Non-Ergodic Behavior in a Financial Market With Interacting Investors*

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Abstract

We identify possible long-run market shares and the long-run asset price dynamics of financial markets with heterogenous interacting agents. This involves stability conditions for a class of difference equation in a random environment, where the random environment is endogenously generated by agents' investment behavior. Depending on the evaluation of a performance measure of an investment, asset prices may behave in a non-ergodic manner. That is, the price processes converge in distribution, but the limiting distribution is not necessarily uniquely determined. The long-run market shares of two competing financial mediators may strongly depend on the random environment which is endogenously generated by a noise traders.

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1 Introduction and Overview

In recent years models with interacting agents have increasingly attracted attention in focusing on trader heterogeneity as a central building block of a descriptive theory of financial markets. It is a commonly accepted view that agents in financial markets trade since either preferences, beliefs, or wealth positions differ. Their attitudes towards risk and their beliefs about the future development of prices constitute the key influence on the determination of market prices.

A central issue of financial markets with interacting agents is to model the way in which agents select between different portfolio strategies. From a finance perspective, it is argued that professional traders use technical trading rules (Allen & Taylor 1992), because they are profitable, e.g., see Brock, Lakonishok & LeBaron (1992) or Lo, Mamaysky & Wang (2000). In many models such as the Santa Fe artificial stock market (LeBaron, Arthur & Palmer 1999) and in the *adaptive beliefs* framework of Brock & Hommes (1997b, 1998) and others, various forms of chartism and trend chasing are applied according to their profitability.

An old conjecture which dates back to Alchian (1950) and Friedman (1953) states that investors who are unable to predict future outcomes sufficiently accurately will eventually be driven out of the market. This conjecture is a main pillar of the rational-expectations paradigm. It implies that agents either learn to make accurate predictions or will not survive. Recent literature, however, has questioned this theory. In a series of papers, De Long, Shleifer, Summers & Waldmann (1989, 1990, 1991) showed that agents with incorrect beliefs may, on average, earn higher returns than agents who are able to correctly predict the future. Addressing the same issue, the seminal papers by Blume & Easley (1992, 2005) and Sandroni (2000) investigate the determinants of scenarios in which agents with incorrect beliefs may or may not accumulate more wealth than those with accurate predictions and may even be driven out of the market.

In the context of a CAPM with interacting investors, Alchian and Friedman's conjecture suggests that only rational investors who hold mean-variance efficient portfolios will survive in the long run. A caveat of the traditional CAPM is the assumption of homogeneous investors holding self-fulfilling expectations on future returns. This rules out heterogeneous expectations and the possibility of misspecified beliefs including their effects on portfolio decisions. Its static nature leaves unexplained why different agents may hold portfolios that differ considerably from the proposed market portfolio. A recent generalization of the classical security market line result in Wenzelburger (2004) overcomes this problem. It states that regardless of the diversity of beliefs, the portfolios that are collinear to some *reference portfolio* are (mean-variance) efficient in the classical sense of CAPM theory. The reference portfolio may be seen as a 'modified' market portfolio that accounts for discrepancies from incorrect beliefs. It coincides with the market portfolio if beliefs are homogeneous. It turns out that portfolios of investors with linear mean-variance preferences *and* rational expectations are efficient.

The modeling framework for this result is based on Böhm, Deutscher & Wenzelburger (2000), Böhm & Chiarella (2005) and Brock & Hommes (1998). Its prerequisite is the notion of a perfect forecasting rule for first and second moments in the presence of possibly non-rational beliefs of other market participants. These forecasting rules provide correct first and second moments of the price process conditional on the available information, and in this sense generate rational expectations equilibria. If the Alchian and Friedman's conjecture were true in the context of such a dynamic CAPM, then any sequence of *realized returns* associated with an efficient portfolio should statistically prove to be superior to any other non-efficient portfolio. However, first simulation results of the empirical performance of efficient portfolios in Böhm & Wenzelburger (2005) indicate that non-rational investors neither necessarily attain a lower market share than a rational investor, nor are they necessarily driven out of the market.

This paper provides a rigorous mathematical framework within which to study the long-run behavior of both market shares and asset prices in a dynamic capital asset pricing model. We prove that the financial market dynamics is *ergodic* if the interaction between households is sufficiently weak. In this case, market shares settle down to a unique equilibrium. Ergodicity breaks down if interactive complementarities become too powerful. While market shares and asset prices still converge, their asymptotics is *random* and depends on noise trader transactions. We consider an agent-based model of financial markets where the demand for multiple risky assets comes form a large set of households. Rather than investing directly in the financial market households select between two mediators. Mediators are characterized by their ability to forecast future asset prices. Following on from Frankel and Froot (1986) and Brock & Hommes (1997, 1998) the beliefs are summarized as 'chartist' and 'expert trader' views. Chartists base their trading strategies upon observed historical price patterns such as trends. Following Wenzelburger (2004), we assume that an expert trader are able to correctly predict the first two moments of the price process and holds mean-variance efficient portfolios. We analyze the question to what extent boundedly rational consumers are able to identify the mediator holding mean-variance efficient portfolios by means of simple empirical performance measures.

The joint dynamics of asset prices and forecasts are described by a deterministic recursion in a random environment. The environment is generated by an *exogenous* stochastic process that describes the effects of noise trading and by a process that describes the fluctuations in the mediators' market shares. This process is assumed to be *endogenous*. Households evaluate the mediators' performance before making their investment decisions. Their propensity to follow a specific mediator depends on her performance as measured by the empirical return or Sharpe ratio associated to her trading strategy. The dependence of the households' choices on performances introduces a feedback from past asset prices into the random environment. This feedback distinguishes our model from the work of Blume & Easley (1992, 2005), Föllmer and Schweizer (1993), Sandroni (2000), Horst (2005a) Estigneev, Hens & Schenk-Hoppé (2005), or Hens & Schenk-Hoppé (2005) where investors stick to their portfolio decisions. Brock & Hommes (1997,1998) allow for feedback effects but largely rely on numerical simulations. Our model generates a rich dynamics while still being amenable to analytic solutions.

In the context of financial markets with interacting agents a natural equilibrium notion is not a particular state, but rather a distribution of states reflecting the proportion of time the economy spends in each of the states (Kirman 1992). This idea calls for an ergodicity result for asset prices and has recently been applied by Föllmer, Horst & Kirman (2005). They give sufficient conditions which guarantee convergence of the sequence of empirical distributions associated with the price process in financial markets with interacting agents to a unique limit. We extend their convergence result beyond the regime of ergodicity. Depending on the switching behavior of households, asset prices may behave in a non-ergodic manner. While the price process converges in distribution, the limiting distribution is not necessarily uniquely determined and may well be random. When ergodicity breaks down, 'history matters,' and the long-run market shares of competing financial mediators are path dependent. The choice behavior of households thus constitutes an *endogenously* generated source of uncertainty for the long run evolution of asset prices. Our approach may thus be viewed as a first step to bridge the gap between the deterministic models pioneered by Brock and Hommes (1997a) with their rich dynamics but inherent inaccessibility to analytical solutions, and the probabilistic frameworks of Föllmer, Horst & Kirman (2005) and Horst (2005a) which allows for mathematical results but rules out non-ergodic dynamics.

It will turn out that the joint dynamics of asset prices and forecasts of our model can be described by multi-dimensional linear recursive relation of the form

$$X_{t+1} = A(\eta_t)X_t + B(\eta_t, \varepsilon_t)$$

where η_t and ε_t denote the market share of chartists and the demand coming from noise traders in period t, respectively. The mediators' performances z_t are functions of the empirical distribution of asset prices and forecasts such as empirical returns of average Sharpe ratios. Households choose a mediator at random depending on their actual performance. In the limit of an infinite set of households, market shares are determined by performances whose dynamics takes the form

$$z_{t+1} = z_t + \xi_t(g(z_t) + \beta_t).$$

Approximation results for such difference equations have been established by, e.g., Kushner & Yin (2003). They have previously been applied in various economic contexts such as in the (non-) parametric learning theory, e.g., see Kuan & White (1994) or Chen & White (1998) for a recent application. Under suitable regularity conditions on the function g and constants ξ_t, the set of accumulation points of the process $\{z_t\}_{t\in\mathbb{N}}$ can be described by the limiting points of an ordinary differential equation if the impact of the noise terms β_t is asymptotically negligible. In our case, this is the delicate part. In order to estimate the impact of the noise terms we establish a uniform moderate deviations principle for parameterized linear stochastic difference equations of the form

$$X^{\eta}_{t+1} = A(\eta)X_t + B(\eta, \varepsilon_t).$$

Such difference equation are of importance in financial time series analysis (Embrechts, Klüppelberg & Mikosch 1997). In our model they describe the dynamics of asset prices and beliefs in benchmark models with 'frozen' market shares. Our uniform moderate deviation principle allows us to approximate the sequence $\{z_t\}_{t\in\mathbb{N}}$ by the trajectory of an ordinary differential equation. It turns out that asset prices display an ergodic behavior if the limiting ODE has a unique and globally stable steady state. When performances are measured by empirical returns, we link the number of possible steady states to the strength of interaction. Our results are consistent with many results in the social interaction literature which assert uniqueness of equilibria if interactions are weak (Blume 1993, Horst & Scheinkman 2005) whereas powerful interactive complementarities often generate non-ergodic dynamics (Brock & Durlauf 2001, Durlauf 1997, Horst 2005b). The sequence $\{z_t\}_{t\in\mathbb{N}}$ eventually stays in the basin of attraction of *some* steady sate and settles down along the trajectories of the ODE. In contrast to Bisin &

Verdier (2001), the initial value of our limiting ODE is random and endogenous. Hence the long run behavior of the market shares and asset prices is not determined by the initial conditions alone. It may also depend on noise trader transactions.

We introduce our model in Section 2. The convergence results are stated in Section 3 and illustrated by numerical simulations in Section 4. All proofs are carried out in the appendix.

2 The Model

We investigate a dynamic financial market model in which the demand for multiple risky assets stems from many boundedly rational agents. Instead of making direct investments in the financial markets, agents invest through financial mediators. Mediators are characterized by mean variance preferences and heterogenous beliefs for future asset prices. Based on these beliefs they form their demand functions; the actual asset price is determined by market clearing conditions. Asset prices are driven by an exogenous stochastic process describing the agents' liquidity demand and by an *endogenously* generated process that specifies the evolution of the distribution of agents' individual choices of a mediator, i.e., the mediators' market shares.

2.1 Interacting investors

We consider a large number of households who transfer wealth into the next by investing a fixed amount $e > 0$ of their exogenously given endowment into K risky assets and a bond in each period. Households are boundedly rational in the sense of Simon (1982) and delegate[1] their portfolio transactions to one of two mediators $i = 1, 2$ who carry out portfolio transactions on their behalf. In terms of her respective market share $\eta_t^{(i)} \in [0, 1]$, mediator i receives $W_t^{(i)} = \eta_t^{(i)} e$ units of per capita resources from investing households in period t. The bond pays the risk-less rate $r > 0$. Aggregate per-capita repayment obligations from investing in a portfolio $x_{t-1}^{(i)} \in \mathbb{R}^K$ of risky assets and $y_{t-1}^{(i)} \in \mathbb{R}$ risk-less bonds to households in period t amount to $(1 - \delta^{(i)}) \left[p_t^\top x_{t-1}^{(i)} + (1 + r) y_{t-1}^{(i)} \right]$, where $0 \leq \delta^{(i)} \leq 1$ stipulates the income share of mediator i. Here, $p_t \in \mathbb{R}_+^K$ denotes the vector of current asset prices. Thus, given a vector p of proposed asset prices,

[1] We refer to Sims (2003) and references therein for discussion of limitations of the paradigm of rational, computationally unconstrained households akin to many macroeconomic models when agents information processing abilities are limited.

mediator i's budget constraints read

$$W_t^{(i)} = p^\top x^{(i)} + y^{(i)}.$$

We assume that the mediators are myopic mean-variance optimizers. Thus, their demand for the risky assets is solely based on their coefficient of risk aversion $\alpha^{(i)}$ along with their subjective assessments $(q_t^{(i)}, V_t^{(i)})$ of the mean value of covariance matrix of the asset prices in the subsequent period $t+1$. Abstracting from short-sell constraints, the per-capita aggregate demand function for risky assets of all households who employ i is

$$x^{(i)}(p) = \frac{\eta_t^{(i)}}{\alpha^{(i)}} V_t^{(i)-1}[q_t^{(i)} - (1+r)p].$$

Let us denote by $\bar{x} \in \mathbb{R}_+^K$ the number of tradeable risky assets in per capita terms and by ε_t the (per-capita) portfolio holdings of noise traders after trading in period t. Given the mediators' *beliefs* $(q_t^{(i)}, V_t^{(i)})$, the market-clearing condition in period t takes the form

$$\frac{\eta_t^{(1)}}{\alpha^{(1)}} V_t^{(1)-1}[q_t^{(1)} - (1+r)p_t] + \frac{\eta_t^{(2)}}{\alpha^{(2)}} V_t^{(2)-1}[q_t^{(i)} - (1+r)p_t] + \varepsilon_t \stackrel{!}{=} \bar{x}.$$

Solving for the market-clearing price p_t, we obtain a temporary equilibrium map

$$p_t := \Gamma_t^{(1)}\, q_t^{(1)} \;+\; \Gamma_t^{(2)}\, q_t^{(2)} \;-\Gamma_t(\bar{x} - \varepsilon_t), \tag{1}$$

where, for $i = 1, 2$, we put

$$\Gamma_t^{(i)} := \tfrac{\eta_t^{(i)}}{\alpha^{(i)}} \Gamma_t V_t^{(i)-1} \quad \text{with} \quad \Gamma_t := \tfrac{1}{(1+r)} \left(\tfrac{\eta_t^{(1)}}{\alpha^{(1)}} V_t^{(1)-1} + \tfrac{\eta_t^{(2)}}{\alpha^{(2)}} V_t^{(2)-1} \right)^{-1}.$$

Assuming that all covariance matrices $V_t^{(i)}$ are positive definite, Γ_t is well defined, symmetric, and positive definite, and we obtain a sequence of temporary price equilibria driven by the evolution of the mediators' market shares and noise trader transactions.

2.2 The feedback of subjective beliefs on asset prices

The mediators are assumed to be boundedly rational in sense of Sargent and use forecasting rules for first and second moments to update their subjective beliefs. Mediator 1 is assumed to be a *chartist* or *trend chaser*. She bases her forecasts for the future asset prices on past observations and applies a simple *technical trading rule* of the form

$$q_t^{(1)} := \sum_{l=1}^{L} D^{(l)}\, p_{t-l} \tag{2}$$

where $D^{(1)}, \dots, D^{(L)}$ denote her expected impact of the past prices $p_{t-1}, \dots, p_{t-L}$ on p_{t+1}. For simplicity, we assume that mediator 1 never updates second moment beliefs and uses constant subjective variance-covariance matrices, denoted by $V^{(1)}$. While the beliefs of mediator 1 may well be incorrect, suppose now that mediator 2 is able to correctly predict the first two moments of the price process, conditional on all available information. As a short hand, we will use the term *rational expectations* to describe the situation in which the first two moments of mediator 2's subjective distributions of asset prices, i.e., the conditional mean values and the conditional covariance matrices, coincide with the respective moments of the true distributions. Assuming that the first moment beliefs $q_t^{(2)}$ of mediator 2 are unbiased, it is shown in Wenzelburger (2004) that they are determined by an *unbiased forecasting rule* which takes the form

$$q_t^{(2)} := \Gamma_t^{(2)-1}\Big[q_{t-1}^{(2)} - \Gamma_t^{(1)} q_t^{(1)} + \Gamma_t\big(\bar{x} - \mathbb{E}_{t-1}[\varepsilon_t]\big)\Big], \tag{3}$$

where $\mathbb{E}_{t-1}$ denotes the conditional expectation with respect to all the information available in period $t-1$. Notice that the inverse $\Gamma_t^{(2)-1}$ is well defined, because $\Gamma_t^{(2)}$ is symmetric and positive definite. The forecasting rule (3) provides unbiased forecasts of asset prices for mediator 2 in the sense that $q_{t-1}^{(2)}$ is the best least-squares prediction for p_t, given the available information. Indeed, it is straightforward to verify that almost surely $\mathbb{E}_{t-1}[p_t - q_{t-1}^{(2)}] = 0$ for all times t when the forecast $q_t^{(2)}$ is given by (3).

Remark 2.1 *Notice that mediator* 2 *anticipates the presence of technical traders in the financial market. As a result, her forecasting rule (3) depends also on past price patterns. This amplifies the impact of trend chasing on asset price dynamics. Wenzelburger (2004) shows how (3) can be estimated from the excess demand function.*

To focus on the effects of heterogeneity in the mediators' beliefs about expected future asset prices and on the interplay between rational expectations and trend chasing, we assume from now on that the mediators' beliefs for second moments coincide and are correct and that they share a common coefficient of risk aversion, i.e., $\alpha^{(1)} = \alpha^{(2)} = \alpha$. All results carry over to the case with differing risk aversions $\alpha^{(1)} \neq \alpha^{(2)}$, see Appendix C. Assuming the covariance matrix $\mathbb{V}_\varepsilon$ of the noise trader transactions to be constant over time, the beliefs take the form

$$V_t^{(i)} \equiv \left(\tfrac{1+r}{\alpha}\right)^2 \mathbb{V}_\varepsilon^{-1}. \tag{4}$$

The concept is similar to the concept for unbiased forecasting rules, see Wenzelburger (2004) for details. Put $\eta_t = \eta_t^{(1)}$ and $\eta_t^{(2)} = 1 - \eta_t$ for the remainder of the paper.

Inserting the forecasts (2)-(4) into (1), the resulting process of asset prices and forecasts is given by a set of stochastic difference equations

$$\begin{cases} p_t &= q_{t-1}^{(2)} + \frac{1+r}{\alpha}\mathbb{V}_\varepsilon^{-1}\big(\varepsilon_t - \mathbb{E}_{t-1}[\varepsilon_t]\big), \\ q_t^{(1)} &= \sum_{j=1}^J D^{(j)} p_{t-j}, \\ q_t^{(2)} &= \frac{1+r}{1-\eta_t}\, q_{t-1}^{(2)} - \frac{\eta_t}{1-\eta_t}\, q_t^{(1)} + \frac{(1+r)^2}{(1-\eta_t)\alpha}\,\mathbb{V}_\varepsilon^{-1}\left(\bar{x} - \mathbb{E}_{t-1}[\varepsilon_t]\right), \end{cases} \tag{5}$$

where the last equation corresponds to the unbiased forecasting rule (3). Observe that the difference equations (5) are linear if market shares η_t were constant over time. Households' decisions which determine the market share of a mediator will be based on performance measures. The households evaluate the mediators before making their investment decision. The dependence of households' choices on performances generates a feedback effect from the sequence of asset prices and forecasts into the evolution of market shares. We will introduce our performance measures in the following section.

Remark 2.2 *The model can easily be imbedded into an OLG framework without changing the key equations. With the focus directed towards the long-run behavior of a financial market, the present setup appears to be more plausible. A numerical analysis of an extension to investors with multiperiod planning horizons as in Hillebrand & Wenzelburger (2004) are available. While our mathematical framework is flexible enough to capture planning horizons beyond the simple two period case, it renders the notation needlessly cumbersome.*

2.3 Performance measures and choice rules

In our model the dynamics of asset prices and forecasts are generated by two underlying stochastic processes, $\{\varepsilon_t\}_{t\in\mathbb{N}}$ and $\{\eta_t\}_{t\in\mathbb{N}}$, describing the evolution of noise-trader transactions and mediators' market shares, respectively. In the spirit of De Long, Shleifer, Summers & Waldmann (1989) we assume throughout that the noise trader portfolios $\{\varepsilon_t\}_{t\in\mathbb{N}}$ are an *exogenous* i.i.d. process with mean $\bar{\varepsilon}$ and a non-degenerate variance matrix $\mathbb{V}_\varepsilon$. We can then represent the joint dynamics of asset prices and forecasts in terms of a linear difference equation in a random environment. To this end, we put $X_t := (q_t^{(2)}, q_{t-1}^{(2)}, p_t, \dots, p_{t-J-1}) \in \mathbb{R}^d$ with $d = K(J+4)$ as well as

$$\begin{aligned} a_0(\eta) &:= \tfrac{1+r}{1-\eta}, & a_j(\eta) &:= \tfrac{\eta}{1-\eta} D^{(j)},\ j = 1,\dots,J \\ b_0(\eta) &:= \tfrac{(1+r)^2}{(1-\eta)\alpha}\mathbb{V}_\varepsilon^{-1}\left(\bar{x} - \bar{\varepsilon}\right), & b_1(\varepsilon) &:= \tfrac{1+r}{\alpha}\mathbb{V}_\varepsilon^{-1}\left(\varepsilon - \bar{\varepsilon}\right). \end{aligned}$$

Then the process $X = \{X_t\}_{t\in\mathbb{N}}$ as defined by (5) then takes the linear form

$$X_t = A(\eta_t)X_{t-1} + B(\eta_t, \varepsilon_t) \qquad (t \in \mathbb{N}) \tag{6}$$

where the $d \times d$ matrix $A(\eta_t)$ and the vector $B(\eta_t, \varepsilon_t) \in \mathbb{R}^d$ are given by

$$A(\eta_t) := \begin{pmatrix} a_0(\eta_t) & 0 & a_1(\eta_t) & \cdots & a_J(\eta_t) & 0 & 0 \\ I & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ I & 0 & \ddots & & & & \vdots \\ 0 & 0 & I & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & I & 0 \end{pmatrix} \text{ and } B(\eta_t, \varepsilon_t) := \begin{pmatrix} b_0(\eta_t) \\ 0 \\ b_1(\varepsilon_t) \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix},$$

respectively. We are now going to specify a probabilistic framework within which to analyze the long-run behavior of market shares and asset prices when households' decisions are based on the perceived performance of mediators. To this end, let $\{\varrho_t\}_{t\in\mathbb{N}}$ be the sequence of empirical distributions associated to the process $\{X_t\}_{t\in\mathbb{N}}$, i.e.,

$$\varrho_t := \frac{1}{t}\sum_{i=0}^{t-1} \delta_{X_i} \quad \text{so that} \quad \varrho_t(f) := \int f \mathrm{d}\varrho_t = \frac{1}{t}\sum_{i=0}^{t-1} f(X_i) \tag{7}$$

for any bounded map $f : \mathbb{R}^d \to \mathbb{R}$ where δ_x denotes the Dirac measure that puts all mass on x. A household's propensity to invest through a specific mediator will depend on the performance associated to the mediator's investment strategy.

Definition 2.3 *Let $f^l : \mathbb{R}^d \to \mathbb{R}$ $(l = 1, 2, \ldots, L)$ be a given list of bounded measurable functions. A performance measure is a Lipschitz continuous function $\psi : \mathbb{R}^L \to \mathbb{R}^2$. The list of performances in period t is given by*

$$\pi_t = \psi\left(\varrho_t(f^1), \cdots, \varrho_t(f^L)\right).$$

Notice that the performance of a mediator at time t depends on the entire empirical distribution of asset prices and forecasts up to time $t - 1$. Since the functions f^l are fixed, it is convenient to interpret the performance measure ψ as function from the set of all probability distributions on $\mathbb{R}^d$ to $\mathbb{R}^2$ so that $\pi_t = \psi(\varrho_t)$. Brock & Hommes (1997a) and many other authors associate performances with current profits. Föllmer, Horst & Kirman (2005) consider the case where performances are given by the sum of discounted profits a mediators investment strategy would have generated in the past. Our focus is on performance measures that depend on empirical statistics of asset prices and forecasts such as average returns of empirical Sharpe ratios. It turns out that such a dependence of asset prices on the entire past is capable of generating a rich dynamics, while still being amenable to analytic solutions.

Example 2.4 *Suppose that a mediator's performance is measured by historically realized returns* $\{R_s^{(i)}\}_{s=0}^t$, $i = 1, 2$ *on investment. Having invested the amount* $W_t^{(i)} = \eta_t e$, *mediator* i*'s return from selling the portfolio* $x_t^{(i)} := \frac{\eta_t^{(i)}}{\alpha} V_t^{(i)-1}[q_t^{(i)} - (1+r)p_t]$ *in period* $t+1$ *is*

$$R_{t+1}^{(i)} = r + \frac{1}{\alpha e} [p_{t+1} - (1+r)p_t]^\top V_t^{(i)-1} [q_t^{(i)} - (1+r)p_t]. \tag{8}$$

Since second moment beliefs are constant, realized returns at time t *take the form*

$$R_t^{(i)} = f^i(X_t)$$

for suitably defined functions $f^i : \mathbb{R}^{2d} \to \mathbb{R}$, $i = 1, 2$. *The specific case where* ψ *denotes the identity matrix yields the performance measure*[2]

$$\pi_t = \left(\varrho_t(f^1), \varrho_t(f^2)\right)^\top .$$

Empirical Sharpe ratios also fit into our framework.

Example 2.5 *Based on Example 2.4, an alternative performance measure is the differences of empirical* Sharpe ratios *associated to the two times series* $\{R_s^{(i)}\}_{s=0}^t$, $i = 1, 2$. *To this end, we define two continuous functions* $f^{i+2} : \mathbb{R}^{2d} \to \mathbb{R}$, $i = 1, 2$ *by*

$$\left(R_t^{(i)}\right)^2 = f^{i+2}(X_t), \quad i = 1, 2.$$

The empirical Sharpe ratios associated with the each of the mediators $i = 1, 2$ *is*

$$\frac{\varrho_t(f^{i+2}) - r}{\sqrt{\varrho_t(f^{i+2}) - \left[\varrho_t(f^i)\right]^2}}, \quad i = 1, 2,$$

respectively. With f^1 *and* f^2 *as in Example 2.4 a performance measure based on mediators' empirical Sharpe ratios is given by*

$$\pi_t = \left(\frac{\varrho_t(f^1) - r}{\sqrt{\varrho_t(f^3) - \left[\varrho_t(f^1)\right]^2}}, \frac{\varrho_t(f^2) - r}{\sqrt{\varrho_t(f^4) - \left[\varrho_t(f^2)\right]^2}} \right)^\top . \tag{9}$$

[2]There is no a-priori reason to assume that returns can be represented by *bounded* functions. For the ergodic theorem to be applicable we thus need to specify an upper and lower bound for the earnings reported by the mediators. This can be justified when households do not trust unusually high earnings to prevail. Similar considerations apply to Example 2.5.

We assume that a household's propensity to employ a mediator depends on the current performances. While our framework is flexible enough to allow for a wide variety of choice probabilities, following the standard approach in the social interactions literature (Blume 1993, Brock & Durlauf 2001,2002), we assume that, given the performance π_t, households act conditionally independent of each other so that an individual household employs mediator 1 with probability

$$\Phi(\pi,\beta) := \left[\overline{\eta}-\underline{\eta}\right]\frac{\exp(\beta\pi_1)}{\exp(\beta\pi_1)+\exp(\beta\pi_2)}+\underline{\eta} = \frac{\overline{\eta}-\underline{\eta}}{\exp(\beta(\pi_2-\pi_1))+1}+\underline{\eta}. \tag{10}$$

Here $\beta > 0$ specifies the dependence of the agents' choices on the mediators' performances, and $0 \leq \underline{\eta} \leq \overline{\eta} \leq 1$ are upper and lower bounds for the realized market share η. In the limit of an infinite number of households the chartist's market share at time t are deterministic. This implies the existence of a continuous "choice function" $F : \mathbb{R}^L \to \mathbb{R}$ such that

$$\eta_t = F\big(\varrho_t(f^1),\ldots,\varrho_t(f^L)\big) := \Phi\big(\psi\big(\varrho_t(f^1),\ldots,\varrho_t(f^L)\big),\beta\big). \tag{11}$$

Thus, the dynamics of asset prices and beliefs can be described by *path-dependent stochastic difference equation*. Its dynamics is analyzed in the following section.

Remark 2.6 *It follows from a generalized security market line result (Wenzelburger 2004, Thm. 3.1) that the portfolios of investors with rational expectations are mean-variance efficient in the ex-ante sense of classical CAPM theory. Transposed into the present setting, this theorem implies that the conditional* Sharpe ratio *of mediator* 2 *will always be greater than the conditional Sharpe ratio of mediator* 1*, so that a.s.*

$$\frac{\mathbb{E}_t[R^{(1)}_{t+1}]-r}{\sqrt{\mathbb{V}_t[R^{(1)}_{t+1}]}} \leq \frac{\mathbb{E}_t[R^{(2)}_{t+1}]-r}{\sqrt{\mathbb{V}_t[R^{(2)}_{t+1}]}} \quad \textit{for all times } \ t.$$

From this result one might expect that at least in the scenario of Example 2.5 the mean-variance efficient portfolios of mediator 2 *will always empirically outperform the inefficient portfolios of mediator* 1*. A first simulation analysis in Böhm & Wenzelburger (2005), however, indicates that this conjectures does not necessarily hold and that noise traders or chartists may in fact outperform investors holding mean-variance efficient portfolios. These findings suggest that mediators holding mean-variance efficient portfolios may not be identified. We will provide a rigorous analysis of this phenomenon.*

3 Convergence to equilibrium

In this section we state conditions on the behavior of households and mediators which guarantee that asset prices converge in distribution, albeit a random one. To this end, we shall first give sufficient conditions for almost sure convergence of the mediators' market shares to a discrete random variable η_*. Numerical simulations suggest that the distribution of η_*, i.e., the distribution of long-run market shares depends on the initial condition as well as on the strength of interactions; η_* turns out to be a constant if households choose the mediators more or less independently of their respective performances. If the interaction is strong enough, η_* is a proper random variable. While the trend chasers' forecasts may be considered inaccurate, the feedback effects from household behavior into the dynamics of asset prices and forecasts may prevent trend chasers from being driven our of the market. In fact, if the interaction effects are strong, trend chasers and 'rational' mediators typically coexist. It is this coexistence that distinguishes our model from, for instance, Sandroni's where 'markets favor agents that make accurate predictions.' Moreover, while asset prices convergence in distribution, the limiting distribution may be random. Randomness in the limiting distribution may be viewed as an *endogenous* source of randomness originating from interaction and imitation effects.

3.1 Benchmark models driven by independent noise

Trend chasers may have a destabilizing affect on asset prices. Without any bound on their impact of asset prices there is no reason to believe that prices and forecasts are stable in the long run. In fact, whenever trend chasers predominate prices will start deviating from their long-run averages and will eventually grow without bounds. In order to guarantee long-run stability, we therefore introduce a condition which limits the impact of trend chasing.

Assumption 3.1 *Households employ mediators* 1*, the trend chaser, at least with some probability* $\underline{\eta}$ *and at most with probability* $\overline{\eta}$*. In particular, the choice function* F *in (11) is such that*

$$\eta_t \in [\underline{\eta}, \overline{\eta}]. \tag{12}$$

Let us now introduce, for any market share $\eta \in [\underline{\eta}, \overline{\eta}]$ the process $X^\eta = \{X_t^\eta\}$ defined by the linear recursive relation

$$X_t^\eta = A(\eta) X_{t-1}^\eta + B(\eta, \varepsilon_t) \qquad (t \in \mathbb{N}). \tag{13}$$

The process X^η describes the evolution of asset prices and forecasts in a benchmark model with market shares "frozen" at the level η. The long-run behavior of such sequences has been extensively investigated under a contraction condition on $A(\eta)$. We need a slightly stronger condition.

Assumption 3.2 *(i) The map $\eta \mapsto A(\eta)$ is Lipschitz continuous:*

$$\|A(\eta) - A(\hat{\eta})\| \leq a|\eta - \hat{\eta}|.$$

(ii) The interval $[\underline{\eta}, \overline{\eta}]$ is chosen such that the eigenvalues of all the matrices $A(\eta)$ with $\eta \in [\underline{\eta}, \overline{\eta}]$ lie uniformly within the unit circle.

(iii) The function $B(\cdot, \cdot)$ is bounded, $|B(\eta, \varepsilon)| \leq B$, and uniformly Lipschitz continuous in its first argument, i.e.,

$$\sup_{\varepsilon} |B(\eta, \varepsilon) - B(\hat{\eta}, \varepsilon)| \leq b|\eta - \hat{\eta}|. \tag{14}$$

Notice that our system (6) satisfies the Lipschitz conditions of Assumption 3.2. For fixed $\eta \in [\underline{\eta}, \overline{\eta}]$, the stochastic difference equation (13) under Assumption 3.2, has unique stationary solution, i.e., there exists a unique stationary and ergodic process $x^\eta = \{x_t^\eta\}_{t \in \mathbb{N}}$ that satisfies (13). For any starting point x, the distribution μ_t^η of X_t^η converges weakly to the distribution μ^η of x_0^η, and X^η is bounded; see Theorem A.1 in Section A for more details on stochastic difference equations.

The following proposition, whose proof is also given in Section A, shows that Assumption 3.2 also guarantees boundedness of the original sequence X. It is in this sense that (12) prevents prices from exploding by limiting the impact of trend chasers.

Proposition 3.3 *Under Assumption 3.2 the sequence $\{X_t\}$ is almost surely bounded. More precisely, for any initial value $X_0 = x$ there exists a constant M_x such that*

$$\mathbb{P}_x \left[\sup_t |X_t| \leq M_x \right] = 1. \tag{15}$$

Here $\mathbb{P}_x$ denotes the probability measure on the canonical path space induced by the process X with initial state x.

It will be convenient to write $\mu^\eta(f)$ for the integral of a bounded function f with respect to the unique stationary distribution μ^η of the process X^η defined by (13).

3.2 Characterization of equilibria

As a first step towards a general convergence result, we are now going to characterize all possible long-run distributions of the sequence $\{X_t\}_{t\in\mathbb{N}}$ by means of a fixed point property. The key assumption is that the empirical process $\{\eta_t\}_{t\in\mathbb{N}}$ converges almost surely as $t \to \infty$. The proof requires some preparation and will be carried out in Section A.

Theorem 3.4 *Suppose that Assumption 3.2 is satisfied and that the empirical process $\{\eta_t\}_{t\in\mathbb{N}}$ converges almost surely to some random variable η_*. Then the sequence of empirical averages $\{\varrho_t\}_{t\in\mathbb{N}}$ converges almost surely weakly to the random limiting measure μ^{η_*}. More precisely, for all bounded continuous functions f,*

$$\mathbb{P}\left[\lim_{t\to\infty} \varrho_t(f) = \mu^{\eta_*}(f)\right] = 1.$$

The previous theorem states the distribution of X_t converges weakly to a *random* limiting measure if the sequence of market shares settles down to a random limit in the long run. The result imposes a consistency condition on limiting market shares and thus allows us to characterize the class of asymptotic market shares. The limiting market shares have to be consistent with the market shares induced by the limiting empirical distributions of X through the choice function F in (11). In order to make this more precise, we define a map $\zeta : [\underline{\eta}, \overline{\eta}] \to \mathbb{R}^L$ by

$$\zeta(\eta) := \left(\mu^\eta(f^1), \ldots, \mu^\eta(f^L)\right) \in \mathbb{R}^L. \tag{16}$$

This map assigns the *long-run empirical averages* of the Markov process X^η with fixed η to the market shares η. The question of existence and uniqueness of long-run equilibria of the process X can now be reduced to a simple fixed point condition.

Corollary 3.5 *Under the assumptions of Theorem 3.4, the random variable η_* takes values in the set*

$$E := \left\{\eta \in [\underline{\eta}, \overline{\eta}] : \; \eta = F \circ \zeta(\eta)\right\}, \tag{17}$$

i.e., it almost surely satisfies the fixed point condition $\eta = F \circ \zeta(\eta)$. The long-run empirical averages of the process X are given by $\zeta(\eta_*)$ and take values in the set

$$S := \left\{z \in \mathbb{R}^L : \; z = \zeta \circ F(z)\right\}. \tag{18}$$

It is well known that the map $\zeta : [\underline{\eta}, \overline{\eta}] \to \mathbb{R}^L$ is continuous. Typically, however, no analytical expression will be available. The map requires knowledge about the structure of the stationary distributions μ^η for the Markov processes X^η for which, in general, no closed form representation will be available. However, it can easily be simulated. Thus, if we can prove the convergence of market shares, then the possible equilibria can be identified by means of simple and relatively fast Monte Carlo simulations; see Figure 3 below for a graphical representation of the fixed point condition when performances are measured by Sharpe rations and the choice probabilities take the modified logit form (10). A purely numerical analysis of the sequence of market shares, on the other hand, is not always appropriate because the speed of convergence of the sequence $\{\eta_t\}_{t\in\mathbb{N}}$ is very slow. In fact, consider

$$\eta_t = F\left(\varrho_t(f^1), \ldots, \varrho_t(f^L)\right)$$

for bounded Lipschitz continuous functions f^l and F with constant c_F. For $t, T \in \mathbb{N}$ we have

$$\begin{aligned} \left|\varrho_{T+t}(f^l) - \varrho_T(f^l)\right| &= \frac{1}{T+t}\left|\frac{T+t}{T}\sum_{j=1}^{T} f^l(X_j) - \sum_{j=1}^{T+t} f^l(X_j)\right| \\ &\leq 2\|f^l\|_\infty \frac{t}{T+t} \end{aligned} \tag{19}$$

and the difference $|\eta_{T+t} - \eta_T|$ is of the order $\frac{t}{T+t}$. More precisely:

$$|\eta_{T+t} - \eta_T| \leq c_F \max_{l=1,2,\ldots,L} \|f^l\|_\infty \frac{t}{T+t}. \tag{20}$$

A numerical analysis may thus become extremely time consuming and could easily be misleading.

3.3 Convergence of market shares

Having stated our characterization result of long-run market shares, it remains to state conditions which guarantee that the sequence $\{\eta_t\}_{t\in\mathbb{N}}$ converges almost surely. It will turn out that the long-run behavior of empirical averages and hence the asymptotics of market shares can be analyzed by means of an ordinary differential equation. To this end, we recall that $z_t = \left(\varrho_t(f^1), \ldots, \varrho_t(f^L)\right)$, rewrite our stochastic difference equation (6) and (11) as

$$\begin{aligned} X_t &= A\big(F(z_t)\big)X_{t-1} + B\big(F(z_t), \varepsilon_t\big), \\ z_t &= \frac{t-1}{t} z_{t-1} + \frac{1}{t}\big(f^1(X_{t-1}), \ldots, f^L(X_{t-1})\big), \end{aligned} \tag{21}$$

and define a map $g : \mathbb{R}^L \to \mathbb{R}^L$ by setting

$$g(z) := \zeta \circ F(z) - z. \tag{22}$$

The zeros of the map g are given by the set S defined in (18). Continuity of F and ζ implies continuity of g.

We are now in position to state our main convergence result. The proof will be given in Section B below.

Theorem 3.6 *Suppose that the following hypotheses are satisfied:*

1. *The map g is Lipschitz continuous such that for each initial condition $z_0 \in \mathbb{R}^L$, the ODE*
$$\dot{z} = g(z) \tag{23}$$
admits a unique solution.

2. *Let*
$$S^* := \{s_1, \ldots, s_N\} \subset S := \{z \in \mathbb{R}^L : \ g(z) = 0\},$$
be set of asymptotically stable steady states with corresponding basins of attraction $DA(s_i)$, $i = 1, \ldots, N$, such that for any $z_0 \in DA(s_i)$, the solution $z(\tau, z_0)$ of (23) with initial condition $z(0, z_0) = z_0$ converges to the steady state $s_i \in S^*$ as $\tau \to \infty$.

If the sequence $\{z_t\}_{t\in\mathbb{N}}$ of empirical averages visits a compact subset of some $DA(s_i)$, $i = 1, \ldots, N$ infinitely often with probability $\mathbf{p} > 0$, then the following holds:

(i) The sequence $\{z_t\}_{t\in\mathbb{N}}$ converges to s_i, i.e.,
$$\lim_{t\to\infty} |z_t - s_i| = 0 \quad \textit{with at least probability } \mathbf{p}.$$

(ii) The discrete-time stochastic process $\{\eta_t\}_{t\in\mathbb{N}}$ of market shares converges with at least probability $\mathbf{p}$ to a stationary value $F(s_i) \in E$.

The problem of convergence of market shares and hence stock prices can thus be reduced to establishing convergence of ordinary differential equations on the level of empirical averages. In particular, market shares converge almost surely to some constant if the ODE (22) has a unique, globally asymptotically stable steady state. This will be the case if, for instance, performances are measured by empirical returns and the dependence of household choices on performances is sufficiently weak. If the interactive effects are too strong, ergodicity breaks down and market shares converge to a random limit as we illustrate in the following section.

4 Convergence for returns and Sharpe ratios

We apply Theorem 3.6 to investigate whether the rational mediator who holds mean-variance efficient portfolios will attain larger market shares than the chartist. Since trading of assets takes place before households can observe the relevant returns, the *empirical performance* of a portfolio has to rely on estimates. These estimates are reflected by the performance measure. The superiority of a mean-variance efficient portfolio will only show if the estimators involved in the performance measure are consistent. It is intuitively clear that for inconsistent estimators portfolios other than the efficient portfolio could appear to perform better. In fact, we find that technical and 'rational' traders often coexist although we bound the the probability with which households follow the chartist.

Remark 4.1 *We assume throughout that the risk-fee is* $r = 1\%$*. In this case the stability condition for our model is satisfied for* $\underline{\eta} = 0.04$ *and* $\overline{\eta} = 0.36$*. Thus, in order to prevent prices from exploding we allow for no more than 36% trend chasers.*

4.1 Empirical returns as performance measures

Suppose that the mediators' performance is measured in average returns as in Example 2.4. In this case the difference in the performances is of the form $z_t = \varrho_t(f^1 - f^2)$ for suitable bounded continuous functions f^i on $\mathbb{R}$.

4.1.1 Analytical results

A numerical approximation of the function g defining the ODE (23) is depicted in Figure 1(a). It indicates that g has at least three steady states, two asymptotically stable ones and one unstable in the middle. To see whether or not more steady states exist, notice that in the present case $\eta_t = F(z_t)$ is a diffeomorphism, i.e., an invertible map with differentiable inverse F^{-1}. In this case the ODE (23) is topological conjugate (Arrowsmith & Place 1994) to the ODE for market shares

$$\dot{\eta} = h(\eta) \tag{24}$$

where $h := (F' \circ F^{-1})(g \circ F^{-1}) = (F' \circ F^{-1})\big(\zeta - F^{-1}\big)$. The conjugacy implies that taking empirical averages as performance measures the behavior of (23) and (24) is qualitatively the same. In particular, the long-run behavior of empirical averages is precisely described by the long-run behavior of market shares. Notice that

$$h(\underline{\eta}) = h(\overline{\eta}) = 0 \quad \text{because} \quad (F' \circ F^{-1})(\eta) = -\beta(\overline{\eta} - \eta)(\eta - \underline{\eta}). \tag{25}$$

We infer from and the conjugacy of the two ODEs and Figure 1 that h has *five* steady states when $\beta = 2$; the three steady states corresponding to g along with $\underline{\eta}$ and $\overline{\eta}$. Four equilibria are clearly visible; the fifth lies close to $\underline{\eta}$. The solution to (23) exists for all times, and standard monotonicity arguments show that all solutions converge to one of the two asymptotically stable fixed points. The respective basins of attraction are simply separated by the unstable fixed point. Now Theorems 3.4 and 3.6 guarantee convergence of both market shares and asset prices. The respective limits, however, are not necessarily unique. They may depend on the random environment generated by noise trader transactions.

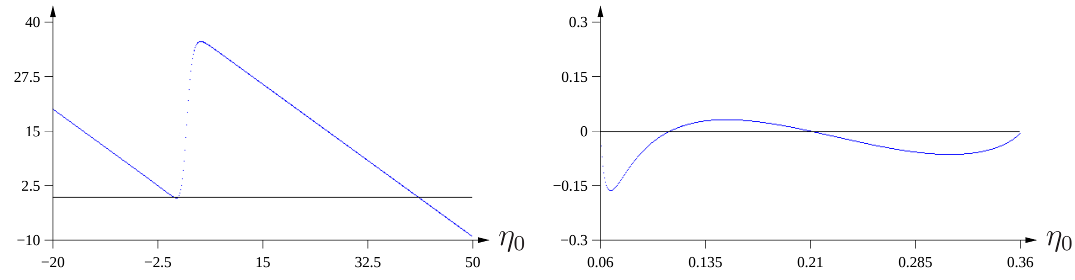
Remark 4.2 *(i) The ODE (24) has a unique globally asymptotically stable steady if the map $\eta \mapsto \zeta(\eta) - F^{-1}(\eta)$ has a unique zero. Thus, our financial market dynamics are ergodic if the dependence of households' choices on performances is sufficiently weak; for small β the inverse choice function F^{-1} is essentially a vertical line. Ergodicity breaks down if the dependence of households' choices on the mediators' performance is too strong. In this case the limiting behavior of market shares and price distributions is random.*

(ii) The lowest and highest possible market share of chartists are always unstable under the dynamics of the ODE. This implies that the long-run market share of the chartists will always be strictly above $\underline{\eta}$ and strictly smaller than $\overline{\eta}$. Hence, under the given restriction of the model, the chartist will never be completely driven out of the market.

4.1.2 Numerical results

To further illustrate the result of Theorem 3.6 we simulate the non-linear model (21) with empirical averages as the performance measure.[3] We find that the long-run market shares are in fact random. The probability with which they converge to the possible steady states depends both on the initial condition η_0 and the intensity of choice. Figure 2 shows the empirical distribution of market shares after $T = 10.000$ periods for $N = 100.000$ independent samples of η_T when chartists initially have a market share of

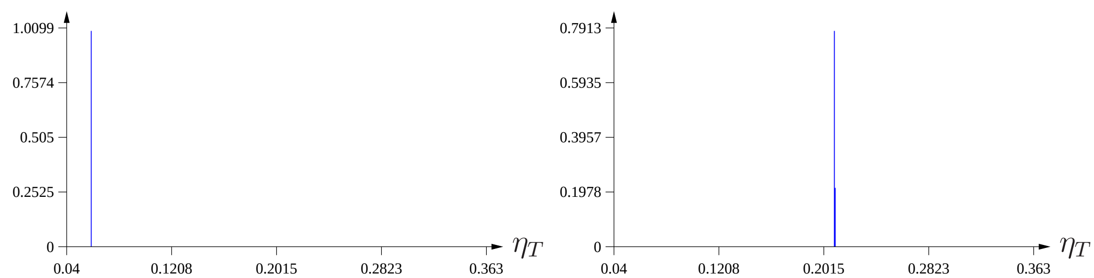
[3] We used the program package MACRODYN for all simulation results.



(a) ODE $\dot{z} = g(z)$ (b) ODE $\dot{\eta} = h(\eta)$

Figure 1: Approximated ODEs for $T = 10000$, $\beta = 2$

6.5% and 15%, respectively. The empirical distributions roughly coincide with Dirac measure concentrated at the left- and rightmost steady state of the ODE of in Figure 1(b), respectively. This suggests that the long-run market shares depend significantly on the initial condition.



(a) Initial market share $\eta_0 = .065$ (b) Initial market share $\eta_0 = .15$

Figure 2: Empirical distributions of market shares for $\beta = 2$.

4.2 Empirical Sharpe ratios as performance measures

The analysis of asymptotic market shares becomes more involved if the mediators' performance is measured by historical Sharpe ratios rather than average returns: in this case the households' choice function as given by (11) is no longer invertible.

4.2.1 Analytical results

If the mediators' performance is measured in historical Sharpe ratios as in Example 2.5, then $z_t = (\varrho_t(f^1), \ldots, \varrho_t(f^4))$ is a 4-dimensional vector. The long-run behavior of

z_t can be described by the asymptotic behavior of an ordinary differential equation of the form

$$\dot{z} = g(z) \tag{26}$$

for a suitable function $g : \mathbb{R}^4 \to \mathbb{R}^4$. Contrary to the previous Example 4.1, however, the dynamics of (26) cannot be described in terms of market shares alone because the choice function F as given by (11) is no longer invertible. The set of asymptotic market shares as given in Corollary 3.5 allows the representation

$$E = \left\{\eta \in [\underline{\eta}, \overline{\eta}] : \Phi^{-1}(\eta, \beta) = \Psi \circ \zeta(\eta)\right\}, \tag{27}$$

where $\Phi^{-1}(\cdot, \beta)$ is the inverse of the logit function (10) and $\Psi \circ \zeta$ describes the stationary difference in Sharpe ratios of the two mediators. While the inverse $\Phi^{-1}(\cdot, \beta)$ is analytically available, the map $\Psi \circ \zeta$ can only be obtained by simulating the benchmark models (13). As before, the market share η is ranging in $\eta \in [\underline{\eta}, \overline{\eta}]$ with $\underline{\eta} = .06$, $\overline{\eta} = .36$ and $r = 1\%$. Figure 3 indicates that the two functions in (27) have three intersection points, provided that the intensity of choice β is sufficiently large, whereas the leftmost intersection point ($\beta = 2$) is hardly visible but exists, because $\Phi^{-1}(\cdot, \beta)$ has two vertical asymptotes at $\underline{\eta}$ and $\overline{\eta}$. These intersection points characterize the possible long-run market shares of the chartist. In particular, market shares and hence asset prices converge to a unique limit if the dependence of household decisions on the mediators' performance is sufficiently weak.

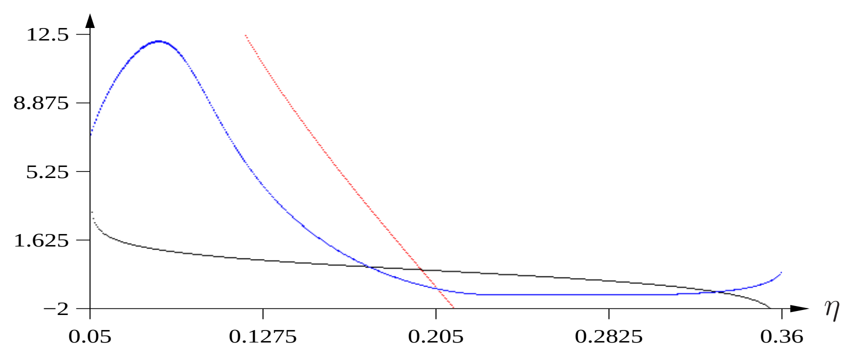
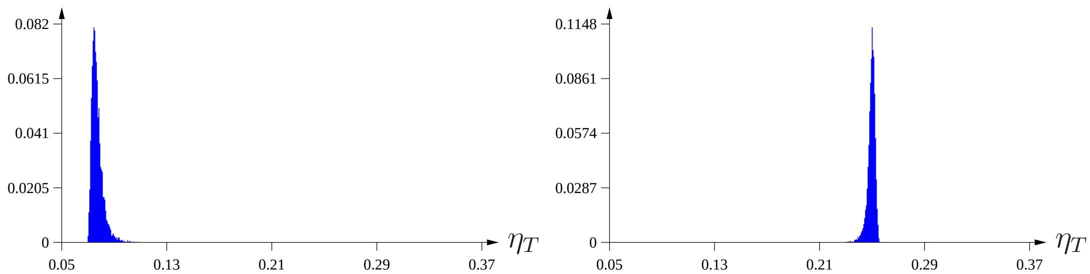


Figure 3: Long-run Sharpe ratios versus logit function; $\beta = 0.5$ (red) and $\beta = 2$ (blue).

4.2.2 Numerical results

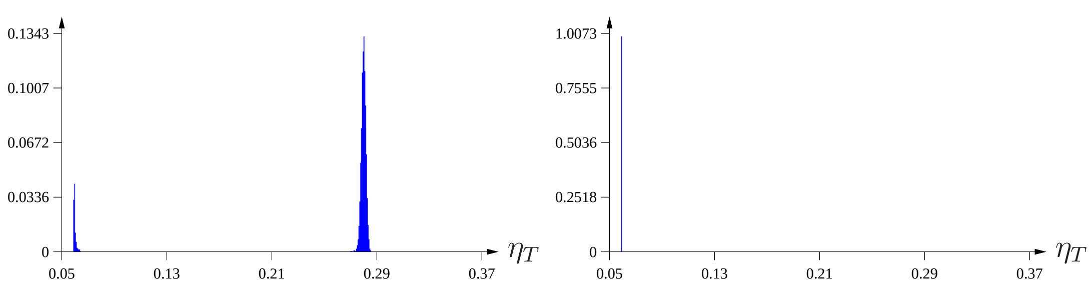
When simulating the non-linear model (21) we find again that the asymptotics of market shares are random and depend on the initial condition and the intensity of choice. For $\beta = 1$ and a sample of $N = 10.000$ independent repetitions, Figure 4 shows the empirical distribution of market shares after $T = 10.000$ periods when the chartists initially have a market share of 6.5% and 35.5%, respectively. The empirical average of this distributions roughly coincide with the right- and leftmost intersection point of the two functions depicted in Figure 3, respectively. We deduce that the long-run market shares are sensitive to initial conditions.



(a) Initial market share $\eta_0 = 0.065$ (b) Initial market share $\eta_0 = 0.355$

Figure 4: Empirical distribution of market shares for $\beta = 1$.

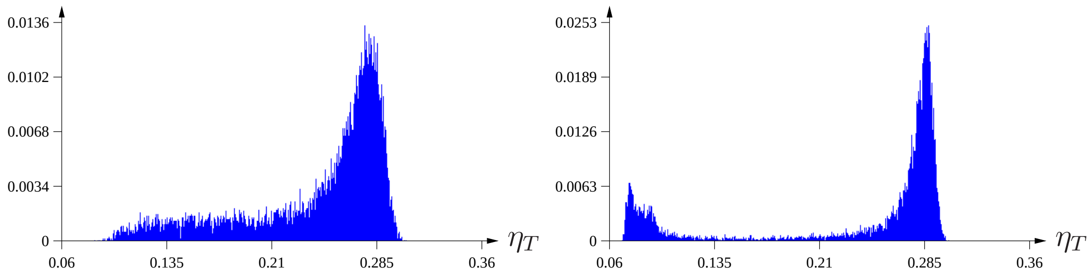
Similar observations are made for $\beta = 2$. In this case, however, the empirical distribution of market shares after $T = 10.000$ periods is either unimodal or bimodal depending on the initial market shares. Figure 5(a) shows the empirical distribution for $\eta_0 = 0.355$. In this case the chartists almost die out while for $\eta_0 = .065$ the empirical distribution of asymptotic market shares is bimodal with two peaks which are approximately located at the outer intersection points of the two functions depicted in Figure 3. If the chartists initially have a sufficiently high market share, they 'survive' with high probability.



(a) Initial market share $\eta_0 = .355$ (b) Initial market share $\eta_0 = .065$

Figure 5: Empirical distribution of market shares for $\beta = 2$.

If the interaction between household is strong, the long-run market shares not only depend on the initial condition, but the specific transactions of noise traders. In both scenarios the limiting distribution of asset prices depends on the realized long-run market shares. Figure 6 displays the empirical densities of market shares after $T = 500$ and $T = 1000$ periods, respectively. We see that the empirical distribution converges only very slowly to the bimodal shape shown in Figure 5(a).



(a) Distribution after 500 periods. (b) Distribution after 1000 periods.

Figure 6: Empirical distribution of market shares for $\beta = 2$ and $\eta_0 = .355$.

5 Conclusions

This paper provided a rigorous mathematical characterization of the long-run behavior of asset prices in a financial market with interacting agents demonstrating that the limiting distribution is not necessarily uniquely determined. As long-run market shares are random, the limiting distribution of asset prices itself is random. Asset prices may behave in a non-ergodic manner in that the price processes converge in distribution, but the limiting distribution is not necessarily uniquely determined. Economically, this implies that the long-run market shares of two competing financial mediators depend strongly on the random environment of the market which is created by the switching behavior of households. This switching behavior constitutes a source of randomness and uncertainty for the evolution of asset prices.

The paper also casts new doubts on Alchian's and Friedman's conjecture that investors who fail to make accurate predictions about the future will be driven out of the market. Taking either empirical average returns or Sharpe ratios as a performance measure, mean-variance efficient portfolios may fail to empirically outperform inefficient portfolios. Hence, chartists may attain higher market shares than mediators holding mean-variance efficient portfolios, so that rational mediators may not be identified. Instead, empirical performance measures of portfolios may be highly misleading. Overall these findings lead us to the conclusion that whether or not investors are driven out of the market depends highly on the random environment of a market.

A Proof of Theorem 3.4

We assume that The Markov processes X^η and the sequence X are defined on the canonical path space $(\Omega, \mathcal{F})$. We denote by $\mathbb{P}^\eta_\mu$ and $\mathbb{P}_\mu$ the law of X^η and X with initial distribution μ, respectively, and put $\mathbb{P}^\eta_x := \mathbb{P}^\eta_{\delta_x}$ and $\mathbb{P}_x := \mathbb{P}_{\delta_x}$. The respective expectations are denoted $\mathbb{E}^\eta_\mu$ and $\mathbb{E}^\eta_x$. The sequences of empirical distributions associated to X^η and X are denoted by $\{\varrho^\eta_t\}_{t\in\mathbb{N}}$ and $\{\varrho_t\}_{t\in\mathbb{N}}$, respectively.

A.1 Some prerequisites

Extending $\{\varepsilon_t\}_{t\in\mathbb{N}}$, to a sequence of i.i.d. random variables on $\mathbb{Z}$, the stationary solution $\{x^\eta_t\}_{t\in\mathbb{N}}$ of (13) may be viewed as the Markov chain x^η with initial value

$$x^\eta_0 = \sum_{j=1}^{\infty} A^j(\eta) B(\eta, \varepsilon_{-j}). \tag{28}$$

More precisely we have the following result. For a proof we refer the reader to Brandt (1986) or Embrechts, Klüppelberg & Mikosch (1997).

Theorem A.1 *Under Assumption 3.2 the following holds:*

(i) The difference equation (13) has a unique stationary solution $\{x^\eta_t\}_{t\in\mathbb{N}}$, and

$$\lim_{t\to\infty} |X^\eta_t - x^\eta_t| = 0 \quad a.s.$$

(ii) The Markov chain X^η has a unique stationary distribution μ^η. For any starting point, the sequence $\{\mu^\eta_t\}$ converges weakly to μ^η:

$$\lim_{t\to\infty} \int f d\mu^\eta_t \to \int f d\mu^\eta \quad \text{for any bounded continuous function } f : \mathbb{R}^d \to \mathbb{R}.$$

(iii) The stationary distribution μ^η depends continuously on η, i.e., if $\lim_{n\to\infty} \eta_n = \eta$, then

$$\lim_{n\to\infty} \int f d\mu^{\eta_n} \to \int f d\mu^\eta \quad \text{for any bounded continuous function } f : \mathbb{R}^d \to \mathbb{R}.$$

Since X^η is stationary and ergodic under $\mathbb{P}^\eta_{\mu^\eta}$, the empirical averages converge $\mathbb{P}^\eta_{\mu^\eta}$-almost surely to their expected values under the unique stationary measure:

$$\lim_{t\to\infty} \int f d\varrho_t^\eta = \int f d\mu^\eta \qquad \mathbb{P}^\eta_{\mu^\eta}\text{-a.s.} \tag{29}$$

for any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$. In the sequel we shall consider the dynamics of empirical averages given an arbitrary initial condition $x \in \mathbb{R}^d$. For this, we need the following perturbation of (29). Its proof follows from standard arguments given in, e.g., Föllmer, Horst & Kirman (2005).

Lemma A.2 *Under the assumptions of Theorem A.1,*

$$\lim_{t\to\infty} \int f d\varrho_t^\eta = \int f d\mu^\eta \qquad \mathbb{P}^\eta_{\mu}\text{-a.s.}$$

for any bounded continuous *function f, each initial distribution μ, and each $\eta \in [\underline{\eta}, \overline{\eta}]$.*

A.2 Proof of Proposition 3.3

Notice first that the process X admits the explicit representation

$$X_{t+1} = \sum_{j=0}^{t} \left(\prod_{i=t-j+1}^{t} A(\eta_i) \right) B(\eta_{t-j}, \varepsilon_{t-j}) + \prod_{i=0}^{t} A(\eta_i) X_0. \tag{30}$$

To establish a.s. boundedness of (30), we will provide an estimate for the involved random products of the matrices $A(\eta)$. Based on these estimates we shall then establish a contraction property for the process X.

(i) Let $\lambda(\eta)$ be the maximal eigenvalue of the matrix $A(\eta)$ with multiplicity $p(\eta)$. By Theorem 3.1 in Varga (1962), there exists a constant $r(\eta)$ such that

$$\|A^n(\eta)\| \sim r(\eta) \begin{pmatrix} n \\ p(\eta)-1 \end{pmatrix} \lambda_1^{n-p(\eta)+1}(\eta) \quad \text{as} \quad n \to \infty.$$

As all the eigenvalues of the matrices $A(\eta)$ lie uniformly within the unit circle there exists, for any $\alpha_0 < 1$ a smallest constant $N(\eta) \in \mathbb{N}$ which satisfies

$$\|A^{N(\eta)}(\eta)\| < \alpha_0 \quad \text{and so} \quad \|A^{nN(\eta)}(\eta)\| < \alpha_0^n \quad \text{for all} \quad n \in \mathbb{N}.$$

That is, the map $X \mapsto A(\eta)X$ is a contraction of order $N(\eta)$. Since the entries of the matrices $A(\eta)$ are uniformly bounded, we obtain constants $C(\eta)$ such that

$$\|A^n(\eta)\| < C(\eta) \alpha_0^{\left[\frac{n}{N(\eta)}\right]} \quad \text{for all} \quad n \in \mathbb{N} \tag{31}$$

where $[x]$ denotes the largest integer less than or equal to $x \in \mathbb{R}^+$; see also (35) below.

In order to show that $N(\eta)$ and $C(\eta)$ are uniformly bounded, we assume to the contrary that there exists a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ such that $N(\eta_n) \uparrow \infty$ as $n \to \infty$. Since the market shares take values in a compact set we may with no loss of generality assume that the sequence $\{\eta_n\}_{n\in\mathbb{N}}$ converges to $\tilde{\eta}$ as $n \to \infty$. Moreover, the maps

$$\eta \mapsto \|A(\eta)^n\| \qquad (n \in \mathbb{N}) \tag{32}$$

are continuous because entries of the matrices $A(\cdot)$ depend continuously on η. Thus $\|A(\eta_n)^{N(\tilde{\eta})}\| < \alpha_0$. for all sufficiently large $n \in \mathbb{N}$. As a result, $N(\eta_n) \leq N(\tilde{\eta})$ contradicting $N(\eta_n) \uparrow \infty$. This implies

$$N := \sup_{\eta\in[\underline{\eta},\overline{\eta}]} N(\eta) < \infty. \tag{33}$$

(ii) For any $n \in \mathbb{N}$ and each $N(\eta) \in \{1, 2, \ldots, N\}$ we have a representations of the form

$$N = i_1 N(\eta) + i_2 \quad \text{and} \quad n = j_1 N + j_2$$

for suitable $i_1 \geq j_1 \geq 0$ and $i_2 \in \{0, 1, \ldots, N(\eta) - 1\}$ and $j_2 \in \{0, 1, \ldots, N - 1\}$. In particular,

$$n = (j_1 i_1 + j_3) N(\eta) + j_4$$

where $j_3 = \left[\frac{j_1 i_2 + j_2}{N(\eta)}\right]$ and $j_4 = j_1 i_2 + j_2 - j_3 N(\eta) \in \{0, 1, \ldots, N - 1\}$. Introducing the constants

$$C_n := \sup\left\{\|A(\eta_1)\| \cdots \|A(\eta_{n-1})\| \ : \ \eta_i \in [\underline{\eta}, \overline{\eta}]\right\} \qquad (n \in \mathbb{N}) \tag{34}$$

we thus obtain for each $\eta \in [\underline{\eta}, \overline{\eta}]$

$$\|A^n(\eta)\| = \|A^{(j_1 i_1 + j_3) N(\eta) + j_4}(\eta)\| \leq C_N \alpha_0^{j_1} = C_N \alpha_0^{\left[\frac{n}{N}\right]}. \tag{35}$$

(iii) The mapping (32) is continuous and hence *uniformly* continuous because market shares take values in a compact set. In view of (33), this allows us to obtain an estimate of the form (35) for time-varying market shares. To this end, we first choose ε and $\delta = \delta(\varepsilon)$ such that

$$\beta_0 := \alpha_0 + \varepsilon < 1$$

and such that for all η and each η_i with $|\eta_i - \eta| < \delta$, $i = 1, 2, \ldots, N(\eta)$, we have

$$\|A(\eta_1) \cdots A(\eta_{N(\eta)})\| \leq \|A^{N(\eta)}(\eta)\| + \varepsilon \leq \beta_0. \tag{36}$$

By 19 there exists a constant $\tau(\varepsilon) \in \mathbb{N}$ that satisfies

$$\sup_{t \geq \tau(\varepsilon)} \max_{i=1,\ldots,N} |\eta_t - \eta_{t+i}| < \delta,$$

almost surely and so (36) yields

$$\sup_{t \geq \tau(\varepsilon)} \|A(\eta_{t+1}) \cdots A(\eta_{t+N(\eta_t)})\| \leq \|A^{N(\eta_t)}(\eta_t)\| + \varepsilon \leq \beta_0. \tag{37}$$

In terms of the constants defined by (34) this gives us

$$\left\| \prod_{i=0}^{t} A(\eta_i) \| \right\| \leq \tilde{C} \beta_0^{\left[\frac{t-\tau(\varepsilon)}{N}\right]} \quad \text{where} \quad \tilde{C} := C_{\tau(\varepsilon)} C_N. \tag{38}$$

(iv) We now establish a.s. boundedness of (30). In view of step (iii)

$$\left\| \prod_{i=0}^{t} A(\eta_i) X_0 \right\| \leq \tilde{C} |X_0|.$$

As the random variables $B(\eta, \varepsilon)$ are uniformly bounded, $|B(\eta, \varepsilon)| \leq B$, it is hence enough to prove that

$$\sup_t \sum_{j=0}^{t} \left\| \prod_{i=t-j+1}^{t} A(\eta_i) \right\| \leq C$$

for some $C < \infty$. For this, note that $t - j + 1 \geq \tau(\varepsilon)$ if $j \leq t - \tau(\varepsilon) + 1$. Thus, by analogy to (37) and (38) we have

$$\left\| \prod_{i=t-j+1}^{t} A(\eta_i) \right\| \leq C_N \beta_0^{\left[\frac{j-1}{N}\right]}$$

for $j \leq t - \tau(\varepsilon) + 1$. This yields

$$\begin{aligned}
\sum_{j=0}^{t} \left\| \prod_{i=t-j+1}^{t} A(\eta_i) \right\| &= \sum_{j=0}^{t-\tau(\varepsilon)} \left\| \prod_{i=t-j+1}^{t} A(\eta_i) \right\| + \sum_{j=t-\tau(\varepsilon)+1}^{t} \left\| \prod_{i=t-j+1}^{t} A(\eta_i) \right\| \\
&\leq C_N \sum_{j=1}^{\infty} \beta_0^{\left[\frac{j-1}{N}\right]} + \tau(\varepsilon) C_{\tau(\varepsilon)} \\
&\leq C_N N \sum_{j=1}^{\infty} \beta_0^j + \tau C_{\tau(\varepsilon)} \\
&=: C.
\end{aligned}$$

□

The same arguments as in the proof of the previous proposition can also be applied to show that family of Markov chains X^η ($\eta \in [\underline{\eta}, \overline{\eta}]$) is uniformly bounded.

Corollary A.3 *(i) For any compact set D of initial values, there exists a constant M_D such that*

$$\mathbb{P}\left[\sup_{t,\eta} |X_t^\eta| \leq M_D \mid X_0^\eta \in D\right] = 1.$$

(ii) For any two processes X^η and Y^η with initial values x and y, respectively,

$$\sup_{\eta,t} |X_t^\eta - Y_t^\eta| \leq C\alpha^{\left[\frac{t}{N}\right]}|x-y|.$$

In particular, $\lim_{t\to\infty} \sup_\eta |X_t^\eta - Y_t^\eta| = 0$ almost surely.

The proof of Proposition 3.3 shows that the sequence $X_T, X_{T+1}, \ldots$ has a contraction property for all sufficiently large T. Specifically, by (35) there exists $\hat{N} \in \mathbb{N}$ such that

$$\sup_\eta \|A^{\hat{N}}(\eta)\| < 1,$$

and so X has, asymptotically, a contraction property of order $\hat{N}$ for all sufficiently. The arguments given in the proof of Proposition 3.3 also show that the random variable x^η defined in (28) is almost surely bounded and that the bound can be chosen independently of η. In particular, the all the stationary distributions μ^η are concentrated on a common compact set K. Thus, for any compact set of initial values D, the Markov chains may be viewed as Markov chains on a compact state space K_D. That is, we may assume that the transition kernels Π_η satisfy

$$\Pi_\eta(x; K_D) = 1 \quad \text{for all} \quad x \in K_D.$$

A.3 Proof of Theorem 3.4

In this section we show that almost sure convergence of market shares implies convergence in distribution of asset prices and forecasting rules. To ease notational complexity, we prove Theorem 3.4 under the simplifying assumption that

$$\alpha := \sup\{\|A(\eta)\| : \underline{\eta} \leq \eta \leq \overline{\eta}\} < 1, \quad \text{i.e., that} \quad \hat{N} = 1. \tag{39}$$

The general case $\hat{N} > 1$ is follows from straightforward, but tedious, modification of the arguments given below. At this point it is also convenient to recall that the *Vasserstein metric*

$$d(\mu, \nu) := \sup\{|f(\mu) - f(\nu)| \ : \ \|f\|_\infty \leq 1, \ f \text{ Lipschitz with constant } 1\} \tag{40}$$

induces the weak topology on the class of all probability measures on $\mathbb{R}^d$. In terms of this metric, almost sure convergence of empirical distributions to their expected value under the unique stationary measure translates into

$$\lim_{t\to\infty} d\left(\varrho_t^\eta, \mu^\eta\right) = 0 \qquad \mathbb{P}_x^\eta\text{-a.s.}$$

Proof of Theorem 3.4: Let us introduce, for any $T \in \mathbb{N}$ a benchmark processes $\overline{X}^T$ by

$$\overline{X}_t^T = X_t \quad \text{for} \quad t \le T \quad \text{and} \quad \overline{X}_{t+1}^T = A(\eta_T)\overline{X}_t^T + B(\eta_T, \varepsilon_t) \quad \text{for} \quad t > T.$$

In view of our simplifying condition (39) and Assumption 3.2 we obtain

$$\begin{aligned} |X_{T+t} - \overline{X}_{T+t}^T| &\le \alpha|X_{T+t-1} - \overline{X}_{T+t-1}^T| + C\sup_{t\ge T}|\eta_t - \eta_T| \\ &\le \frac{C}{1-\alpha}\sup_{t\ge T}|\eta_t - \eta_T|. \end{aligned}$$

Notice now that for any bounded Lipschitz continuous function g with constant 1,

$$\sup_{t\ge T}|\varrho_t(g) - \overline{\varrho}_t^T(g)| \le \sup_{t\ge T}\frac{1}{t-T}\sum_{i=T+1}^{t}|X_i - \overline{X}_i^T|.$$

This implies

$$\lim_{T\to\infty}\mathbb{P}_x\left[\sup_{t\ge T} d(\varrho_t, \overline{\varrho}_t^T) \ge \varepsilon\right] = 0. \tag{41}$$

Moreover, almost sure convergence of the sequence of market shares $\{\eta_t\}_{t\in\mathbb{N}}$ to η_* along with Theorem A.1 (ii) yields

$$\mathbb{P}_x\left[\lim_{T\to\infty} d(\mu^{\eta_T}, \mu^{\eta_*}) = 0\right] = 1. \tag{42}$$

Since the random variables η_T and $\varepsilon_{T+1}, \varepsilon_{T+2}, ...$ are independent and all the eigenvalues of $A(\eta_T)$ lie inside the unit circle, $\overline{X}^T$ is an ergodic Markov chain with invariant distribution μ^{η_T}. By Lemma A.2, the associated sequence of empirical distributions $\{\overline{\varrho}_t^T\}_{t\ge T}$ converges almost surely weakly to μ^{η_T}:

$$\mathbb{P}_x\left[\lim_{t\to\infty} d(\overline{\varrho}_t^T, \mu^{\eta_T}) = 0\right] = 1. \tag{43}$$

Now, our assertion follows from (41)-(43) because

$$\sup_{t\ge T} d(\varrho_t, \mu^{\eta_*}) \le \sup_{t\ge T} d(\varrho_t, \overline{\varrho}_t^T) + \sup_{t\ge T} d(\overline{\varrho}_t^T, \mu^{\eta_T}) + d(\mu^{\eta_T}, \mu^{\eta_*}).$$

□

B Proof of the convergence result

This section proves our convergence result for market shares stated in Theorem 3.6. The arguments will be based on a stochastic approximation result for stochastic difference equations and a uniform large deviations principle for stable autoregressive processes.

B.1 Stochastic approximation

Our goal is to apply a stochastic approximation result of Kushner & Yin (2003) in which the limiting behavior of empirical averages $\{z_t\}_{t\in\mathbb{N}}$ is described in terms of convergence properties of a suitably defined ordinary differential equation. To this end we first rewrite the second equation in (21). For each $T \in \mathbb{N}$, set

$$z_T^t := \frac{1}{t}\sum_{s=T+1}^{T+t}\big(f^1(X_{s-1}),\ldots,f^L(X_{s-1})\big),$$

such that empirical averages up to time $T+t$ take the form

$$z_{T+t} = \frac{T}{T+t}z_T + \frac{t}{T+t}z_T^t. \tag{44}$$

Using (22), we obtain the following representation of z_{T+t}:

$$z_{T+t} = z_T + \frac{t}{T+t}\big[g(z_T) + \beta_T^t\big] \quad \text{with} \quad \beta_T^t = z_T^t - \zeta\big(F(z_T)\big). \tag{45}$$

Let us also introduce a different time scale by defining $\{t_n\}_{n\in\mathbb{N}}$ and $\{T_n\}_{n\in\mathbb{N}}$ by

$$t_n := n^4, \quad T_n := \sum_{i=1}^{n} t_{i-1}.$$

Setting

$$\hat{\theta}_n := z_{T_n} \quad \text{and} \quad \beta_n := \beta_{T_n}^{t_n}, \tag{46}$$

the random sequence $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ satisfies the recursive relation

$$\hat{\theta}_{n+1} = \hat{\theta}_n + \varepsilon_n\big[g(\hat{\theta}_n) + \beta_n\big], \tag{47}$$

where

$$\varepsilon_n := \frac{t_n}{T_n + t_n}, \quad n \in \mathbb{N}. \tag{48}$$

Since T_n is of the order n^5, ε_n is of the order n^{-1}. This allows us to apply a stochastic approximation algorithm to the sequence $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ defined in (47), if the "error terms" β_n converge to zero sufficiently fast.

The ODE method for approximating the dynamics of the discrete time process (47) uses a continuous time interpolation of the sequence $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$. A natural time scale for the interpolation is defined in terms of the step-size sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$. Specifically, let us define

$$\tau_0 = 0 \quad \text{and} \quad \tau_n := \sum_{i=0}^{n-1} \varepsilon_i$$

and the continuous time interpolation $\theta^0 = (\theta^0(t))_{t\geq 0}$ of the discrete time process $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ by $\theta^0(0) = \hat{\theta}_0$ and

$$\theta^0(t,\omega) = \hat{\theta}_n(\omega) \quad \text{for} \quad \tau_n \leq t < \tau_{n+1}.$$

Furthermore, we introduce the left-shifts θ^n of θ^0 by $\theta^n(t,\omega) = \theta^0(\tau_n + t, \omega)$. If the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ satisfies

$$\sum_{n\geq 0} \varepsilon_n = \infty \quad \text{and} \quad \sum_{n\geq 0} \varepsilon_n^2 < \infty$$

and if the "error terms" β_n are asymptotically negligible in the sense that

$$\sum_{n\geq 0} \varepsilon_n |\beta_n| < \infty \tag{49}$$

almost surely, then the functions $\theta^n(\cdot,\omega)$ are equicontinuous for almost every ω. If, in addition the discrete time sequence $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ is bounded with probability one, then any limit $\theta(\cdot,\omega)$ of some convergent subsequence $\{\theta^{n_k}(\cdot,\omega)\}_{k\in\mathbb{N}}$ of $\{\theta^n(\cdot,\omega)\}_{n\in\mathbb{N}}$ is a trajectory of the ordinary differential equation

$$\dot{\theta} = g(\theta). \tag{50}$$

More precisely, the functions $\{\theta^{n_k}(\cdot,\omega)\}_{k\in\mathbb{N}}$ converge to the unique solution of the ODE (50) with initial condition $\theta(0,\omega)$, given by

$$\theta(t,\omega) = \theta(0,\omega) + \int_0^t g(\theta(s,\omega))\, ds,$$

uniformly on compact time intervals. For all $T < \infty$ we have

$$\lim_{k\to\infty} \sup_{0\leq t\leq T} |\theta^{n_k}(t,\omega) - \theta(t,\omega)| = 0. \tag{51}$$

This approximation result allows us to analyze the asymptotics of the sequence $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ by means of the long-run behavior of the ODE (50). Theorem 2.1 in Kushner & Yin (2003, Chapter 5) states that if $\{\hat{\theta}_n(\omega)\}_{n\in\mathbb{N}}$ enters infinitely often some compact subset C_i of a basin of attraction $DA(s_i)$ of a fixed point $s_i = g(s_i)$, then $\lim_{n\to\infty} \hat{\theta}_n(\omega) = s_i$. We summarize this result in the following proposition.

Proposition B.1 *Suppose that the following hypotheses are satisfied:*

(i) The sequence of positive numbers $\{\varepsilon_n\}_{n\in\mathbb{N}}$ *fulfills*

$$\varepsilon_n \to 0, \quad \sum_{n\geq 0} \varepsilon_n = \infty, \quad \textit{and } \sum_{n\geq 0} \varepsilon_n^2 < \infty.$$

(ii) $\sum_{n\geq 0} \varepsilon_n |\beta_n| < \infty$ *with probability 1.*

(iii) Let $g : \mathbb{R}^L \to \mathbb{R}^L$ *be Lipschitz continuous and*

$$S^* := \{s_1, \ldots, s_N\} \subset S := \{\hat{\theta} \in \mathbb{R}^L : \; g(\hat{\theta}) = 0\},$$

be set of asymptotically stable steady states with corresponding basins of attraction $DA(s_i)$, $i = 1, \ldots, N$, *such that for each* $i = 1, \ldots, N$ *and any* $\hat{\theta}_0 \in DA(s_i)$, *the solution* $\theta(\tau, \omega)$ *of* $\dot{\theta} = g(\theta)$ *with initial condition* $\theta(0, \omega) = \hat{\theta}_0$ *converges to* $s_i \in S^*$.

For each i, *fix a compact subset* $C_i \subset DA(s_i)$. *If the discrete-time process* $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ *visits* C_i *infinitely often with probability* $\mathbf{p} > 0$, *then*

$$\lim_{n\to\infty} |\hat{\theta}_n - s_i| = 0 \quad \textit{with at least probability} \quad \mathbf{p}.$$

While condition (i) is satisfied by construction, the difficult part is to verify condition (ii) which will be established in Lemma B.6 of Section B.3 below. Assuming that condition (ii) holds, we will first complete the proof of Theorem 3.6.

B.2 Proof of Theorem 3.6

Presuming that condition (ii) holds all hypotheses of Proposition B.1 are satisfied, and the discrete-time process $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ defined in (47) converges to some asymptotically stable fixed point $s_i \in S^*$ of the associated ODE with at least probability $\mathbf{p}$.

(i) We are left to show convergence of $\{z_n\}_{n\in\mathbb{N}}$. For this, recall that $\hat{\theta}_n = z_{T_n}$ as defined in (46) with a subsequence $\{T_n\}_{n\in\mathbb{N}}$. By Proposition 3.3 and (44), this yields convergence of the entire sequence $\{z_n\}_{n\in\mathbb{N}}$ with probability $\mathbf{p}$, because $|\hat{\theta}_{n+1} - \hat{\theta}_n| = O(n^{-1})$ and hence

$$\max\{|z_{T_n} - z_t| \; : \; t = T_n + 1, T_n + 2, \ldots, T_{n+1}\} = O(n^{-1}).$$

(ii) We have $\eta_t = F(z_t)$, and so continuity of the choice function yields

$$\lim_{t\to\infty} |F(s_i) - F(z_t)| = 0$$

with at least probability $\mathbf{p} > 0$.

□

B.3 A uniform large deviations principle

For a given market share η the Markov chain X^η with transition probability Π_η is ergodic; time averages converge almost surely to surely to their expected value under the unique stationary measure μ^η. The large deviation principle provides a measure for the speed of convergence.

Definition B.2 *A sequence $\{M_t\}_{t\in\mathbb{N}}$ of random variables defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfies a large deviations principle with speed $a_t \uparrow \infty$ $(t \to \infty)$ and rate function I if*

$$\limsup_{t\to\infty} \frac{1}{a_t} \log \mathbb{P}\left[|M_t| \in F\right] \leq -\inf\{I(u) : u \in F\}$$

for any closed set F and

$$\liminf_{t\to\infty} \frac{1}{a_t} \log \mathbb{P}\left[|M_t| \in U\right] \geq -\inf\{I(u) : u \in U\}$$

for any open set U. The sequence $\{M_t\}_{t\in\mathbb{N}}$ satisfies a moderate deviations principle with speed $a_t = o(t)$ and rate function $\hat{I}$ if the sequence $\left\{\sqrt{\frac{t}{a_t}} M_t\right\}_{t\in\mathbb{N}}$ satisfies a large deviations principle with speed a_t and rate function $\hat{I}$.

If the sequence of random variables $\{M_t\}_{t\in\mathbb{N}}$ satisfy a moderate deviations principle with speed a_t and rate function $\hat{I}$, then

$$\mathbb{P}\left[|M_t| \geq \sqrt{\frac{a_t}{t}}\right] = \mathbb{P}\left[\sqrt{\frac{t}{a_t}}|M_t| \geq 1\right] \leq e^{-a_t \inf\{\hat{I}(u):|u|\geq 1\}} \tag{52}$$

for all sufficiently large $t \in \mathbb{N}$. While the rate function associated to a large deviations principle can only rarely be given in closed form, the rate function $\hat{I}$ associated to a moderate deviations principle typically has a simpler structure. If, for instance, $\{M_t\}_{t\in\mathbb{N}}$ is driven by an underlying Markov process, we often have that

$$\inf\{\hat{I}(u) : |u| \geq 1\} = \frac{1}{2c} \quad \text{for some} \quad c > 0.$$

B.3.1 A uniform moderate deviations principle for empirical averages

In the sequel we derive a *uniform* moderate deviations principle for the random variables

$$M_t^\eta := \varrho_t^\eta(f) - \mu^\eta(f) \tag{53}$$

for a bounded Lipschitz continuous function f.

Proposition B.3 *There exists $c > 0$ such that for any compact set D of initial values*

$$\limsup_{t\to\infty} \frac{1}{\sqrt{t}} \log \sup_{\eta, x\in D} \mathbb{P}_x^\eta \left[|M_t^\eta| \geq t^{-1/4}\right] \leq -\frac{1}{2c}. \tag{54}$$

As an immediate corollary we then obtain that uniformly in all the possible market shares the random variable $M_{t^4}^\eta$ exceeds the values t^{-1} only finitely often.

Corollary B.4 *For any compact set of initial values D and all $\eta \in [\underline{\eta}, \overline{\eta}]$, we have that*

$$\mathbb{P}_x^\eta \left[|M_{t^4}^\eta| \geq \frac{1}{t} \text{ infinitely often} \right] = 0.$$

Proof: The assertion follows from the Borel-Cantelli Lemma because

$$\sup_{\eta, x\in D} \mathbb{P}_x^\eta \left[|M_{t^4}^\eta| \geq \frac{1}{t}\right] \leq e^{-\frac{t^2}{2c}}$$

for all sufficiently large t and because $\sum_{t\geq 0} e^{-\frac{t^2}{2c}} < \infty$. □

The proof of Proposition B.3 requires some preparation. From

$$M_t^\eta \leq \sqrt{N} \sum_{j=0}^{N-1} |M_t^{\eta,j}| \quad \text{where} \quad M_t^{\eta,j} := \frac{1}{\sqrt{[t/N]}} \sum_{i=0}^{[t/N]} \left(f(X_{Ni+j}^\eta) - \mu^\eta(f)\right)$$

we obtain

$$\left\{|M_t^\eta| \leq t^{-1/4}\right\} \supset \left\{|M_t^{\eta,j}| \leq \frac{1}{N^{3/2}} t^{-1/4} \text{ for } j = 1, 2, \ldots, N-1\right\}.$$

As a result, it suffices to prove Proposition B.3 for the Markov chain $\{X_{Nt}^\eta\}_{t\in\mathbb{N}}$. In view of the discussion at the end of Section A.2 we may as well assume that the Markov chains X^η are contractions uniformly in $\eta \in [\underline{\eta}, \overline{\eta}]$, i.e., that $\sup_\eta \|A(\eta)\| < 1$. It then follows from Worms (1999, Thm. 2), that the sequence

$$\left\{\sqrt{\frac{t}{a_t}} M_t^\eta\right\}_{t\in\mathbb{N}}$$

satisfies a large deviation principle with speed $\{a_t\}_{t\in\mathbb{N}}$ if $a_t = o(t)$. Furthermore, the rate function $\hat{I}^\eta$ can be given in closed form:

$$\hat{I}^\eta(u) = \sup_\theta \left\{ u\theta - \frac{1}{2}\theta^2 c_\eta \right\} = \frac{1}{2}\frac{u^2}{c_\eta}. \tag{55}$$

The constant c_η is given in terms of the solution G_η to the *Poisson equation* associated to f and the transition operator Π_η of the Markov chain X^η. More specifically, there exist functions G_η which solve

$$f - \mu^\eta(f) = G_\eta - \Pi_\eta G_\eta, \tag{56}$$

and Lipschitz continuity of f yields Lipschitz continuity of G_η and $G_\eta - \Pi_\eta G_\eta$ with the *same* constant, cf. Duflo (1997, Chap. 6). The normalized functions $G_\eta(x) - G_\eta(0)$ also satisfy the Poisson equation (56) and share the same Lipschitz constant. Thus, we may with no loss of generality assume that $G_\eta(0) = 0$. In this case, the functions G_η are uniformly bounded and equicontinuous because, for any compact set D of initial values, the processes X^η may be viewed as a Markov processes on a common compact state space. In terms of G_η the constant c_η and the random variable M_t^η are given by, respectively,

$$c_\eta = \int \left[G_\eta^2 - (\Pi_\eta G_\eta)^2\right] d\mu^\eta = \mathbb{E}_{\mu^\eta}\left[G_\eta^2 - (\Pi_\eta G_\eta)^2\right] \tag{57}$$

and

$$M_t^\eta = \frac{1}{t}\sum_{s=1}^{t}\left[G_\eta(X_s^\eta) - \Pi_\eta G_\eta(X_{s-1}^\eta)\right] + \frac{1}{t}G_\eta(X_0) - \frac{1}{t}G_\eta(X_t^\eta). \tag{58}$$

Since all the functions G_η are uniformly bounded, the constants c_η are uniformly bounded, too, i.e.,

$$c := \sup\left\{c_\eta : \underline{\eta} \le \eta \le \overline{\eta}\right\} < \infty. \tag{59}$$

Moreover, if $\{\mathcal{F}_t^\eta\}_{t\in\mathbb{N}}$ denotes the filtration generated by the Markov chain X^η, then

$$\mathbb{E}_x^\eta\left[G_\eta(X_t^\eta) - \Pi_\eta G_\eta(X_{t-1}^\eta) | \mathcal{F}_{t-1}^\eta\right] = 0.$$

Thus, the deviation of empirical averages from their expected values under the stationary measure can be described in terms of a *martingale difference* sequence; see Duflo (1997, Theorem 6.3.20) for details.

Remark B.5 *The Markov property of X^η implies that*

$$\begin{aligned}\mathbb{E}_x^\eta\left[G_\eta(X_{t+j}^\eta)\Pi_\eta G_\eta(X_{t+j-1}^\eta)|\mathcal{F}_j^\eta\right] &= \mathbb{E}_{X_j^\eta}^\eta\left[G_\eta(X_t^\eta)\Pi_\eta G_\eta(X_{t-1}^\eta)\right] \\ &= \mathbb{E}_{X_j^\eta}^\eta\left[\mathbb{E}_{X_j^\eta}^\eta\left[G(X_t^\eta)\Pi_\eta G_\eta(X_{t-1}^\eta)|X_{t-1}^\eta\right]\right] \\ &= \mathbb{E}_{X_j^\eta}^\eta\left[(\Pi_\eta G_\eta)^2(X_{t-1}^\eta)\right] \\ &= \mathbb{E}_x^\eta\left[(\Pi_\eta G_\eta)^2(X_{t+j-1}^\eta)|\mathcal{F}_j^\eta\right].\end{aligned}$$

From this we obtain

$$\mathbb{E}_x^\eta\left[G_\eta^2(X_{t+j}^\eta)-(\Pi_\eta G_\eta)^2(X_{t+j-1}^\eta)|\mathcal{F}_t^\eta\right] = \mathbb{E}_x^\eta\left[\left(G_\eta(X_{t+j}^\eta)-\Pi_\eta G_\eta(X_{t+j-1}^\eta)\right)^2|\mathcal{F}_t^\eta\right].$$

Since μ^η is the stationary distribution for X^η, a similar argument shows that

$$c_\eta = \mathbb{E}_{\mu^\eta}\left[G_\eta^2(x_j^\eta)-(\Pi_\eta G_\eta)^2(x_{j-1}^\eta)\right] = \mathbb{E}_{\mu^\eta}\left[G_\eta(x_j^\eta)-(\Pi_\eta G_\eta)(x_{j-1}^\eta)\right]^2.$$

These representations enable us to estimate the speed of convergence of empirical averages by means of a uniform moderate deviation principle for martingale difference sequences established in Gao (1996).

With the specific choice $a_t = \sqrt{t} = o(t)$, it follows from the results in Worms (1999) along with (52) and (59) that

$$\sup_x \mathbb{P}_x^\eta\left[|M_t^\eta| \geq \frac{1}{t^{1/4}}\right] \leq e^{-\frac{1}{2c}\sqrt{t}} \quad \text{for all} \quad t \geq T_\eta. \tag{60}$$

For our purposes we need the latter estimate to be uniform in all the possible market shares. To this end, recall first that the stationary solution x^η may be viewed as the Markov chain X^η with initial value x_0^η defined in (28). The arguments given in the proof of Proposition 3.3 show that x_0^η is bounded uniformly in $\eta \in [\underline{\eta}, \overline{\eta}]$. In view of Corollary A.3 (ii) this yields a constant C depending on D, but not on η such that

$$\sup_{x\in D}|X_t^\eta - x_t^\eta| \leq C\alpha^t$$

almost surely. Since the functions $G_\eta - \Pi_\eta G_\eta$ are uniformly bounded and uniformly Lipschitz continuous, so are G_η^2 and $(\Pi_\eta G_\eta)^2$. Thus, in view of Remark B.5 there exists a constant $L < \infty$ such that

$$\begin{aligned}&\left|\mathbb{E}_x^\eta\left[\left(G_\eta(X_{t+j}^\eta)-(\Pi_\eta G_\eta)(X_{t+j-1}^\eta)\right)^2|\mathcal{F}_t^\eta\right]-c_\eta\right| \\ =\ &\left|\mathbb{E}_x^\eta\left[G_\eta^2(X_{t+j}^\eta)-(\Pi_\eta G_\eta)^2(X_{t+j-1}^\eta)|\mathcal{F}_t^\eta\right]-\mathbb{E}_{\mu^\eta}\left[G_\eta^2(x_j^\eta)-(\Pi_\eta G_\eta)^2(x_{j-1}^\eta)\right]\right| \\ \leq\ & L\alpha^{-(j-1)}\end{aligned}$$

uniformly in all the possible market shares and in $x \in D$. This yields

$$\lim_{j\to\infty, \frac{j}{T}\to 0} \sup_{\eta, x\in D} \left\| \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}_x^\eta \left[\left(G_\eta(X_{t+j}^\eta) - \Pi_\eta G_\eta(X_{t+j-1}^\eta) \right)^2 | \mathcal{F}_t^\eta \right] - c_\eta \right\|_{L^\infty(\mathbb{P}_x^\eta)} = 0.$$

Hence it follows from Theorem 1.1 in Gao (1996) that

$$\limsup_{T\to\infty} \frac{1}{\sqrt{T}} \log \sup_{\eta, x\in D} \mathbb{P}_x^\eta \left[\frac{1}{T^{3/4}} \left| \sum_{t=1}^{T} G_\eta(X_t^\eta) - \Pi_\eta G_\eta(X_{t-1}^\eta) \right| \geq 1 \right] \leq -\frac{1}{2c}.$$

In view of (58) and because the functions G_η are uniformly bounded we obtain

$$\limsup_{T\to\infty} \frac{1}{\sqrt{T}} \log \sup_{\eta, x\in D} \mathbb{P}_x^\eta \left[|M_T| \geq T^{-1/4} \right] \leq -\frac{1}{2c}.$$

This finishes the proof of Proposition B.3.

B.3.2 Bounding error terms by a uniform moderate deviations principle

Our uniform large deviations principle allows us to prove that the error terms β_n in the representation (47) of the sequence $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ vanish sufficiently fast. Specifically, we will show that almost surely $|\beta_n| > \frac{1}{n}$ only finitely often.

Lemma B.6 *We have that*

$$\mathbb{P}\left[\sum_{n\geq 0} \frac{t_n}{T_n + t_n} |\beta_n| < \infty \right] = 1. \tag{61}$$

Proof: Recall that $\{\overline{\varrho}_t^T\}_{t\geq T}$ denotes the sequence of empirical distributions of a process $\overline{X}^T$ starting at time T in X_T which evolves according to the linear recursive relation (13) with fixed market shares η_T. Write

$$\overline{z}_t^T := \left(\overline{\varrho}_t^T(f^1), \ldots, \overline{\varrho}_t^T(f^L) \right)$$

and recall (45) to get

$$\beta_n = \left(z_{T_n}^{t_n} - \overline{z}_{t_n}^{T_n} \right) + \left(\overline{z}_{t_n}^{T_n} - \zeta(\eta_{T_n}) \right).$$

The decomposition of β_n into two terms can now be estimated separately.

(i) Independence of the random variables η_{T_n} and $\varepsilon_{T_n+1}, \varepsilon_{T_n+2}, \ldots$ yields

$$\mathbb{P}_x\left[\left|\overline{z}_{t_n}^{T_n} - \zeta(\eta_{T_n})\right| \geq \frac{1}{n}\right] \quad = \quad \mathbb{P}_{X_{T_n}}^{\eta_{T_n}}\left[\left|z_{t_n}^{\eta_{T_n}} - \zeta(\eta_{T_n})\right| \geq \frac{1}{n}\right].$$

Given an initial value x, the process $\{X_t\}_{t\in\mathbb{N}}$ is almost surely bounded, due to Proposition 3.3. In particular, there exists a compact set D such that

$$\mathbb{P}_x\left[X_t \in D \text{ for all } t \in \mathbb{N}\right] = 1.$$

Thus, Proposition B.3 applied to the Lipschitz continuous functions f^i ($i = 1, 2, \ldots, L$) yields a constant $N \in \mathbb{N}$ such that

$$\begin{aligned}\mathbb{P}_x\left[\left|\overline{z}_{t_n}^{T_n} - \zeta(\eta_{T_n})\right| \geq \frac{1}{n}\right] \quad &\leq \quad \sup_{x\in D}\sup_{\eta\in[\underline{\eta},\overline{\eta}]} \mathbb{P}_x^{\eta}\left[\max_{i=1,2,\ldots,L}\left|\varrho_{t_n}^{\eta}(f^i) - \mu^{\eta}(f^i)\right| \geq \frac{1}{n}\right] \\ &\leq \quad Le^{-\frac{1}{2c}n}\end{aligned}$$

for all $n \geq N$. By the Borel-Cantelli Lemma

$$\mathbb{P}_x\left[\left|\overline{z}_{t_n}^{T_n} - \zeta(\eta_{T_n})\right| \geq \frac{1}{n} \text{ infinitely often }\right] = 0,$$

and so

$$\sum_{n\geq 1}\frac{t_n}{T_n + t_n}\left|\overline{z}_{t_n}^{T_n} - \zeta(\eta_{T_n})\right| < \infty \qquad \mathbb{P}_x\text{-a.s.}$$

(ii) We can apply similar arguments as in the second and third part of the proof of Theorem 1.4 to deduce that

$$\begin{aligned}\left|X_{T_n+t_n} - \overline{X}_{T_n+t_n}^{T_n}\right| \quad &\leq \quad \alpha_0^{\left[\frac{t_n}{N}\right]}\sup_{0\leq k\leq N}\left|X_{T_n+k} - \overline{X}_{T_n+k}^{T_n}\right| + C_N \sup_{T_n\leq t\leq T_n+t_n}\left|\eta_t - \eta_{T_n}\right| \\ &\leq \quad \alpha_0^{\left[\frac{t_n}{N}\right]}\sup_{0\leq k\leq N}\left|X_{T_n+k} - \overline{X}_{T_n+k}^{T_n}\right| + \bar{C}\frac{t_n}{T_n + t_n}\end{aligned}$$

for some $\bar{C} < \infty$. Here the second inequality follows from (20). Since the random variable $\sup_{0\leq k\leq N}\left|X_{T_n+k} - \overline{X}_{T_n+k}^{T^n}\right|$ is almost surely bounded by some constant that depends only on the starting point of X, we see that

$$\left|X_{T_n+t_n} - \overline{X}_{T_n+t_n}^{T_n}\right| = O\left(n^{-1}\right) \qquad \mathbb{P}_x\text{-a.s.}$$

Hence uniform continuity of the maps $f^1, \ldots, f^l$ and the choice function F yields

$$\left|\overline{z}_{t_n}^{T_n} - z_{t_n}^{\eta_{T_n}}\right| = O\left(n^{-1}\right) \qquad \mathbb{P}_x\text{-a.s.}$$

and so

$$\sum_{n\geq 1}\frac{t_n}{T_n + t_n}\left|\overline{z}_{t_n}^{T_n} - z_{t_n}^{\eta_{T_n}}\right| < \infty \qquad \mathbb{P}_x\text{-a.s.}$$

□

C Heterogeneous risk aversion

Let us briefly outline the case in which mediators have heterogeneous risk aversions $\alpha^{(1)} \neq \alpha(2)$. We will show that only the inhomogeneous part of the system (6) will change so that the results of the paper will not be affected. Inserting the two forecasting rules (2) and (3) into the temporary equilibrium map (1), we see that the covariance matrix of the asset prices is given by

$$\mathbb{V}_{t-1}[p_t] = \Gamma_t \mathbb{V}_\varepsilon \Gamma_t \quad (t \in \mathbb{N}).$$

To obtain correct subjective second moments, choose an initial symmetric positive matrix V_0 which is similar to $\mathbb{V}_\varepsilon$, i.e., diagonalizable by the same coordinate transformation so that $V_0 \mathbb{V}_\varepsilon^{-1} = \mathbb{V}_\varepsilon^{-1} V_0$. To form second moment beliefs V_t, choose the forecasting rule

$$V_t := (1+r)\left(\tfrac{\eta_t}{\alpha^{(1)}} + \tfrac{1-\eta_t}{\alpha^{(2)}}\right)\sqrt{\mathbb{V}_\varepsilon^{-1} V_{t-1}} \quad (t \in \mathbb{N}), \tag{62}$$

where $\sqrt{B}$ denotes the quare root of a symmetric positive definite matrix B. Then

$$\Gamma_t = \left[(1+r)\left(\tfrac{\eta_t}{\alpha^{(1)}} + \tfrac{1-\eta_t}{\alpha^{(2)}}\right)\right]^{-1} V_t$$

and it is straightforward to verify that

$$\mathbb{V}_{t-1}[p_t] = V_{t-1} \quad (t \in \mathbb{N}),$$

implying that both mediators are able to correctly predict second moments of the price process. Thus (62) is a perfect forecasting rule for second moments in the sense of Wenzelburger (2004). The resulting process of prices and forecasts then takes the form

$$\begin{cases} p_t &= q_{t-1}^{(2)} + \sqrt{\mathbb{V}_\varepsilon^{-1} V_{t-1}}\big(\varepsilon_t - \mathbb{E}_{t-1}[\varepsilon_t]\big), \\ q_t^{(1)} &= \sum_{j=1}^{J} D^{(j)} p_{t-j}, \\ q_t^{(2)} &= \frac{(1+r)\eta_t}{1-\eta_t}\frac{\alpha^{(2)}}{\alpha^{(1)}} q_{t-1}^{(2)} - \frac{\eta_t}{1-\eta_t}\frac{\alpha^{(2)}}{\alpha^{(1)}} q_t^{(1)} + \frac{(1+r)\eta_t}{1-\eta_t}\frac{\alpha^{(2)}}{\alpha^{(1)}}\sqrt{\mathbb{V}_\varepsilon^{-1} V_{t-1}}\,\big(\bar{x} - \mathbb{E}_{t-1}[\varepsilon_t]\big), \\ V_t &= (1+r)\left(\frac{\eta_t}{\alpha^{(1)}} + \frac{1-\eta_t}{\alpha^{(2)}}\right)\sqrt{\mathbb{V}_\varepsilon^{-1} V_{t-1}} \end{cases} \tag{63}$$

with compared to (5) has an additional fourth describing the updating of second moment beliefs V_t, as given by (62). Notice that, apart from the market shares, this equation decouples from the other equations in (63). Hence (63) takes the form (6), where the inhomogeneous term $B(\eta_t, \varepsilon_t)$ has to be changed to some suitable term $B(\eta_t, \varepsilon_t, V_{t-1})$ which, in addition, depends on subjective second moments. If we can show that $\{B(\eta_t, \varepsilon_t, V_{t-1})\}_{t\in\mathbb{N}}$ is bounded, all results of the paper hold for the case with heterogeneous risk aversion. Observe to this end that all $\{V_t\}_{t\in\mathbb{N}}$ are diagonalizable by the same coordinate transformation. Since

$$0 < \frac{1}{\max\{\alpha^{(1)},\alpha^{(2)}\}} \leq \frac{\eta_t}{\alpha^{(1)}} + \frac{1-\eta_t}{\alpha^{(2)}} \leq \frac{1}{\min\{\alpha^{(1)},\alpha^{(2)}\}} \quad (t \in \mathbb{N}),$$

it follows from the definition of the square root of a matrix that all $\{V_t\}_{t\in\mathbb{N}}$ remain bounded and symmetric positive definite for all times $t \in \mathbb{N}$. This proves our claim.

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