Debt Dilution and Maturity Structure of Sovereign Bonds

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Abstract

We develop a dynamic model of sovereign default and renegotiation to study how expectations of default and debt restructuring in the near future affect the ex ante maturity structure of sovereign debts. This paper argues that the average maturity is shorter when a country is approaching financial distress due to two risks: default risk and "debt dilution" risk. Long-term yield is generally higher than short-term yield to reflect the higher default risk incorporated in long-term debts. When default risk is high and long-term debt is too expensive to afford, the country near default has to rely on short-term debt. The second risk, "debt dilution" risk, is the focus of this paper. It arises because there is no explicit seniority structure among different sovereign debts, and all debt holders are legally equal and expect to get the same haircut rate in the post-default debt restructuring. Therefore, new debt issuances around crisis reduce the amount

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that can be recovered by existing earlier debt-holders in debt restructuring, and thus "dilute" existing debts. As a result, investors tend to hold short-term debt which is more likely to mature before it is "diluted" to avoid the "dilution" risk. Model features non-contingent bonds of two maturities, endogenous default and endogenous hair cut rate in a debt renegotiation after default. We show that "debt dilution" effect is always present and is more severe when default risk is high. When default is a likely event in the near future, both default risk and "dilution" risk drive the ex ante maturity of sovereign debts to be shorter. In a quantitative analysis, We try to calibrate the model to match various features of the recent crisis episode of Argentina. In particular, we try to account for the shifts in maturity structure before crisis and the volatility of long-term and short-term spreads observed in the prior default episode of Argentina.

Keywords: Maturity Structure, Debt Dilution, Sovereign Default, Debt Renegotiation

1 Introduction

The last decade has witnessed recurrent large-scale sovereign debt crises in many emerging markets, and most of them were resolved by debt renegotiations after default. These observances have aroused much interest in how composition and maturity structure of sovereign debts affect a country's default probability and debt renegotiation outcome[1].

However, few have studied this problem the other way around. That is, when

[1] For example, many authors (Cole and Kehoe (1996), Sachs, Tornell, and Velasco (1996), Furman and Stiglitz (1998), etc.) have argued that excessive reliance on short-term debt increases a country's vulnerability to sudden capital reversals and liquidity crisis, and affects the depth of crisis when it happens.

an emerging market is near financial distress, how do expectations of future default and debt renegotiations affect sovereign debt composition and maturity structure ex ante? This paper aims to answer this question and in particular, we study the effects of expected future default and debt restructuring on the ex ante maturity structure of sovereign debts.

It is already a well documented fact that the maturity structure of emerging market debt issuances correlates with their domestic conditions. That is, emerging markets issue long-term debts more in tranquil times, and issue short-term debts more when they are near crisis. Long-term spread is generally higher than short-term spread and this difference increases as the country approaches crisis (Broner, Lorenzoni and Schumukler (2005)). This paper constructs a dynamic model of sovereign borrowing, default and renegotiation to explain why expectations of default and debt restructuring in the near future drive the ex ante average debt maturity to be shorter. In this model, we emphasize two risks that affect debt maturity structure: default risk and "debt dilution" risk. Default risk comes from the well-known willingness-to-pay problem and long-term debts usually bear higher default risk than short-term debts, since the latter are more likely to mature before crisis actually happens. Therefore, long-term spreads are generally higher than short-term spreads and the differences are even larger when default probability is high. Long-term debts can be too expensive to afford when a country is around crisis, and the country has to rely on short-term debts.

In addition to default risk, we go further and analyze another risk, which has not been studied much in the literature: "debt dilution" risk. "Debt dilution" risk arises when default is resolved by debt restructuring in an environment without explicit seniority structure among different types of sovereign debts[2]. In such an environment,

[2]Although there are no legally binding priority rules, most sovereigns do respect a number of informal rules. Debt from creditors like the IMF, the World Bank, and other multilateral development

most sovereign debts rank as legally equal or, *pari passu*, and all debt holders expect to get the same haircut rate during the post-default debt restructuring. Thus, new debt issuances before crisis reduce the amount that can be recovered by existing debt-holders in a debt renegotiation in case of default, and hence, "dilute" existing debts. That is, each new debt issuance incurs a potential capital loss to existing debt holders. This "debt dilution" effect is always present whenever a country issues new debt, but it becomes a main concern to investors only when default risk is high and debt restructuring is a likely event. Country around financial distress has incentive to issue large amount of new debts in order to postpone or to avoid crisis, and it is able to do so to some extent, since new creditors will not charge prohibitive interest rates given that they can effectively obtain a share of the existing creditors debt recovery value. As a result, existing debts can be "diluted" intensively when the country is around crisis. In order to forestall debt dilution, investors tend to hold short-term debt when crisis is around the corner.

In the model, a risk-averse country and risk-neutral competitive international investors trade short-term and long-term bonds. Facing a stochastic endowment stream, the country chooses to repay or to default optimally. Default results in exclusion from international capital markets and proportional output loss, but it can be resolved by debt renegotiation between the country and its debt holders. If agreement of debt reduction is reached, all debt holders get the same haircut rate and by repaying the reduced amount of debt, the country regains access to capital markets. The endogenously determined haircut rate affects the country's ex ante default probability. And expected default probability and debt haircut rate together

banks (MDBs) almost always has *de facto* seniority, in part because these international financial institutions (IFIs) usually refinance their maturing debt rather than demand full payment after a default. Trade credits also enjoy *de facto* seniority. In this paper, however, we focus on privately held sovereign bonds, among which there are no formal or informal priority rules.

affect the ex ante average maturity and bond spreads of different maturities.

We analytically characterize the model equilibrium and establish that "debt dilution" effect is always present and is most severe when default risk is high. And when default is a likely event, both default risk and "dilution" risk drive the ex ante maturity of sovereign debts to be shorter. In a quantitative analysis, We try to calibrate the model to match various features of the recent crisis episode of Argentina. In particular, we try to account for the shifts in maturity structure before crisis and the volatility of long-term and short-term spreads observed in the prior default episode of Argentina.

This paper builds on several strands of literature. One strand of literature studies the impacts of debt renegotiation in event of sovereign default. Bulow and Rogoff (1989b) present a model with continuous debt renegotiation, through which direct sanctions can be lifted. Yue (2005) models debt renegotiation explicitly and characterize the endogenously determined debt recovery schedule. Our paper studies the impacts of debt renegotiation from a different perspective and analyzes how debt renegotiations affects ex ante debt maturity structure.

The second strand of literature is on seniority structure and debt dilution effect. Debt dilution problem was initially addressed in corporate finance literature by Fama and Miller (1972). White (1980) and Schwartz (1989) then explore the optimal seniority structure in the corporate debt context. Hart (1995), Hart and Moore (1995) argues that debt dilution problem in corporate finance arises mainly due to the agency problem, but not the absence of explicit priority rules, since seniority structure does exist in corporate debts by contract or statute. In sovereign debt context, the role of seniority structure has been analyzed by Detragiache (1994), Roubini and Setser (2004), among others. Dooley (2000) and Saravia (2003) study the conflict between official and private lenders in the competition for repayments. Formal studies on debt dilution effect, however, are relatively underdeveloped. Cohen (1991) presents a

3-period model of sovereign debt dilution and notes the resulting inefficiency. Bolton and Jeanne (2004) is closely related to this paper and they argue that debt dilution problem led to the shift in sovereign debt composition from bank loans to bonds from 1980s to 1990s. However, the above papers on debt dilution problem are based on a static one-shot borrowing framework. Therefore, a country's consideration for its future access to capital markets and consumption smoothing plays no role in the renegotiation. Our paper improves on this point by incorporating endogenous default and renegotiation into an infinite-horizon dynamic model and studies their impacts on ex ante debt maturity structure.

This paper is also related to Arellano (2005) and Broner, Lorenzoni and Schumukler (2005), both of which study the optimal maturity structure of sovereign debts. Arellano (2005) focuses on the default risk and analyzes the role of long-term borrowing. While this paper focuses on an additional risk: "debt dilution" risk, which interacts with default risk and both of them affect ex ante maturity structure. Broner, Lorenzoni and Schumukler (2005) places more emphasis on the lender's side. They assume risk-averse lenders and argue that short-term debts do not include compensation for varying short rate when lenders face liquidity needs, and thus short-term debts are cheaper. That's why emerging markets borrow short term around crisis. Our framework focuses on the borrower's side and bond spreads in this model reflect the endogenous default probability and debt recovery rate.

The remainder of the paper is organized as follows. In the next section, the model environment is described and sovereign country's problem, renegotiation problem and investors' problem are discussed in details in three subsections. We then define the model equilibrium and characterize the equilibrium properties in section 3. Section 4 provides our plan of model calibration and quantitative analysis. The proofs are in the Appendix.

2 The Model Environment

Model features 2 types of agents: a small open economy and infinite number of international investors. In each period, the economy receives a stochastic stream of non-storable consumption goods y_t. The stochastic endowment y_t is drawn from a compact set Y, and $\mu(y_t|y_{t-1})$ is the probability distribution function of a shock y_t conditional on the previous realization y_{t-1}. The sovereign government of this economy is risk averse and aims to maximize the expected lifetime utility of a representative domestic resident. The preference of the sovereign government is given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

where $0 < \beta < 1$ is the discount factor, and c_t denotes the consumption in period t. The period utility function $u(c_t)$ is continuous, strictly increasing, strictly concave, and satisfies the Inada conditions.

International investors are risk neutral and behave competitively on the international capital markets. They can borrow whatever amount they want in the international capital markets at the world risk-free interest rate r, which we assume to be a constant. And their borrowing and lending cannot affect the risk-free interest rate r. Investors have perfect information on the country's asset holdings and endowment streams. When the sovereign government issues bonds, an investor will be randomly chosen to trade with the government.

Capital markets are incomplete. The sovereign government and international investors can only trade non-contingent zero-coupon bonds with short (one-period) and long (two periods) maturities. The face value of a discount bond issued in period t and maturing in period s is denoted as b_s^t, which is the amount to be repaid in period s, and s can be $t+1$ or $t+2$. If b_s^t is positive, then it's a saving by the government; if it's negative, then it's a borrowing from investors. The price of a bond with face value b_s^t is denoted as q_s^t, which is a function of current endowment y_s, bond face value b_s^t and the value of other existing bonds. Bond prices will be determined in equilibrium and the explicit price function will be described in details later.

Furthermore, we assume that investors always commit to repay their debts, while the sovereign government can choose to default on its debts rather than repay in full, whenever the former generates higher expected lifetime utility. We assume that once the government defaults, it defaults on all existing debts. Default is costly in two ways: one is that when the country is in default, it suffers a proportional output loss, γy, since defaulting country may not obtain advanced technology, direct investment, or foreign aid from other countries, which reduces its output[3]; the other cost is that default incurs exclusion from international capital markets, and thus the country cannot save or borrow in capital markets while it's in default[4]. However, financial exclusion in this model can be temporary instead of being permanent as in Eaton and Gersovitz (1981). The defaulting country can regain access to the international capital markets after debt restructuring. That is, it can renegotiate with its debt holders about a debt reduction[5]. Once renegotiation agreement has been reached and the government repays the reduced debt arrears in full, it can return to the international capital markets with a clean record. So in this model, regaining access to the international capital markets is endogenous, depending on the renegotiation process, the total amount of defaulted debt and the country's streams of output. Thus, this paper is distinct from models with an exogenous probability for the defaulting country to re-access capital markets, as studied in Arellano (2005) and Aguiar and Gopinath (2004). We use a discrete state variable $s = \{0, 1\}$ to denote the country's credit standing at the beginning of each period. If $s = 0$, it means that

[3]Reputation spillover analyzed in Cole and Kehoe (1998) also lead to output loss.

[4]This assumption can be rationalized if the creditors can seize the country's assets accumulated in the default periods, or the creditors can collude, as in Wright (2002).

[5]In the real life, debt restructurings can be quite complicated. Here for simplicity, we assume that debt restructuring takes its simplest form, debt reduction.

the country inherits a good credit standing from the last period, and it's current on its debt service. If $s = 1$, then the country is in default and inherits a bad credit standing from the last period. Then the government needs to renegotiate with debt holders in this period in order to settle its defaulted debt. If the government and the debt holders never reach agreement, then the country stays in autarky forever. In the next 3 subsections, we describe the sovereign government's problem, renegotiation process and investors' problem in details.

2.1 Sovereign government's Problem

At the beginning of period t, an endowment shock y_t realizes, and the country inherits a credit standing s_t and a set of existing assets B_t from the last period. B_t consists of 3 bonds: the long-term bond issued 2 periods ago, b_t^{t-2}, the short-term bond issued in the last period, b_t^{t-1}, and the long-term bond issued in the last period, b_{t+1}^{t-1}. Let $V(y_t, s_t, B_t)$ be the country's lifetime value function from period t on with current endowment y_t, credit standing s_t and set of existing assets B_t. The sovereign government makes decisions depending on the current states.

If $s = 0$, then the country has a good credit standing and the amount of maturing assets in this period is $(b_t^{t-2} + b_t^{t-1})$. When $b_t^{t-2} + b_t^{t-1} \geq 0$, the government won't choose to default since it has non-negative savings.[6] While if $b_t^{t-2} + b_t^{t-1} < 0$, that is, when the government has maturing debts, it chooses to repay or to default. And

[6] When both b_t^{t-2} and b_t^{t-1} are savings, it's obviously the case that the government doesn't default since it won't refuse to receive payments from international investors. However, when one of them is a saving and the other is a debt, one option might be the government defaults on its debt while getting returns from its savings. But this situation is ruled out in this model by assuming that once the government defaults, it defaults on all its assets. So when saving is enough to cover maturing debt, it's optimal for the government to repay debt and receive returns from its savings.

thus, its value function is

$$V(y_t, 0, B_t) = max\{V^R, V^D\} \tag{2}$$

where V^R is the value function if the government doesn't default (either when $b_t^{t-2} + b_t^{t-1} \geq 0$ or when $b_t^{t-2} + b_t^{t-1} < 0$ and the government chooses to repay). Then it becomes a standard consumption-saving problem. That is, the government repays its maturing debt (or receive net payments from international investors) and then decide how much short-term and long-term bonds to issue in this period. And then the government will start the next period with a good credit standing $s_{t+1} = 0$.

$$V^R(y_t, 0, B_t) = \max_{b_{t+1}^t, b_{t+2}^t} u(c_t) + \beta EV(y_{t+1}, 0, B_{t+1}) \tag{3}$$

$$\begin{aligned} s.t. \quad c_t &= y_t - q_{t+1}^t(y_t, b_{t+1}^t, B_t)b_{t+1}^t - q_{t+2}(y_t, b_{t+2}^t, B_t^t)b_{t+2}^t + b_t^{t-1} + b_t^{t-2} \\ B_{t+1} &= \{b_{t+1}^t, b_{t+1}^{t-1}, b_{t+2}^t\} \end{aligned}$$

where b_{t+1}^t and b_{t+2}^t are short-term bond and long-term bond issued in this period, respectively. $q_{t+1}^t(y_t, b_{t+1}^t, B_t)$ and $q_{t+2}^t(y_t, b_{t+2}^t, B_t)$ are prices corresponding to the short-term and long-term bonds. B_{t+1} is the set of existing assets at the beginning of period $t+1$.

If the government chooses to default, then its value function V^D is given by

$$V^D(y_t, 0, B_t) = u(c_t) + \beta EV(y_{t+1}, 1, B_{t+1}) \tag{4}$$

$$\begin{aligned} s.t. \quad c_t &= (1-\gamma)y_t \\ B_{t+1} &= (b_t^{t-1} + b_t^{t-2})(1+r) + b_{t+1}^{t-1} \end{aligned}$$

We assume that once the government defaults, it defaults on all its existing debts, including debts maturing today and debts maturing in the future. In addition, the defaulting country suffers a proportional output loss γy_t, and it cannot save or borrow while it is in default. Thus, consumption in this period is only $(1-\gamma)y_t$ and the country enters the next period with a bad credit standing and defaulted debt B_{t+1}. When the country is in default, B is no longer a vector of existing debts, instead, it's the present value of all defaulted debt.

If the country starts a period with a bad credit standing $s = 1$, it means that the government defaulted in some previous period and the defaulted debt has not been settled yet. Then in this period, the sovereign government negotiates with its debt holders for a debt reduction and tries to determine an endogenous haircut rate $(1 - \alpha(y, B)$. In other words, $|\alpha(y, B)B|$ is the amount of debt the sovereign government has to repay in order to settle its defaulted debt, according to the renegotiation agreement. Potentially, the defaulting country can choose to stay in autarky instead of initiating a debt renegotiation. In this model, staying in autarky corresponds to the case that the government always initiates a debt renegotiation but never agrees on any haircut rate. Since it is assumed that debt renegotiation incurs no cost to either the sovereign government or to the debt holders, and both parties are indifferent between participating or not participating in it. What matters here is whether or not an agreement can be reached and then be carried out. Therefore, we assume that defaulting country and its debt holders always participate in debt renegotiation but they can choose when to reach agreement and what the haircut rate is.

If agreement has been reached in the current period, then the country repays its reduced debt arrears according to the agreement, and the value function is

$$V(y_t, 1, B_t) = u((1-\gamma)y_t + \alpha_t(y_t, B_t)B_t) + \beta EV(y_{t+1}, 0, 0) \tag{5}$$

Thus, the country still suffers an output loss γy_t and repays reduced debt arrears $\alpha_t(y_t, B_t)B_t$. Then it enters the next period with a good credit standing and no debt arrear. $\alpha_t(y_t, B_t)$ is determined endogenously in the debt renegotiation, which is modeled explicitly in subsection 2.2.

In case that no agreement has been reached this period, both parties enter the next period and continue negotiation. Then the value function of the sovereign government is

$$V(y_t, 1, B_t) = u((1-\gamma)y_t) + \beta EV(y_{t+1}, 1, (1+r)B_t) \tag{6}$$

The country consumes $(1-\gamma)y_t$, cannot borrow or save, and enters the next period with still a bad credit standing and unpaid debt arrears $B_{t+1} = (1+r)B_t$.

Default is optimal for the sovereign government when $V^D \geq V^R$. So given debt position B, we can define the default set $y^D(B) \subseteq Y$. This is a set of endowment shocks under which default is optimal given debt position B.

$$y^D(B) = \{y \in Y : V^D(y, 0, B) \geq V^R(y, 0, B)\}$$

We can also define the probability of default $\theta(B)$ as the probability that the endowment shock falls into the default set given debt position B.

$$\theta(B_t) = Pr\{y_t \subseteq y^D(B_t)\}$$

Since B consists of both debts maturing in the current period and debt maturing in the future, the probability of default $\theta(B_t)$ is affected not only by the total stock of debt, but also by the composition of debt, i.e., how much maturing debt relative to the total stock of debt. Hence choices of maturity structure have important effect on the probability of default, and changes in the probability of default, in turn, affects choices of ex ante maturity structure.

2.2 Debt Renegotiation Problem

Once the sovereign government and its debt-holders enter the stage of debt renegotiation, they need to determine an endogenous debt recovery rate $\alpha(y, B) \in [0, 1]$ (or a haircut rate $(1 - \alpha(y, B)) \in [0, 1]$), given the current endowment shock y and defaulted debt B. We model this renegotiation problem using Nash Bargaining[7]. Because of the static nature of Nash Bargaining, the outcome of this bargaining game is either agreeing on a positive debt recovery rate immediately ($\alpha > 0$) or never reaching agreement ($\alpha = 0$)[8]. Never reaching agreement is the threat point of the game, in which case the country stays in autarky forever and its debt holders receive no repayment at all. Since there is no explicit seniority structure among different debt issues and thus all debt holders should be treated legally equally, we assume that debt holders all get the same haircut rate, and they can behave like a representative debt holder in the post-default renegotiation[9].

The reservation value for the country is to stay in autarky forever, which is given by

$$V^A(y_t) = E\sum_{i=t}^{\infty}\beta^{i-t}u((1-\gamma)y_i)$$

That is, the country faces a proportional output loss every period and has no access to capital markets. We denote the country's surplus in the Nash bargaining by Δ_B, which is the difference between the expected value of accepting some optimal positive debt recovery rate α_t and the expected value of rejecting it (and thus stays in autarky

[7]We assume Nash Bargaining mainly because it keeps the model tractable. And furthermore, equilibrium obtained in Nash Bargaining can be supported by more complicated and realistic game structures, such as the continuous bargaining Rubinstein game. Therefore, Nash Bargaining is a reasonable benchmark to model renegotiation problem.

[8]By using Nash Bargaining we cannot generate delays in reaching renegotiation agreement, which we always observed in the real life. While in this model, the focus is not to study delays occurring in debt renegotiation. All we need from debt renegotiation problem is the endogenous debt recovery rate α, and Nash Bargaining is enough to generate a close approximation of that debt recovery rate. My second project is focused on studying delays in debt renegotiation after default, where we use a more complicated and realistic game structure to model debt renegotiation.

[9]By that assumption, we rule out the strategic "hold-outs" behavior of creditors in the post-default debt renegotiation. This assumption is reasonable here, since the interests of all creditors are perfectly in line with each other.

forever).

$$\Delta_B(y_t, B_t, \alpha_t) = u((1-\gamma)y_t + \alpha_t B_t) + \beta EV(y_{t+1}, 0, 0) - E\sum_{i=t}^{\infty}\beta^{i-t}u((1-\gamma)y_i) \quad (7)$$

This surplus can be positive because by accepting a positive debt recovery rate, the country suffers output loss and being excluded from the international capital markets just temporarily, instead of permanently, as in the case of staying in autarky forever.

The reservation value for the representative debt holder is 0, i.e., the value of receiving no repayment. And thus, the surplus to the debt holder is the present value of recovered debt. Let Δ_L denotes the representative creditor's surplus, and it's given by

$$\Delta_L(y_t, B_t, \alpha_t) = -\alpha_t B_t \quad (8)$$

The debt holder has positive surplus as long as a positive debt recovery rate can be agreed upon. And how the total surplus is divided between the country and debt holders depend on their bargaining powers. The bargaining power parameter captures all institutional factors in the renegotiation in a simple way. The higher bargaining power one party has, the more surplus it can extract. We denote the country's bargaining power as $\phi \in [0, 1]$, and then the debt holder's bargaining power is $(1-\phi)$. The Nash Bargaining problem is given by

$$\alpha^*(y_t, B_t) = argmax \Delta_B(y_t, B_t, \alpha_t)^{\phi}\Delta_L(y_t, B_t, \alpha_t)^{1-\phi} \quad (9)$$

$$\begin{aligned} s.t. \quad \Delta_B(y_t, B_t, \alpha_t) &\geq 0 \\ \Delta_L(y_t, B_t, \alpha_t) &\geq 0 \end{aligned}$$

2.3 International Investors' Problem

International investors are risk neutral and behave competitively. In each period, if the sovereign government has a good credit standing and repays its maturing debts in full, it can trade with a randomly chosen international investor via one-period and two-period bonds. In period t, taking the bond price functions as given, the chosen international investor chooses the short-term and long-term bonds to maximize the expected profit.

$$\max_{b_{t+1}^t, b_{t+2}^t} E(\pi_t^s + \pi_t^l) \tag{10}$$

where π_t^s and π_t^l are profits from short-term and long-term bonds, respectively. More explicitly, $E(\pi_s)$ can be expressed as

$$E(\pi_t^s) = \begin{cases} q_{t+1}^t b_{t+1}^t - \frac{b_{t+1}^t}{1+r} & \text{if } b_{t+1}^t \geq 0 \\ q_{t+1}^t b_{t+1}^t - [\frac{1-\theta_{t+1}}{1+r} + \frac{\theta_{t+1} E(\alpha_{t+2})}{(1+r)^2}] b_{t+1}^t & \text{if } b_{t+1}^t < 0 \end{cases} \tag{11}$$

When $b_{t+1}^t \geq 0$, the sovereign government is the lender and there is no default risk since the investor always repays his debt. While in the other case, the sovereign government is the borrower and there is default risk. θ_{t+1} denotes the probability of default in period $t+1$. If the government defaults in period $t+1$, then it initiates debt renegotiation in period $t+2$. Because of the assumption of Nash Bargaining in debt renegotiation, agreement is always reached in period $t+2$. And thus, α_{t+2} denotes the agreed proportion of defaulted debt to be repaid in period $t+2$. The expression $[\frac{1-\theta_{t+1}}{1+r} + \frac{\theta_{t+1} E(\alpha_{t+2})}{(1+r)^2}]$ captures the expected proportion of repayment in present value terms. Similarly, expected profit from long-term bond $E(\pi_l)$ can be expressed as:

$$E(\pi_t^l) = \begin{cases} q_{t+2}^t b_{t+2}^t - \frac{b_{t+2}^t}{(1+r)^2} & \text{if } b_{t+2}^t \geq 0 \\ q_{t+2}^t b_{t+2}^t - \frac{E(H)}{(1+r)^2} b_{t+2}^t & \text{if } b_{t+2}^t < 0 \end{cases} \tag{12}$$

where $E(H)$ is the expected proportion of repayment.

$$H = \prod_{i=1}^{2} (1 - \theta_{t+i}) + \theta_{t+1} \alpha_{t+2} + \frac{(1 - \theta_{t+1}) \theta_{t+2} \alpha_{t+3}}{1 + r}$$

From the first order conditions, prices of short-term bonds can be expressed as:

$$q_{t+1}^t = \begin{cases} \frac{1}{1+r} & \text{if } b_{t+1}^t \geq 0 \\ \frac{1-\theta_{t+1}}{1+r} + \frac{\theta_{t+1} E(\alpha_{t+2})}{(1+r)^2} & \text{if } b_{t+1}^t < 0 \end{cases} \tag{13}$$

When $b^t_{t+1} \geq 0$, there is no default risk, and then the price of short-term sovereign bond is equal to that of a risk-free bond, $\frac{1}{1+r}$. When the country is the borrower $b^t_{t+1} < 0$, there exist risks of default and debt restructuring. Then the sovereign bond is priced to compensate the lenders for bearing both risks. By the same token, prices of long-term bonds can be expressed as:

$$q^t_{t+2} = \begin{cases} \frac{1}{(1+r)^2} & \text{if } b^t_{t+2} \geq 0 \\ \frac{E(H)}{(1+r)^2} & \text{if } b^t_{t+2} < 0 \end{cases} \tag{14}$$

3 Equilibrium

We define the recursive equilibrium of this model as follows:

Definition 1. *A* ***recursive equilibrium*** *is a list of allocations for (i) consumption $c(y, s, B)$, short-term bond holdings $b^1(y, B)$, long-term bond holdings $b^2(y, B)$, default set $y^D(B)$; (ii) pricing function for short-term bonds $q^1(y, b^1, B)$ and pricing function for long-term bonds $q^2(y, b^2, B)$; (iii) debt recovery rate $\alpha(y, B)$ such that:*

1. *Taking as given the short-term and long-term bonds' pricing functions $q^1(y, b^1, B)$ and $q^2(y, b^2, B)$, as well as the debt renegotiation outcome $\alpha(y, B)$, the country's asset holdings $b^1(y, B)$, $b^2(y, B)$, consumption $c(y, s, B)$ and default set $y^D(B)$ satisfy the sovereign government's optimization problem.*

2. *Given the renegotiation outcome $\alpha(y, B)$ and the sovereign country's optimal policy, the bond pricing functions $q^1(y, b^1, B)$ and $q^2(y, b^2, B)$ satisfies investors' maximization problem.*

3. *Given the bond pricing functions $q^1(y, b^1, B)$ and $q^2(y, b^2, B)$, the debt recovery rate $\alpha(y, B)$ solves the post-default debt renegotiation problem.*

3.1 Properties of Equilibrium

We first analyze the equilibrium debt recovery rate in case of default.

Proposition 1. *As long as the sovereign government's bargaining power* $\phi \in [0, 1)$*, the equilibrium debt recovery rate* $\alpha^*(y, B)$ *is always positive.*

Proof. See Appendix. ■

If the sovereign government has all the bargaining power, $\phi = 1$, then it will get complete debt reduction, and thus the equilibrium debt recovery rate $\alpha^* = 0$. As we have discussed above, when the country chooses to stay in autarky, α^* is also zero. However, the complete debt reduction case is different from that of staying in autarky, because in the former case, the country regain access to the capital markets without repaying anything, and complete debt reduction is a renegotiation agreement; while in the latter case, the country faces permanent proportional output loss and permanent exclusion from the capital markets, and $\alpha^* = 0$ is due to no agreement being achieved. So ex post, if the government has all the bargaining power in a debt renegotiation, it can experience complete debt relief. But ex ante, the extremely strong position of the country in debt renegotiation will greatly limit the country's ability to borrow, and the country's debt level cannot be higher than the expected proportional output loss in case of default. In the other extreme, when the representative debt holder has all the bargaining power, it will extract the total surplus and get such a large repayment that the county is indifferent between accepting it and rejecting it. However, we never observed the above two extreme cases happening. The more realistic case is that $\phi \in (0, 1)$, that is both parties have some bargaining powers and the equilibrium debt recovery rate is always positive. The debt holders obviously welcome a positive repayment after default, and the country is also willing to repay some of its defaulted debt in order to avoid permanent output loss and have the chance to smooth consumption in the future. Thus, when $\phi \in [0, 1)$, agreeing upon a positive debt recovery rate is mutually beneficial. This result shows that debt restructuring in case of default is a rational choice for both sovereign borrower and its debt holders, and thus allowing debt renegotiation in case of default is a more reasonable assumption than the traditional assumption of permanent exclusion after default.

Since the equilibrium debt recovery rate is determined endogenously in debt renegotiation, it is a function of current endowment shock and defaulted debt. The next proposition characterize the equilibrium amount of debt recovery.

Proposition 2. *For all y and B, there exists a threshold debt recovery value $\overline{B}(y)$, and the equilibrium amount of debt recovery satisfies*

$$\alpha^*(y,B)B = \begin{cases} \overline{B}(y) & \text{if } B \leq \overline{B}(y) \\ B & \text{if } B > \overline{B}(y) \end{cases}$$

Proof. See Appendix. ■

This proposition says when the size of defaulted debt is smaller than the threshold value $\overline{B}(y)$, the country has to repay all its defaulted debt even after debt renegotiation. While when the size of defaulted debt is larger than the threshold value, the country only needs to repay the threshold value $\overline{B}(y)$. This result is obtained because the sovereign government and the debt holders actually care about the absolute size of debt recovery, and due to Nash Bargaining, there is only one optimal value of debt recovery $\overline{B}(y)$ (interior solution) that maximizes the total surplus, and this optimal value of debt recovery is independent of defaulted debt. This proposition can be used to establish the existence of debt dilution effect, which is illustrated more clearly by the following corollary.

Corollary 1. *For all y and $B^1 \leq B^2 < 0$, the equilibrium debt recovery rate satisfies $\alpha^*(y,B^1) \leq \alpha^*(y,B^2)$.*

Proof. See Appendix. ■

This corollary is an immediate result from Proposition 2. It says that the recovery rate is (weakly) decreasing in the amount of defaulted debt. Or in simpler words, the more a sovereign government owes, the (weakly) smaller proportion the government needs to repay in a debt restructuring after default. This corollary demonstrates that the sovereign government enjoys larger haircut rate in the post-default debt restructuring if it defaults with higher level of debt, and this gives the government incentive to issue more debt when it is very near crisis. However, to the existing debt holders, each new debt issue reduces the debt recovery rate in the debt restructuring after default, and thus "dilutes" their debt holdings.

The debt dilution effect illustrated in corollary 1 is largely consistent with what we have observed in the recent sovereign bond exchanges. The following table shows the scale of debt crises and debt recovery rates for Ukraine, Pakistan, Ecuador, Russia, and Argentina. The debt recovery rate is lower for a higher level of defaulted debt, both in dollar amount and relative to the country's output. Thus the model prediction is in line with the empirical observations.

Country	Pakistan	Ukraine	Russia	Ecuador	Argentina
Time of default	Dec. 98	Sep. 98	Nov. 98	Aug. 99	Nov. 01
Defaulted debt (billion $)	0.75	2.7	73	6.6	82.3
Defaulted debt/output	0.10	0.09	0.12	0.46	0.32
Debt recovery rate	100%	100%	64%	60%	30%

Note: Data are from World Bank, Moody's (2003) and Yue(2005).

A last remark regarding results obtained in Proposition 2 is that the simple and clean form of debt recovery function is obtained due to the assumption of Nash Bargaining, as we have explained above[10]. We may get different debt recovery function if

[10] Yue(2005) obtained similar results by using Nash Bargaining, though the purpose of that paper

we use different bargaining structure. However, Nash Bargaining outcome can serve as a benchmark in which the debt dilution effect is the smallest. Because in Nash Bargaining, both parties are cooperative and maximize the total surplus, that is, the sovereign government also cares about the debt holders' surplus and vice versa. Note that the country's surplus function is concave and the debt holders' surplus function is linear, so the country needs to "sacrifice" a lot and choose a relatively big α^* in order to maximize the total surplus. As a result, the investors' debt holdings are not diluted too much under Nash Bargaining. While in more realistic bargaining structures, when agents are non-cooperative and only care about their own surpluses, the sovereign government may achieve higher surplus by choosing a smaller α^* in expense of its debt holders' surplus, and in that case the investors' debt holdings are diluted more severely.

Debt dilution effect is present whenever the sovereign government issues new debt, but it is a big concern only when default probability is very high. We then proceed to characterize factors that affect the default probability. Let B^M denote the amount of maturing debt and B^T the present value of total debt stock. More explicitly, in period t, $B_t^M = b_t^{t-1} + b_t^{t-2}$ and $B_t^T = b_t^{t-1} + b_t^{t-2} + \frac{b_{t+1}^{t-1}}{1+r}$.

Proposition 3. *Given the equilibrium debt recovery rate $\alpha^*(y, B)$, for $B^{1M} \leq B^{2M} < 0$ and $B^{1T} = B^{2T}$, if default is optimal for B^{2M} then default is also optimal for B^{1M}. For $B^{1M} = B^{2M}$ and $B^{1T} \leq B^{2T} < 0$, if default is optimal for B^{2T} then default is also optimal for B^{1T}.*

Proof. See Appendix. ■

The first half of the proposition predicts that given the same total level of debt, the sovereign government is more inclined to default with larger amount of maturing debt. Therefore, maturity structure matters here. For example, if the country doesn't

is not about debt dilution effect.

change the total amount of debt issues B^T, while increasing the ratio of short-term bonds to long-term bonds, then in the next period B^M is going to be higher though B^T doesn't change, and the default probability is higher. Hence, more reliance on short-term debt increases the probability of default. The second half of the proposition predicts that with the same amount of maturing debt, the government tends to default with larger level of total debt stock. This result is consistent with that obtained in existing literature (such as Eaton and Gersovitz (1981), Chatterjee et al. (2002), Arellano (2004) and Yue (2005)), though in those models there are only one-period bonds, and the result reduces to that default probability increases in the level of debt.

In summary, we have the following corollary.

Corollary 2. *Default probability is increasing in both B^M and B^T.*

Next, we compare the default probabilities in two situations: when post-default debt renegotiation is possible and when debt renegotiation is not possible, given the same amount of maturing debt and total debt.

Proposition 4. *Given the same B^M and B^T, if default is optimal when debt renegotiation is not possible ($\alpha^* = 0$), then default is also optimal when debt renegotiation is possible ($\alpha^* > 0$).*

Proof. See Appendix. ■

When post-default renegotiation is a possible option, the sovereign government is more inclined to default than in the case that default is always followed by staying in autarky forever. This is because permanent exclusion from the capital markets is a more severe penalty than debt restructuring in case of default. First, with debt restructuring, the defaulting country is only temporarily excluded from the capital markets and it's able to smooth consumption as soon as it repays the reduced amount of debt. Furthermore, the country suffers the output loss only when it's in default. While if the country has to stay in autarky forever after default, both output loss and exclusion from capital markets are permanent. Thus, by taking into consideration of the endogenous debt restructuring after default, this model can generate higher equilibrium default probability than existing sovereign debt models without debt renegotiation. While failure to generate equilibrium default probability comparable to data is one of the main weaknesses of existing sovereign debt models.

4 Quantitative Analysis (Incomplete)

4.1 Calibration

We plan to solve this model numerically to evaluate its quantitative predictions on the maturity structure of sovereign bonds with default risk and debt dilution risk. Parameters in the model are calibrated to match certain features of the sovereign debt of Argentina.

Utility function used in the quantitative analysis has constant relative risk-aversion (CRRA):

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

where σ is the risk aversion coefficient and it's set to 2, which is a commonly used value in real business cycle studies. We assume that the exogenous endowment stream follows an AR(1) process:

$$y_t = \overline{y} + \rho(y_{t-1} - \overline{y}) + \varepsilon_t, \quad \varepsilon \sim N(0, \sigma_\varepsilon^2)$$

ρ and σ_ε will be estimated to Argentine quarterly output (from MECON), which are de-trended and normalized such that $\overline{y} = 1$. The proportional output loss parameter γ is set to 2% , which is consistent with that estimated by Sturzenegger (2002). The risk- free interest rate r is set to 1%, the average quarterly interest rates on 3 month US treasury bills. The last two parameters are the time discount factor β and the sovereign government's bargaining power θ. The discount factor β will calibrated across experiments so that the default probability in the limiting distribution matches the average default frequency of Argentina. From 1824 to 1999, Argentina defaulted 5 times (Reinhart, Rogoff and Savastano (2003) report four sovereign default episodes of Argentina from 1824 to 1999. And Argentina defaulted the fifth time in 2001). Then the average default frequency in Argentina is 0.69% quarterly. The bargaining power θ will be calibrated to match Argentina's average debt recovery rate in the most recent debt restructuring, which is 28% according to Moody's (2003). The following table summarizes parameter values estimated and to be estimated.

Parameter	Symbol	Value
Coefficient of Risk Aversion	σ	2
Risk Free Interest Rate	r	1%
Output Loss in Default	γ	2%
Mean Endowment	$\overline{y}$	1
Std. Dev. of Endowment shock	σ_ε	to be estimated
Autocorr. Coef. of Endowment	ρ	to be estimated
Discount Factor	β	to be estimated
Borrower's Bargaining Power	ϕ	to be estimated

4.2 Solution Algorithm

The following algorithm outlines the procedures we plan to use to compute the equilibrium of the model.

First, we discretize the spaces of asset holdings and endowment. The limits of the asset space and endowment space are set to ensure that the limits do not bind in equilibrium and big deviations of shocks are possible. We approximate the distribution of endowment shocks by a discrete Markov transition matrix using a quadrature based procedure (Hussey and Tauchen 1991).

Then, we guess an initial debt recovery schedule $\alpha_0 = 1$. Given the initial debt recovery rate α_0, guess an initial price of short-term bonds $q_0^s = \frac{1}{1+r}$, and an initial price of long-term bonds $q_0^l = \frac{1}{(1+r)^2}$.

Thirdly, given initial guess of α_0 and prices q_0^s and q_0^l, solve the country's optimization problem when it repays debt. We use the Bellman equation iteration to find out the value function V_0^R. And then we compute the optimal default choice by comparing V_0^R and V_0^D, and also get the default set. From the default set and initial guess of recovery rate, we can compute the new short-term and long-term prices q_1^s and q_1^l. If the new prices are sufficiently close to the old one, stop iterating on prices and go on, otherwise repeat this step until prices converge.

Fourthly, solve the bargaining problem given the converged prices and compute the new debt recovery schedule α_1 for all y, B^M and B^T. If the new recovery schedule is sufficiently close to the old one, stop iterating. Otherwise, go back to the last step.

4.3 Simulation

We will feed the endowment process to the model and conduct simulations to explore the behavior of the model economy in the stationary distribution.

First, we will compare the model statistics with data statistics. These statistics include average and volatility of interest rates of different maturities, relation between bond spreads, outputs and current accounts.

We then can do the following experiments: plot the time series dynamics of the model prior to a default episode in two cases. One is with debt renegotiation after default and the other without debt renegotiation after default. We would like to compare the shift in maturity structures (changes in the ratio of short-term bonds over long-term bonds), changes in the interest rates of short-term bonds and long-term bonds in the above two cases. This experiment is aim to show that sovereign bonds will be biased to short-term bonds in prior crisis episode in the first case, due to default risk only; and the maturity structure will be biased even more to short-term bonds in the second case in which post-default renegotiation exists, due to the debt dilution effect. We also aim to show that in general long-term interest rate is higher than short-term interest rate, and before crisis this difference can be even larger, which reflects both higher default risk and debt dilution risk incorporated in long-term bonds.

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A appendix

Proof of Proposition 1. The debt renegotiation problem is

$$\begin{aligned}
\alpha^*(y_t, B_t) &= argmax \Delta_B(y_t, B_t, \alpha_t)^{\phi} \Delta_L(y_t, B_t, \alpha_t)^{1-\phi} \\
s.t. \quad \Delta_B(y_t, B_t, \alpha_t) &= u((1-\gamma)y_t + \alpha_t B_t) + \beta \\
& \quad EV(y_{t+1}, 0, 0) - E\sum_{i=t}^{\infty} \beta^{i-t} u((1-\gamma)y_i) \\
\Delta_L(y_t, B_t, \alpha_t) &= -\alpha_t B_t \\
\Delta_B(y_t, B_t, \alpha_t) &\geq 0 \\
\Delta_L(y_t, B_t, \alpha_t) &\geq 0
\end{aligned}$$

Take the first order conditions of the maximization problem, we have

$$\phi \Delta_B^{\phi-1} \Delta_L^{1-\phi} u'((1-\gamma)y_t + \alpha_t^* B_t) B_t = (1-\phi) \Delta_B^{\phi} \Delta_L^{-\phi} B_t \tag{15}$$

First, let's consider the case that $\Delta_B \neq 0$ and $\Delta_L \neq 0$. From the above first order condition, we know that the interior solution is given by

$$\phi \Delta_L u'((1-\gamma)y_t + \alpha_t^* B_t) = (1-\phi)\Delta_B \tag{16}$$

Obviously, the interior solution cannot be $\alpha^* = 0$, otherwise the above equation cannot hold with equality. Then let's consider the case that $\Delta_B = 0$. This is a solution to the Nash Bargaining problem only when the interior solution α^* is beyond the $[0, 1]$ interval, and in this case equilibrium debt recovery rate is 1. As to the case that $\Delta_L = 0$, it has been ruled out since $\phi < 1$, and the debt holders can always extract some positive surplus from the bargaining game. Thus, in equilibrium $\alpha^*(y, B) > 0$. ■

Proof of Proposition 2. Let $B^R = -\alpha_t B_t$ denote the reduced amount of debt repayment. And we rewrite equation (16) as follows:

$$\phi B^R u'((1-\gamma)y_t - B^R) = (1-\phi)u((1-\gamma)y_t - B^R) + \beta EV(y_{t+1}, 0, 0) - E\sum_{i=t}^{\infty} \beta^{i-t} u((1-\gamma)y_i) \tag{17}$$

Take derivative of both sides with respect to B^R:

$$\frac{\partial LHS}{\partial B^R} = \phi u' + \phi B^R u''(-1) > 0$$

$$\frac{\partial RHS}{\partial B^R} = (1-\phi)u'(-1) < 0$$

So the left-hand-side is an increasing function in B^R and the right-hand-side is an decreasing function in B^R. There must be a unique B^{R*} such that equation (17) holds with equality. And that $-B^{R*}$ is the threshold value $\overline{B}$. When $B_t \leq \overline{B}$, then the equilibrium reduced debt repayment is $\overline{B}$. When $B_t > \overline{B}$, there is no debt reduction, since α cannot be larger than 1. This is the case that $\Delta_B = 0$ and $\Delta_L > 0$. ■

Proof of Corollary 1. We can rewrite Proposition 2 as:

$$\alpha^*(y,B) = \begin{cases} \frac{\overline{B}(y)}{B} & \text{if } B \leq \overline{B}(y) \\ 1 & \text{if } B > \overline{B}(y) \end{cases}$$

We need to consider 3 cases.

First when $B^1 \leq B^2 \leq \overline{B}(y)$, then $\alpha^*(y,B^1) = \frac{\overline{B}(y)}{B^1} \leq \frac{B^2}{B^1} = \alpha^*(y,B^2)$. Thus, corollary holds.

Secondly, when $B^1 \leq \overline{B}(y) \leq B^2 < 0$, then $\alpha^*(y,B^1) = \frac{\overline{B}(y)}{B^1} \leq 1 = \alpha^*(y,B^2)$. Thus, corollary holds.

Thirdly, when $\overline{B}(y) \leq B^1 \leq B^2 < 0$, $\alpha^*(y,B^1) = \alpha^*(y,B^2) = 1$. Thus, corollary holds. ■

Proof of Proposition 3. Given the same B^T, it's straight forward to show that the value function $V(y,0,B)$ increases in B^M. The proof is as follows. When $s = 0$,

$$\frac{V^R(y,0,B)}{B^M} = u'(c^*) > 0$$

$$\frac{V^D(y,0,B)}{B^M} = \beta EV'(y',1,B')(1+r) > 0$$

Thus, the value function $V(y,0,B)$ is increasing in B^M, given the same B^T. If default is optimal for B^{2M}, then $V^D(y,0,B^2) \geq V^R(y,0,B^2)$. Since $B^{1M} \leq B^{2M}$, $V^R(y,0,B^1) \leq V^R(y,0,B^2)$. In addition, given $B^{1T} = B^{2T}$, we have $\alpha^*(y,B^1)B^1 = \alpha^*(y,B^2)B^2$ and

$V^D(y,0,B^1) = V^D(y,0,B^2)$. Thus, we have $V^D(y,0,B^1) = V^D(y,0,B^2) \geq V^R(y,0,B^2) \geq V^R(y,0,B^1)$, so default is also optimal for B^{1M}. The first half of the proposition is thus proved.

For the second half, we first rewrite the value function of default.

$$V^D(y,0,B) = u((1-\gamma)y) + \beta E u((1-\gamma)y' + \alpha' B^T(1+r) + \beta^2 EV(y'',0,0)$$

Take derivative of $V^D(y,0,B)$ with respect to B^T:

$$\frac{V^D(y,0,B)}{B^T} = Eu'((1-\gamma)y + \alpha B^T(1+r))(\frac{\partial \alpha}{\partial B} B^T(1+r) + \alpha(1+r)) > 0$$

So $V^D(y,0,B)$ is increasing in B^T. If default is optimal for B^{1T}, then $V^D(y,0,B^2) \geq V^D(y,0,B^1) \geq V^R(y,0,B^1)$. And similarly, we can prove that $V^R(y,0,B^1) \geq V^R(y,0,B^2)$. Thus, default is also optimal for B^{2T}. And the second half of the proposition is proved. ■

Proof of Proposition 4. If debt renegotiation is not a possible option and equilibrium debt recovery rate $\alpha^* = 0$, then default entails permanent autarky. Use V^A to denote the value function of default in this case.

$$V^A(y_t,0,B_t) = u((1-\gamma)y_t) + E\sum_{i=t}^{\infty} \beta^{i-t} u((1-\gamma)y_i).$$

As shown in proof of Proposition 1, $V^D(y,0,B) > V^A(y,0,B)$ when α^* is positive. If default is optimal even without debt renegotiation, then $V^A(y,0,B) > V^{RA}(y,0,B)$, where $V^{RA}(y,0,B)$ denotes the value function of repaying debt when debt renegotiation is not allowed. Since staying in autarky is a more severe penalty to the country, the budget set $B(y,0,\alpha^* = 0) \supseteq B(y,0,\alpha^* > 0)$, and hence, $V^{RA}(y,0,B) > V^R(y,0,B)$ given the same B^M and B^T. Therefore, we have $V^D(y,0,B) > V^R(y,0,B)$ and default is also optimal when debt renegotiation is allowed. Proposition 4 is thus proved.