The College Admissions Problem with Uncertainty[*]

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Abstract

We consider a college admissions problem with uncertainty. Unlike Gale and Shapley (1962), we realistically assume that (*i*) students' college application choices are nontrivial because applications are costly, (*ii*) college rankings of students are noisy and thus uncertain at the time of application, and (*iii*) matching between colleges and students takes place in a decentralized setting. We analyze an equilibrium model where two ranked colleges set admissions standards for student caliber signals, and students, knowing their calibers, decide where to apply to.

Do the best students try to attend the best colleges? While application noise works against this, we show that weaker students may apply more ambitiously than stronger ones, further overturning it. But we prove that a unique equilibrium with assortive matching of student caliber and college quality exists provided application costs are small and the capacity of the lesser college is not too small. We also provide equilibrium comparative static results with respect to college capacities and application costs. Applying the model, we find that racial affirmative action at the better college comes at the expense of diversity at the other college.

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1 Introduction

The college admissions process has been the object of much academic scrutiny lately. This interest in part owes to the strategic nature of college admissions, as schools use the tools at their disposal to attract the best students. Those students, in turn, respond most strategically in making their application decisions. This paper examines the joint behavior of students and colleges in an economic framework.

We develop and flesh out an equilibrium model of the college admissions process, with decentralized matching of students and two colleges — one better and one worse, respectively, called 1 and 2. Better and worse here refers to the payoffs students earn from attending the college. The model captures two unexplored aspects of the 'real-world' problem. First, each college application is costly, and so students must solve a nontrivial portfolio choice problem. Second, colleges only observe a noisy signal of each applicant's caliber, and seek to fill their capacity with the very best students possible.

Our first contribution is methodological: We provide an intuitive graphical analysis of the student choice problem that fully captures the application equilibrium. We hope that this framework will prove a tractable workhorse for future work on this subject. It embeds both the tradeoffs found in search-theoretic problems of Chade and Smith (2006), and the colleges' choice of capacity-filling admission standards.

On the one hand, assuming uncertain college prospects and costly applications is realistic, for college application entails a non-negligible cost and almost no students know which of the better colleges will admit them. Did their essay on how they will change the world go well with Harvard, or ring hollow? On the other hand, absent uncertainty or application cost, the student problem trivializes — as they would either simply apply to the best college that would admit them (the noiseless case) or to both colleges (the costless case). The tandem of noisy caliber and costly applications feeds the intriguing conflict at the heart of the student choice problem: Gamble on Harvard, settle for Michigan, or apply to Harvard while insuring oneself with Michigan.

A central question addressed in this paper is: Does assortative matching between students and colleges emerge? Whether the best students attend the best colleges is far from obvious, because two forces must cooperate. First, student applications must increase in their caliber. Specifically, we argue that this means that: (*i*) the best students apply just to college 1; (*ii*) the middling/strong students insure by applying to both colleges 1 and 2; (*iii*) the middling/weak students apply just to college 2; and finally, (*iv*) the weakest students apply nowhere. There are theoretical reasons for using the strong set order, but notice from an empirical standpoint that it would deliver the desirable property that the expected calibers of students is higher at college 1 than at college 2. The answer is far from obvious, as we show that the student portfolio choice problem can fail the standard conditions for monotone behavior in the student's caliber.

Secondly, does college 1 impose a higher admission standard than 2? This need not hold if 1's capacity is too large for its caliber niche, for then a curious inversion may arise — college 2 may screen applicants more tightly than college 1. This observation that college standards reflect not only their inherent caliber but also their capacity is our second contribution. We show that a unique equilibrium with assortative matching exists when application costs are small and the capacity of the lesser college is not too small. Furthermore, in this equilibrium, the distribution of calibers among the students who enroll in the better college stochastically dominates that of the lesser college.

When matching is assortative, the equilibrium exhibits some interesting comparative statics. In particular, we uncover an externality of the lesser upon the better college. If college 2 raises its capacity, then this lowers the admissions standards at *both* colleges. The reason is that the marginal student that was previously indifferent between just applying to college 2 and adding college 1 as well, now prefers to avoid the extra cost; he applies to college 2 only. This portfolio reallocation pushes down college 1's admission standards. Notice that *both* cost and noise play a role here, for without noise students do not send multiple applications, and without cost all of them trivially apply to both colleges. For instance, our theory predicts that when the University of Chicago substantially raised its college capacity in the 1990's, if its College payoff level remained constant, then better ranked Ivy League schools should have dropped their standards.

We provide an application of our framework by examining the effects of race-based admissions policies. While one college may experience greater diversity of its student body from such a policy, it comes at the cost of reduced diversity at the other school. On balance, we show that the *average* composition of the student body is not necessarily weakened by introducing such a policy, although some weaker students will be admitted.

The paper is related to several strands of literature. Gale and Shapley initiated the college admissions problem in their classic 1962 work in the economics of matching. As the prime example of many-to-one matching, it has long been in the province of cooperative game theory (e.g., Roth and Sotomayor (1990)). Our model varies by introducing the realistic assumption that matching is decentralized and subject to frictions, where the frictions are given by the application cost and the noisy evaluation process.

Whether matching is assortative has been the organizing question of the two-sided matching literature since Becker (1973). This has already been fleshed out in many symmetric one-to-one matching settings. Shimer and Smith (2000) and Smith (1997) characterized it for search frictions, while Anderson and Smith (2005) and Chade (2006) found different answers for incomplete information depending on whether reputations are private or public. But in this many-to-one college matching setting, the sides play different roles, as colleges control standards while students choose application sets.

The student portfolio problem embedded in the model is a special case of the simultaneous search problem solved in Chade and Smith (2006). Here, we use their solution to characterize the optimal student application strategy. However, the acceptance probabilities here are endogenous, since any one student's acceptance probability depends on which of her peers also applies to that school. Thus, this paper is also contributes to the literature on equilibrium models with nonsequential or directed search (e.g., Burdett and Judd (1983), Burdett, Shi, and Wright (2001), and Albrecht, Gautier, and Vroman (2002)), as well as Kircher and Galenianos (2006) and Kircher (2006).

A related paper to ours is Nagypál (2004). She assumes that colleges precisely know each student's caliber, and students only imperfectly so — they observe normal signals and update their beliefs before applying. We believe that assuming noisy college assessments is more realistic. Also, it affords a definition of assortative matching that is in line with literature cited above.[1] Arguably, neither students themselves nor colleges know the true talent; however, we feel that students have the informational edge.[2]

The rest of the paper is organized as follows. The model is found in Section 2. Section 3 turns to the main results — an analysis first of the student portfolio choice problem, and next of the equilibria. We focus on their assortative matching and comparative statics properties. For clarity of exposition, the main results are derived in the text for a uniform signal distribution, but we show in the appendix that they extend to a large class of monotone likelihood signals, as well as conditionally correlated signals. Section 5 applies our framework for race-based admissions policies. Section 6 concludes.

[1]In Nagypál (2004), a student's caliber is his posterior belief over possible qualities, which is parameterized not only by the posterior mean but also by the posterior variance. This makes the definition and interpretation of what constitutes assortative matching less clear cut.

[2]In a richer model, students and colleges alike would have signals of the student's true caliber. Colleges then have signals of signals, and our assorting conclusions should extend.

2 The Model

We impose very little structure in order to focus on the essential features of the problem. There are two colleges 1 and 2, and a continuum of students with measure equal to one. Thus, student choice matters, but there is no *ex-post* enrollment uncertainty. We assume common values: Students agree that college 1 is the best (payoff 1), and college 2 not as good (payoff $u \in (0,1)$), but still better than not attending college (payoff 0).

SIGNALS OF STUDENT CALIBER. Any given student is equally desired by both colleges. However, there is an informational friction here, as colleges only observe noisy signals of any student's caliber. The *caliber* x is described by the atomless density $f(x)$ with support $[0,\infty)$. While students know their caliber, colleges do not. Colleges 1 and 2 each observe a noisy conditionally i.i.d. signal of each applicant's caliber.[3]

To simplify the presentation of the main results, we assume in the text that each college observes the signal σ drawn from a uniform distribution whose support depends on the student's caliber x: The conditional density is $g(\sigma|x) = 1/x$ on $[0,x]$, and zero elsewhere. The corresponding cdf is then $G(\sigma|x) = \sigma/x$ when $0 \leq \sigma \leq x$, and later 1. Later on, we extend our results to an arbitrary signal density $g(\sigma|x)$ with the monotone likelihood ratio property (MLRP).

STUDENT AND COLLEGE ACTIONS. Students may apply to either, both, or neither college. Each application costs $c > 0$, and applications must be sent simultaneously. To avoid trivialities, we assume $u > c$. Students maximize their expected college payoff less application costs. Colleges accept or reject each student, and seek to fill their *capacities* $\kappa_1, \kappa_2 \in (0,1)$ with the best students. We focus on the interesting case in which limited college capacity does not afford room for all students to attend college, as $\kappa_1 + \kappa_2 < 1$.

In this setting, students' strategies can be summarized by a correspondence $S(x)$, which selects, for each caliber x, a college application menu in $\{\varnothing, \{1\}, \{2\}, \{1,2\}\}$. College $i = 1,2$ must set an "admission standard" $\underline{\sigma}_i \in [0,\infty)$ for signals, above which it accepts students.[4] College 1 knows that it will be accepted by any student admitted to both colleges, while college 2 knows that it will be rejected in that event. Thus, student x's acceptance chance at college i is given by $\alpha_i(x) = 1 - G(\underline{\sigma}_i|x)$. Since a higher caliber student x generates stochastically higher signals, $\alpha_i(x)$ is increasing in x.

[3]Such a signal may possibly arise from a college-specific essay or interview. If a common SAT score is taken into consideration, then the signals are perfectly correlated. We analyze this case in §7.

[4]Such a rule is optimal for colleges that seek to maximize the expected caliber of their student body. We will show this formally in the Appendix; for now, we will assume it.

Equilibrium. A *college equilibrium* is a pair $(S^e(\cdot), (\underline{\sigma}^e_1, \underline{\sigma}^e_2))$ such that

(*a*) Given $(\underline{\sigma}^e_1, \underline{\sigma}^e_2)$, $S^e(x)$ is an optimal college application portfolio for each x,

(*b*) Given $S^e(\cdot)$, college $i = 1, 2$, exactly fills its student capacity given $\underline{\sigma}^e_i$.

An equilibrium exhibits positive assortative matching (PAM) if colleges and students' strategies are *monotone*. This means that $\underline{\sigma}^e_1 > \underline{\sigma}^e_2$ and that $S^e(x)$ is increasing in x — namely, under the "strong set order" ranking $\varnothing \prec \{2\} \prec \{1, 2\} \prec \{1\}$. In this way, better colleges are more selective and higher caliber students choose better portfolios. Furthermore, for any distribution over students' calibers, the average student caliber applying to college 1 exceeds that applying to college 2. This empirical relevance is perhaps more convincing than the coincidence with the strong set order.

3 Equilibrium Analysis for Two Benchmark Cases

Two realistic modifications of the Gale-Shapley model play a key role in all our results. First, applications are costly, and second signals of students' calibers are noisy. To appreciate the role of each, we now analyze the model with each feature separately.

3.1 The Noiseless Case

Assume that colleges directly observe students' calibers x. Absent uncertainty, students will simply apply to the best school to accept them, and colleges will set thresholds $\underline{\sigma}_1, \underline{\sigma}_2$ to fill their capacities. In this case, we cannot have $\underline{\sigma}_1 \leq \underline{\sigma}_2$, for then no student would apply to the worse college 2 — impossible in an equilibrium. Assuming $\underline{\sigma}_1 > \underline{\sigma}_2$:

$$S(x) = \begin{cases} \{1\} & \text{if } x \geq \underline{\sigma}_1 \\ \{2\} & \text{if } \underline{\sigma}_2 \leq x < \underline{\sigma}_1 \\ \varnothing & \text{if } x < \underline{\sigma}_2 \end{cases}$$

Thus, given $\kappa_1, \kappa_2 \in (0, 1)$, an equilibrium is simply a pair of thresholds $\underline{\sigma}_1, \underline{\sigma}_2$ solving:

$$\kappa_1 = \int_{\underline{\sigma}_1}^{\infty} f(x)dx \qquad \text{and} \qquad \kappa_2 = \int_{\underline{\sigma}_2}^{\underline{\sigma}_1} f(x)dx \tag{1}$$

where the right sides in (1) measure the students who apply to each college.

Theorem 1 (Noiseless Case) *Let $\kappa_1 \in (0,1)$ and $\kappa_2 \in (0, 1-\kappa_1)$. Then there exists a unique equilibrium $(\underline{\sigma}^e_1, \underline{\sigma}^e_2)$, and it exhibits PAM. Furthermore, $\underline{\sigma}^e_1$ falls in κ_1, and is independent of κ_2 and c; $\underline{\sigma}^e_2$ falls in κ_1 and κ_2, and is independent of c.*

3.2 The Costless Case

Now assume a zero application cost. Everyone applies to both colleges, $S(x) \equiv \{1,2\}$. Student behavior is then trivially (weakly) monotone. However, the college acceptance thresholds may satisfy $\underline{\sigma}_1 > \underline{\sigma}_2$ or $\underline{\sigma}_2 > \underline{\sigma}_1$, depending on the relative capacities κ_1, κ_2.

An equilibrium with $\underline{\sigma}_1 > \underline{\sigma}_2$ entails the capacity equations:

$$\kappa_1 = \int_{\underline{\sigma}_1}^{\infty} \left(1 - \frac{\underline{\sigma}_1}{x}\right) f(x)dx \tag{2}$$

$$\kappa_2 = \int_{\underline{\sigma}_2}^{\underline{\sigma}_1} \left(1 - \frac{\underline{\sigma}_2}{x}\right) f(x)dx + \int_{\underline{\sigma}_1}^{\infty} \frac{\underline{\sigma}_1}{x} \left(1 - \frac{\underline{\sigma}_2}{x}\right) f(x)dx, \tag{3}$$

where the right side of (2)–(3) is the measure of students that apply to, are accepted by, and enroll in each college. For instance, the second term on the right side of (3) states students with $\sigma > \underline{\sigma}_1$ that will end up in college 2 consists of those students who are rejected by college 1 and accepted by college 2.

Likewise, an equilibrium with $\underline{\sigma}_2 > \underline{\sigma}_1$ obeys (2) and

$$\kappa_2 = \int_{\underline{\sigma}_2}^{\infty} \frac{\underline{\sigma}_1}{x} \left(1 - \frac{\underline{\sigma}_2}{x}\right) f(x)dx. \tag{4}$$

Theorem 2 (Costless Case) *For each $\kappa_1 \in (0,1)$, there exists $\underline{\kappa}_2 \in (0, 1-\kappa_1)$ s.t.*
(a) If $\kappa_2 \in (\underline{\kappa}_2, 1-\kappa_1)$, then there is a unique equilibrium, and it has $\underline{\sigma}^e_1 > \underline{\sigma}^e_2$.
(b) If $\kappa_2 \in (0, \underline{\kappa}_2)$, then there is a unique equilibrium, and it has $\underline{\sigma}^e_2 > \underline{\sigma}^e_1$.
(c) The threshold $\underline{\sigma}^e_1$ falls in κ_1, and is independent of κ_2; $\underline{\sigma}^e_2$ falls in both κ_1 and κ_2.

Summarizing, equilibrium in the noiseless case *always* exhibits PAM, while PAM can *fail* in the costless case since college behavior need not be monotone. For if college 2's capacity is sufficiently small, it may end up setting a higher threshold than college 1. Also, greater capacity at college 2 has no effect whatsoever on college 1. Finally, the students' behavior is straightforward in both cases (trivial in the costless case). As we see below, the results are drastically different when both cost *and* noise are present.

4 Student Portfolio Choice

Consider the student portfolio choice problem with costly applications and noisy signals. College thresholds $\underline{\sigma}_1$ and $\underline{\sigma}_2$ induce acceptance chances $\alpha_1(x)$ and $\alpha_2(x)$ for every caliber x. Taking these acceptance chances as given, each student of caliber x chooses a portfolio of colleges to apply to. They accept the best school accepts them.

Application Sets for a Given Student. Students' optimal choice sets are:

$$S(x) = \text{argmax}\{0,\, \alpha_1(x) - c,\, \alpha_2(x)u - c,\, \alpha_1(x) + (1 - \alpha_1(x))\alpha_2(x)u - 2c\} \tag{5}$$

Here, $\alpha_1(x) - c$ is the expected payoff of portfolio $\{1\}$, $\alpha_2(x)u - c$ is the expected payoff of $\{2\}$, and $\alpha_1(x) + (1 - \alpha_1(x))\alpha_2(x)u - 2c$ is the expected payoff of $\{1, 2\}$.

This optimization problem admits an illuminating and rigorous graphical analysis. Consider a given student facing acceptance chances α_1 and α_2. Figure 1 depicts:

$$\alpha_2 u \;=\; \alpha_1 \tag{6}$$

$$MB_{21} \equiv (1 - \alpha_1)\alpha_2 u \;=\; c \tag{7}$$

$$MB_{12} \equiv \alpha_1(1 - \alpha_2 u) \;=\; c, \tag{8}$$

where MB_{ij} is the marginal (gross) benefit of adding college i to a portfolio of college j. In Figure 1, (6) is the line $\alpha_2 = \alpha_1/u$, (7) is the concave curve $\alpha_2 = c/[u(1 - \alpha_1)]$, and (8) is the convex curve $\alpha_2 = (1 - c/\alpha_1)/u$.

Marginal analysis reveals that, as a function of α_1 and α_2, the optimal application strategy is given by the following rule (which follows from Chade and Smith (2006)):

(a) Apply just to college 1, if that beats applying just to college 2 or nowhere, and if adding college 2 is then worse: $\alpha_1 \geq \max\langle c, \alpha_2 u\rangle, MB_{21} < c \Rightarrow$ apply to $\{1\}$

(b) Apply to both colleges if choosing just college 1 beats choosing just college 2 or applying nowhere, and if adding college 2 is then better, or vice versa: $\alpha_1 \geq \max\langle c, \alpha_2 u\rangle$ & $MB_{21} \geq c$ or $\alpha_2 u \geq \max\langle c, \alpha_1\rangle$ & $MB_{12} \geq c \Rightarrow$ apply to $\{1, 2\}$

(c) Apply just to college 2, if that beats applying just to college 1 or nowhere, and if adding college 1 is then worse: $\alpha_2 u \geq \max\langle c, \alpha_1\rangle$, $MB_{12} < c] \Rightarrow$ apply to $\{2\}$

(d) Apply nowhere if no solo application is profitable: $\alpha_1 < c$, $\alpha_2 u < c \Rightarrow$ apply to $\varnothing$.

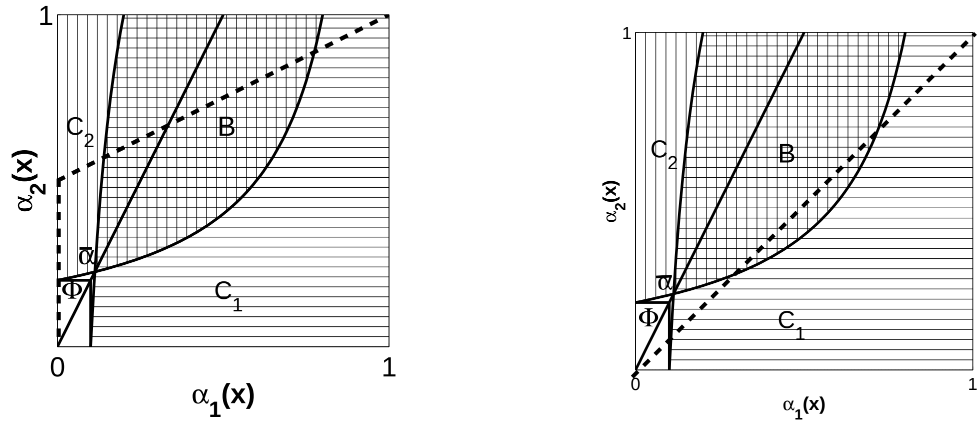

Figure 1: **Optimal Decision Regions.** A student *applies to college 2 only* in the vertical shaded region C_2; *to both colleges* in the hashed shaded region B; finally, *to college 1 only* in the horizontal shaded region C_1. The dashed line acceptance relation ψ relates college acceptance chances. In the left panel, as their caliber rises, students apply first to college 2, then both, and then only college 1. The right panel shows non-monotone behavior, where weak students aspire for college 1, and no one just applies to college 2. This happens here as colleges set exactly the same thresholds, and so acceptance chances for a given caliber coincide across colleges.

Cases (a)–(d) partition the unit square into regions of α_1 and α_2 that correspond to each portfolio choice. These application regions are shaded in Figure 1.

To simplify matters and avoid uninteresting cases, we require that $\alpha_2 = c/u(1-\alpha_1)$ and $\alpha_2 = (1-c/\alpha_1)/u$ cross only once in the unit square (as seen in Figure 1). Insisting that $\alpha_2 \leq 1$ in (7) and (8) easily yields an upper bound on costs (hereafter assumed):

$$c \leq u(1-u). \tag{9}$$

The College Acceptance Relation. A higher caliber student x generates stochastically higher signals, and so a higher acceptance chance at each college. A monotone relation is thus induced between α_1 and α_2 across students for every pair of thresholds $\underline{\sigma}_1, \underline{\sigma}_2$. More precisely, this is constructed by inverting $\alpha_1 = 1-G(\underline{\sigma}_1|x)$ in x, and inserting the resulting expression in $\alpha_2 = 1 - G(\underline{\sigma}_2|x)$. In particular, if $\underline{\sigma}_1 \geq \underline{\sigma}_2$, then the *acceptance relation* is piecewise linear, and is given by

$$\alpha_2 = \psi(\alpha_1, \underline{\sigma}_1, \underline{\sigma}_2) \equiv \begin{cases} \left(1 - \dfrac{\underline{\sigma}_2}{\underline{\sigma}_1}\right) + \dfrac{\underline{\sigma}_2}{\underline{\sigma}_1}\alpha_1 & \text{if } \alpha_1 > 0 \\ \left[0, 1 - \dfrac{\underline{\sigma}_2}{\underline{\sigma}_1}\right] & \text{if } \alpha_1 = 0 \end{cases} \tag{10}$$

Application Sets Across Students. Given acceptance rates $\alpha_1(x), \alpha_2(x)$, the acceptance relation implies an optimal student correspondence $S(x)$. In a monotone portfolio, higher caliber students apply to better schools, as in the left panel of Figure 1.

$$S(x) = \begin{cases} \{1\} & \text{if } x \geq \xi_1 \\ \{1,2\} & \text{if } \xi_B \leq x < \xi_1 \\ \{2\} & \text{if } \xi_2 \leq x < \xi_B \\ \varnothing & \text{if } 0 \leq x < \xi_2 \end{cases} \tag{11}$$

Here, the thresholds $\xi_2 < \xi_B < \xi_1$ are implicitly defined by the intersection of the acceptance relation with c/u, $\alpha_2 = c/[u(1-\alpha_1)]$, and $\alpha_2 = (1 - c/\alpha_1)/u$, respectively.

Lemma 1 (Thresholds) *Assuming* (15), *the thresholds* ξ_2, ξ_B, *and* ξ_1 *in* (11) *obey:*

$$\xi_2 = \frac{\underline{\sigma}_2}{1 - c/u} \tag{12}$$

$$\xi_B = \frac{2u\underline{\sigma}_1\underline{\sigma}_2}{u\underline{\sigma}_2 - (1-u)\underline{\sigma}_1 + \sqrt{((1-u)\underline{\sigma}_1 + u\underline{\sigma}_2)^2 - 4cu\underline{\sigma}_1\underline{\sigma}_2}} \tag{13}$$

$$\xi_1 = \frac{2\underline{\sigma}_2}{1 - \sqrt{1 - 4c\underline{\sigma}_2/(u\underline{\sigma}_1)}} \tag{14}$$

With both costs and noise present, student behavior need *not* be monotone in their caliber. It is easy to construct examples in which $S(x)$ fails to be monotone *even* with $\underline{\sigma}_1 \geq \underline{\sigma}_2$. The right panel of Figure 1 provides such an example (explored in §5). In this case, behavior is not monotone since it is possible to find a low caliber student who applies only to college 1 and a higher caliber student that applies to both.

When is student behavior monotone? Inspecting Figure 1, a necessary and sufficient algebraic condition for an increasing student strategy $S(x)$ such as (11) is that the acceptance relation crosses above the intersection $(\overline{\alpha}_1, \overline{\alpha}_2)$ of (6)–(8). Given $\overline{\alpha}_2 = \overline{\alpha}_1/u$, this holds if and only if the college thresholds obey the following condition:

$$\underline{\sigma}_2 \leq \left(\frac{1 - \bar{\alpha}_1/u}{1 - \bar{\alpha}_1}\right)\underline{\sigma}_1 \equiv \eta\underline{\sigma}_1 \qquad \text{where} \qquad \bar{\alpha}_1 = (1 - \sqrt{1-4c})/2 \tag{15}$$

Since $1 - \bar{\alpha}_1/u < 1 - \bar{\alpha}_1$, the threshold of college 2 must lie *sufficiently* below college 1's. Theorem 4 produces conditions on primitives that deliver a monotone equilibrium.

5 Equilibrium Analysis

We suggestively denote the application sets, suppressing thresholds $\underline{\sigma}_1, \underline{\sigma}_2$, by:

$$\Phi = S^{-1}(\varnothing),\ C_2 = S^{-1}(\{2\}),\ B = S^{-1}(\{1,2\}),\ C_1 = S^{-1}(\{1\})$$

By analogy to equations (2)–(3), the *enrollment* at the colleges is then given by

$$E_1(\underline{\sigma}_1, \underline{\sigma}_2) = \int_{B \cup C_1} \left(1 - \frac{\underline{\sigma}_1}{x}\right) f(x)\, dx \tag{16}$$

$$E_2(\underline{\sigma}_1, \underline{\sigma}_2) = \int_{C_2} \left(1 - \frac{\underline{\sigma}_2}{x}\right) f(x)\, dx + \int_B \frac{\underline{\sigma}_1}{x} \left(1 - \frac{\underline{\sigma}_2}{x}\right) f(x)\, dx. \tag{17}$$

Given κ_1 and κ_2, a college equilibrium is then a pair of thresholds $\underline{\sigma}_1, \underline{\sigma}_2$ that satisfy the enrollment 'market clearing' conditions $\kappa_1 = E_1(\underline{\sigma}_1, \underline{\sigma}_2)$ and $\kappa_2 = E_2(\underline{\sigma}_1, \underline{\sigma}_2)$.

Theorem 3 (Existence) *Let $c > 0$. For any $\kappa_1 \in (0,1)$ and $\kappa_2 \in (0, \overline{\kappa}_2(\kappa_1))$, with $0 < \overline{\kappa}_2(\kappa_1) < 1 - \kappa_1$, an equilibrium exists.*

Inspection of Figure 1 reveals that set C_1 is always nonempty. The capacity of college 2 must then be bounded away from $1 - \kappa_1$, since a positive measure of students applies *only* to college 1. Otherwise, college 2 would be unable to fill its capacity.

Theorem 3 does *not* assert that any equilibrium exhibits monotone behavior. For assume both colleges use the same standard $\underline{\sigma}^e_1 = \underline{\sigma}^e_2 = \underline{\sigma}^e$. Then the acceptance relation (10) reduces to the 'diagonal' $\alpha_2 = \alpha_1$. As $\alpha_1 > \alpha_2 u$ for every x, any student applying somewhere will apply to college 1, as in Figure 1. If $c < u/4$, then $\Phi = [0, \xi_0)$, $C_2 = \varnothing$, $B = [\xi_b, \xi_1)$, and $C_1 = [\xi_0, \xi_b) \cup [\xi_1, \infty)$, where (6)–(8) yield respective thresholds:

$$\xi_0 = \frac{\underline{\sigma}^e}{1-c} \quad \text{and} \quad \xi_b, \xi_1 = \frac{2\underline{\sigma}^e}{1 \pm \sqrt{1 - 4c/u}}$$

The equations $\kappa_1 = E_1(\underline{\sigma}_1, \underline{\sigma}_2)$ and $\kappa_2 = E_2(\underline{\sigma}_1, \underline{\sigma}_2)$ then yield parameters for which the resulting equilibrium *fails* to exhibit PAM, for any signal distribution. If it is perturbed, the same non-monotone student behavior emerges in equilibrium for an open set of (κ_1, κ_2). Unlike the failure of PAM for the costless case, this owes to the lack of 'monotonicity' in x of the nontrivial student portfolio optimization problem. In other words, the best students need not even attend the colleges not only because of noise, but also because students application strategies need not be monotone in their calibers.

Lemma 2 (Application Thresholds)
(i) ξ_2 *is independent of* $\underline{\sigma}_1$*, rises in* $\underline{\sigma}_2$*, and rises in* c*, with* $\lim_{c\to 0}\xi_2 = \underline{\sigma}_2$.
(ii) ξ_B *rises in* $\underline{\sigma}_1$*, falls in* $\underline{\sigma}_2$*, and rises in* c*, with* $\lim_{c\to 0}\xi_B = \underline{\sigma}_1$.
(iii) ξ_1 *rises in* $\underline{\sigma}_1$*, falls in* $\underline{\sigma}_2$*, and falls in* c*, with* $\lim_{c\to 0}\xi_1 = \infty$.

The proof in the appendix differentiates the closed form expressions (12)–(14).

When college 1 raises its admission standards, ceteris paribus, fewer students apply to college 1 (as ξ_B rises) and more apply to college 2 (as ξ_1 rises). This reallocation improves the applicant pool at both colleges. When college 2 raises its admission standards, ceteris paribus, fewer students apply to college 2 (as ξ_2 rises and ξ_1 falls) and more apply to college 1 (as ξ_B falls). So the applicant pool at college 2 need not improve, while the pool for college 1 worsens as more students find it optimal to include it in their portfolios.

Let us re-write the enrollment equations (16)–(17), making explicit the dependence of E_i on the application cost c. Further substitute the monotonicity conditions, putting $B\cup C_1 = [\xi_B, \infty)$ into (16) and $C_2 = [\xi_2, \xi_B)$ and $B = [\xi_B, \xi_1)$ into (17). By differentiation:

Lemma 3 (Enrollment) *Enrollment* E_1 *at college* 1 *falls in* $\underline{\sigma}_1$*, rises in* $\underline{\sigma}_2$*, and falls in* c*, while enrollment* E_2 *at college* 2 *rises in* $\underline{\sigma}_1$*, falls in* $\underline{\sigma}_2$*, and is ambiguous in* c*.*

The cost comparative statics follow as students insure themselves less often when it is more costly — the set B shrinks on both ends, as ξ_B rises and ξ_1 falls. This harms enrollment at college 1. Altogether, $B\cup C_1$ shrinks. The effect on college 2 is uncertain, as its marginal applicant ξ_2 also rises — so that the set C_2 expands, while B shrinks.

We now impose the necessary and sufficient condition (15) for monotone student behavior, as it depends on the endogenous variables $\underline{\sigma}_1$ and $\underline{\sigma}_2$, and not obviously on the capacities. So a *monotone equilibrium* entails a pair of college thresholds that obey:

$$\kappa_1 = E_1(\underline{\sigma}_1, \underline{\sigma}_2, c) \tag{18}$$

$$\kappa_2 = E_2(\underline{\sigma}_1, \underline{\sigma}_2, c) \tag{19}$$

$$0 \leq \underline{\sigma}_2 \leq \left(\frac{1-\bar{\alpha}_1/u}{1-\bar{\alpha}_1}\right)\underline{\sigma}_1 \tag{20}$$

Reformulate (18) and (19) as explicit functions relating the two college thresholds:

$$\underline{\sigma}_1 = H_1(\underline{\sigma}_2, \kappa_1, c) \tag{21}$$

$$\underline{\sigma}_2 = H_2(\underline{\sigma}_1, \kappa_2, c), \tag{22}$$

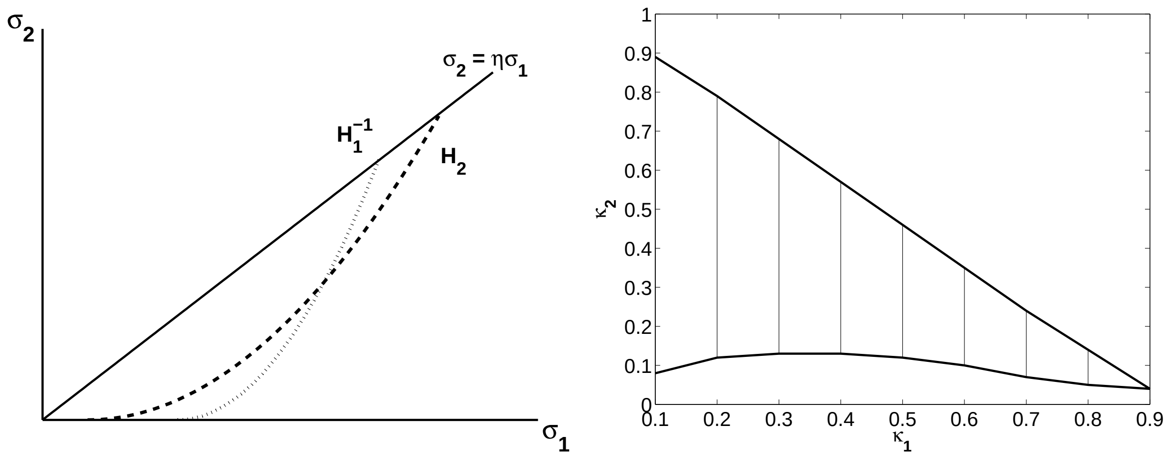

Figure 2: **Monotone Equilibrium.** The functions H_1 and H_2 in the left panel give pairs of threshold levels that allow colleges 1 and 2 to fill their capacities. A monotone equilibrium obtains if H_1 and H_2 cross below the straight line with slope η, from (15). The right panel describes the range of feasible capacities κ_2 for each κ_1 for which a monotone equilibrium exists — computed assuming $u = 0.5$, $c = 0.1$, and exponentially distributed calibers.

where (21) rises in $\underline{\sigma}_2$ and falls in κ_1, while (22) rises in $\underline{\sigma}_1$ and falls in κ_2. A monotone equilibrium is then a pair $(\underline{\sigma}^e_1, \underline{\sigma}^e_2)$ that satisfies (20)–(22).

Theorem 4 (Existence and Uniqueness) *Let $\kappa_1 \in (0,1)$. Then there exists a cost $c(\kappa_1) > 0$ and capacities $0 < \underline{\kappa}_2(\kappa_1) < \overline{\kappa}_2(\kappa_1) < 1 - \kappa_1$, such that a unique monotone equilibrium exists for all $(\kappa_1, \kappa_2) \in (0,1) \times (\underline{\kappa}_2(\kappa_1), \overline{\kappa}_2(\kappa_1))$ and $c \in (0, c(\kappa_1))$.*

Figure 2 illustrates the set $(0,1) \times (\underline{\kappa}_2(\kappa_1), \overline{\kappa}_2(\kappa_1))$ for which our equilibrium obtains.

For the same reason as in Theorem 3, capacity $\kappa_2 \ll 1 - \kappa_1$. For different reasons, monotonicity requires that $\kappa_2 \gg 0$. Only 'intermediate' values of κ_2 are consistent with a monotone equilibrium. For further insights, recall that equilibrium is unique in the costless case with $\underline{\sigma}_1 > \underline{\sigma}_2$ (Theorem 2 (*i*)). Since (18)–(20) are continuous in c, a monotone equilibrium is unique for c sufficiently small, i.e., when $c \in (0, c(\kappa_1))$.

Let $F_i(x|\underline{\sigma}^e_1, \underline{\sigma}^e_2)$ be the cumulative distribution of calibers accepted by college i in equilibrium, $i = 1, 2$. In a monotone equilibrium, one would expect that the distribution of calibers accepted by college 1 is 'better' than that of college 2 in the sense of first order stochastic dominance (FSD). We now formalize this important property of PAM:

Theorem 5 (PAM and Distribution of Types) *In any monotone equilibrium, the distribution of student calibers at college 1 is stochastically higher, in the sense of FSD.*

Changes in the model parameters κ_1, κ_2, and c affect both equilibrium student behavior and college thresholds. The comparative statics of the model are summarized:

Theorem 6 (Equilibrium Comparative Statics)
(*i*) *An increase in* κ_1 *decreases both college thresholds* $\underline{\sigma}^e_1$ *and* $\underline{\sigma}^e_2$*.*
(*ii*) *An increase in* κ_2 *decreases both college thresholds* $\underline{\sigma}^e_1$ *and* $\underline{\sigma}^e_2$*.*
(*iii*) *An increase in* c *decreases college threshold* $\underline{\sigma}^e_1$ *but has an ambiguous effect on* $\underline{\sigma}^e_2$*.*

The first of these comparative statics seems intuitive, and it is also found in the costless and noiseless cases (see Theorem 1 (*ii*) and Theorem 2 (*iii*)). The other two results, however, only obtain when both cost and noise are present.

Consider the effects of a rising capacity of the lower ranked college 2. Given $\underline{\sigma}_1$, greater κ_2 reduces $\underline{\sigma}_2$, and this increases ξ_B (Lemma 2 (*ii*)), and thereby pushes down $\underline{\sigma}_1$. So the marginal caliber that was indifferent between applying just to college 2 and adding college 1 as well, now prefers to avoid the extra cost and applies to college 2 only. This portfolio reallocation occasions a drop in the admission standards of college 1.

Consider now the effects of greater cost c. Given $\underline{\sigma}_1$, the effect on $\underline{\sigma}_2$ is ambiguous, since ξ_2 and ξ_B rise while ξ_1 falls (Lemma 2). The rise in ξ_2 and fall in ξ_1 shrinks the applicant pool at college 2, but with greater ξ_B, some students now prefer to apply only to college 2. Notice, however, that greater c necessarily pushes down $\underline{\sigma}_1$ (since ξ_B rises), which in turn decreases $\underline{\sigma}_2$. But the net effect on $\underline{\sigma}_2$ is ambiguous. As an illustration, the decrease in mailing costs, information gathering, and online applications have reduced the cost of applying to college. This, *ceteris paribus*, induces colleges to change their admission standards, which may actually increase the probability of being rejected by all the colleges in any given portfolio.

6 Application: Race-Based Admissions

In this section, we provide a simple application of our model to the topic of race-based admissions policies at top schools. This issue has been particularly topical since the Supreme Court cases *Gratz v. Bollinger* and *Grutter v. Bollinger* in which the University of Michigan was sued for its use of race as a factor in its admissions process. The controversy centers on whether the university's interest in promoting a diverse student body justifies the use of a discriminatory admissions process. The aim of our analysis

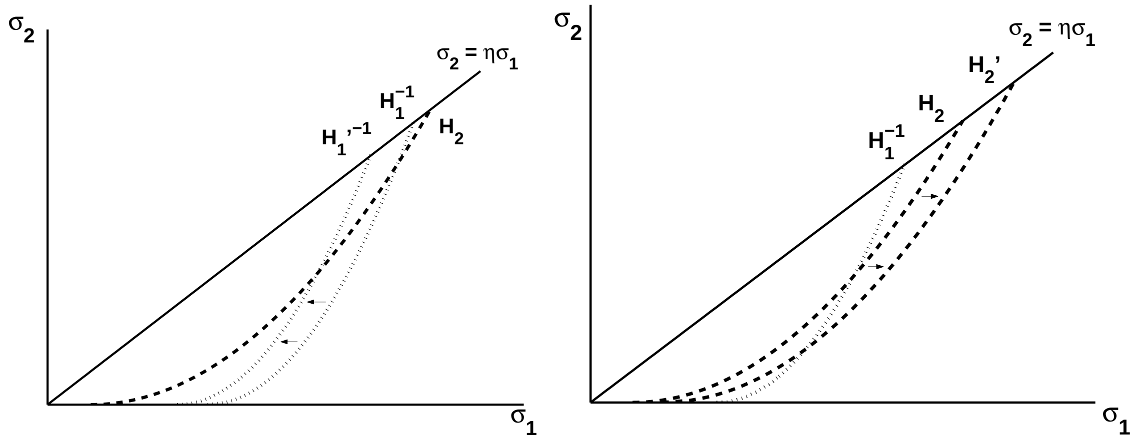

Figure 3: **Equilibrium Comparative Statics.** The two functions H_1 (dotted) and H_2 (dashed) are drawn, as well as the diagonal where (15) binds. This illustrates how the equilibrium is affected by changing capacities κ_1, κ_2. The left panel considers a rise in κ_1, shifting H_1 left, thereby lowering both college thresholds. The right panel depicts a rise in κ_2, shifting H_2 right, thereby lowering both college thresholds.

is not to provide a comprehensive treatment of this issue. Instead, we seek to provide a few simple insights into the effect of these policies using our equilibrium model.

We take as our starting point a race-based admissions policy implemented at the better college, college 1. Students of a minority group receive an additional Δ points on their applications, so that they are admitted if their signal σ is greater than $\underline{\sigma}_1 - \Delta$. This follows closely the actual undergraduate admissions policy of the University of Michigan, which was struck down by the Supreme Court. The underlying population consists of a proportion ρ of students from a minority group, and $1 - \rho$ from the majority group. The caliber distribution is identical for all population members.

Students of the minority group now have additional incentive to apply to college 1, as for any caliber x their probability of admission at that college has increased. By Lemma 2, ξ_B and ξ_1 will both fall for minority students, whereas ξ_2 is unaffected since it does not depend on college 1's admissions standard. The shift in the acceptance relation for minority students is illustrated in Figure 4.

This shift in student behavior disrupts the initial equilibrium, $(\underline{\sigma}_1^0, \underline{\sigma}_2^0)$. After the introduction of the policy, applications to college 1 from minority students will increase, and this leads to an increase in its admissions threshold. This effect is depicted in the left panel of Figure 5 by a rightward shift in the H_1^{-1} function. Yet the increase in $\underline{\sigma}_1$ must be less than Δ, for otherwise enrollment would be strictly lower than it was

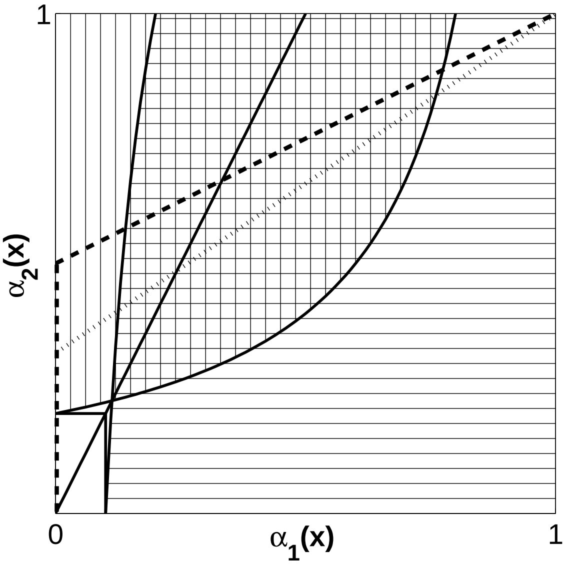

Figure 4: **Student Behavior with Race Based Admissions.** With the introduction of a race based admissions policy, students in the minority group are more likely to get into college 1 than before. As a result, their acceptance relation shifts downward. Their new acceptance relation is indicated by the dotted line, while the old one is dashed. Note that minority students start applying to college 1 at lower calibers than was previously the case.

initially. Hence the policy raises the admissions standard at college 1 for members of the majority group, and lowers it for those in the minority. The size of the effect depends on the fraction of minority students in the population: for ρ low, the shift in H_1^{-1} will be small and the new threshold $\underline{\sigma}_1^1$ will only be marginally higher. This means that $\underline{\sigma}_1^1 - \Delta$ will be close to $\underline{\sigma}_1^0 - \Delta$, and thus the minority group will receive the full benefit of the policy. By contrast, for ρ high, the admissions standard will increase markedly, and the new threshold for the minority group will not be that much lower. Moreover, that for members of the majority group will be considerably higher.

To complete the analysis, we note that college 2 must similarly drop its admissions threshold relative to before, as it is losing minority applicants to college 1. This is shown in the right panel of Figure 5 by a downward shift in the H_2 function. Yet since college 1 is simultaneously raising its applications threshold, thereby raising demand for places at college 2 from students of the majority group, the overall effect on the applications threshold of college 2 is unclear. We summarize our results in a proposition:

Theorem 7 (Race-Based Admissions) *Consider a monotone equilibrium and let college 1 implement a race-based admissions system parameterized by Δ. Then,*

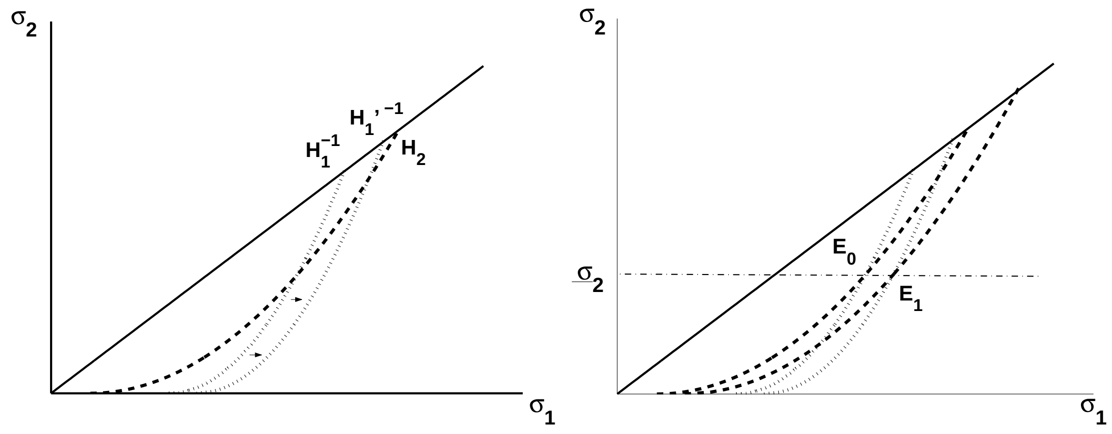

Figure 5: **Equilibrium under Race Based Admissions** The left panel shows the effect of increased applications by minority students to college 1. The H_1^{-1} functions shifts rightwards, intersecting H_2 at a higher level of both $\underline{\sigma}_1$ and $\underline{\sigma}_2$. The right panel shows the rightward shift of H_2 that results from college 2 losing minority applicants. The equilibrium shifts from E_0 to E_1, with a clear increase in the applications threshold at college 1, but an ambiguous effect on the applications threshold of college 2.

(i) The proportion of minority students at college 1 increases, and at college 2 falls;

(ii) The new threshold set by college 1 is higher, and is increasing in ρ;

(iii) The new threshold set by college 2 is the same as the old one.

These results provide some novel insights into the effects of these policies. The first is that there is an implicit tradeoff between the level of diversity across colleges. Making one college more diverse by attracting and admitting minority applicants through a race-based admissions policy must in turn limit the fraction of minority applicants at other schools. Moreover, these minority applicants may have less chance of being admitted at other schools as they are crowded out by strong majority applicants who were discouraged from applying to the college with the race-based admissions policy. This will be the case when the minority group is a small fraction of the population. So even if achieving diversity across colleges is considered an appropriate aim, there is a need to be sure that the schools that implement it are less diverse than their competitors.

The second insight relates to the composition of the student body at college 1. The policy allows weaker students from the minority group admission into the school. But it also restricts enrollment to stronger applicants from the majority group, and thus the effects on the average caliber of students in the school are unclear. To the extent that there are benefits to having a diverse student body, these benefits will be experienced by all. The question then becomes whether the caliber of the learning experience is dependent on the average caliber the student body, or is shaped by the weakest caliber in the student population. If the former, there is no real tradeoff and the policy is good for college 1; while if the latter, the effects of the policy are more tricky to pin down.

7 Different Signals of Caliber

7.1 Conditionally iid MLRP Signals

For simplicity, we have derived all the results under the assumption that the conditional distribution of the signal for a student of caliber x is uniform on $[0, x]$. We now show that all the results extend to a large class of signal distributions with the MLRP.

Let $g(\sigma|x)$ be a continuous density function that satisfies the MLRP, with $\sigma \in [\underline{\sigma}, \bar{\sigma}]$, $-\infty \leq \underline{\sigma} < \bar{\sigma} \leq \infty$, $x \in [\underline{x}, \infty)$, and $\underline{x} \geq -\infty$. Let $G(\sigma|x)$ be its cdf, which is assumed to be twice differentiable in x. Since G is monotone in each variable, we can take the inverse function with respect to each of them. We shall denote by φ the inverse of G with respect to x, and by ϕ the corresponding one with respect to σ.

Inverting $\alpha_1(x) = 1 - G(\underline{\sigma}_1|x)$ with respect to x and inserting the resulting expression in $\alpha_2(x) = 1 - G(\underline{\sigma}_2|x)$ yields the following *acceptance relation*:

$$\alpha_2 = \psi(\underline{\sigma}_1, \underline{\sigma}_2, \alpha_1) \equiv 1 - G(\underline{\sigma}_2|\varphi(1 - \alpha_1, \underline{\sigma}_1)). \tag{23}$$

Inspired by the properties of the acceptance relation in the uniform case, we shall impose the following assumptions on (23): (a) ψ is increasing in α_1; (b) $\psi(0, \underline{\sigma}_1, \underline{\sigma}_2) \geq 0$, $\psi(1, \underline{\sigma}_1, \underline{\sigma}_2) = 1$, and $0 < \psi < 1$ for all $\alpha_1 \in (0, 1)$; (c) ψ is concave in α_1 if $\underline{\sigma}_1 \geq \underline{\sigma}_2$. Under these conditions, we get an acceptance relation of the form depicted in Figure 6.

The following result provides a set of sufficient conditions on the family of signal distributions that engender properties (a), (b), and (c).

Lemma 4 (Signal Distributions) *$\psi(\underline{\sigma}_1, \underline{\sigma}_2, \alpha_1)$ satisfies (a), (b), and (c), if:*

(i) $g(\sigma|x)$ satisfies MLRP;

(ii) For all interior σ and x, $G(\sigma|x) > 0$, $\lim_{x \to \underline{x}} G(\sigma|x) = 1$ and $\lim_{x \to \infty} G(\sigma|x) = 0$;

(iii) $-G_x(\sigma|x)$ is log-supermodular in (σ, x).

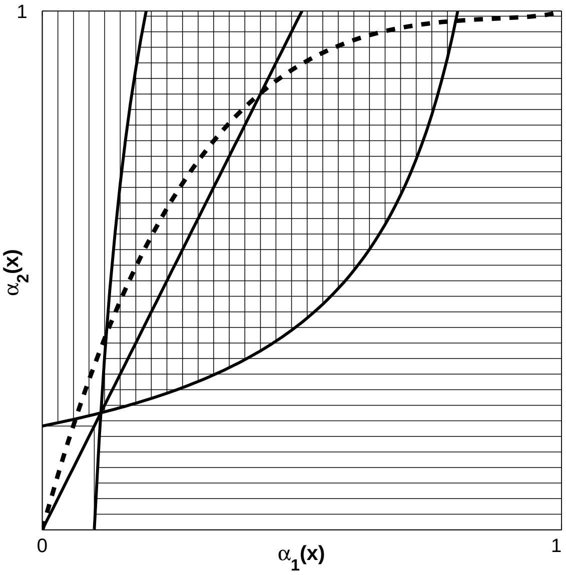

Figure 6: **Concave acceptance relation.** The figure depicts an example of an acceptance relation that satisfies conditions (a), (b), and (c), for which student behavior is monotone.

MLRP implies property (a), while (ii) yields the boundary conditions embedded in (b). In turn, log-supermodularity of $-G_x$ ensures the concavity property asserted in (c).

We will henceforth assume (i), (ii), and (iii). To be sure, this defines a *large* class of signal distributions. One can easily show that it includes the *location families* $G(\sigma|x) = G(\sigma - x)$ (e.g., normal), the *scale families* $G(\sigma|x) = G(\sigma/x)$ (e.g., uniform and exponential), and also the oft-used in applications *power family* $G(\sigma|x) = G(\sigma)^x$.

Careful inspection of the proofs of Theorems 1–3 reveal that they only make use of properties (a) and (b) of the uniform distribution, and thus they readily extend to the class of signal distributions defined in Lemma 4. Also, the construction of the non-monotone equilibrium was independent of the signal distribution.

Notice that extension of Lemma 2 implies that of Lemma 3, since its proof does not depend on the uniform distribution. Similarly, extension of Theorem 4 implies that of Theorems 5 and 6, and Proposition 7, for their proofs do not depend on the uniform distribution either. Thus, it only remains to extend Lemma 2 and Theorem 4. The details can be found in the Appendix, but here is an overview of the main issues.

Regarding Lemma 2, the only problem that arises is that ξ_B need not be unique, as it is defined by the solution of the acceptance relation (23) and the marginal benefit function (8), both of which are concave functions. We show that the following condition on the information structure is *sufficient* (but by no means *necessary*) for ξ_B to be unique for c sufficiently small: If $\underline{\sigma}_1 \geq \underline{\sigma}_2 > \underline{\sigma}$, then

$$\lim_{x \to \underline{x}} (1 - G(\underline{\sigma}_1|x)) G_x(\underline{\sigma}_2|x) / G_x(\underline{\sigma}_1|x) = 0. \tag{24}$$

This is a relatively mild condition that is satisfied by most of the aforementioned examples (e.g., uniform, exponential, and product family).

Regarding Theorem 4, the analog to condition (15) in this case is:

$$1 - G(\underline{\sigma}_2|\varphi(1 - \overline{\alpha}_1, \underline{\sigma}_1)) \geq \frac{1}{u}\overline{\alpha}_1,$$

which is equivalent to

$$\underline{\sigma}_2 \leq \phi(1 - \frac{1}{u}\overline{\alpha}_1, \varphi(1 - \overline{\alpha}_1, \underline{\sigma}_1)). \tag{25}$$

This condition is now necessary but not sufficient for monotone behavior, since we also need to ensure that ξ_B is unique. It is easy to show that the right side of (25) is increasing in $\underline{\sigma}_1$, strictly less than $\underline{\sigma}_1$ for all $c > 0$, and converges to $\underline{\sigma}_1$ as c vanishes.[5]

Assuming that ξ_B is unique, a monotone equilibrium is a pair of college thresholds that satisfy equations (18) and (19) (with $G(\underline{\sigma}_i|x)$ instead of $\underline{\sigma}_i/x$, $i = 1, 2$), as well as condition (25). We show that if the signal distribution satisfies (24), then the existence part of Theorem 4 holds. The uniqueness part holds if, in addition, it satisfies the following condition: If $\underline{\sigma}_1 \geq \underline{\sigma}_2 > \underline{\sigma}$, then

$$\lim_{x \to \underline{x}} (1 - G(\underline{\sigma}_1|x)) g(\underline{\sigma}_2|x) / G_x(\underline{\sigma}_1|x) = 0. \tag{26}$$

Like (24), condition (26) is also satisfied by most of the examples mentioned above.[6]

In short, *all* the insights extend to a large class of information structures that satisfy MLRP plus some regularity conditions. The role of the latter is simply to ensure that student behavior will be monotone when college behavior is monotone.

[5]For example, in the location families with $\underline{x} = -\infty$, $\underline{\sigma} = -\infty$, and $\overline{\sigma} = \infty$, the right side of (25) becomes $\underline{\sigma}_1 - \left(G^{-1}(1 - \underline{\alpha}_1) - G^{-1}(1 - \underline{\alpha}_1/u)\right)$, while it is equal to $\left(1 - \frac{\log u}{\log \underline{\alpha}_1}\right)\underline{\sigma}_1$ if $g(\sigma|x) = \frac{1}{x}e^{-\sigma/x}$, $\underline{x} = \underline{\sigma} = 0$, and $\overline{\sigma} = \infty$. It is straightforward to check the aforementioned properties.

[6]Notice that (24) and (26) are satisfied if $G_x(\sigma|\underline{x}) \neq 0$ and finite for all $\sigma > \underline{\sigma}$, as in $G(\sigma|x) = G(\sigma)^x$.

7.2 Perfectly Correlated Signals

So far we have assumed that if a student of caliber x applies to both colleges, they observe independent signals drawn from $g(\sigma|x)$. We now study the polar case in which the signals observed by the colleges are perfectly (positively) correlated (i.e., both observe the *same* realization). For simplicity, we present the main results assuming that signals are uniformly distributed, and then sketch their straightforward generalization to the class of signal distributions of Lemma 4. The key feature of this case is that if a student is accepted at the more selective college, then he is also accepted at the less selective one. This immediately implies that $\underline{\sigma}_1 > \underline{\sigma}_2$ in equilibrium, for otherwise nobody would apply to college 2. That is, in any equilibrium college behavior *must* be monotone.

The expected utility of applying to just one college is $\alpha_1(x) - c$ and $\alpha_2(x)u - c$, respectively. Since $\underline{\sigma}_1 > \underline{\sigma}_2$, applying to both yields $\alpha_1(x) + (\alpha_2(x) - \alpha_1(x))u - 2c$. Thus, the student's optimal strategy is characterized by the following conditions:

$$\begin{aligned} \alpha_2 u &= \alpha_1 \\ MB_{21} \equiv (\alpha_2 - \alpha_1)u &= c \\ MB_{12} \equiv \alpha_1(1-u) &= c. \end{aligned}$$

Graphically, the following straight lines delimit the regions of the student's application strategy: $\alpha_1 = c/(1-u)$, $\alpha_2 = \alpha_1 + c/u$, $\alpha_2 = \alpha_1/u$, $\alpha_1 = c$, and $\alpha_2 = c/u$. Since the acceptance relation (10) is also linear, student optimal behavior is always monotone. To avoid the trivial case without multiple applications, we impose the condition that the acceptance relation crosses above the point $(\hat{\alpha}_1, \hat{\alpha}_2) = (c/(1-u), c/u(1-u))$. This yields

$$\underline{\sigma}_2 \leq \left(\frac{1 - c/u(1-u)}{1 - c/(1-u)} \right) \underline{\sigma}_1. \tag{27}$$

The thresholds ξ_2, ξ_B, and ξ_1 solve, respectively, $\alpha_2(\xi_2) = c/u$, $\alpha_2(\xi_B) = (1 - \frac{\underline{\sigma}_2}{\underline{\sigma}_1}) + \frac{\underline{\sigma}_2}{\underline{\sigma}_1}\frac{c}{1-u}$, and $\alpha_2(\xi_1) = (1 - \frac{\underline{\sigma}_2}{\underline{\sigma}_1}) + \frac{\underline{\sigma}_2}{\underline{\sigma}_1}(\alpha_2(\xi_1) - c/u)$. Easy algebra yields

$$\begin{aligned} \xi_2 &= \frac{\underline{\sigma}_2}{1 - c/u} \\ \xi_B &= \frac{\underline{\sigma}_1}{1 - c/(1-u)} \\ \xi_1 &= \frac{u}{c}(\underline{\sigma}_1 - \underline{\sigma}_2). \end{aligned}$$

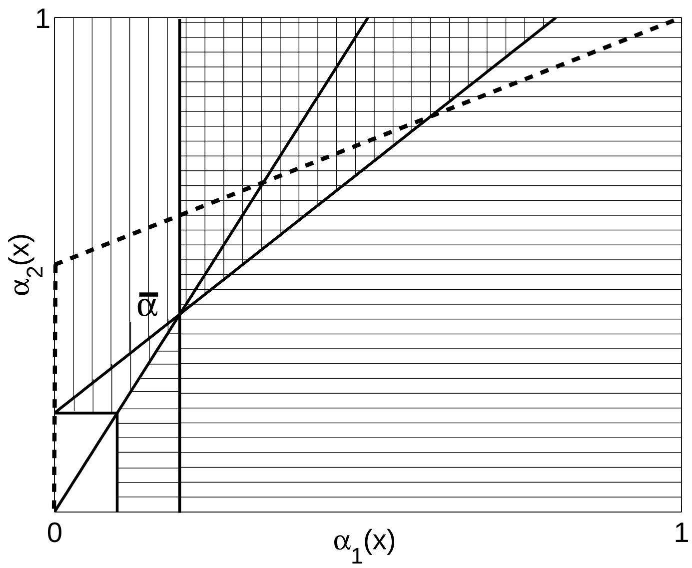

Figure 7: **Student Behavior with perfectly correlated signals.** With perfectly correlated signals, we get different acceptance regions from those in the independent case.

Notice that these thresholds satisfy all the properties stated in Lemma 2 except for one: ξ_B does *not* depend $\underline{\sigma}_2$. This dramatically simplifies the equilibrium analysis, for it makes the determination of the acceptance threshold $\underline{\sigma}_1$ *independent* of $\underline{\sigma}_2$.

A monotone equilibrium is a pair of college thresholds that satisfies:

$$\kappa_1 = E_1(\underline{\sigma}_1) \tag{28}$$

$$\kappa_2 = E_2(\underline{\sigma}_1, \underline{\sigma}_2) \tag{29}$$

$$0 \leq \underline{\sigma}_2 \leq \left(\frac{1 - c/u(1-u)}{1 - c/(1-u)}\right)\underline{\sigma}_1, \tag{30}$$

where

$$E_1(\underline{\sigma}_1) = \int_{\xi_B(\underline{\sigma}_1)}^{\infty} (1 - \frac{\underline{\sigma}_1}{x}) f(x)dx$$

$$E_2(\underline{\sigma}_1, \underline{\sigma}_2) = \int_{\xi_2(\underline{\sigma}_2)}^{\xi_B(\underline{\sigma}_1)} (1 - \frac{\underline{\sigma}_2}{x}) f(x)dx + \int_{\xi_B(\underline{\sigma}_1)}^{\xi_1(\underline{\sigma}_1, \underline{\sigma}_2)} \frac{\underline{\sigma}_1 - \underline{\sigma}_2}{x} f(x)dx$$

Equations (28)–(29) define, implicitly, the functions $\underline{\sigma}_1 = H_1(\kappa_1)$ and $\underline{\sigma}_2 = H_2(\underline{\sigma}_1, \kappa_2)$.

Theorem 8 (Perfectly Correlated Signals) *Let $\kappa_1 \in (0, 1)$ and $c < u(1-u)$.*

(i) There is an interval $(\underline{\kappa}_2(\kappa_1), \overline{\kappa}_2(\kappa_1))$, with $0 < \underline{\kappa}_2(\kappa_1) < \overline{\kappa}_2(\kappa_1) < 1 - \kappa_1$, such that, for all $(\kappa_1, \kappa_2) \in (0,1) \times (\underline{\kappa}_2(\kappa_1), \overline{\kappa}_2(\kappa_1))$, a unique equilibrium exists.

(ii) The comparative static properties with respect to κ_1 and c are the same as in Theorem 6, while an increase in κ_2 reduces $\underline{\sigma}^e_2$ but has no effect on $\underline{\sigma}^e_1$.

Notice that an increase in the capacity of college 2 has *no* effect on the acceptance threshold of college 1. This is because the marginal benefit of adding college 1 to a portfolio that already contains college 2 is independent of $\underline{\sigma}_2$. Thus, ξ_B, which determines the caliber above which students apply to college 1, does *not* depend on $\underline{\sigma}_2$ either, thereby insulating college 1's applicant pool from the acceptance decisions of college 2 (i.e., the H_1^{-1} function is a 'vertical' line). But this result *only* holds in the perfectly correlated case and hence it is *not* robust. To see this, note that in the conditional independent case $MB_{12} = \alpha_1 - \alpha_1\alpha_2 u$; i.e., adding college 1 increases expected utility by α_1 but it decreases it by $\alpha_1\alpha_2 u$, since acceptance by college 1 has an 'opportunity cost' of $\alpha_2 u$. In the perfectly correlated case, that opportunity cost is equal to u, as the probability of being accepted at college 2 conditional on being accepted at college 1 is equal to one, and thus it is independent of $\underline{\sigma}_2$. But if correlation is less than perfect, this conditional probability is less than one and depends on $\underline{\sigma}_2$. Therefore, ξ_B will *depend* on $\underline{\sigma}_2$ as well. Graphically, this means that the slope of H_1^{-1} will be positive but less than infinity *unless* the signals of a student observed by colleges are perfectly correlated.

Proposition 8 easily extends to the class of $g(\sigma|x)$ of Lemma 4; just replace (27) by

$$\underline{\sigma}_2 \leq \phi(1 - c/u(1-u), \varphi(1 - c/(1-u), \underline{\sigma}_1)). \tag{31}$$

The only additional insight that emerges in the general case is that, if ψ is strictly concave, then there exists nonmonotone equilibria when condition (31) does not hold.

8 Conclusion

Assume, simply for the sake of argument, that Harvard is indeed the best college. Does that mean that Harvard attracts the best students to apply to it? When the University of Chicago increased the size of its college by a third, what should we expect were the effects on its student caliber, or on other competing colleges? What are the effects of the recent advances in technology that have reduced college application costs? How do race-based admissions policies affect student body composition? These are some of the issues this paper has been designed to answer, in an extremely stylized environment.

We have provided an intuitive graphical solution to the nontrivial student portfolio choice problem, which clearly illustrates the lack of a natural single crossing property to guarantee monotone behavior as a function of calibers. We have also explored assortive matching in this context, showing that it arises provided the application cost is small and the capacity of the lesser college is neither too large nor too small. We have shown that there are also equilibria that are non-monotone; surprisingly, they exist even when the better college sets a higher admission threshold than the lesser college. We have provided equilibrium comparative static results with respect to college capacities and application cost. Crucially, we have shown that noisy signals *and* student application costs introduce an externality of the worse-ranked college upon the better one. This only arises when *both* noise and cost are present, for it is driven by the subtle student portfolio reallocation that ensues when a parameter changes. We have shown the robustness of our main results by generalizing our model to a large class of signal distributions, explored the case with correlated signals, and provided a foundation for our reduced-form model of college behavior (see Appendix). Finally, we have presented an application to the case of race-based admissions, and showcased some of the tradeoffs that arise there.

There are many natural avenues for future research. Obviously, as we have proceeded with just two colleges, this does not realistically capture the far richer world. Also, college caliber is in the long-term endogenously determined by the caliber of students attending. Finally, we have assumed that student preferences are homogeneous. In Chade, Lewis, and Smith (2005) we allow for heterogenous preferences and investigate — theoretically and empirically — the informational content of several equilibrium statistics (e.g., acceptance rates, yields, etc.) that are commonly used as a basis for the construction of college rankings.

A Appendix: Omitted Proofs

A.1 Proof of Theorem 1 (Noiseless Case)

(i) We can solve (1) sequentially as follows. For any $\kappa_1 \in (0,1)$, there is a unique $\underline{\sigma}_1^e$ that solves the κ_1 equation in (1), which is independent of $\underline{\sigma}_2$ and is decreasing in κ_1. Inserting this solution in the κ_2 equation, we find that the right side is decreasing in $\underline{\sigma}_2$, and its maximum value is equal to $\int_0^{\underline{\sigma}_1^e} f(x)dx = 1 - \kappa_1$. Thus, if $\kappa_2 \in (0, 1-\kappa_1)$, then there is a unique $\underline{\sigma}_2^e$ that solves the second equality in (1). Hence, there is a unique pair $(\underline{\sigma}_1^e, \underline{\sigma}_2^e)$ that solves both equations in (1).

(ii) Notice that equations (1) do not depend on c. Next, differentiate (1). □

A.2 Proof of Theorem 2 (Costless Case)

(i) We can solve (2)–(3) sequentially as follows. For any $\kappa_1 \in (0,1)$, there is a unique $\underline{\sigma}_1^e$ that solves equation (2), which is independent of $\underline{\sigma}_2$ and is decreasing in κ_1. Inserting this solution in equation (3), we find that the right side is decreasing in $\underline{\sigma}_2$, and its maximum value — i.e., when $\underline{\sigma}_2 = 0$ — is equal to $\int_0^{\underline{\sigma}_1^e} f(x)dx + \int_{\underline{\sigma}_1^e}^{\infty} \frac{\underline{\sigma}_1^e}{x} f(x)dx = 1 - \kappa_1$. Since the largest value $\underline{\sigma}_2$ can assume in this case is $\underline{\sigma}_1^e$, it follows that the smallest feasible value of the right side of (3) is equal to $\int_{\underline{\sigma}_1^e}^{\infty} \frac{\underline{\sigma}_1^e}{x}(1 - \frac{\underline{\sigma}_1^e}{x}) f(x)dx$. Call this value $\underline{\kappa}_2(\kappa_1)$. Then, if $\kappa_2 \in (\underline{\kappa}_2(\kappa_1), 1-\kappa_1)$, there is a unique $\underline{\sigma}_2^e$ that solves (3). Hence, there is a unique pair of college thresholds $(\underline{\sigma}_1^e, \underline{\sigma}_2^e)$ that solves (2)–(3).

(ii) Let $\kappa_1 \in (0,1)$. Proceeding as in (i) and inserting the unique solution $\underline{\sigma}_1^e$ of (2) in (4), it follows that there is a unique solution $\underline{\sigma}_2^e$, with $\underline{\sigma}_1^e < \underline{\sigma}_2^e$, so long as $\kappa_2 \in (0, \underline{\kappa}_2(\kappa_1))$.

(iii) Notice that equations (2)–(3) (and (4)) do not depend on c. The rest follows by straightforward differentiation of equations (2) and (3) (or (4)). □

A.3 Proof of Theorem 3 (Equilibrium Existence)

The enrollment functions have the following properties: (a) $E_1(0, \underline{\sigma}_2) = 1$; (b) $E_1(\infty, \underline{\sigma}_2) = 0\ \forall \underline{\sigma}_2$; (c) $E_2(\underline{\sigma}_1, \infty) = 0\ \forall \underline{\sigma}_1$; (d) E_1 is decreasing in $\underline{\sigma}_1$ and increasing in $\underline{\sigma}_2$; (e) E_2 is increasing in $\underline{\sigma}_1$ and decreasing in $\underline{\sigma}_2$. Now, (a), (b) and (d) imply that there exists a unique $\underline{\sigma}_1$ for any κ_1 such that $E_1(\underline{\sigma}_1, \underline{\sigma}_2) = \kappa_1$. To get a similar existence statement for $E_2(\underline{\sigma}_1, \underline{\sigma}_2)$ we must bound $\underline{\sigma}_1$ away from 0 (otherwise college 2 will be unable to fill its capacity). Let $\underline{\sigma}_1^L(\kappa_2)$ be defined by $E_2(\underline{\sigma}_1^L, 0) = \kappa_2$. Then by (e) for $\underline{\sigma}_1 \geq \underline{\sigma}_1^L(\kappa_2)$

there exists a unique $\underline{\sigma}_2 \geq 0$ such that $E_2(\underline{\sigma}_1, \underline{\sigma}_2) = \kappa_2$. But since we restrict college 1 to a threshold of at least $\underline{\sigma}_1^L(\kappa_2)$, it is clear that it cannot have 'large capacity.' More precisely, let $\kappa_1^H(\kappa_2) = E_1(\underline{\sigma}_1^L(\kappa_2), 0)$. Then for any $\kappa_1 \leq \kappa_1^H(\kappa_2)$, there will be a unique solution for $\underline{\sigma}_1$, with $\underline{\sigma}_1 \geq \underline{\sigma}_1^L(\kappa_2)$ for all $\underline{\sigma}_2$. We may show that this restriction on κ_1 is equivalent to requiring κ_2 to be less than $\bar{\kappa_2}(\kappa_1)$ for $\bar{\kappa_2}(\kappa_1) = E_2(H_1(0, \kappa_1), 0)$.

Note that the colleges cannot set thresholds above a certain level. Certainly, college 1 cannot set a threshold higher than $\underline{\sigma}_1^H$, where $\underline{\sigma}_1^H$ (which determines $\alpha_1(x)$) solves

$$\int_0^\infty \alpha_1(x) f(x) dx = \kappa_1,$$

for the set of calibers applying to college 1 is a subset of $[0, \infty)$. Similarly, we define $\underline{\sigma}_2^H$.

So fix κ_1 and κ_2 as in the statement, and define the space $S = [\underline{\sigma}_1^L, \underline{\sigma}_1^H] \times [0, \underline{\sigma}_2^H]$, which is compact, convex and nonempty. Define a vector-valued function $T : S \to S$ by $T(\underline{\sigma}_1, \underline{\sigma}_2) = (\tilde{\sigma}_1, \tilde{\sigma}_2)$, where $\tilde{\sigma}_1$ satisfies $E_1(\tilde{\sigma}_1, \underline{\sigma}_2) = \kappa_1$ and $\tilde{\sigma}_2$ satisfies $E_2(\underline{\sigma}_1, \tilde{\sigma}_2) = \kappa_2$. T is well-defined on S, as shown by the earlier analysis. Further, T is continuous, as the demand functions are continuous in both arguments. The latter follows since the caliber distribution is atomless, and both the application sets and acceptance probabilities are smooth functions of the thresholds. Thus we may apply Brouwer's fixed point theorem to deduce that T has a fixed point — which immediately satisfies $\kappa_1 = E_1(\underline{\sigma}_1, \underline{\sigma}_2)$ and $\kappa_2 = E_2(\underline{\sigma}_1, \underline{\sigma}_2)$. □

A.4 Proof of Lemma 1 (Thresholds for Monotone Strategies)

The threshold ξ_2 solves $\alpha_2(\xi_2) = c/u$, which immediately yields (12).

Threshold ξ_B is derived in two steps. We first find the value of α_1 at which the acceptance relation $\alpha_2 = \psi(\underline{\sigma}_1, \underline{\sigma}_2, \alpha_1)$ intersects $\alpha_2 = (1 - c/\alpha_1)/u$. Call this value $\hat{\alpha}_1$, which is the unique solution in $(0, 1)$ (i.e., the smallest solution) to the equation

$$\alpha_1^2 - \left(1 + \frac{(1-u)}{u} \frac{\underline{\sigma}_1}{\underline{\sigma}_2}\right) \alpha_1 + \frac{c}{u} \frac{\underline{\sigma}_1}{\underline{\sigma}_2} = 0.$$

Given $\hat{\alpha}_1$, we then find ξ_B as $\hat{\alpha}_1 = 1 - G(\underline{\sigma}_1 | \xi_B)$, which yields $\xi_B = \underline{\sigma}_1/(1 - \hat{\alpha}_1)$. Expression (13) follows by simple algebraic manipulation.

Finally, ξ_1 is obtained as follows. We first find the value of α_1 at which the acceptance relation $\alpha_2 = \psi(\underline{\sigma}_1, \underline{\sigma}_2, \alpha_1)$ intersects $\alpha_2 = c/(1 - \alpha_1)u$. Call this value $\tilde{\alpha}_1$, which is the unique solution in $(0,1)$ (i.e., the largest solution) to the equation

$$(1-\alpha_1)\left(1-\frac{\underline{\sigma}_2}{\underline{\sigma}_1}(1-\alpha_1)\right)-\frac{c}{u}=0.$$

Given $\tilde{\alpha}_1$, we then find ξ_1 as $\tilde{\alpha}_1 = 1-G(\underline{\sigma}_1|\xi_1)$, which yields $\xi_1 = \underline{\sigma}_1/(1-\tilde{\alpha}_1)$. Expression (14) follows by simple algebraic manipulation.

A.5 Proof of Lemma 2 (Monotone Student Behavior)

Since the proof amounts to straightforward and tedious differentiation and limit-taking of the closed form expressions (12)–(14), we shall prove parts (*ii*) and (*iii*) in a way that generalizes beyond the uniform distribution. Firstly, (*i*) is obvious from (12)

(*ii*) Notice that $\hat{\alpha}_1$ satisfies

$$\hat{\alpha}_1 = \frac{c}{1-u\psi(\underline{\sigma}_1,\underline{\sigma}_2,\hat{\alpha}_1)}. \tag{32}$$

Denote the right side of (32) by $z(\hat{\alpha}_1,c,u,\underline{\sigma}_1,\underline{\sigma}_2)$. It is easy to show that

$$0<z(0,c,u,\underline{\sigma}_1,\underline{\sigma}_2)<z(1,c,u,\underline{\sigma}_1,\underline{\sigma}_2)<1,$$

$z(\hat{\alpha}_1,c,u,\underline{\sigma}_1,\underline{\sigma}_2)$ is strictly increasing in $\hat{\alpha}_1$ and, under the uniform distribution, it is also strictly convex in $\hat{\alpha}_1$. Thus, there is a unique $\hat{\alpha}_1$ that satisfies (32). Moreover, $z(\hat{\alpha}_1,c,u,\underline{\sigma}_1,\underline{\sigma}_2)$ decreases $\underline{\sigma}_2$, increases in c and $\underline{\sigma}_1$, and converges to zero as c vanishes. Hence, $\hat{\alpha}_1$ exhibits this behavior as well. Since $\xi_B = \underline{\sigma}_1/(1-\hat{\alpha}_1)$, the properties stated in part (*ii*) now follow easily.

(*iii*) Notice that $\tilde{\alpha}_1$ satisfies

$$\tilde{\alpha}_1 = 1-\frac{c}{u\psi(\underline{\sigma}_1,\underline{\sigma}_2,\tilde{\alpha}_1)}. \tag{33}$$

Denote the right side of (33) by $r(\tilde{\alpha}_1,c,u,\underline{\sigma}_1,\underline{\sigma}_2)$. It is easy to show that (recall (15))

$$\overline{\alpha}_1<r(\overline{\alpha}_1,c,u,\underline{\sigma}_1,\underline{\sigma}_2)<r(1,c,u,\underline{\sigma}_1,\underline{\sigma}_2)<1,$$

$r(\tilde{\alpha}_1,c,u,\underline{\sigma}_1,\underline{\sigma}_2)$ is strictly increasing in $\tilde{\alpha}_1$ and, since ψ is concave in $\tilde{\alpha}_1$ under the uniform distribution, $r(\tilde{\alpha}_1,c,u,\underline{\sigma}_1,\underline{\sigma}_2)$ is also strictly concave in $\hat{\alpha}_1$. Thus, there is a unique $\tilde{\alpha}_1 \in (\overline{\alpha}_1, 1)$ that satisfies (33). Moreover, $r(\tilde{\alpha}_1, c, u, \underline{\sigma}_1, \underline{\sigma}_2)$ decreases in c and $\underline{\sigma}_2$, increases in $\underline{\sigma}_1$, and converges to one as c vanishes. Hence, $\tilde{\alpha}_1$ exhibits these properties as well. Since $\xi_1 = \underline{\sigma}_1/(1-\tilde{\alpha}_1)$, part (iii) is now straightforward. □

A.6 Proof of Theorem 4 (Monotone Equilibrium)

Let $\kappa_1 \in (0,1)$ be given, and let $\eta \equiv \left(\frac{1-\tilde{\alpha}_1/u}{1-\tilde{\alpha}_1}\right)$. We shall denote by $(\hat{\underline{\sigma}}_1, \hat{\underline{\sigma}}_2)$ the unique solution to $\kappa_1 = E_1(\underline{\sigma}_1, \underline{\sigma}_2, c)$ and $\underline{\sigma}_2 = \eta\underline{\sigma}_1$. Obviously, $(\hat{\underline{\sigma}}_1, \hat{\underline{\sigma}}_2)$ depend on κ_1 and c.

Existence. To prove existence, we proceed in four steps. First, we show that there is an interval of κ_2 such that the value of $\underline{\sigma}_1$ at which $H_2(\underline{\sigma}_1, \kappa_2, c) = 0$ is less than the value of $\underline{\sigma}_1$ at which $\underline{\sigma}_1 = H_1(0, \kappa_2, c)$. Second, we show that there exists an interval of κ_2 such that, at $\hat{\underline{\sigma}}_1$, $H_2(\hat{\underline{\sigma}}_1, \kappa_2, c) < \hat{\underline{\sigma}}_2$. Third, we prove that if c is sufficiently small, then the intersection of the aforementioned intervals is nonempty. Fourth, we use the continuity of the functions H_1 and H_2 to complete the existence proof.

Step 1: *There is an interval of κ_2 such that $H_2^{-1}(0, \kappa_2) < H_1(0, \kappa_1)$.*

To see this, note first that $\underline{\sigma}_1 = H_1(0, \kappa_1, c)$ is the unique solution to $\kappa_1 = E_1(\underline{\sigma}_1, 0, c)$. Also, it is easy to show that, at $\underline{\sigma}_2 = 0$, $\xi_2 = 0$, $\xi_B = \underline{\sigma}_1/(1-c/(1-u))$, and $\xi_1 = \underline{\sigma}_1 u/c$. Thus, $\kappa_2 = E(\underline{\sigma}_1, 0, c)$ is given by

$$\kappa_2 = \int_0^{\frac{\underline{\sigma}_1(1-u)}{1-u-c}} f(x)dx + \int_{\frac{\underline{\sigma}_1(1-u)}{1-u-c}}^{\frac{\underline{\sigma}_1 u}{c}} \frac{\underline{\sigma}_1}{x} f(x)dx. \tag{34}$$

Since $E_2(0,0,c) = 0$ and $\partial E_2/\partial\underline{\sigma}_1 > 0$, it follows that if $\kappa_2 \in (0, E_2(H_1(0,\kappa_1,c),0,c))$, then the unique solution to (34) satisfies $H_2^{-1}(0, \kappa_2, c) < H_1(0, \kappa_1, c)$.

Step 2: *There is an interval of κ_2 such that $H_2(\hat{\underline{\sigma}}_1, \kappa_2, c) < \hat{\underline{\sigma}}_2 = \eta\hat{\underline{\sigma}}_1$.*

Since $\partial E_2/\partial\underline{\sigma}_2 < 0$, it follows that if $\kappa_2 \in (E_2(\hat{\underline{\sigma}}_1, \hat{\underline{\sigma}}_2, c), E_2(\hat{\underline{\sigma}}_1, 0, c))$, then the unique $\underline{\sigma}_2$ that solves $\kappa_2 = E_2(\hat{\underline{\sigma}}_1, \underline{\sigma}_2, c)$ belongs to the interval $(0, \hat{\underline{\sigma}}_2)$, and thus $H_2(\hat{\underline{\sigma}}_1, \kappa_2, c) < \hat{\underline{\sigma}}_2$.

Step 3: *For c small, the intersection of the two intervals of κ_2 is an interval.*

We will show that $E_2(\hat{\underline{\sigma}}_1, \hat{\underline{\sigma}}_2, c) < E_2(H_1(0,\kappa_1,c),0,c))$ for $c > 0$ sufficiently small. Let us write $\hat{\underline{\sigma}}_1(\kappa_1, c)$ and $\hat{\underline{\sigma}}_2(\kappa_1, c)$ to emphasize their (continuous) dependence on κ_1 and c. Since $\overline{\alpha}_1 = (1-\sqrt{1-4c})/2$, $\lim_{c\to 0}\overline{\alpha}_1 = 0$ and thus $\hat{\underline{\sigma}}_2(\kappa_1, 0) = \hat{\underline{\sigma}}_1(\kappa_1, 0)$. Notice also that $\kappa_1 = E_1(\hat{\underline{\sigma}}_1(\kappa_1, 0), \hat{\underline{\sigma}}_1(\kappa_1, 0), 0)$ implies that $\hat{\underline{\sigma}}_1(\kappa_1, 0) > 0$, as $E_1(0, \underline{\sigma}_2, c) = 1 > \kappa_1$. Moreover, $H_1(0, \kappa_1, 0) = \hat{\underline{\sigma}}_1(\kappa_1, 0)$, for ξ_B is equal to $\underline{\sigma}_1$ at $c = 0$ (Lemma 2 (ii)) and thus is independent of $\underline{\sigma}_2$.[7] Using these results and Lemma 2, it follows that

$$\begin{aligned}
\lim_{c\to 0} E_2(\underline{\hat{\sigma}}_1(\kappa_1,c), \underline{\hat{\sigma}}_2(\kappa_1,c), c) &= \int_{\underline{\hat{\sigma}}_1(\kappa_1,0)}^{\infty} \frac{\underline{\hat{\sigma}}_1(\kappa_1,0)}{x}(1-\frac{\underline{\hat{\sigma}}_1(\kappa_1,0)}{x})f(x)dx \\
\lim_{c\to 0} E_2(H_1(0,\kappa_1,0),0,c) &= \int_0^{\underline{\hat{\sigma}}_1(\kappa_1,0)} f(x)dx + \int_{\underline{\hat{\sigma}}_1(\kappa_1,0)}^{\infty} \frac{\underline{\hat{\sigma}}_1(\kappa_1,0)}{x} f(x)dx.
\end{aligned}$$

Thus, $E_2(\underline{\hat{\sigma}}_1(\kappa_1,0), \underline{\hat{\sigma}}_2(\kappa_1,0), 0) < E_2(\underline{\hat{\sigma}}_1(\kappa_1,0),0,0)$. By continuity, $E_2(\underline{\hat{\sigma}}_1, \underline{\hat{\sigma}}_2, c) < E_2(H_1(0,\kappa_1,c),0,c))$ for $c>0$ sufficiently small.

Step 4: *Given κ_1, an equilibrium exists for an interval of c and an interval of κ_2.* Define $\underline{\kappa}_2(\kappa_1) \equiv E_2(\underline{\hat{\sigma}}_1, \underline{\hat{\sigma}}_2, c)$ and $\overline{\kappa}_2(\kappa_1) \equiv E_2(H_1(0,\kappa_1,c),0,c)$. Thus far we have shown that if $c \in (0, c_0(\kappa_1))$ and $\kappa_2 \in (\underline{\kappa}_2(\kappa_1), \overline{\kappa}_2(\kappa_1))$, then $H_2^{-1}(0,\kappa_2) < H_1(0,\kappa_1)$ (Step 1) and $H_2(\underline{\hat{\sigma}}_1, \kappa_2, c) < H^{-1}(\underline{\hat{\sigma}}_1, \kappa_1, c)$ (Step 2 plus the definition of $\underline{\hat{\sigma}}_1$). In words, within the set of $(\underline{\sigma}_1, \underline{\sigma}_2)$ such that $0 \leq \underline{\sigma}_2 \leq \eta\underline{\sigma}_1$, the function $\underline{\sigma}_2 = H_1^{-1}(\underline{\sigma}_1, \kappa_1, c)$ lies 'below' the function $\underline{\sigma}_2 = H_2(\underline{\sigma}_1, \kappa_2, c)$ for low values of $\underline{\sigma}_1$ (Step 1) and 'above' when $\underline{\sigma}_1 = \hat{\sigma}_1$. Since H_1 and H_2 (as well as their inverses) are continuous, it follows that a monotone equilibrium $(\underline{\sigma}_1^e, \underline{\sigma}_2^e)$ exists.

Uniqueness. We now show that if $\kappa_2 \in (\underline{\kappa}_2(\kappa_1), \overline{\kappa}_2(\kappa_1))$ and c is small, then the slope of $\underline{\sigma}_2 = H_2(\underline{\sigma}_1, \kappa_2, c)$ is smaller than that of $\underline{\sigma}_2 = H_1^{-1}(\underline{\sigma}_1, \kappa_1, c)$, thereby implying that the equilibrium is unique. Formally, we need to show that $\partial H_1/\partial\underline{\sigma}_2 \times \partial H_2/\partial\underline{\sigma}_1 < 1$, or

$$\partial E_1/\partial\underline{\sigma}_1 \times \partial E_2/\partial\underline{\sigma}_2 - \partial E_1/\partial\underline{\sigma}_2 \times \partial E_2/\partial\underline{\sigma}_1 > 0. \tag{35}$$

Differentiation of expressions (12)–(14) and (16)–(17) yields after tedious algebra

$$\begin{aligned}
\lim_{c\to 0} \partial E_1/\partial\underline{\sigma}_2 &= 0 \\
\lim_{c\to 0} \partial E_2/\partial\underline{\sigma}_1 &= \int_{\underline{\sigma}_1}^{\infty} (1-\frac{\underline{\sigma}_2}{x})\frac{f(x)}{x}dx \\
\lim_{c\to 0} \partial E_1/\partial\underline{\sigma}_1 &= -\int_{\underline{\sigma}_1}^{\infty} \frac{f(x)}{x}dx \\
\lim_{c\to 0} \partial E_2/\partial\underline{\sigma}_2 &= -\int_{\underline{\sigma}_2}^{\underline{\sigma}_1} \frac{f(x)}{x}dx - \int_{\underline{\sigma}_1}^{\infty} \frac{\underline{\sigma}_1}{x}\frac{f(x)}{x}dx.
\end{aligned}$$

Hence, (35) holds at $c=0$. By continuity, the result also holds for $c \in (0, c_1(\kappa_1))$.

[7] Graphically, $\underline{\sigma}_2 = H_1^{-1}(\underline{\sigma}_1, \kappa_1, c)$ becomes a 'vertical line' as c goes to zero; recall the costless case.

Let $\overline{c}(\kappa_1) = \min\{c_0(\kappa_1), c_1(\kappa_1)\}$. Given $\kappa_1 \in (0,1)$, we have thus proved that there is a unique monotone equilibrium $(\underline{\sigma}^e_1, \underline{\sigma}^e_2)$ if $\kappa_2 \in (\underline{\kappa}_2(\kappa_1), \overline{\kappa}_2(\kappa_1))$ and $c \in (0, \overline{c}(\kappa_1))$. □

A.7 Proof of Theorem 5 (Types under PAM)

Let $f_1(x)$ and $f_2(x)$ be the densities of calibers accepted at colleges 1 and 2, respectively, where we have omitted $(\underline{\sigma}^e_1, \underline{\sigma}^e_2)$ to simplify the notation. Formally,

$$f_1(x) = \frac{\alpha_1(x)f(x)}{\int_{\xi_B}^{\infty} \alpha_1(t)f(t)dt} I_{[\xi_B,\infty)}(x) \tag{36}$$

$$f_2(x) = \frac{I_{[\xi_2,\xi_B]}(x)\alpha_2(x)f(x) + (1 - I_{[\xi_2,\xi_B]}(x))\alpha_2(x)(1-\alpha_1(x))f(x)}{\int_{\xi_2}^{\xi_B} \alpha_2(s)f(s)ds + \int_{\xi_B}^{\xi_1} \alpha_2(s)(1-\alpha_1(s))f(s)ds} I_{[\xi_2,\xi_1]}(x), \tag{37}$$

where I_A is the indicator function of the set A.

We shall show that, if $x_L, x_H \in [0,\infty)$, with $x_H > x_L$, then $f_1(x_H)f_2(x_L) \geq f_2(x_H)f_1(x_L)$; i.e., $f_i(x)$ is log-supermodular in $(-i, x)$, or it satisfies MLRP. Since MLRP of the densities implies that the cdfs are ordered by FSD, the theorem follows.

Using (36) and (37), $f_1(x_H)f_2(x_L) \geq f_2(x_H)f_1(x_L)$ is equivalent to

$$\begin{aligned} &\alpha_{1H} I_{[\xi_B,\infty)}(x_H)\left(I_{[\xi_2,\xi_B]}(x_L)\alpha_{2L} + (1 - I_{[\xi_2,\xi_B]}(x_L))\alpha_{2L}(1-\alpha_{1L})\right) I_{[\xi_2,\xi_1]}(x_L) \geq \\ &\alpha_{1L} I_{[\xi_B,\infty)}(x_L)\left(I_{[\xi_2,\xi_B]}(x_H)\alpha_{2H} + (1 - I_{[\xi_2,\xi_B]}(x_H))\alpha_{2H}(1-\alpha_{1H})\right) I_{[\xi_2,\xi_1]}(x_H), \end{aligned} \tag{38}$$

where $\alpha_{ij} = \alpha_i(x_j)$, $i = 1,2$, $j = L, H$. It is easy to show that the only nontrivial case to consider is when $x_L, x_H \in [\xi_B, \xi_1]$ (in all the other cases, either both sides are zero, or only the right side is). If $x_L, x_H \in [\xi_B, \xi_1]$, then (38) becomes $\alpha_{1H}\alpha_{2L}(1-\alpha_{1L}) \geq \alpha_{1L}\alpha_{2H}(1-\alpha_{1H})$, or

$$\begin{aligned} &(1 - G(\underline{\sigma}_1 \mid x_H))(1 - G(\underline{\sigma}_2 \mid x_L))G(\underline{\sigma}_1 \mid x_L) \geq \\ &(1 - G(\underline{\sigma}_1 \mid x_L))(1 - G(\underline{\sigma}_2 \mid x_H))G(\underline{\sigma}_1 \mid x_H). \end{aligned} \tag{39}$$

Since $g(\sigma \mid x)$ satisfies MLRP, it follows that (i) $G(\sigma \mid x)$ is decreasing in x, and hence $G(\underline{\sigma}_1 \mid x_L) \geq G(\underline{\sigma}_1 \mid x_H)$; (ii) $1 - G(\sigma \mid x)$ is log-supermodular in (x, σ), and therefore $(1 - G(\underline{\sigma}_1 \mid x_H))(1 - G(\underline{\sigma}_2 \mid x_L)) \geq (1 - G(\underline{\sigma}_1 \mid x_L))(1 - G(\underline{\sigma}_2 \mid x_H))$, for $\underline{\sigma}_1 > \underline{\sigma}_2$ in a monotone equilibrium. Thus, (39) is satisfied, thereby proving that $f_i(x)$ is log-supermodular, which in turn implies that F_1 dominates F_2 is the sense of FSD. □

A.8 Proof of Theorem 6 (Comparative Statics)

(i) In equilibrium, $\kappa_1 = E_1(\underline{\sigma}^e_1, \underline{\sigma}^e_2, c)$ and $\kappa_2 = E_2(\underline{\sigma}^e_1, \underline{\sigma}^e_2, c)$. Differentiating this system with respect to κ_1, yields

$$\frac{\partial \underline{\sigma}^e_1}{\partial \kappa_1} = \frac{\frac{\partial E_2}{\partial \underline{\sigma}^e_2}}{\Delta} < 0 \qquad \frac{\partial \underline{\sigma}^e_2}{\partial \kappa_1} = \frac{-\frac{\partial E_2}{\partial \underline{\sigma}^e_1}}{\Delta} < 0,$$

where $\Delta = \partial E_1/\partial \underline{\sigma}^e_1 \times \partial E_2/\partial \underline{\sigma}^e_2 - \partial E_2/\partial \underline{\sigma}^e_1 \times \partial E_1/\partial \underline{\sigma}^e_2 > 0$ (see Theorem 4).

(ii) Differentiating $\kappa_1 = E_1(\underline{\sigma}^e_1, \underline{\sigma}^e_2, c)$ and $\kappa_2 = E_2(\underline{\sigma}^e_1, \underline{\sigma}^e_2, c)$ with respect to κ_2 yields

$$\frac{\partial \underline{\sigma}^e_1}{\partial \kappa_2} = \frac{-\frac{\partial E_1}{\partial \underline{\sigma}^e_2}}{\Delta} < 0 \qquad \frac{\partial \underline{\sigma}^e_2}{\partial \kappa_2} = \frac{\frac{\partial E_1}{\partial \underline{\sigma}^e_1}}{\Delta} < 0,$$

where $\Delta = \partial E_1/\partial \underline{\sigma}^e_1 \times \partial E_2/\partial \underline{\sigma}^e_2 > 0$.

(iii) Differentiating $\kappa_1 = E_1(\underline{\sigma}^e_1, \underline{\sigma}^e_2, c)$ and $\kappa_2 = E_2(\underline{\sigma}^e_1, \underline{\sigma}^e_2, c)$ with respect to c yields

$$\frac{\partial \underline{\sigma}^e_1}{\partial c} = \frac{-\frac{\partial E_1}{\partial c}\frac{\partial E_2}{\partial \underline{\sigma}^e_2} + \frac{\partial E_2}{\partial c}\frac{\partial E_1}{\partial \underline{\sigma}^e_2}}{\Delta} < 0 \qquad \frac{\partial \underline{\sigma}^e_2}{\partial c} = \frac{-\frac{\partial E_1}{\partial \underline{\sigma}^e_1}\frac{\partial E_2}{\partial c} + \frac{\partial E_2}{\partial \underline{\sigma}^e_1}\frac{\partial E_1}{\partial c}}{\Delta} \gtrless 0$$

To see this, notice that the numerator of $\partial \underline{\sigma}^e_1/\partial c$ is negative, since it is given by

$$(1 - \frac{\underline{\sigma}_1}{\xi_B}) f(\xi_B) \left(-A(1 - \frac{\underline{\sigma}_2}{\xi_2}) f(\xi_2) + B\frac{\underline{\sigma}_1}{\xi_1}(1 - \frac{\underline{\sigma}_2}{\xi_1}) f(\xi_1) - \int_{\xi_B}^{\xi_1} \frac{f(x)}{x} dx - \underline{\sigma}_1 \int_{\xi_B}^{\infty} \frac{f(x)}{x^2} dx \right),$$

where (using Lemma 2)

$$\begin{aligned} A &= \frac{\partial \xi_B}{\partial c}\frac{\partial \xi_2}{\partial \underline{\sigma}_2} - \frac{\partial \xi_B}{\partial \underline{\sigma}_2}\frac{\partial \xi_2}{\partial c} > 0 \\ B &= \frac{\partial \xi_B}{\partial c}\frac{\partial \xi_1}{\partial \underline{\sigma}_2} - \frac{\partial \xi_B}{\partial \underline{\sigma}_2}\frac{\partial \xi_1}{\partial c} < 0. \end{aligned}$$

In turn, the numerator of $\partial \underline{\sigma}^e_2/\partial c$ is given by

$$C(1 - \frac{\underline{\sigma}_1}{\xi_B}) f(\xi_B) \frac{\underline{\sigma}_1}{\xi_1}(1 - \frac{\underline{\sigma}_2}{\xi_1}) f(\xi_1) - D(1 - \frac{\underline{\sigma}_1}{\xi_B}) f(\xi_B) + E \int_{\xi_B}^{\infty} \frac{f(x)}{x} dx,$$

where (using Lemma 2)

$$
\begin{aligned}
C &= \frac{\partial \xi_B}{\partial \underline{\sigma}_1}\frac{\partial \xi_1}{\partial c} - \frac{\partial \xi_B}{\partial c}\frac{\partial \xi_1}{\partial \underline{\sigma}_1} < 0 \\
D &= (1-\frac{\underline{\sigma}_2}{\xi_2})f(\xi_2)\frac{\partial \xi_2}{\partial c}\frac{\partial \xi_B}{\partial \underline{\sigma}_1} + \left(\int_{\xi_B}^{\xi_1}(1-\frac{\underline{\sigma}_2}{x})\frac{f(x)}{x}dx - (1-\frac{\underline{\sigma}_2}{\xi_B})\int_{\xi_B}^{\infty}\frac{f(x)}{x}dx\right)\frac{\partial \xi_B}{\partial c} \gtrless 0 \\
E &= -(1-\frac{\underline{\sigma}_2}{\xi_2})f(\xi_2)\frac{\partial \xi_2}{\partial c} + \frac{\underline{\sigma}_1}{\xi_1}(1-\frac{\underline{\sigma}_2}{\xi_1})f(\xi_1)\frac{\partial \xi_1}{\partial c} < 0,
\end{aligned}
$$

thereby revealing that the sign of $\partial \underline{\sigma}_2^e/\partial c$ is ambiguous. □

A.9 Proof of Theorem 7 (Race Based Admissions)

Let $E_i^m(\underline{\sigma}_1 - \Delta, \underline{\sigma}_2)$ be the minority students enrollment at college i, and $E_i^M(\underline{\sigma}_1, \underline{\sigma}_2)$ be the majority students enrollment at college i, $i = 1, 2$. In (a monotone) equilibrium

$$
\begin{aligned}
\kappa_1 &= \rho E_1^m(\underline{\sigma}_1^e - \Delta, \underline{\sigma}_2^e) + (1-\rho)E_1^M(\underline{\sigma}_1^e, \underline{\sigma}_2^e) && (40) \\
\kappa_2 &= \rho E_2^m(\underline{\sigma}_1^e - \Delta, \underline{\sigma}_2^e) + (1-\rho)E_2^M(\underline{\sigma}_1^e, \underline{\sigma}_2^e). && (41)
\end{aligned}
$$

To analyze the effects of introducing Δ, we will analyze the equilibrium comparative static with respect to Δ evaluated at $\Delta = 0$. Notice that, at $\Delta = 0$, $E_i^m = E_i^M$, $i = 1, 2$, and the same is true with their derivatives.

(i) Note that, with the introduction of Δ, ξ_B and ξ_1 fall for minority students. Since (ξ_B, ∞) and (ξ_2, ξ_1) are, respectively, the sets of minority students calibers applying to college 1 and to college 2, it follows that more of them now apply to college 1 and fewer to college 2. Moreover, each minority student from the original equilibrium now has a higher chance of getting into college 1, and the same chance of getting into school 2 as their majority counterparts. Combining these effects (i.e. the application sets and the acceptance probabilities), the result follows.

(ii) Differentiating (40)–(41) with respect to Δ, and evaluating the resulting expressions at $\Delta = 0$ yields $\partial \underline{\sigma}_1^e/\partial \Delta = \rho$, which is positive, less than one, and increasing in ρ.

(iii) Proceeding as in (ii), we obtain that, evaluated at $\Delta = 0$, $\partial \underline{\sigma}_2^e/\partial \Delta = 0$. □

A.10 Proof of Lemma 4 (MLRP Families)

(a) ψ is increasing in α_1, for

$$\frac{\partial\psi}{\partial\alpha_1} = G_x(\underline{\sigma}_2|\phi(1-\alpha_1,\underline{\sigma}_1))/G_x(\underline{\sigma}_2|\phi(1-\alpha_1,\underline{\sigma}_1)),$$

and MLRP implies that $G_x < 0$.

(b) $0 < \psi < 1$ for all $\alpha_1 \in (0,1)$ follows from $G(\sigma|x) > 0$ for every interior σ, $\underline{x} < \phi(1-\alpha_1,\underline{\sigma}_1) < \infty$, and $\underline{\sigma}_1 \geq \underline{\sigma}_2$. To show that $\psi(0,\underline{\sigma}_1,\underline{\sigma}_2) \geq 0$, notice that ψ is positive, single-valued, and continuous for all $\alpha_1 \in (0,1)$. Finally, $\lim_{x\to\infty} G(\sigma|x) = 0$ implies that $\lim_{\alpha_1\to 1}\phi(1-\alpha_1,\underline{\sigma}_1) = \infty$, and thus $\psi(1,\underline{\sigma}_1,\underline{\sigma}_2) = \lim_{x\to\infty} 1-G(\underline{\sigma}_2|x) = 1$.

(c) ψ is concave in α_1, for

$$\frac{\partial^2\psi}{\partial\alpha_1^2} = \frac{G_x(\underline{\sigma}_2|\varphi(1-\alpha_1,\underline{\sigma}_1))}{G_x(\underline{\sigma}_1|\varphi(1-\alpha_1,\underline{\sigma}_1))^2}\left(\frac{G_{xx}(\underline{\sigma}_1|\varphi(1-\alpha_1,\underline{\sigma}_1))}{G_x(\underline{\sigma}_1|\varphi(1-\alpha_1,\underline{\sigma}_1))} - \frac{G_{xx}(\underline{\sigma}_2|\varphi(1-\alpha_1,\underline{\sigma}_1))}{G_x(\underline{\sigma}_2|\varphi(1-\alpha_1,\underline{\sigma}_1))}\right),$$

and this derivative is nonpositive if the expression in parenthesis is nonnegative. A sufficient condition for this to hold is that $G_{xx}(\sigma|x)/G_x(\sigma|x)$ be increasing in σ or, equivalently, that $-G_x(\sigma|x)$ be log-supermodular. $\square$

A.11 Proof of Theorem 8 (Correlated Signals)

(i) It is easy to show that $\frac{\partial E_1}{\partial\underline{\sigma}_1} < 0$, $\frac{\partial E_2}{\partial\underline{\sigma}_2} < 0$, and $\frac{\partial E_2}{\partial\underline{\sigma}_1} > 0$. Fix any $\kappa_1 \in (0,1)$, and let $\lambda \equiv \left(\frac{1-c/u(1-u)}{1-c/(1-u)}\right)$. Then there exists a unique $\underline{\sigma}_1^e$ that solves (28), which is decreasing in κ_1 and independent of $\underline{\sigma}_2$. Inserting this solution in equation (29), we find that the right side is decreasing in $\underline{\sigma}_2$, and its maximum value — i.e., when $\underline{\sigma}_2 = 0$ — is equal to $E_2(\underline{\sigma}_1^e, 0)$. Call this value $\overline{\kappa}_2(\kappa_1)$. Since the largest value $\underline{\sigma}_2$ can assume in this case is $\lambda\underline{\sigma}_1^e$, it follows that the smallest feasible value of the right side of (29) is equal to $E_2(\underline{\sigma}_1^e, \lambda\underline{\sigma}_1^e)$. Call this value $\underline{\kappa}_2(\kappa_1)$. Then, if $\kappa_2 \in (\underline{\kappa}_2(\kappa_1), \overline{\kappa}_2(\kappa_1))$, there is a unique $\underline{\sigma}_2^e$ that solves (29). Hence, there is a unique pair of college thresholds $(\underline{\sigma}_1^e, \underline{\sigma}_2^e)$ that solves (28)–(30). It is straightforward to check that $0 < \underline{\kappa}_2(\kappa_1) < \overline{\kappa}_2(\kappa_1) < 1-\kappa_1$.

(ii) In equilibrium, $\kappa_1 = E_1(\underline{\sigma}_1^e, c)$ and $\kappa_2 = E_2(\underline{\sigma}_1^e, \underline{\sigma}_2^e, c)$. Differentiating this system with respect to κ_i, $i = 1, 2$, yields

$$\frac{\partial \underline{\sigma}^e_1}{\partial \kappa_1} = \frac{\frac{\partial E_2}{\partial \underline{\sigma}^e_2}}{\Delta} < 0 \qquad \frac{\partial \underline{\sigma}^e_2}{\partial \kappa_1} = \frac{-\frac{\partial E_2}{\partial \underline{\sigma}^e_1}}{\Delta} < 0$$
$$\frac{\partial \underline{\sigma}^e_1}{\partial \kappa_2} = 0 \qquad \frac{\partial \underline{\sigma}^e_2}{\partial \kappa_2} = \frac{\frac{\partial E_1}{\partial \underline{\sigma}^e_1}}{\Delta} < 0,$$

where $\Delta = \partial E_1/\partial \underline{\sigma}^e_1 \times \partial E_2/\partial \underline{\sigma}^e_2 > 0$. Differentiation with respect to c yields

$$\frac{\partial \underline{\sigma}^e_1}{\partial c} = \frac{\frac{\partial E_1}{\partial c}\frac{\partial E_2}{\partial \underline{\sigma}_2}}{\Delta} < 0 \qquad \frac{\partial \underline{\sigma}^e_2}{\partial c} = \frac{-\frac{\partial E_1}{\partial \underline{\sigma}^e_1}\frac{\partial E_2}{\partial c} + \frac{\partial E_2}{\partial \underline{\sigma}^e_1}\frac{\partial E_1}{\partial c}}{\Delta} \gtrless 0$$

since $\partial E_1/\partial c < 0$ but the sign of $\partial E_2/\partial c$ is ambiguous. □

B Appendix: General Signal Structure and PAM

Let $x^u(\underline{\sigma}_i) = \max\{x \in [\underline{x}, \infty) | \, G(\underline{\sigma}_i|x) = 1\}$, $i = 1, 2$ (e.g., in the uniform, $x^u(\underline{\sigma}_i) = \underline{\sigma}_i$).

Consider the extension of Lemma 2. Part (i) follows easily, since $\alpha_2(\xi_2) = c/u$ implies that $\xi_2 = \phi(1 - c/u, \underline{\sigma}_2)$, which is increasing in c and in $\underline{\sigma}_2$. Moreover, since $\xi_2 \in [x^u(\underline{\sigma}_2), \infty)$ and is continuous in c, $\lim_{c\to 0} \xi_2 = x^u(\underline{\sigma}_2)$. The proof of part (iii) is exactly the same as before, except that now $\xi_1 = \phi(1 - \tilde{\alpha}, \underline{\sigma}_1)$. Regarding part (ii), the problem in the general case is that, although there is a solution $\hat{\alpha}_1$ to equation (32) with the properties stated in the proof, it need not be unique. Hence, $\xi_B = \phi(1-\hat{\alpha}_1, \underline{\sigma}_1)$ need not be unique either. It is clear from (32) that a sufficient condition for uniqueness is that the slope of $z(\hat{\alpha}_1, c, u, \underline{\sigma}_1, \underline{\sigma}_2)$ be less than one at any solution $\hat{\alpha}$ to (32). Formally,

$$\frac{\partial z}{\partial \alpha_1} = \frac{u\hat{\alpha}_1 \frac{\partial \Psi}{\partial \alpha_1}}{1 - u\Psi} < 1,$$

where we have omitted the arguments of the functions to simplify the notation. Now, as c vanishes, the denominator converges to a positive number, and the numerator vanishes if $\hat{\alpha}_1 \partial \Psi/\partial \alpha_1$ goes to zero. But this is equivalent to

$$\lim_{c\to 0}(1 - G(\underline{\sigma}_1|\xi_B))\frac{G_x(\underline{\sigma}_2|\xi_B)}{G_x(\underline{\sigma}_1|\xi_B)} = 0. \tag{42}$$

Since $\xi_B \in [x^u(\underline{\sigma}_1), \infty)$ and is continuous in c, it follows that $\lim_{c\to 0} \xi_B = x^u(\underline{\sigma}_1) \geq \underline{x}$.

Thus, a sufficient condition for (42) is that condition (26) holds. Hence, ξ_B is unique for an interval of $c > 0$ sufficiently small. The rest of the proof of (ii) is the same as before.

Let us turn now to Theorem 4. Assume for a moment that ξ_B is unique.

As regards to existence, the proof goes through with the following minor modifications. In Step 1, replace $\underline{\sigma}_2 = 0$ by $\underline{\sigma}_2 = \underline{\sigma}$, $\underline{\sigma}_1/x$ by $G(\underline{\sigma}_1|x)$, and use $\xi_2 = \underline{x}$, $\xi_B = \phi(1-c/u, \underline{\sigma}_1)$, and $\xi_1 = \phi(c/u, \underline{\sigma}_1)$. In Step 2, replace $\eta\underline{\sigma}_1$ by $\phi(1-\frac{1}{u}\overline{\alpha}_1, \varphi(1-\overline{\alpha}_1, \underline{\sigma}_1))$, and 0 by $\underline{\sigma}$. In Step 3, replace $\underline{\sigma}_i = 0$ (whenever it appear) by $\underline{\sigma}_i = \underline{\sigma}$, $i = 1, 2$; replace $\xi_B = \underline{\sigma}_1$ at $c = 0$ by $\xi_B = x^u(\underline{\sigma}_1)$ at $c = 0$, and $\frac{\hat{\sigma}_1(\kappa_1,0)}{x}$ by $G(\hat{\underline{\sigma}}_1(\kappa_1, 0)|x)$. Finally, in Step 4, replace 0 by $\underline{\sigma}$ inside E_2, H_1, and H_2^{-1}, and $\eta\underline{\sigma}_1$ by $\phi(1-\frac{1}{u}\overline{\alpha}_1, \varphi(1-\overline{\alpha}_1, \underline{\sigma}_1))$.

Regarding uniqueness, differentiation of $\xi_1 = \phi(1-\tilde{\alpha}, \underline{\sigma}_1)$ and $\xi_B = \phi(1-\hat{\alpha}_1, \underline{\sigma}_1)$ reveals, after some manipulation, that $\partial\xi_1/\partial\underline{\sigma}_1 \times \partial\xi_B/\partial\underline{\sigma}_2 = \partial\xi_1/\partial\underline{\sigma}_2 \times \partial\xi_B/\partial\underline{\sigma}_1$. Using this result, one can show after much algebra that the slope condition (35) becomes:

$$\frac{-(1-G(\underline{\sigma}_1|\xi_B))f(\xi_B)\frac{\partial\xi_B}{\partial\underline{\sigma}_2}\int_{\xi_B}^{\xi_1} g(\underline{\sigma}_1|x)(1-G(\underline{\sigma}_2|x))f(x)dx}{-\frac{\partial E_2}{\partial\underline{\sigma}_2}\int_{\xi_B}^{\infty} g(\underline{\sigma}_1|x)f(x)dx + D} < 1, \tag{43}$$

where D is a sum of positive terms. Since $-\frac{\partial E_2}{\partial\underline{\sigma}_2} > \int_{\xi_B}^{\xi_1} G(\underline{\sigma}_1|x)g(\underline{\sigma}_2|x)f(x)dx$, $D > 0$, and $\int_{\xi_B}^{\xi_1} g(\underline{\sigma}_1|x)(1-G(\underline{\sigma}_2|x))f(x)dx > \int_{\xi_B}^{\infty} g(\underline{\sigma}_1|x)f(x)dx$, it follows that a sufficient condition for (43) to hold is that

$$\frac{-(1-G(\underline{\sigma}_1|\xi_B))f(\xi_B)\frac{\partial\xi_B}{\partial\underline{\sigma}_2}}{\int_{\xi_B}^{\xi_1} G(\underline{\sigma}_1|x)g(\underline{\sigma}_2|x)f(x)dx} < 1. \tag{44}$$

If condition (26) holds, then the numerator of (44) converges to zero as c vanishes, for $\frac{\partial\xi_B}{\partial\underline{\sigma}_2} = (c\,u\,g(\underline{\sigma}_2|\xi_B))/(G_x(\underline{\sigma}_1|\xi_B)((1-u\Psi)^2 - cu\partial\Psi/\partial\alpha_1))$. The denominator of (44) converges to $\int_{x^u(\underline{\sigma}_1)}^{\infty} G(\underline{\sigma}_1|x)g(\underline{\sigma}_2|x)f(x)dx > 0$. Hence, (44) holds at $c = 0$ and, by continuity, for small $c > 0$. Thus, (43) holds for small $c > 0$, and uniqueness follows.

So far we have assumed that ξ_B is unique. But this is true for small $c > 0$ if condition (42) holds, since the equilibrium thresholds are continuous in c.

C Appendix: The College Objective Function

We now show that the reduced-form model of college behavior used in the paper can be rationalized as a game between the two colleges, in which their objective is to maximize the total expected caliber of the student body, subject to a capacity constraint.

Let $B = \{x \mid S(x) = \{2\}\}$, $C = \{x \mid S(x) = \{1, 2\}\}$, $D = \{x \mid S(x) = \{1\}\}$, and let $A_1 = C \cup D$ and $A_2 = B \cup C$ be the number of students who apply to college 1 and college 2. To avoid trivialities, let $\int_{A_i} f(x)dx > \kappa_i$, $i = 1, 2$; i.e., each college receives more applicants than the number of slots available. For simplicity, we assume that $\underline{x} \geq 0$.

Let $g_i(\sigma) = \int_{A_i} g(\sigma|x)f(x)dx$ be the density of signals given the set of applicants A_i, $i = 1, 2$, where $g(\sigma|x)$ is *any* density with the MLRP. Also, let $f_i(x|\sigma) = \frac{g(\sigma|x)f(x)}{g_i(\sigma)}$ be the conditional density of x given σ and A_i, $i = 1, 2$.

With some abuse of notation, let $a_i(\sigma)$ be the probability that a student who has applied to i and whose signal was σ accepts college i, and let $\gamma_i(\sigma)$ be the expected value of the student's caliber given that he applies to i, his signal is γ, and he accepts college i.

A strategy for college i, $i = 1, 2$, is a (measurable) indicator function $\chi_i : [\underline{\sigma}, \overline{\sigma}] \rightarrow \{0, 1\}$; i.e., a college chooses the set of signal realizations that it will accept. Then college i's optimization problem is given by

$$\max_{\chi_i(\sigma)} \int_{\underline{\sigma}}^{\overline{\sigma}} \chi_i(\sigma)a_i(\sigma)\gamma_i(\sigma)g_i(\sigma)d\sigma \tag{45}$$

subject to

$$\int_{\underline{\sigma}}^{\overline{\sigma}} \chi_i(\sigma)a_i(\sigma)g_i(\sigma)d\sigma \leq \kappa_i. \tag{46}$$

Note that constraint (46) must be binding at the optimum, for all students admitted add to total caliber. Let λ_i, $i = 1, 2$, be the multiplier associated with (46). Forming the Lagrangian and optimizing pointwise reveals that the optimal strategy is

$$\chi_i(\sigma) = \begin{cases} 1 & \text{if } \gamma_i(\sigma) \geq \lambda_i \\ 0 & \text{if } \gamma_i(\sigma) < \lambda_i, \end{cases} \tag{47}$$

where λ_i is the 'shadow price' of each slot available at college i.

If we can show that $\gamma_i(\sigma)$ increases in σ for each college i, then the optimal $\chi_i(\sigma)$ will be represented by a threshold $\underline{\sigma}_i$ such that each college accepts students whose signals are above $\underline{\sigma}_i$, $i = 1, 2$. And since (46) holds with ecaliber for each college, this will immediately justify the use of the reduced-form model.

Consider college 1. Since it is the best college, obviously $a_1(\sigma) = 1$ for all σ. Thus,

$\gamma_1(\sigma) = \int_{A_1} x f_1(x|\sigma)dx$, for being accepted by a student is uninformative. The MLRP of $g(\sigma|x)$ implies the MLRP of $f_1(x|\sigma)$, and therefore $\gamma_1(\sigma)$ is increasing in σ and college 1's optimal strategy is summarized by a threshold $\underline{\sigma}_1$. Notice that, for given applicant pools, the determination of $\underline{\sigma}_1$ is *independent* of the strategy used by college 2.

Consider college 2. Now $a_2(\sigma)$ is given by

$$a_2(\sigma) = \int_B f_2(x|\sigma)dx + \int_C G(\underline{\sigma}_1|x) f_2(x|\sigma)dx. \tag{48}$$

That is, a given signal σ could come from a student who only applied to college 2, or from one who applied to both colleges and was rejected from college 1. Therefore,

$$\gamma_2(\sigma) = \frac{\int_B x f_2(x|\sigma)dx + \int_C x G(\underline{\sigma}_1|x) f_2(x|\sigma)dx}{a_2(\sigma)}. \tag{49}$$

It is easy to show that $\gamma_2(\sigma) < \int_{A_2} x f_2(x|\sigma)dx$. This is because being accepted by a student reduces college 2's estimate of his caliber, as there is a positive probability that the student was rejected by college 1; i.e., college 2 suffers an *acceptance curse* effect.

Notice that equation (49) can be written as $\gamma_2(\sigma) = \int_{A_2} x h_2(x|\sigma)dx$, where

$$h_2(x|\sigma) = \frac{(I_B(x) + I_C G(\underline{\sigma}_1|x)) f_2(x|\sigma)}{\int_{A_2} (I_B(x) + I_C G(\underline{\sigma}_1|x)) f_2(x|\sigma)dx}, \tag{50}$$

where $I_B(x)$ ($I_C(x)$) is the indicator function of the set B (C). It is immediate that the 'density' $h_2(x|\sigma)$ has the MLRP if $f_2(x|\sigma)$ does, which in turn follows from the MLRP of $g(\sigma|x)$. Therefore, $\gamma_2(\sigma)$ increases in σ, and the optimal $\chi_2(\sigma)$, given $\underline{\sigma}_1$, is summarized by a threshold $\underline{\sigma}_2$ such that students are accepted if only if their signal σ is above $\underline{\sigma}_2$.

References

Albrecht, J., P. Gautier, and S. Vroman (2002): "Equilibrium Directed Search with Multiple Applications," Georgetown mimeo.

Anderson, A., and L. Smith (2005): "Assortative Matching and Reputation," University of Michigan mimeo.

Becker, G. (1973): "A Theory of Marriage: Part I," *Journal of Political Economy*, 81, 813–846.

Burdett, K., and K. Judd (1983): "Equilibrium Price Dispersion," *Econometrica*, 51(4), 955–69.

Burdett, K., S. Shi, and R. Wright (2001): "Pricing and Matching with Frictions," *Journal of Political Economy*, 109(5), 1060–1085.

Chade, H. (2006): "Matching with Noise and the Acceptance Curse," forthcoming, *Journal of Economic Theory.*

Chade, H., G. Lewis, and L. Smith (2005): "What's in a Ranking?," work in progress, University of Michigan.

Chade, H., and L. Smith (2006): "Simultaneous Search," forthcoming in *Econometrica.*

Gale, D., and L. Shapley (1962): "College Admissions and Stability of Marriage," *American Mathematical Monthly*, 69(1), 9–15.

Kircher, P. (2006): "Simultaneous Directed Search with Recall," mimeo, University of Bonn.

Kircher, P., and M. Galenianos (2006): "Directed Search with Multiple Job Applications," mimeo, University of Bonn.

Nagypál, E. (2004): "Applications with Incomplete Information," mimeo, Northwestern University.

Roth, A., and M. Sotomayor (1990): "The College Admissions Problem Revisited," *Econometrica*, 57(3), 559–570.

Shimer, R., and L. Smith (2000): "Assortative Matching and Search," *Econometrica*, 68(2), 343–369.

Smith, L. (1997): "The Marriage Model with Search Frictions," MIT working Paper.