannot jump to another value where his incentive constraint binds in some state.

It also follows that agent A 's Stackelberg value V_{stack} is the only value for which agent A 's incentive constraint binds in all states. Assume that agent A 's incentive constraints bind for W and \tilde{W} . Then we know that the consumption levels and the continuation utilities will be the same across states and across ex-ante values. In this case, agent A cannot provide two distinct promised utilities to agent B , that is, the promise keeping constraint will be violated.

Proposition 6 proves the above claims.

Proposition 6 *Assume that $t_s^{st} \geq 0$ for all s with equality for some s , then the following holds.*

- i. If $s \in S_a^W$, then $s \in S_a^{\tilde{W}}$ for all $\tilde{W} \geq W$.*
- ii. If $s \in S_a^W$, and $t_s^{st} = 0$, then for any j such that $Y_j^a > Y_s^a$, we have $j \in S_a^{\tilde{W}}$.*
- iii. If s lies in S_a^W and r lies in $S_a^{\tilde{W}}$, and $t_s^{st} \geq t_r^{st} = 0$, then $a_s \geq a_r$ and $V(W_s) \geq V(\tilde{W}_r)$.*

Proof. In Appendix A. ■

We now turn to the states where agent B 's incentive constraint binds.

3.2.3 States where agent B 's incentive constraint binds and she is the transferrer

The consumption levels and the continuation values are given by the following.

$$\begin{aligned}
 c_s &= \alpha Y \\
 m_s &= \frac{M_s^b}{1 + M_s^b} (1 - \alpha) Y \\
 f_s &= \frac{1}{1 + M_s^b} (1 - \alpha) Y \\
 t_s^b &= \frac{1 - \alpha}{1 + M_s^b} Y_s^b - \frac{\alpha + M_s^b}{1 + M_s^b} Y_s^a \\
 V'(W_s) &= -\mu - \lambda_s^b.
 \end{aligned}$$

where $M_s^b = (\mu + \lambda_s^b)^{\frac{1}{\alpha}}$. Since $M_s^b > M_0$, compared to the constrained efficient levels, agent B 's consumption must be higher and agent A 's consumption lower, shrinking the wedge of agent B 's intra-temporal optimality condition and expanding that of agent A . Again by concavity of the frontier V , $W_s \geq W$, so agent B 's continuation value increases and that of agent A decreases.

Once more, the consumption of the public good is equal to the efficient value, and is independent of state and promised utility.

In principle, the agent B 's incentive constraint can bind whether she is a net transferrer or a net receiver. However, given that agent B transfers a positive amount in a state to provide insurance to agent A against his low income shock, she is more likely to transfer to agent A in states where her income is high or when her promised utility is low. So for the states where she is a transferrer, her incentive constraint is more likely to bind when her income is high or her value is low. If agent B 's incentive constraint binds and she is transferring in a state s for a given promised utility, her incentive constraint binds in state s for all lower promised utilities and in all states where she has higher income as well. There is then a $W_{s_{\max}}^1$ which is the maximum continuation value corresponding to agent B 's maximum income $Y_{s_{\max}}^b$, and beyond which no incentive constraint of agent B binds with $t_s^b > 0$.

It is also straightforward to show that if agent B is the transferrer in a state where her incentive constraint binds in the contract, the transfer in autarky for that state will be equal to zero.⁶ This implies that the period deviation utility of the agents in these states are state-dependent. Hence, if agent B is transferring in a state where her incentive constraint binds, her deviation utility, thus her consumption values and continuation utilities, are state dependent. More precisely, they are increasing in her income. State dependency immediately implies that they are constant across promised utilities for which her incentive constraint binds in a given

⁶See Appendix A.

state.

Figure $S_{b_1}^W$ depicts a presentation of this set over her income levels and values.

Proposition 7 *i. If $s \in S_{b_1}^W$, then $s \in S_{b_1}^{\tilde{W}}$ for all $\tilde{W} \leq W$.*

ii. If $s \in S_{b_1}^W$, then for any j such that $Y_j^b > Y_s^b$, we have $j \in S_{b_1}^W$.

iii. If s lies in $S_{b_1}^W$ and in $S_{b_1}^{\tilde{W}}$, then $b_s = b_s$ and $W_s = \tilde{W}_s$.

iv. If s and r lie in $S_{b_1}^W$, and $Y_s^b > Y_r^b$, then $b_s > b_r$ and $W_s > W_r$.

Proof. In Appendix A. ■

If Agent B 's incentive constraint binds in all states, and she is the transferrer in all of them, then her lifetime utility has to be the lowest sustainable lifetime utility, W_{stack} . Proposition 8 proves this claim formally.

Proposition 8 *If $S_{b_1}^W = \{1, 2, \dots, S\}$ and $t_s^b > 0$ for all s , then $W = W_{stack}$.*

Proof. In Appendix A. ■

Finally, we turn to the states where agent B 's incentive constraint binds and she's a net receiver. This is maybe the most interesting case, since it captures the incentive problem related to the provision of the public good, as opposed to the insurance problem. It is also empirically the most relevant case.

3.2.4 States where agent B 's incentive constraint binds and agent A is the transferrer

The results for the above sets will not easily extend for this set. We were able to prove above that if the incentive constraints of an agent were binding in all of the states for a given promised utility, that agent was receiving his or her autarky lifetime utility. This result is specific to the cases where the agent whose incentive constraint binds is the transferrer. Here, there are a set of continuation utilities for which agent B 's incentive constraint binds for all states. This is a direct implication of agent B 's deviation utility being a function of the transfer dictated by the contract. Remember that agent B 's disposable income in these states is given by $Y_s^b + t_s^a$. Hence, different transfers will imply different deviation utilities for agent B . In the case where agent B 's incentive constraints bind in all the states and she is the receiver in all the states, she will typically receive a higher lifetime utility than her Stackelberg lifetime utility.

Across the states where agent B 's incentive constraint binds, agent A transfers an amount in each state such that he can smooth out the consumption levels and the continuation utilities (Proposition 4). To be able to equate the consumptions and continuation values for different states, agent A transfers first in the low income states for agent B , then for higher income states.

The important characteristic of these states is the following. Agent B is already receiving transfers, hence agent A is insuring her against her low income shocks. If agent B 's incentive constraint is binding, it has to be because she needs to be compensated for spending the “adequate” amount on the public good. As the promised utility to agent B increases, she needs to be compensated for higher income levels. Figure $S_{b_2}^W$ depicts this.

If she is finding it difficult to comply with the spending dictated by the contract when agent A is transferring, she finds it difficult when she herself is the one who is transferring.

Proposition 9 *i. If $t_s^{aFB} > 0$ and agent B 's incentive constraint binds in state s , then $t_s^a > 0$.*

ii. If s lies in $S_{b_2}^W$, then for all states r such that $t_r^a > 0$, r lies in $S_{b_2}^W$. Moreover, $b_s = b_r$, $c_s = c_r$, and $W_s = W_r$.

iii. If s lies in $S_{b_2}^W$, then for all states j such that $t_j^b > 0$, j lies in $S_{b_1}^W$.

iv. If s lies in $S_{b_2}^W$ and r lies in $S_{b_2}^{\tilde{W}}$ where $W > \tilde{W}$ then $b_s > b_r$, $c_s > \tilde{c}_r$ and $W_s < \tilde{W}_r$.

Proof. In Appendix A. ■

As we pointed out, the first best allocation provides her with a lower period utility than deviating and spending the transfer as her individual optimality dictates. Asking her to spend even more on the public good would further hurt her incentives making defaulting more likely for her. Hence, the public good consumption is lower than the first best amount, $c < \alpha Y$, in these states.

Agent A can compensate agent B in three ways: By increasing her continuation utility, transferring a larger amount (to be spent on either b_s or c_s or both), or for a given transfer, shifting her expenditure from c_s to b_s . If the first best allocation dictated a transfer from agent A to agent B , agent A will find it less costly to decrease the public good consumption from its first best level on the margin than to transfer a higher amount to agent B and ask her to keep the public good consumption at the first best level.

On the other hand, if the first best allocation dictated a transfer from agent B to agent A , it implies that agent A 's relative income is lower in this state, and hence agent A would benefit from insurance. However, in the contract, it is still him who makes the transfer. This implies that he values the public good more than insurance. Then, it is not clear which compensation would outweigh the others in terms of utility for agent A .

Consider the following scenario: There is a state s in which agent A is the transferrer and agent B 's incentive constraint binds for a set of values of agent B . Assume for simplicity that s is the only state for which this is true. Let $W_1 > W_2$ be any two promised utilities from this set. The contract can dictate the same consumption levels and continuation utilities for W_1 and

W_2 . Then, to be able to deliver a higher promised utility, agent A needs compensate agent B in states other than s . These states can either be in S_{b1}^W or in S_0^W . But, we proved that agent B 's consumption and continuation values are constant for a given state across promised utilities if her incentive constraint binds for these states and she is the transferrer. As for the states in S_0^W , we know that the consumption levels and the continuation utilities of agent B are increasing in her promised utility.

3.3 Long Run Equilibrium

Denote the highest continuation value for which agent B 's incentive constraint is binding in at least one state by \underline{W} . The concavity of the frontier implies that if agent B 's incentive constraint is binding in a state s , at a value of agent B W , her continuation value will increase, $W_s > W$. For a given $W < \underline{W}$, this implies that agent B 's continuation value in any state s will satisfy $W_s \geq W$. Hence, if agent B starts to the left of \underline{W} , her continuation values will increase until the agents enter the interval where the first best is sustainable. A similar argument for agent A will show that if his incentive constraint is binding, his continuation values will increase until the first best interval. Note that this result hinges on the two intervals of W 's, namely $[W_{stack}, \underline{W}]$ and $[\overline{W}, V_{stack}]$ not overlapping.

We can now state the following proposition about the long run equilibrium of the model. See Appendix A for a discussion of the existence of a sustainable first best.

Proposition 10 *If the set of values W for which S_{b2}^W is nonempty is distinct from the set of values for which S_a^W is nonempty, and there exists an interval of the Pareto frontier where the first best is implementable, this interval constitutes the long run equilibrium of the contract. Let \underline{W} and \overline{W} be the smallest and largest values, and $V(\underline{W})$ and $V(\overline{W})$ the corresponding values for agent A , for which the first best allocation is implementable. If initial promised utility to agent B $W_0 < \underline{W}$, agents' values converge to $(\underline{W}, V(\underline{W}))$. If initial promised utility to agent B $W_0 > \overline{W}$, agents' values converge to the pair $(\overline{W}, V(\overline{W}))$. If agent B starts with any promised utility W_0 between \underline{W} and \overline{W} , agents' values stay at $(W_0, V(W_0))$.*

Proof. In Appendix A. ■

If agent B 's incentive constraint binds in a given state when agent A is the transferrer, then it binds in all states where he is the transferrer (Proposition 9 (ii)) as well as in the states where agent B is the transferrer (Proposition 9 (iii)). We also know that both incentive constraints cannot bind in a given state. Hence, we can conclude that for a given promised utility, if agent B 's incentive constraint binds in states where she is the receiver and hence in states where she is the transferrer, then, agent A 's incentive constraint cannot bind for any state for this promised

utility. Moreover, agent A 's incentive constraint cannot bind in any state s for any lower promised utility to agent B since this would violate Proposition 6 (i). Figure $S_{b_2}^W$ and Figure S_a^W .

Proposition 11 *The set of values W for which $S_{b_2}^W$ is nonempty is distinct from the set of values for which S_a^W is nonempty.*

Proof. In Appendix A. ■

Proposition 11 proves that both of the incentive constraints of the agents cannot bind for a given promised utility of agent B . If we are at an interval where agent B 's values are relatively low, then only agent B 's incentive constraints can bind in this interval. If we are at an interval where agent A 's values are relatively low, then only agent A 's incentive constraints can bind in this interval.

We conjecture the following result. Assume that there are no values for which the incentive constraints do not bind in any of the states. Let \tilde{W} be the highest value of agent B for which her incentive constraint binds and agent A is the transferrer, then $\tilde{W} + 1$ is the lowest value of agent B for which agent A 's incentive constraint binds. For $W \leq \tilde{W}$, $W_s > W$, but W_s is decreasing in W . For $W \geq \tilde{W} + 1$, $W_s < W$. We expect agent B 's continuation utility to oscillate.

4 Comparative Statics

Given that our analytical results are specific to limited cases, it is useful to compute the ex-ante frontier. We compute the frontier for CES utility functions, of which Cobb Douglas constitutes a subset. We start by Cobb Douglas utility where the agents have different preference parameters, and then move on to CES utility functions where the agents have identical preferences once again, but the degree of substitutability of the private and the public good is different than in Cobb Douglas parametrization we use.

Then we focus on the Cobb Douglas utility function and impose strict public good enforcement policies in autarky.

There are some regularities that come out of these various computations. The first best is sustainable for a wide region of frontier. As we proved above in Section 3, in most cases, agent A 's incentive constraint binds only at the last point of the frontier. The simulations show that the Cobb Douglas and the CES utility functions generate similar behavior in this respect.

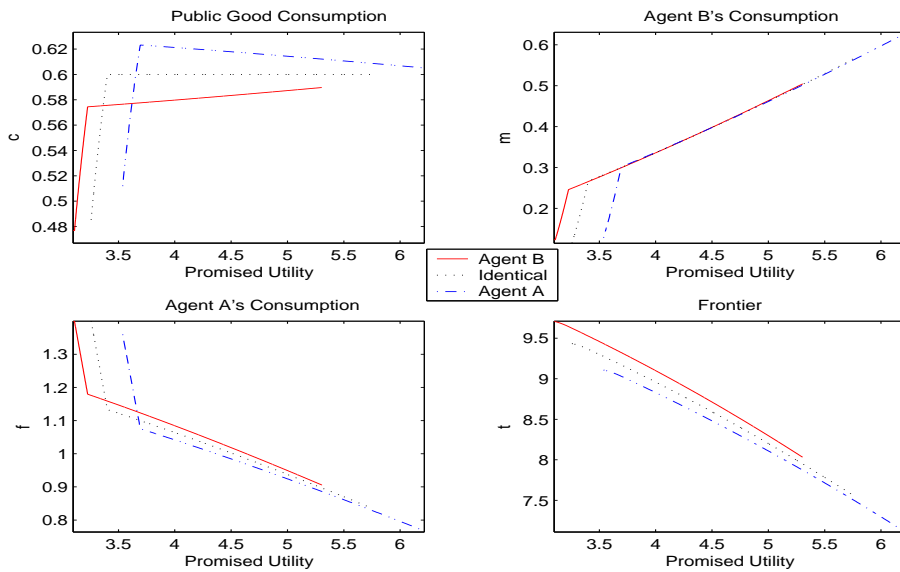


Figure 1: Different Preferences

These imply that in most of the cases, the only region where the constrained frontier will differ from the first best frontier will be where agent B 's incentive constraint is binding. Even then, the contract will provide comparable lifetime utilities to both agents. On the other hand, the policy functions display sizeable changes in the consumption levels of the agents. One important finding here is that even if the public good's consumption is different than its first best value, it changes considerably less across different promised utilities to agent B when compared to either agent's consumption levels.⁷

As for the case where agent B 's incentive constraint is binding and agent A is the transferrer, the consumption levels in the contract compared to the first best levels move in the following manner. agent A is consuming less under the contract than he is under the first best, and his consumption decreases as the promised utility to agent B increases. agent B consumes more under the contract than she does in the first best, and her consumption increases with the promised utility. The public good consumes less than the first best, however, his consumption increases as the promised utility to agent B increases.

⁷Remember that except for the case where the agent B 's incentive constraint is binding and she is receiving the transfers, the child consumes the first best level.

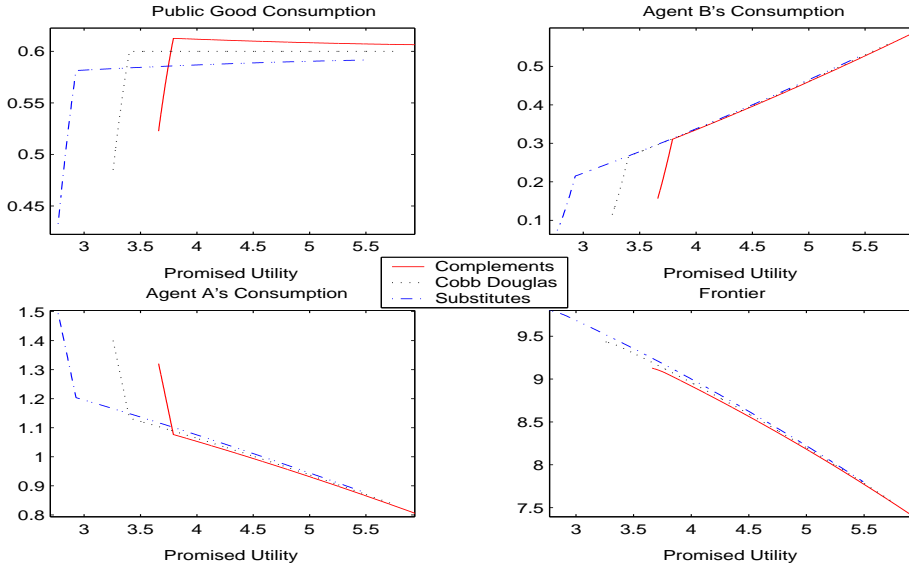


Figure 2: Cobb Douglas vs. CES

First, we allow the agents to have different preferences. Figure 1 depicts the results. Let the benchmark be the case where the agents have identical preferences. We consider two cases; one where agent B cares more and agent A cares less about the public good and vice versa. If agent B cares more and agent A less, the set of subgame perfect equilibria shifts to the left, implying that in this case, compared to the benchmark case, lower values of agent B can be sustained in the optimal contract, whereas the opposite will hold for agent A .

The analysis of the first order conditions of the general model suggested that one way to compensate the agent whose incentive constraint was binding was to diminish the wedge in his or her individual intra-temporal optimality condition. This helps us understand the mechanics of the contract in the case where the agents' preferences are not aligned.

First, remember that in autarky, agent B can actually set this wedge equal to zero whereas agent A 's wedge is be zero in autarky. In the contract, if agent B 's incentive constraint is binding, she enjoys higher consumption to compensate her for a positive wedge. Hence, if she cares more about the public good, she has stronger incentives to accept the transfer and spend it as dictated by the contract. On the other hand, if agent A cares less about the public good, then the fact that agent B 's spending is not aligned with his preferences does not hurt him as much. Therefore, his autarky utility increases, rendering his lower values unsustainable in the contract. An analogous argument holds for the case where agent A cares more about the public good, and agent B less,

compared to the benchmark case.

From these graphs we can also observe how much the agents decide to allocate to the public good under each scenario. When the agents' preferences are identical, we see that the public good's consumption is constant across values of the agents, except for the interval where agent B 's incentive constraint is binding. When agent B cares more, as her value increases, so does the public good's consumption. When agent A cares more, public good's consumption decreases with agent B 's lifetime utility. We also see that agent B 's consumption and her values are positively correlated. This also holds for agent A , given that his value is decreasing in agent B 's.

Figure 2 summarizes the results for Cobb Douglas utility function and for a more general CES form. Assume that the agents are identical. Let the benchmark be the Cobb Douglas utility function. Consider two different parameterizations of the CES utility, one where the private consumption and the public good's consumption is more complementary than the Cobb Douglas specification we consider, the other more supplementary.

When private consumption and public consumption are more complementary compared to the Cobb Douglas case, agent A transfers more in autarky. This increases agent B 's autarky lifetime utility, and decreases that of agent A . The frontier shifts right and down.

When there is higher substitutability between the private and public consumption, agent A will not be transferring as much to agent B in autarky, and the frontier shifts left and up.

5 Conclusion

In this paper, we expand the model of mutual insurance in the absence of commitment to include a public good. The basic premise is that only agent A can spend on the public good, while agent B can merely make indirect contributions through transfers to agent A . This adds a new layer of incentive constraint, as agent B is required not only to insure agent A whenever he has low income, but also to provide the level of public good dictated by the contract.

For general utility functions, we show that in most cases, consumption levels are invariant to the distribution of income and depend only on the level of aggregate income. We build on the conventional result on the public good provision in a Nash equilibrium, and apply it to our dynamic environment with uncertainty.

We fully characterize the transitory and long term equilibria in the special case where the agents have identical Cobb-Douglas utility functions and iid income shocks. We find that the consumption of the public good is remarkably stable across states and across agents' values, while agents are compensated at the expense of each other's private consumptions.

For the long run equilibrium, if there exists an allocation such that no incentive constraints bind (first best), the long run equilibrium will converge to an efficient allocation. If no such alloca-

tion is implementable, agents' utilities and consumptions will converge to a limiting distribution which can possibly depend on their initial values, and hence, not be unique.

We, then, compute the model to allow for differences in the agents' preferences for Cobb Douglas utility functions as well as CES utility functions. We find that if agent B cares more, or if the private good and the public good are more supplementary, the set of subgame perfect equilibria will expand to include lower lifetime utilities of agent B (hence higher values of agent A). If agent B cares more about the child than agent A, the punishment is harsher. Similarly, if the goods are more supplementary, agent A can substitute his private consumption for the public good, hence will transfer less to agent B. This also decreases the value of autarky for agent B. A similar argument applies for agent A. If the autarky provides a lower lifetime utility, either because agent B cares less about the child, or because the goods are more complementary, the set of subgame perfect equilibria will expand to include lower values of agent A's lifetime utility (hence higher values of agent B).

As for future research, an immediate extension to this model is to allow for income shocks that are serially correlated. In the empirical direction, future work should focus on the testable implications of the optimal dynamic contract built here. Concerning the child support application, one could use panel data which contains information on child support obligations and payments, child expenditure, and whether or not there is an enforcement order (wage withholding, tax return interception, etc.) against the father, to test for the implications of the model. One panel data set that provides such variables (or proxies for them) is the Survey of Income and Program Participation.

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A Proofs of Propositions

Proposition 1 *An allocation $\{t_j^a, t_j^b, c_j\}_{j=\tau}^{\infty}$ is subgame-perfect if and only if it satisfies*

$$u(Y_{\tau}^a - t_{\tau}^a + t_{\tau}^b, c_{\tau}) + E_{\tau} \sum_{r=1}^{\infty} \beta^r u(Y_{\tau+r}^a - t_{\tau+r}^a + t_{\tau+r}^b, c_{\tau+r}) \geq u(Y_{\tau}^a - \hat{t}_{\tau}^a + t_{\tau}^b, \hat{c}_{\tau}) + \beta V_{stack}$$

$$z(Y_{\tau}^b + t_{\tau}^a - t_{\tau}^b - c_{\tau}, c_{\tau}) + E_{\tau} \sum_{r=1}^{\infty} \beta^r z(Y_{\tau+r}^b + t_{\tau+r}^a - t_{\tau+r}^b - c_{\tau+r}, c_{\tau+r}) \geq z(Y_{\tau}^b + t_{\tau}^a - c_{\tau}^*, c_{\tau}^*) + \beta W_{stack}$$

for all dates and states, where \hat{c}_{τ} , \hat{t}_{τ}^a and c_{τ}^* are the optimal deviation transfer and consumptions defined as follows

$$\begin{aligned} \hat{c}_{\tau} &= \arg \max_C z(Y_{\tau}^b + \hat{t}_{\tau}^a - t_{\tau}^b - C, C) \\ c_{\tau}^* &= \arg \max_C z(Y_{\tau}^b + t_{\tau}^a - C, C) \\ \hat{t}_{\tau}^a &= \arg \max_T u(Y_{\tau}^a - T + t_{\tau}^b, \hat{c}_{\tau}) \end{aligned}$$

Proof. Consider an allocation $\{t_j^a, t_j^b, c_j\}_{j=\tau}^{\infty}$ satisfying the conditions above, and let the agents follow a strategy whereby they transfer and consume the amounts dictated by the allocation as long as both have done so in the past, otherwise, they revert to the Stackelberg equilibrium. We say that these strategies form a contract.

Consider the deviation strategies of the agents first. Agent A is required only to make a decision about the transfer to agent B. Knowing that agent B would reply to his deviation by applying the Stackelberg rule to her final income, the best agent A can do is transfer \hat{t}^a , to which agent B optimally responds by allocating \hat{c} to the consumption of the public good.

Agent B can deviate in three ways: first, by not transferring the right amount to agent A; second, by not allocating the correct amount to the public good, and third, by combining both deviations. Since the continuation value from any of the three deviations is the same (the Stackelberg utility), agent B maximizes her period utility by transferring zero to agent A, and applying the Stackelberg consumption rule.

One can easily see that the strategies described above form a subgame-perfect equilibrium.

On the other hand, let $\{t_j^a, t_j^b, c_j\}_{j=\tau}^\infty$ be a subgame-perfect allocation. Then given agent A's transfer strategy, the said allocation should provide agent B with at least as high a payoff as any other allocation, including the one resulting from her optimal deviation. The same reasoning applies to agent A. Hence, $\{t_j^a, t_j^b, c_j\}_{j=\tau}^\infty$ should satisfy the above two equations. ■

Proposition 2 *In the optimal contract, in a given state s , either $t_s^a \geq 0$ and $t_s^b = 0$ or $t_s^a = 0$ and $t_s^b \geq 0$.*

Proof. Suppose the vector $\{t_s^a, t_s^b, c_s, W_s\}_{s \in S}$ is a solution to the program above, and that $t_j^a > 0$ and $t_j^b > 0$ for some j . Take another vector $\{\tilde{t}_s^a, \tilde{t}_s^b, c_s, W_s\}_{s \in S}$ where $\tilde{t}_j^a = \max\{0, t_j^a - t_j^b\}$ and $\tilde{t}_j^b = \max\{0, t_j^b - t_j^a\}$, so $\tilde{t}_j^a - \tilde{t}_j^b = t_j^a - t_j^b$. Then the vector of new transfers and old consumptions and continuation utilities gives both agents as high payoffs as the old vector:

$$\begin{aligned} z(Y_j^b + t_j^a - t_j^b - c_j, c_j) + \beta W_j &= z(Y_j^b + \tilde{t}_j^a - \tilde{t}_j^b - c_j, c_j) + \beta W_j \\ u(Y_j^a - t_j^a + t_j^b, c_j) + \beta V(W_j) &= u(Y_j^a - \tilde{t}_j^a + \tilde{t}_j^b, c_j) + \beta V(W_j) \end{aligned}$$

thus leaving intact the left hand side of the incentive constraints.

On the other hand, the new vector implies a smaller constraint set, as the deviation budget constraints are tighter now. Remember that when agent B deviates, she does not make the transfer that the contract dictates, but accepts the transfer that agent A makes to her. If a new contract dictates a transfer that consists of only the difference, agent B will receive a weakly lower transfer from agent A under the new contract, implying lower benefits from deviation. An analogous argument holds for agent A.

More explicitly,

$$Y_s^b + t_s^a > Y_s^b + \tilde{t}_s^a \quad \text{and} \quad Y_s^a + t_s^b > Y_s^a + \tilde{t}_s^b$$

which implies that

$$u(Y_s^a - \hat{t}_s^a + t_s^b, \hat{c}_s) \geq u(Y_s^a - \tilde{t}_s^a + t_s^b, \hat{c}_s)$$

where $u(Y_s^a - \hat{t}_s^a + t_s^b, \hat{c}_s)$ is agent A's utility from deviating from the prescribed \tilde{t}_s^a , and

$$z(Y_s^b + t_s^a - c_s^*, c_s^*) \geq z(Y_s^b + \tilde{t}_s^a - \tilde{c}_s^*, \tilde{c}_s^*)$$

where $z(Y_s^b + \tilde{t}_s^a - \tilde{c}_s^*, c_s^*)$ is agent B's utility from deviating from the prescribed \tilde{t}_s^b, c_s .

As a result, the new vector $\{\tilde{t}_s^a, \tilde{t}_s^b, c_s, W_s\}_{s \in S}$ allows for a wider set of implementable allocations. So any allocation that was available and incentive compatible under the old solution is also under the new solution, and the new contract is weakly better than the old contract. ■

Lemma 3 *Agent A's incentive constraint does not bind in a state where he is a net receiver.*

Proof. Suppose that at some point the contract dictated a positive transfer t_s^b from agent B to agent A. Suppose further that agent A found it profitable to deviate by transferring a positive amount. Then we would have

$$u(Y_s^a + t_s^b, c_s) + \beta V(W_s) = u(Y_s^a + t_s^b - \hat{t}_s^a, \hat{c}_s) + \beta V_{stack}.$$

Since $V(W_s) \geq V_{stack}$, and $Y_s^a + t_s^b \geq Y_s^a + t_s^b - \hat{t}_s^a$, this would only happen if $c_s \leq \hat{c}_s$. Then we can find another contract which dictates the transfer \hat{t}_s^a to agent B and the consumption \hat{c}_s to the public good, with continuation values W_s and $V(W_s)$. agent A is clearly better off with the new allocation, and so is agent B. She is left with a higher disposable income to divide according to her optimal rule. Hence, the initial contract could not have been optimal. ■

Proposition 4 *In the states where agent A is the net transferrer, the equilibrium transfer, the public good consumption and continuation values are invariant to redistribution of income (as long as agent A is still the transferrer after the redistribution). For the states where agent B is the net transferrer, this will not generally hold.*

Proof. Suppose that for a certain promised value W to agent B, and for certain levels of income $\{Y_s^b\}_{s \in S}$ and $\{Y_s^a\}_{s \in S}$, $\{t_s^a, t_s^b, c_s, W_s\}_{s \in S}$ is a solution to the program above and provides agent A with value V . Now assume that there is a redistribution of income in some state j which leaves the agents with new income levels $Y_j^{b'} = Y_j^b - \Delta$ and $Y_j^{a'} = Y_j^a + \Delta$, where Δ can be positive or negative. Suppose that we start with $t_j^a > 0$ and $t_j^b = 0$.

Then as long as $\Delta > -t_j^a$, i.e. agent A is still the transferrer after the redistribution, the new vector $\{t_s^{a'}, t_s^b, c_s, W_s\}_{s \in S}$, where the j th component of $\{t_s^a\}_{s \in S}$ is replaced by $t_j^{a'} = t_j^a + \Delta$, is a solution to the new program. This means that the new vector leaves the agents with the same post-transfer incomes and utilities. It remains to show that the new vector also satisfies all incentive constraints, and there is no other vector that provides one agent with the same utility level, while providing the other with a strictly higher utility level.

To check that $(t_j^{a'}, 0, c_j, W_j)$ is incentive compatible, first note that we have $Y_j^b + t_j^a = Y_j^{b'} + t_j^{a'}$, so agent B's utility from deviating in the first period is the same. In addition, recall a general result by Warr (1983) and Bergstrom et al. (1986) which states that, within a Nash equilibrium, any redistribution of income among the set of contributors to a public good which leaves the set

of contributors unchanged, leaves the total expenditure on the public good as well as the welfare levels of all contributors unchanged. This result should hold immediately for the Stackelberg equilibrium consumption and continuation values, so $(u(Y_j^{a'} - \hat{t}_j^a, \hat{c}_j) + \beta V_{stack})$ and W_{stack} are also intact.

Now suppose there is another tuple $(\tilde{t}_j^a, 0, \tilde{c}_j, \tilde{W}_j)$ that is incentive compatible, and gives agent B the same value W while offering agent A strictly more than V ,

$$\begin{aligned} z(Y_j^{b'} + \tilde{t}_j^a - \tilde{c}_j, \tilde{c}_j) + \beta \tilde{W}_j &\geq z(Y_j^{b'} + \tilde{t}_j^a - c_j^*, c_j^*) + \beta W_{stack} \\ z(Y_j^b + t_j^a - c_j, c_j) + \beta W_s &= z(Y_j^{b'} + \tilde{t}_j^a - \tilde{c}_j, \tilde{c}_j) + \beta \tilde{W}_j \\ u(Y_j^{a'} - t_j^{f'}, c_j) + \beta V(W_j) &< u(Y_j^{a'} - \tilde{t}_j^a, \tilde{c}_j) + \beta V(\tilde{W}_j). \end{aligned}$$

But then, as long as $\Delta > -t_j^a$, it is easy to see that the same tuple $(\tilde{t}_j^a, 0, \tilde{c}_j, \tilde{W}_j)$ was also available under the old income levels. Just set the old transfer $t_{j,old}^j = \tilde{t}_j^a - \Delta$, and specify the public good expenditure and continuation value to be \tilde{c}_j, \tilde{W}_j . This leaves agent B with the same budget and incentive constraints. Hence, the vector $\{t_s^a, t_s^b, c_s, W_s\}_{s \in S}$ could not have been a solution to the original problem.

Note that the theorem by Warr and Bergstrom et al. does not hold when $t_j^b > 0$, since in that case the only contributor to the public good is agent B (so any redistribution of income will alter the income available to the set of contributors).

More generally, assume that the redistribution of income is in favor of agent B, so the new income levels are $Y_j^{b'} = Y_j^b + \Delta$ and $Y_j^{a'} = Y_j^a - \Delta$, where Δ is a positive number. It is clear that $W'_{stack} > W_{stack}$, where W'_{stack} is agent B's new Stackelberg value. Moreover, suppose the contract dictated a transfer $t_s^{b'} = t_s^b + \Delta$, and consider agent B's incentive constraints. Before the income shock agent B's income was satisfied, so

$$z(Y_j^b - t_j^b - c_j, c_j) + \beta W_s \geq z(Y_j^b - c_j^*, c_j^*) + \beta W_{stack}$$

but we have no guarantee that the following will hold

$$z(Y_j^{b'} - t_j^{b'} - c_j, c_j) + \beta W_s \geq z(Y_j^{b'} - c_j^*, c_j^*) + \beta W'_{stack}$$

■

A.1 Characterization of S_a^W

We need the following lemmas to prove Proposition 5.

Lemma a *If $t_s^{st} > 0$ for some s , then $t_s^a > 0$.*

Proof. If $t_s^{st} > 0$, then

$$\frac{Y_s^a}{Y_s^b} > \frac{1 - \alpha}{\alpha} > \frac{1 - \alpha}{\alpha + M_s^a}.$$

■

Lemma b *If $t_s^{st} > 0$ and $s \in S_a^W$ then for any j such that $t_j^{st} > 0$, we have $j \in S_a^W$.*

Proof. For this part, it is enough to see that when $t_s^{st} > 0$ and $t_j^{st} > 0$, agent A's value in the first best is the same in state j as in s , as is the utility from deviating:

$$u(Y_s^a - \hat{t}_s^a, \hat{c}_s) = u(Y_j^a - \hat{t}_j^a, \hat{c}_j) = \alpha^\alpha G(Y_s^a + Y_s^b)$$

If his incentive constraint is binding in s , but not in j , then,

$$\begin{aligned} u(Y_s^a - t_s^a, c) + \beta V(W_s) &= u(Y_s^a - \hat{t}_s^a, \hat{c}_s) + \beta V_{stack} \\ &< u(Y_j^a - t_j^a, c) + \beta V(W_j) \\ &= u(a_W^{FB}, c) + \beta V(W) \end{aligned}$$

But, we know from the first order conditions that $a_s > a_W^{FB}$ and $V(W_s) > V(W)$. This is a contradiction. ■

Lemma c *If s lies in S_a^W and r lies in $S_a^{\tilde{W}}$, and then $a_s = a_r$ and $V(W_s) = V(\tilde{W}_r)$ where $W_s(\tilde{W}_r)$ is the continuation value in state $s(r)$ when the promised utility is $W(\tilde{W})$.*

Proof. As can be seen from agent A's incentive constraint, the deviation utility is the same regardless of agent A's income. For his incentive constraint to be binding in both states, the following should hold

$$u(a_s, c_s) + \beta V(W_s) = u(a_r, c_r) + \beta V(\tilde{W}_r).$$

We know from the first order conditions that $c_s = c_r$. Without loss of generality, suppose we had $a_s > a_r$. This would first imply that $u(a_s, c_s) > u(a_r, c_r)$, and secondly that $M_s^a > M_r^a$, which means that $V(W_s) > V(\tilde{W}_r)$. This contradicts the above equality. We would get a similar contradiction if we started by assuming that $V(W_s) > V(\tilde{W}_r)$. ■

We will prove the proposition below with the two preceding lemmas.

Proposition 5 *If $t_s^{st} > 0$ for all s , then $S_a^W = \{1, 2, \dots, S\}$ for $W = W_{max}$ and $S_a^W = \emptyset$ for all other W .*

Proof. Note that the right hand side of the equation below does not depend on the promised utility to agent B. If it holds with equality in state s for agent B's values W and \tilde{W} , it has to be

that her consumption and continuation values are constant across promised utilities. We have already shown that c will be.

$$a_s^{1-\alpha} c^\alpha + \beta V(W_s) = GY + \beta V_{stack}$$

But if a similar condition holds for all states, that is, if in all the states agent A's consumption and continuation values are constant across promised utilities, then it has to be that

$$\sum_s \pi_s (a_s^{1-\alpha} c^\alpha + \beta V(W_s)) = V_{stack}.$$

Any lower value will cause a deviation by agent A, any higher value to agent A would violate the promise keeping constraint. ■

Proposition 6 *Assume that $t_s^{st} \geq 0$ for all s , then the following holds.*

- i. If $s \in S_a^W$, then $s \in S_a^{\tilde{W}}$ for all $\tilde{W} \geq W$.
- ii. If $s \in S_a^W$, and $t_s^{st} = 0$, then for any j such that $Y_j^a > Y_s^a$, we have $j \in S_a^{\tilde{W}}$.
- iii. If s lies in S_a^W and r lies in $S_a^{\tilde{W}}$, and $t_s^{st} \geq t_r^{st} = 0$, then $a_s \geq a_r$ and $V(W_s) \geq V(\tilde{W}_r)$.

Proof.

- i. Suppose the opposite. Then we would have

$$\begin{aligned} \left(\frac{1}{1 + M_0} \right)^{1-\alpha} G(Y_s^a + Y_s^b) + \beta V(W) &< u(Y_s^a - \hat{t}_s^a, \hat{c}_s) + \beta V_{stack} \\ \left(\frac{1}{1 + \tilde{M}_0} \right)^{1-\alpha} G(Y_s^a + Y_s^b) + \beta V(\tilde{W}) &\geq u(Y_s^a - \hat{t}_s^a, \hat{c}_s) + \beta V_{stack}. \end{aligned}$$

By concavity of V , we know that $V(W) \geq V(\tilde{W})$ and $M_0 \leq \tilde{M}_0$, which is a contradiction.

- ii. There are two different cases to consider: $t_j^{st} > 0$ and $t_j^{st} = 0$. Start with the first one. Note that if $t_j^{st} > 0$, then it has to be the case that $u_j^{dev} > u_s^{dev}$, where u^{dev} is defined by (6). In absence of aggregate uncertainty, setting $t_j^{st} = 0$ is available to agent A, and he has chosen $t_j^{st} > 0$. Hence it has to be that it yields higher utility than zero transfer. Given that $s \in S_a^W$, and assuming that $s \in S_0^W$

$$\begin{aligned} a_s^{1-\alpha} c^\alpha + \beta V(W_s) &= u_s^{dev} + \beta V_{stack} \\ &< u_j^{dev} + \beta V_{stack} \\ &\leq (a_W^{FB})^{1-\alpha} c^\alpha + \beta V(W) \end{aligned}$$

But, we know that $a_s > a^{FB}$ and $V(W_s) > V(W)$ for all $s \in S_a^W$ from the first order conditions, which contradicts the above inequality.

Now, consider the case where $t_j^{st} = 0$ and $Y_j^a > Y_s^a$. Equation (6) implies that agent A's deviation period utility is increasing in Y^a , thus, $u_j^{dev} > u_s^{dev}$.

$$a_s^{1-\alpha} c^\alpha + \beta V(W_s) \leq a_j^{1-\alpha} c^\alpha + \beta V(W_j)$$

It is then easy to see that $j \in S_a^W$ as well.

iii. Given the previous discussion, this part of the proof is trivial.

■

A.2 Characterization of S_{b1}^W

Lemma d *If $s \in S_{b1}^W$ and $t_s^b > 0$, $t_s^{st} = 0$.*

Proof. If $s \in S_{b1}^W$ and $t_s^b > 0$, $t_s^{st} = 0$, then equation (??) dictates the following.

$$\frac{Y_s^b}{Y_s^a} > \frac{\alpha + M_s^b}{1 - \alpha} > \frac{\alpha}{1 - \alpha}$$

which implies that $t_s^{st} = 0$ ■

Lemma e *If s lies in S_{b1}^W and r lies in $S_{b1}^{\tilde{W}}$, and then $b_s = b_r$ and $W_s = \tilde{W}_r$.*

Proof. Analogous to the corresponding proof for S_a^W . ■

Proposition 7 *i. If $s \in S_{b1}^W$, then $s \in S_{b1}^{\tilde{W}}$ for all $\tilde{W} \leq W$.*

ii. If $s \in S_{b1}^W$, then for any j such that $Y_j^b > Y_s^b$, we have $j \in S_{b1}^W$.

iii. If s lies in S_{b1}^W and in $S_{b1}^{\tilde{W}}$, then $b_s = \tilde{b}_s$ and $W_s = \tilde{W}_s$.

iv. If s and r lie in S_{b1}^W , and $Y_s^b > Y_r^b$, then $b_s > b_r$ and $W_s > W_r$.

Proof.

- i. The proof for the first part is mainly the counterpart for the previous proposition's.
- ii. For this part, it is enough to see that agent B's value in the first best is the same in state j as in s , while her gain from deviating is larger when her income is higher.

- iii. The proof for this part is analogous to the previous proposition's. It hinges on agent B's deviation utility and Stackelberg continuation value being the same in the same state for different ex-ante promised utilities. Assuming that either $b_s \neq \tilde{b}_s$ or $W_s \neq \tilde{W}_s$ would result in a contradiction.
- iv. From agent B's incentive constraint, we have

$$\begin{aligned} z(b_s, c_s) + \beta W_s &= GY_s^b + \beta W_{stack} \\ z(b_r, c_r) + \beta W_r &= GY_r^b + \beta W_{stack}. \end{aligned}$$

So we need

$$z(b_s, c_s) + \beta W_s > z(b_r, c_r) + \beta W_r.$$

From the first order conditions we know that $c_s = c_r$. Now suppose we had $b_s \leq b_r$. This would imply first that $z(b_s, c_s) \leq z(b_r, c_r)$, and secondly that $\lambda_s^b \leq \lambda_r^b$, thus $W_s \leq W_r$. Hence, $z(b_s, c_s) + \beta W_s$ cannot be greater than $z(b_r, c_r) + \beta W_r$, which is a contradiction. We get a similar contradiction if we assume that $W_s \leq W_r$. It is unclear how the transfers in these states compare.

■

Proposition 8 *If $S_{b1}^W = \{1, 2, \dots, S\}$ and $t_s^b > 0$ for all s , then $W = W_{stack}$.*

Proof. Note that the right hand side of the equation below does not depend on the promised utility to agent B. If it holds with equality in state s for agent B's values W and \tilde{W} , it has to be that her consumption and continuation values are constant across promised utilities. We have already shown that c will be.

$$b_s^{1-\alpha} c^\alpha + \beta W_s = GY_s^b + \beta W_{stack}$$

But if a similar condition holds for all states, that is, if in all the states agent B's consumption and continuation values are constant across promised utilities, then it has to be that

$$\sum_s \pi_s (b_s^{1-\alpha} c^\alpha + \beta W_s) = W_{stack}.$$

Any higher value will violate the promise keeping constraint, any lower value will cause a deviation by agent B. ■

A.3 Characterization of S_{b2}^W

Proposition 9 *i. If $t_s^{fFB} > 0$ and agent B's incentive constraint binds in state s , then $t_s^a > 0$.*

ii. If s lies in $S_{b_2}^W$, then for all states r such that $t_r^a > 0$, r lies in $S_{b_2}^W$. Moreover, $b_s = b_r$, $c_s = c_r$, and $W_s = W_r$.

iii. If s lies in $S_{b_2}^W$, then for all states j such that $t_j^b > 0$, j lies in $S_{b_1}^W$.

iv. If s lies in $S_{b_2}^W$ and r lies in $S_{b_2}^{\tilde{W}}$ where $W > \tilde{W}$ then $b_s > \tilde{b}_r$, $c_s > \tilde{c}_r$ and $W_s < \tilde{W}_r$.

Proof.

i. Suppose agent B's incentive constraint binds in state s and $t_s^{fFB} > 0$. And suppose that $t_s^b > 0$. This would mean that $b_s > b_s^{fFB}$, while $c_s = c_s^{fFB}$. But this can only happen if agent B's total resources are higher under the contract than under the first best. Hence, it has to be that $t_s^a > 0$.

ii. If s lies in $S_{b_2}^W$, then we have

$$(Y_s^b + t_s^{fFB} - c_s^{fFB})^{1-\alpha} (c_s^{fFB})^\alpha + \beta W < G(Y_s^b + t_s^{fFB}) + \beta W_{stack}$$

Suppose that for some other state j such that $t_j^a > 0$ agent B's incentive constraint did not bind. This means that

$$(Y_j^b + t_j^{fFB} - c_j^{fFB})^{1-\alpha} (c_j^{fFB})^\alpha + \beta W \geq G(Y_j^b + t_j^{fFB}) + \beta W_{stack}$$

We know that $Y_s^b + t_s^{fFB} = Y_j^b + t_j^{fFB}$, so the right hand sides of both equations are equal. We also know that $c_s^{fFB} = c_j^{fFB}$ from the first order conditions, which contradicts the above assumption.

Suppose that $b_s > b_r$. Agent B's incentive constraints would be as follows.

$$\begin{aligned} (Y_s^b + t_s^a - c_s)^{1-\alpha} c_s^\alpha + \beta W_s &= G(Y_s^b + t_s^a) + \beta W_{stack} > \\ G(Y_r^b + t_r^a) + \beta W_{stack} &= (Y_r^b + t_r^a - c_r)^{1-\alpha} c_r^\alpha + \beta W_r \end{aligned}$$

iii. If s lies in $S_{b_2}^W$, then we have

$$(Y_s^b + t_s^{fFB} - c_s^{fFB})^{1-\alpha} (c_s^{fFB})^\alpha + \beta W < G(Y_s^b + t_s^{fFB}) + \beta W_{stack}.$$

Suppose that for some other state j such that $t_j^b > 0$ agent B's incentive constraint didn't bind. This means that

$$(Y_j^b - t_j^{mFB} - c_j^{fFB})^{1-\alpha} (c_j^{fFB})^\alpha + \beta W \geq GY_j^b + \beta W_{stack}.$$

We know that $Y_s^b + t_s^{fFB} = Y_j^b - t_j^{mFB}$ and that $c_s^{fFB} = c_j^{fFB}$ from the first order conditions, so the left hand sides of the equations are equal. Then for the above two inequalities to

hold at the same time, it must be the case that $Y_s^b + t_s^{fFB} > Y_j^b$. Now, $t_j^{mFB} > 0$ implies that $Y_j^b > \frac{1+M_0}{1-\alpha} Y$, while $Y_s^b + t_s^{fFB} = \frac{\alpha+M_0}{1-\alpha} Y < \frac{1+M_0}{1-\alpha} Y$. Hence, we have a contradiction, and agent B's incentive constraint binds in j .

iv. We do not have a proof for this case yet. We have

$$\begin{aligned} (Y_s^b + t_s^a - c_s)^{1-\alpha} (c_s)^\alpha + \beta W_s &= G(Y_s^b + t_s^a) + \beta W_{stack} \\ (Y_r^b + t_r^a - c_r)^{1-\alpha} (c_r)^\alpha + \beta \tilde{W}_r &= G(Y_r^b + t_r^a) + \beta W_{stack} \end{aligned}$$

The right hand side of the incentive constraint depends on the agent B's post transfer income. So hypothetically, agent A could set the transfers such that $Y_s^b + t_s^a = Y_r^b + t_r^a$. By doing this, he induces agent B to spend the same amounts on her private consumption and on the public good in both states, and gives her the same continuation utility. Is that feasible for $W > \tilde{W}$?

We already saw that for states that lie in S_{b1} , consumption and continuation value are only state dependent, so if the same state j lies in S_{b1}^W and in $S_{b1}^{\tilde{W}}$ we should have $W_j = \tilde{W}_j$. Also, that for states that lie in S_a , the continuation value is the same across states and values of the promised utility. Moreover, that agent A's incentive constraint is more likely to bind at W than at \tilde{W} .

Hence, all the difference in the ex-ante promised utility should come from the states that lie in S_0 or in S_{b2} . Suppose for the sake of argument that the set S_0 was empty for both W and \tilde{W} . Then the only way agent A can credibly provide a higher ex-ante value in one case over the other, is by compensating agent B more in states that lie in S_{b2}^W than in $S_{b2}^{\tilde{W}}$. Of course this argument is incomplete because it might very well be the case that S_0^W or $S_0^{\tilde{W}}$ is nonempty.

If a state j lies in S_0^W but not in $S_0^{\tilde{W}}$, then j should belong to $S_a^{\tilde{W}}$, which means that $\tilde{W}_j < \tilde{W}$ which works against the fulfillment of the promise keeping constraint (or which decreases the gap between W and \tilde{W}). If a state j lies in $S_0^{\tilde{W}}$ but not in S_0^W , then j should belong to S_{b2}^W , which means that $W_j > W$ which also decreases the gap between W and \tilde{W} . Some states could lie in both S_0^W and $S_0^{\tilde{W}}$, which doesn't help in filling up the gap either, since they simply keep the values where they are. So, it must be that the states that are in S_{b2} are the ones we are interested in.

When agent B's incentive constraint binds and she's a receiver, we have

$$(Y_s^b + t_s^{fFB} - c_s^{FB})^{1-\alpha} (c_s^{FB})^\alpha + \beta W < G(Y_s^b + t_s^{fFB}) + \beta W_{stack}$$

Agent A can bridge the gap between the right hand side and the left hand side of agent B's incentive constraint in three ways: By increasing her continuation utility, transferring

a larger amount (to be spent on either b_s or c_s or both), or for a given transfer, shifting her expenditure from c_s to b_s . From the envelope condition,

$$V'(W_s) = -\mu - \lambda_s^b$$

we know that agent B's continuation utility increases when her incentive constraint binds, but that is not the only tool agent A uses. We can show that, starting from the efficient values, agent A will prefer decreasing c_s and increasing b_s over increasing the transfer and keeping c_s the same.

Suppose agent A wanted to keep c_s constant, and decided to compensate agent B by giving her more private consumption. The marginal decrease in the gap net of the marginal cost to agent A is be equal to

$$\left(\left(\frac{M_0}{1+M_0} \right)^{-\alpha} - 1 - \left(\frac{1}{1+M_0} \right)^{-\alpha} \right) G.$$

Now if the contract instead dictated a shift in agent B's spending from the public good toward the private good, the marginal decrease in the gap net of the marginal cost to agent A is equal to

$$\left(\left(\frac{M_0}{1+M_0} \right)^{-\alpha} - \left(\frac{M_0}{1+M_0} \right)^{1-\alpha} - \left(\frac{1}{1+M_0} \right)^{1-\alpha} \right) G$$

Clearly, it is more profitable to decrease c_s from its efficient level. This argument holds at the margin, so agent A may use a combination of tools, further decreasing c_s and increasing t_s^a .

Now things look different if we had $t_s^{mFB} > 0$ and $t_s^a > 0$:

$$(Y_s^b - t_s^{mFB} - c_s^{FB})^{1-\alpha} (c_s^{FB})^\alpha + \beta W < GY_s^b + \beta W_{stack}.$$

Repeating the same argument as above, the net benefit of decreasing the required transfer from agent B to agent A is

$$\left(\left(\frac{M_0}{1+M_0} \right)^{-\alpha} - \left(\frac{1}{1+M_0} \right)^{-\alpha} \right) G$$

while the net benefit of a shift in agent B's spending from the public good toward the private good would be

$$\left(\left(\frac{M_0}{1+M_0} \right)^{-\alpha} - \left(\frac{M_0}{1+M_0} \right)^{1-\alpha} - \left(\frac{1}{1+M_0} \right)^{1-\alpha} \right) G.$$

It is not clear which one is the better option.

For a certain value W , if $t_s^{fFB} > 0$ and agent B's incentive constraint binds for state s , it will bind for all states j where she is a receiver and $t_j^{fFB} > 0$, as well as for all states where's she's a transferrer. Her consumption and continuation value will be constant across the states in S_{b2}^W , where the continuation value is increasing in her ex-ante promised utility.

So far, we have not been able to show that if agent B is transferring in the first best equilibrium, she will also be transferring for that state in the contract, though it seems intuitive. Transferring a positive amount to agent B makes the already large right hand side of her incentive constraint even larger, which in turn makes deviating more attractive and the necessary compensation in the contract higher.

To see more precisely whether a positive transfer from agent B in the efficient equilibrium could turn into a positive transfer from agent A in the contract, we try another marginality argument at $t_s^b = t_s^a = 0$. The idea is the following: Suppose that the contract dictated that agent B be required to transfer less to agent A, as a way of compensating her for staying in the contract, while keeping c_s at the efficient level. Suppose further that even upon reaching a zero transfer, agent B's incentive constraint is still violated. Then will it be optimal for agent A to start transferring to agent B, or for the transfer to stay at zero with agent B shifting spending from c_s to b_s (Notice that this is an extreme scenario, since c_s here is at its efficient level, so the marginal utility from a unit of the public good consumption is at its lowest).

If agent B's incentive constraint is still violated when she's transferring a zero amount, then we must have

$$(Y_s^b - c_s^{FB})^{1-\alpha} (c_s^{FB})^\alpha + \beta W < GY_s^b + \beta W_{stack}$$

Suppose agent A were to transfer a positive amount to agent B. The net benefit would be equal to

$$(1 - \alpha)\alpha^\alpha Y^\alpha \left[(Y_s^b - \alpha Y)^{-\alpha} - (Y_s^a)^\alpha \right] - G$$

If the contract instead dictated a shift in agent B's allocation of her income toward more of the private good, the net benefit would be

$$\frac{Y_s^a}{Y} \alpha^\alpha Y^\alpha \left[(Y_s^b - \alpha Y)^{-\alpha} - (Y_s^a)^\alpha \right]$$

We know that for agent B to be transferring in the efficient equilibrium, it must be that

$$\frac{Y_s^a}{Y} \leq \frac{1 - \alpha}{1 + M_0}$$

■

A.4 Long Run Equilibrium

Assume that $t_s^{st} > 0$ for all s . Then, the first best is sustainable if the following holds.

$$\begin{aligned} \frac{1}{1-\beta} \left(\frac{\mu^{1/\alpha}}{1+\mu^{1/\alpha}} \right)^{1-\alpha} GY &\geq \frac{1}{1-\beta} \alpha GY \\ \frac{1}{1-\beta} \left(\frac{1}{1+\mu^{1/\alpha}} \right)^{1-\alpha} GY &\geq \frac{1}{1-\beta} \alpha^\alpha GY. \end{aligned}$$

First, note that whether the first best is sustainable or not does not depend on β if agent A is the transferrer in all the states in autarky, since for these income processes, he does not need insurance, and he is willing to provide insurance to agent B.

Secondly, note that some first best allocations are not be sustainable even in these cases. As agent B's relative Pareto weight μ increases, $\left(\frac{1}{1+\mu^{1/\alpha}} \right) \rightarrow 0$, agent A's utility also approaches 0 and the first best is not sustainable. A similar argument for agent B holds when $\mu \rightarrow 0$.

Proposition 10 *If the set of values W for which $S_{b_2}^W$ is nonempty is distinct from the set of values for which S_a^W is nonempty, and there exists an interval of the Pareto frontier where the first best is implementable, this interval constitutes the long run equilibrium of the contract. Let \underline{W} and \overline{W} be the smallest and largest values, and $V(\underline{W})$ and $V(\overline{W})$ the corresponding values for agent A, for which the first best allocation is implementable. If initial promised utility to agent B $W_0 < \underline{W}$, agents' values converge to $(\underline{W}, V(\underline{W}))$. If initial promised utility to agent B $W_0 > \overline{W}$, agents' values converge to the pair $(\overline{W}, V(\overline{W}))$. If agent B starts with any promised utility W_0 between \underline{W} and \overline{W} , agents' values stay at $(W_0, V(W_0))$.*

Proof. This is a direct result of the frontier being strictly concave. ■

Proposition 11 *The set of values W for which $S_{b_2}^W$ is nonempty is distinct from the set of values for which S_a^W is nonempty.*

Proof. Agent A's incentive constraint will not bind if $t_s^b > 0$. In the set of states S_a^W , $t_s^a > 0$. In the set of states $S_{b_2}^W$, $t_s^a > 0$. But we know by propositions 6 and 9 that if agent A is the transferrer and the one agent's incentive constraint is binding in a state s , then the same agent's incentive constraint binds for all states r where $t_r^a > 0$. Given that both incentive constraints cannot bind at the same time for a given state, we conclude that the set of values W for which $S_{b_2}^W$ is nonempty is distinct from the set of values for which S_a^W is nonempty. ■

B Proof of Concavity

For a general utility function, we will not be able to use the Theorem of the Maximum to prove the convexity of the problem, hence to prove that the first order conditions are necessary and

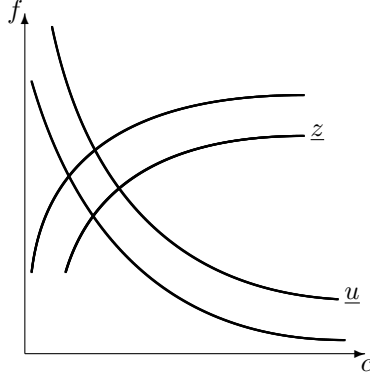


Figure 3: Cobb Douglas Utility

sufficient. Endogenous variables t_s^a and t_s^b enter the right hand side of the incentive constraints. Therefore, the constraint set will not trivially be convex.

However, the value functions for CES utility functions are linear in total wealth. Hence, the deviation utilities of the agents will be linear in their disposable income in the period of deviation, i.e. $z^{dev} = K_1 (Y_s^b + t_s^a)$ and $u^{dev} = K_2 (Y_s^a + t_s^b)$ where K_1 and K_2 are constants. Therefore the constraint set will be convex.

Then Theorem of the Maximum says that the frontier will be strictly concave.

C Existence of a Non-Trivial Contract

We start by showing that the first best utilities are different than the Stackelberg utilities.

The Stackelberg values are as follows.

$$\begin{aligned} c^{st} &= \alpha^2 Y_T \\ b^{st} &= \alpha(1 - \alpha) Y_T \\ a^{st} &= (1 - \alpha) Y_T \end{aligned}$$

The first best values are as follows:

$$\begin{aligned}
c^{FB} &= \alpha Y_T \\
b^{FB} &= \frac{K}{1+K}(1-\alpha)Y_T \\
a^{FB} &= \frac{1}{1+K}(1-\alpha)Y_T
\end{aligned}$$

where $K = \mu^{1/\alpha}$. It is easy to see that the public good consumes a higher fraction of total income in first best, and agent A consumes less compared to their respective Stackelberg levels. As for agent B, it depends on her relative Pareto-weight.

Equating agent A's utility from the Stackelberg equilibrium to the one from the first best, we find the condition on the relative Pareto weight which should satisfy the following:

$$\mu_{fst} = \left(\frac{1}{\alpha^{\frac{1}{1-\alpha}}} - 1 \right)^\alpha$$

The analogous condition when agent B is at her Stackelberg utility is the following:

$$\mu_{bst} = \left(\frac{\alpha^{\frac{1}{1-\alpha}}}{1 - \alpha^{\frac{1}{1-\alpha}}} \right)^\alpha$$

In order to show that there are potential gains from the contract, it is necessary and sufficient that when agent A is receiving the Stackelberg utility, agent B's utility is higher than her Stackelberg, under the first best. In other words, that the relative Pareto weights we found above are not equal, so that $\mu_{mst} > \mu_{fst}$. This implies the following condition on the parameter α :

$$1 < \alpha^{\frac{\alpha}{\alpha-1}} + \alpha^{\frac{1}{\alpha-1}}$$

This provides us with a restriction on α such that the first best utilities are different than Stackelberg utilities. It remains to be proven that the utilities in the contract will be different as well.

Suppose agents are initially playing Stackelberg in all states. Now change the strategies to the following:

Pick a state s for which $t_s^{ast} > 0$. Whenever state s occurs, agent A transfers $t_s^a > t_s^{ast} > 0$, and in return agent B spends $c_s = k c_s^{st}$, where $k > 1$ is chosen to make agent A just indifferent between the old equilibrium and the new one, and agent B better off. Whenever any state other than s occurs, play the Stackelberg strategies.

It is enough that such a state s exists, and is incentive compatible for agent B, for a nontrivial contract to exist under the present utility form.

First, for any given k , find the necessary transfer that would make agent A indifferent between playing the Stackelberg strategy and the new strategy:

$$(Y_s^a - t_s^a)^{1-\alpha} (kc_s^{st})^\alpha = (Y_s^a - t_s^{ast})^{1-\alpha} (c_s^{st})^\alpha$$

which gives us the following transfer equivalence:

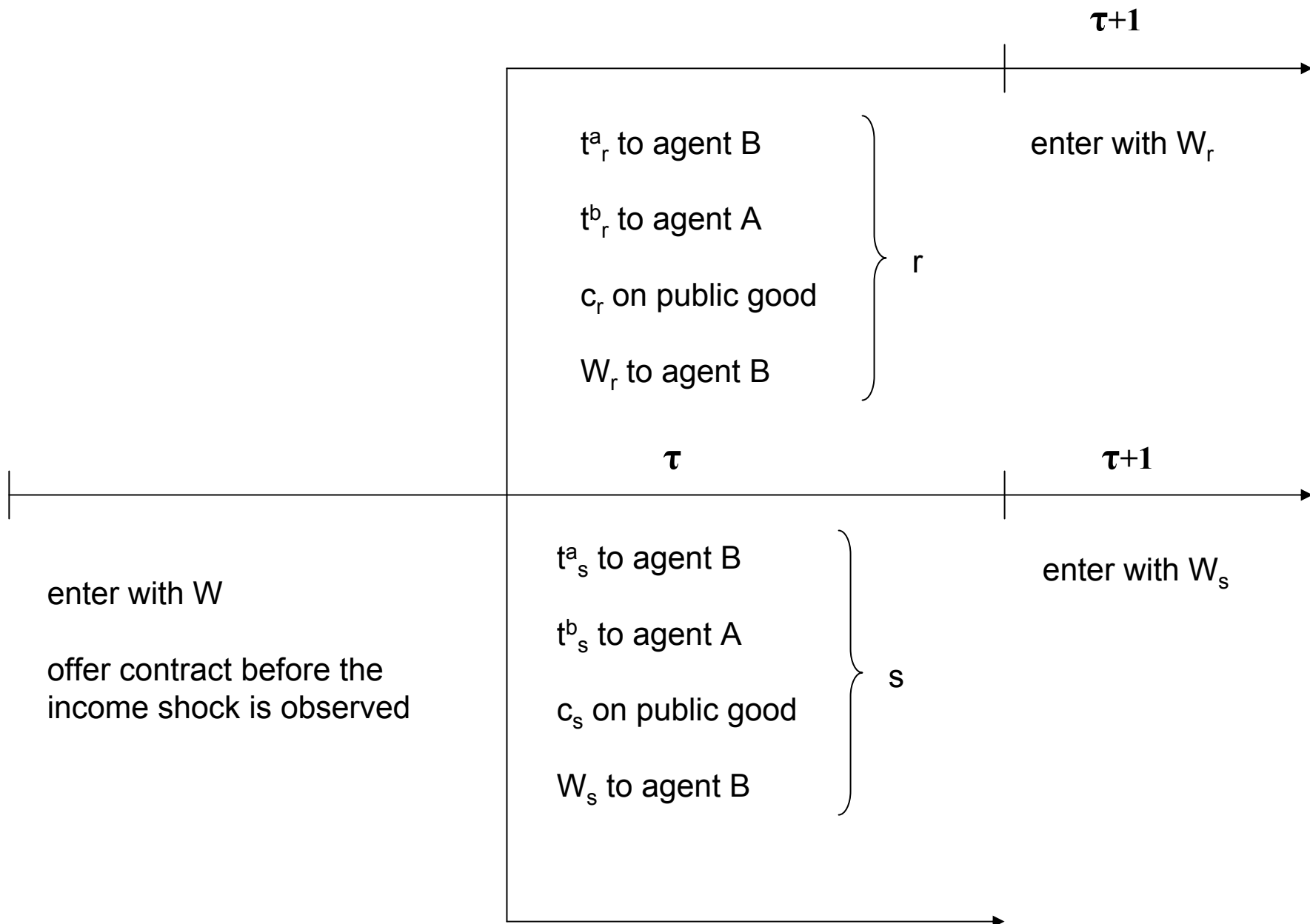
$$t_s^a = Y_s^a - (1 - \alpha)k^{\frac{\alpha}{\alpha-1}}(Y_s^a + Y_s^b)$$

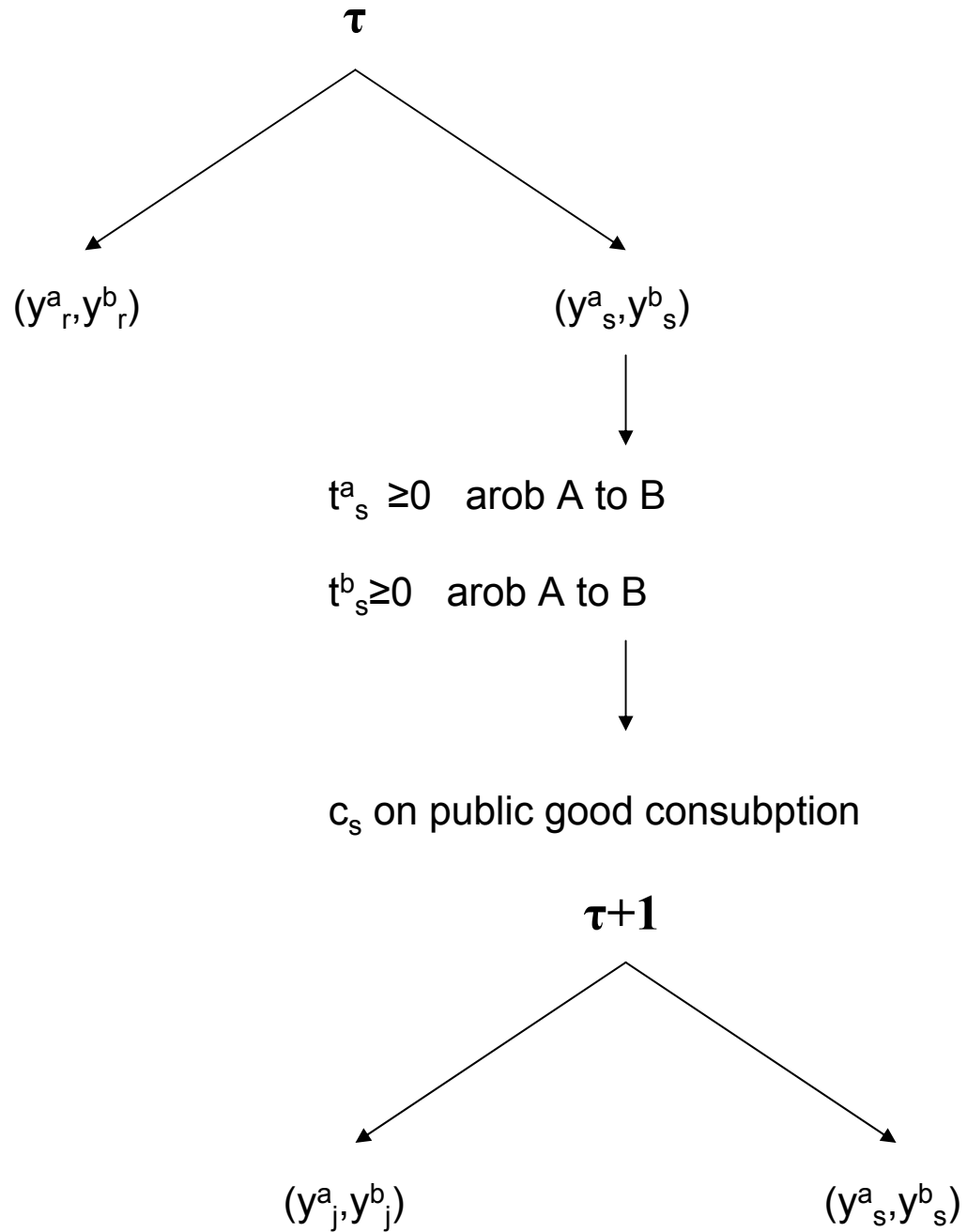
Now we need to check that agent B would not choose to deviate from the new strategy in order to establish that the new contract is incentive compatible and makes agent B better off:

$$\left(1 + \frac{\beta\pi}{1-\beta}\right) (Y_s^b + t_s^a - kc_s^{st})^{1-\alpha} (kc_s^{st})^\alpha \geq G(Y_s^b + t_s^a) + \frac{\beta\pi}{1-\beta}\alpha G(Y_s^a + Y_s^b)$$

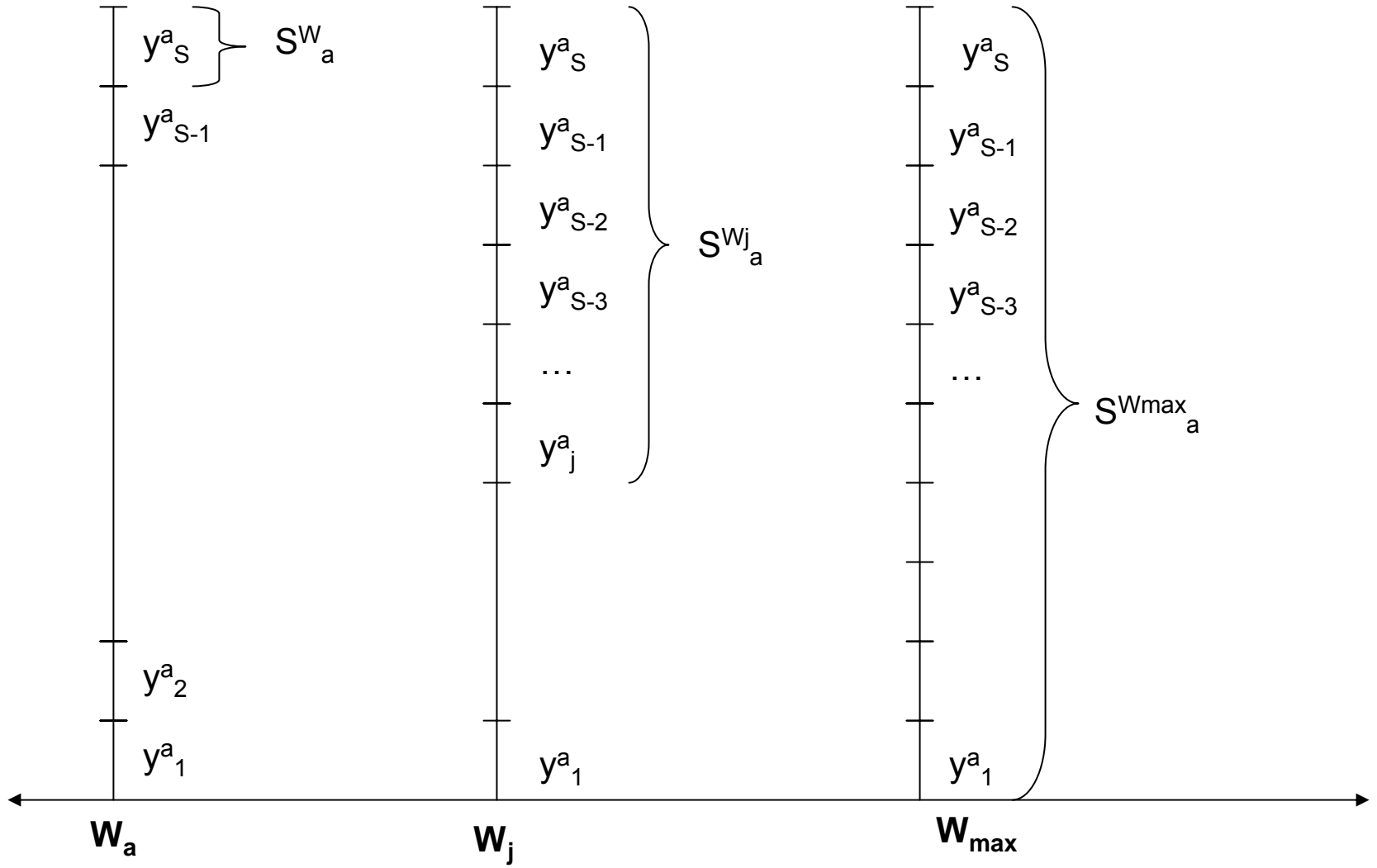
After some manipulation, the term $(Y_s^a + Y_s^b)$ factors out, and we are left with the following expression:

$$\frac{k^\alpha \alpha^2}{(1-\alpha)^{1-\alpha}} \geq \frac{1 - k^{\frac{\alpha}{\alpha-1}}(1-\alpha) - \frac{\beta\pi}{1-\beta}\alpha}{(1 - k^{\frac{\alpha}{\alpha-1}}(1-\alpha) - k\alpha^2)^{1-\alpha} \left(1 - \frac{\beta\pi}{1-\beta}\right)}$$

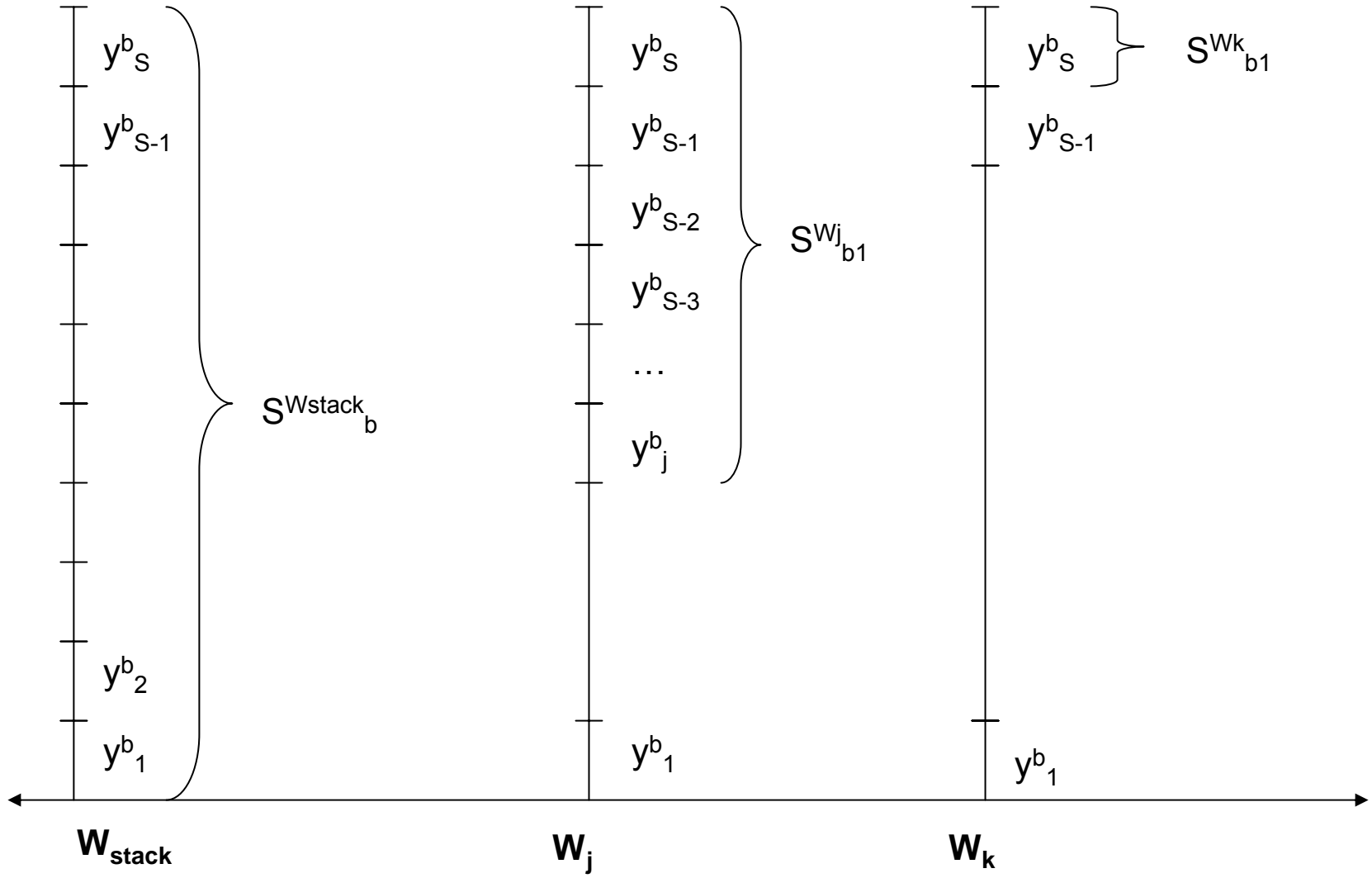


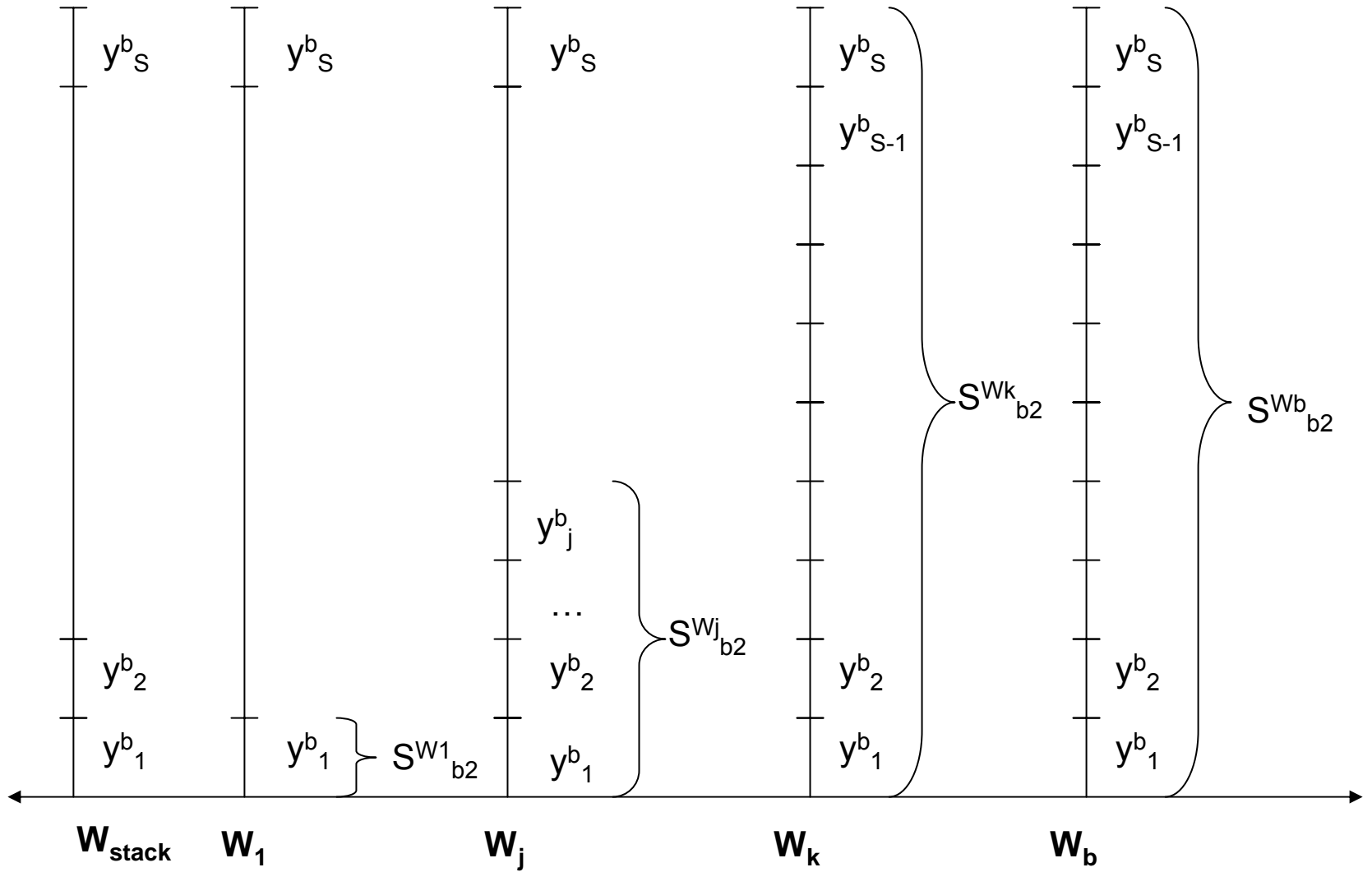


S^W_a

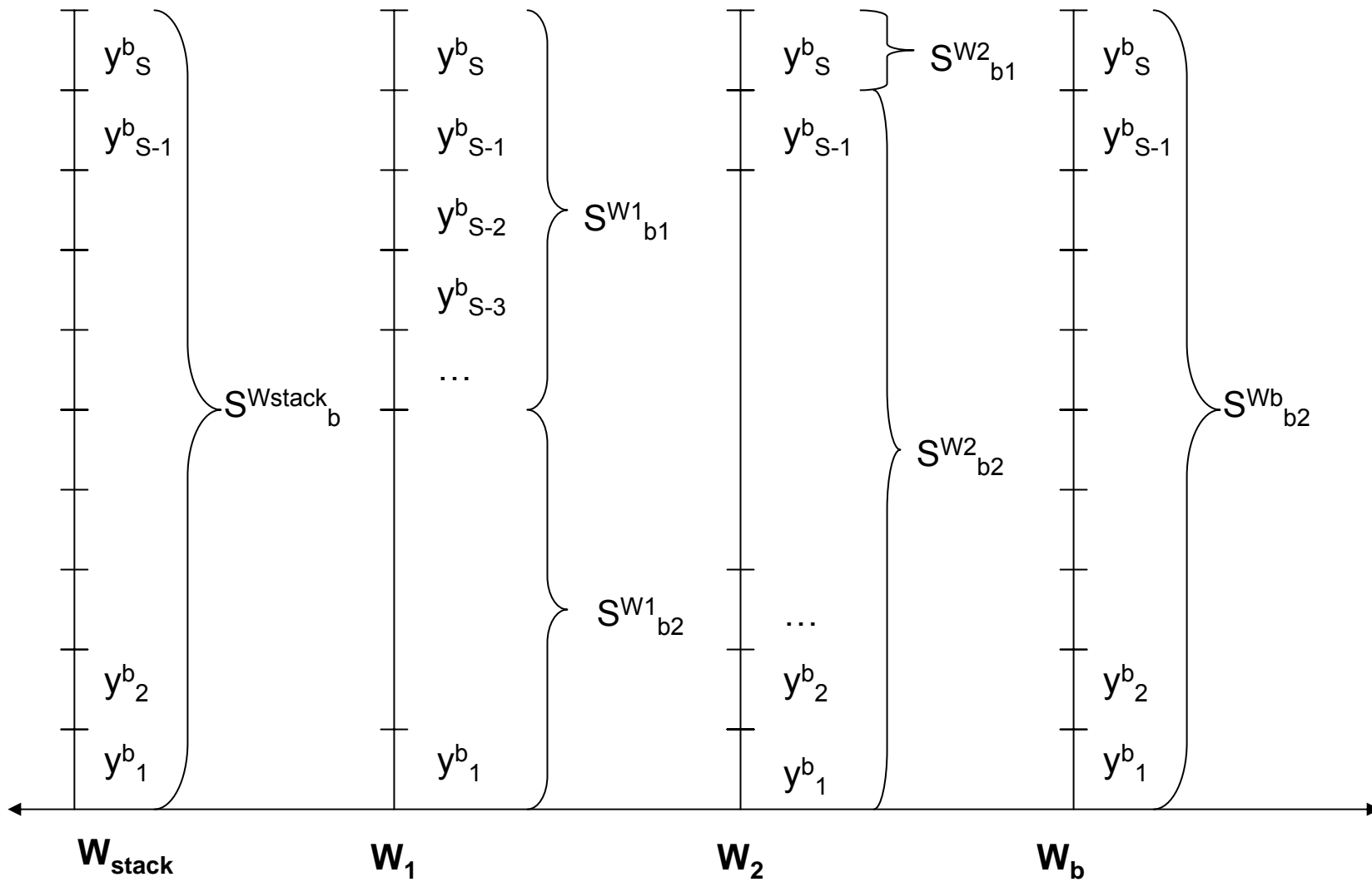


S_{b1}^W



S_{b2}^W 

S_{b1}^W and S_{b2}^W



S_{b2}^W and S_a^W

