Federal Reserve Bank of Minneapolis
Research Department

A Model of Banknote Discounts*

Laurence Ales, Francesca Carapella,
Pricila Maziero, and Warren E. Weber

Working Paper 641

Revised December 2005

ABSTRACT

Prior to 1863, state-chartered banks in the United States issued notes—dollar-denominated promises to pay specie to the bearer on demand. Although these notes circulated at par locally, they usually were quoted at a discount outside the local area. These discounts varied by both the location of the bank and the location where the discount was being quoted. Further, these discounts were asymmetric across locations, meaning that the discounts quoted in location A on the notes of banks in location B generally differed from the discounts quoted in location B on the notes of banks in location A. Also, discounts generally increased when banks suspended payments on their notes. In this paper we construct a random matching model to qualitatively match these facts about banknote discounts. To attempt to account for locational differences, the model has agents that come from two distinct locations. Each location also has bankers that can issue notes. Banknotes are accepted in exchange because banks are required to produce when a banknote is presented for redemption and their past actions are public information. Overall, the model delivers predictions consistent with the behavior of discounts.

JEL: G21,N21,E50

Keywords: banks, random matching, banknotes

*Ales, Carapella, Maziero, Weber—Federal Reserve Bank of Minneapolis and University of Minnesota. Corresponding author Weber (wew@minneapolisfed.org). The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

Between 1783, when the United States won independence from Great Britain, until the passage of the National Currency Act in 1863, the largest component of U.S. currency in circulation was notes issued by state-chartered banks. These banknotes were dollar-denominated promises to pay specie to the bearer on demand, and the notes were distinguishable by issuing bank.[1] Virtually all banks that existed during this period issued banknotes. Since the country had approximately 325 banks in 1820, over 700 banks in 1840, and more than 1,350 in 1860, large numbers of distinct currencies were in circulation in the country throughout the antebellum period.

These banknotes circulated in the location of the issuing bank, and the notes of at least some banks also circulated outside the local area. In the vernacular of the day, such banknotes were known as "foreign notes." We know this from several sources. The balance sheets for virtually every bank during this period have an asset account for notes of other banks, and the balance sheets for banks in several states have separate account listings for out-of-state notes. An 1842 report of banks in Pennsylvania lists the banks' holdings of notes of other banks by bank. It shows that Pennsylvania banks held the notes of banks in at least 61 cities in 15 other states. The clearing system for New England banknotes run by Suffolk Bank in Boston cleared a large volume of notes. Such a system would not have been necessary if the circulation of banknotes had been entirely local. Finally, banknotes bearing the stamp of a business in another location are still in existence.

The circulation of this large number of banknotes prompted the publication of spe-

[1] During this period, a dollar was defined as a governmentally minted coin containing 371.25 grains of silver. Large denomination coins were made of gold. The gold content of the dollar was changed in 1834 and 1837. Banknotes were almost always at least $1 in denomination, and in many cases banks were restricted to issue notes no smaller than $5. Notes of these denominations are roughly comparable to $20 and $100 bills today.

cialized publications, generically called banknote reporters (and counterfeit detectors). They were published at least monthly (or more frequently in some cases). Banknote reporters are known to have been published in many cities, including New York, Philadelphia, Boston, Pittsburgh, Cleveland, Cincinnati, Chicago, and Zanesville (OH). They were usually published by a note broker or in conjunction with one. Each issue of a banknote reporter listed virtually all of the banks in existence in the country at the time and quoted the discount at which the notes of each bank would be exchanged for notes of local banks (banks in the city where the reporter was published). In other words, banknote reporters listed the exchange rates of the notes of each bank in the country in terms of local banknotes.[2]

If we examine the discounts quoted in banknote reporters, five facts emerge:

1. Local banknotes were always quoted at par.
2. "Foreign" banknotes usually were quoted at a discount to local banknotes, and this discount varied by the location of the bank and the reporter.
3. Discounts were asymmetric across locations, meaning that the discounts quoted in location A on the notes of banks in location B generally differed from the discounts quoted in location B on the notes of banks in location A.
4. Discounts on foreign notes were higher when those notes were not being redeemed at par.
5. Local banknotes were quoted at a discount to specie when local banks suspended payment on their notes.

The purpose of this paper is to build a model that can qualitatively match these facts about

[2] Newspapers in many cities also published banknote lists, which contained the same information as in a banknote reporter and were based on information provided by a note broker.

banknote discounts. The model we construct builds on the basic search model of money by Shi (1995) and Trejos and Wright (1995). To attempt to account for the locational differences in banknote discounts, our model has agents that come from two distinct locations, whereas the Trejos-Wright framework implicitly assumes that all agents are in one location. In addition, unlike the model of Trejos-Wright, our model does not have fiat money but instead has banknotes issued by banks in the two locations. We find that an equilibrium exists in which these banknotes are valued and act as media of exchange.

We model the fact that banknotes had to be redeemed in specie on demand by adapting the Cavalcanti and Wallace (1999) innovation that banks are required to produce when a banknote is presented for redemption and their past actions are public information. We present two versions of the model. In the first, banks are threatened with permanent autarky if they ever fail to produce the same quantity of goods when their notes are presented for payment as they obtain when issuing a note. We refer to this case as par redemption. In the second, banks are allowed to suspend payments on their notes by redeeming them below their issue value. This second version of the model is intended to capture a feature of the actual experience of the period under consideration.

By requiring that banks must redeem their notes, we capture the aspect of banknotes that made them like debt instruments as modeled by Gorton (1999). However, Gorton is interested in determining whether prices of banknotes reflected banks' risk-taking behavior in the pre–Civil War period, where a risky bank is one that is likely to overissue banknotes. The mechanism that allows market participants to adequately price these risks is the redemption option: banknotes can be sent for redemption to the issuing bank, and if this is not too costly, then the issuing bank can be discouraged from overissuing by the threat of bank runs. In

Gorton's model, the main determinant of the costs related to sending notes for redemption is the physical distance between the location at which trade occurs and the location of the bank that issued the banknotes used to trade. In our model, banknote prices are instead exclusively related to trading opportunities and the value of each one of these opportunities.

The paper proceeds as follows. In the next section, we document the five facts about banknote discounts that we want to explain. In Section 2, we present the model environment. In Section 3 we define a monetary symmetric steady-state equilibrium for this economy. In Section 4, we show that for certain values of the parameters, we can explain facts 1 through 3 for the case in which bankers are required to redeem their notes at par. In Section 5, we examine the case in which banks have suspended payment on their notes and show that for certain values of the parameters, we can explain facts 4 and 5. The final section concludes.

1. Facts about banknote discounts

In this section we present some documentation for the five facts about banknote discounts that we want our model to be consistent with. The first fact is that banknotes went at par in the local area. An examination of the banknote reporters for New York, Philadelphia, Cleveland, Cincinnati, and Pittsburgh shows that notes of local banks that had not failed were always listed at par. The same examination reveals the second fact: notes of nonlocal banks (i.e., "foreign" banknotes) generally were listed at a discount. For example, Thompson's banknote reporter for August 21, 1854, lists notes of New England banks at a $\frac{1}{4}$ percent discount, those of Virginia banks at a $1\frac{1}{4}$ percent discount, those of Indiana banks at a 2 percent discount, and those of Tennessee banks at a 3 percent discount. For August 13, 1859, they were $\frac{1}{8}$ percent for New England banknotes, $\frac{3}{4}$ percent for those of Virginia

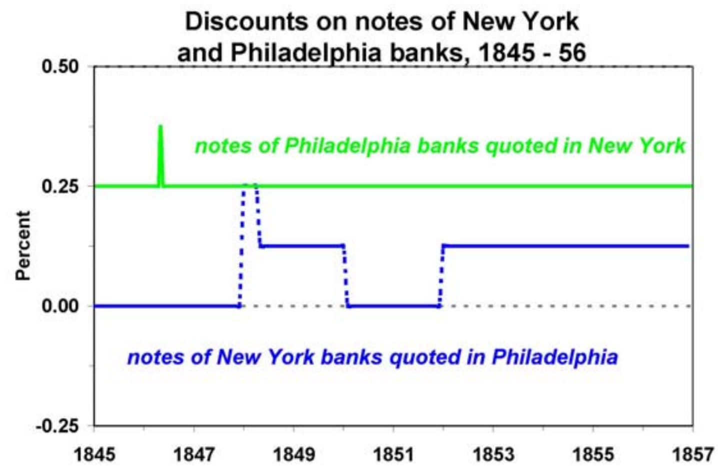

Figure 1: Discounts on notes of New York and Philadelphia banks, 1845–56

banks, $1\frac{3}{4}$ percent for Indiana banks, and $1\frac{1}{2}$ percent for Tennessee banks. These are but a few examples of the discounts for notes of nonlocal banks.

The third fact is that the discounts on banknotes were asymmetric: the discount quoted in location A on notes of banks from location B was not generally the same as the discount quoted in location B on notes of banks from location A. This is shown in Figure 1, where we plot the discounts on New York banknotes as quoted in Philadelphia and the discounts on Philadelphia banknotes as quoted in New York for the period 1845–56. The figure shows that with the exception of a short period in 1847, the notes of New York banks were quoted at a lower discount in Philadelphia than the notes of Philadelphia banks were quoted in New York. Further support for the asymmetry of discounts is given in Figure 2, where we plot the discounts on the notes of Cincinnati banks as quoted in Philadelphia for the

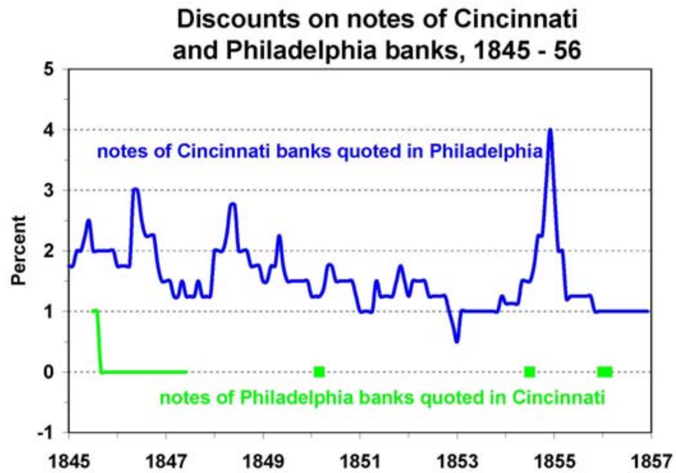

Figure 2: Discounts on notes of Cincinnati and Philadelphia banks, 1845–56

period 1845–56. The notes of Philadelphia banks were always quoted at a lower discount—in most cases, at no discount at all—in Cincinnati, whereas the notes of Cincinnati banks were always quoted at a discount in Philadelphia. We obtain a similar result when we replace Philadelphia with New York for this comparison.

The final two facts we want to explain with our model concern the behavior of banknote discounts when banks suspend, that is, when banks are not redeeming their notes at par on demand. The first of these final two facts is that the discounts on foreign banknotes were higher when banks had suspended payments. Figures 3 and 4 show the behavior of the discounts on notes quoted in New York when banks in Wilmington, NC, and Philadelphia suspended, for the period 1835–60. The figures show that banknote discounts were generally larger for Wilmington, NC, banks and always larger for Philadelphia banks when these banks

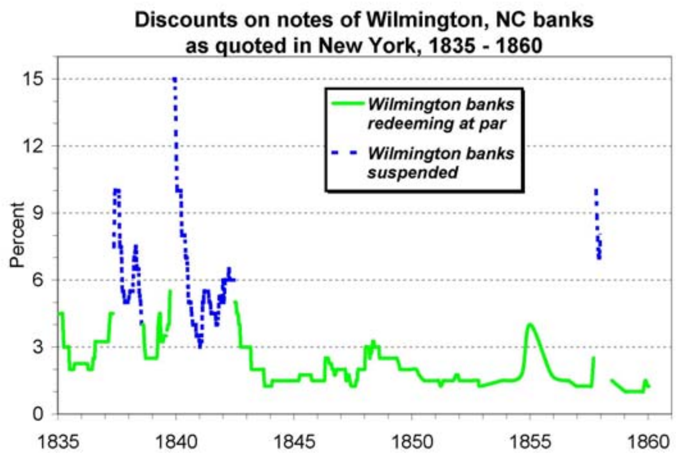

Figure 3: Discounts on notes of Wilmington, NC, banks as quoted in New York, 1835–60

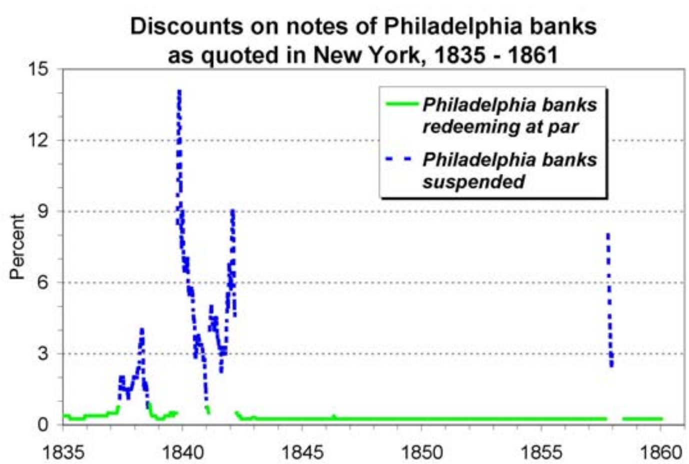

Figure 4: Discounts on notes of Philadelphia banks as quoted in New York, 1835–60

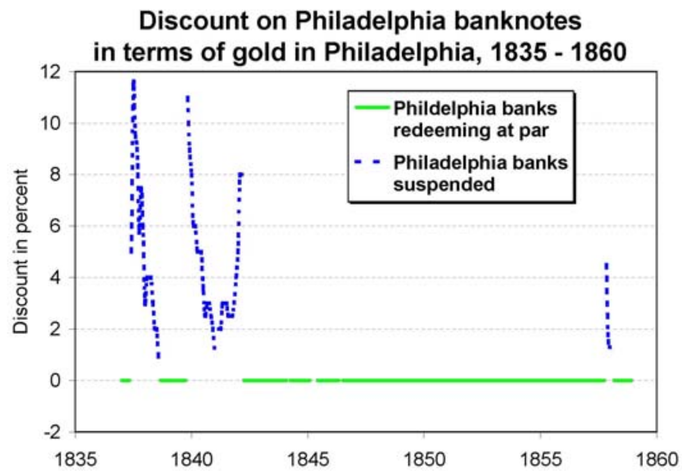

Figure 5: Discounts on Philadelphia banknotes in terms of gold in Philadelphia, 1835–60

were suspended than when they were not. The behavior in these figures is consistent with that of the discounts on the notes of banks in other locations and quotes from Philadelphia instead of New York.

We did find one exception, however. When New York banks suspended in May 1837, the discount on their notes in Philadelphia fell rather than rose. This exception is the only one that we have been able to find.

The second of the final two facts that we want to explain with the model is that banknotes went at a discount against specie locally when banks suspended. This is shown for the case of Philadelphia banknotes in Figure 5.

2. Model environment

The model environment is similar to that of Shi (1995) and Trejos-Wright (1995). Time is discrete and infinite. There is a single, nondurable good that is perfectly divisible. There are two types of agents, private agents and bankers. Both types of agents can produce and consume the nondurable good. Bankers also have access to a technology to produce banknotes—pieces of paper that bear the name of the banker. We assume that private agents are anonymous and their past trading histories are private information. In contrast, bankers are not anonymous and their past histories are public information. All private agents and all bankers are identical.

There are two locations—home and foreign. There is a measure H of private agents from the home location (home agents) and a measure F of agents for the foreign location (foreign agents). There are bankers in both locations, which we refer to as home bankers and foreign bankers. The subscripts H and F refer to home and foreign, respectively.

At the beginning of each period, private agents have a probability π of being a consumer but not a producer, the same probability of being a producer but not a consumer, and a probability $1 - 2\pi$ of being neither. Bankers can always be either a producer or consumer. In every period, private agents from location j have a probability $0 < \theta_j < 1$, $j \in \{H, F\}$ of meeting pairwise with a banker in their location. In other words, we impose the locational restriction that home agents meet only home bankers and foreign agents meet only foreign bankers. The restriction that private agents do not meet bankers in their location with certainty is intended to capture the fact that it was costly for private agents to go to local banks to obtain banknotes (by taking out loans or paying specie) or to redeem banknotes. The restriction that private agents cannot meet bankers from the other location is intended to

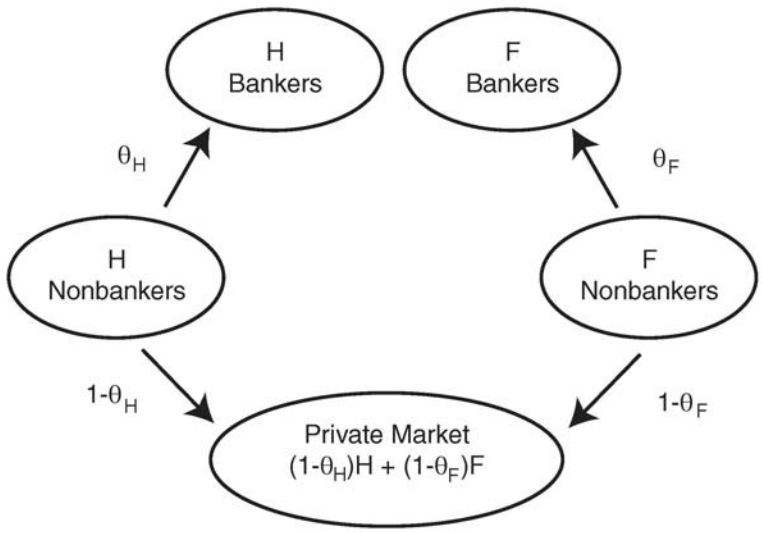

Figure 6: Within-period meeting possibilities for private agents

capture the fact that it was more costly for agents to go to nonlocal bankers than to local ones.

Private agents who do not meet a banker meet randomly pairwise with another private agent with probability δ. Given our random matching assumption, the probability that an agent meets a home agent is

$$P_H = \delta \frac{(1-\theta_H)\frac{H}{F}}{(1-\theta_H)\frac{H}{F} + (1-\theta_F)},$$

and the probability that an agent meets a foreign agent is

$$P_F = \delta \frac{(1-\theta_F)}{(1-\theta_H)\frac{H}{F} + (1-\theta_F)}.$$

The within-period meeting possibilities for private agents are illustrated in Figure 6.

That agents cannot be producers and consumers at the same time introduces a problem of lack of double coincidence of wants into meetings between private agents. The lack of double coincidence of wants and the anonymity of private agents give rise to the need for a medium of exchange. This role is filled by banknotes. We assume that agents can hold the notes of either home or foreign bankers subject to a unit upper-bound restriction on banknote holdings. We assume that private agents can trade banknotes for the nondurable good, but cannot trade notes of banks in one location for notes of banks in the other location. In pairwise meetings between private agents in which one has a banknote and the other does not, we assume that the note holder makes a take-it-or-leave-it (TIOLI) offer to the other agent. We further assume that agents always trade if indifferent.

We now consider the maximization problems of bankers and private agents

A. Bankers

Bankers can produce and consume the nondurable good. In addition, they have the technology to costlessly issue banknotes, which are pieces of paper distinguishable by the issuer. In meetings with private agents, bankers produce a note for $q_j \in \mathbb{R}^+$ units of the nondurable good and redeem a previously issued note for $Q_j \in \mathbb{R}^+$ units of the nondurable good, where the subscript denotes the location of the banker $j \in \{H, F\}$. Thus, q_j is the issue value of a banknote, and Q_j is the redemption value of a banknote. The momentary utility function of a banker is assumed to be

$$u(q_j, Q_j) = u(q_j) - Q_j, \qquad j \in \{H, F\}.$$

When a private producer meets a banker, we assume that q_j is determined by the banker making a TIOLI offer to the producer. Thus, q_j will be the quantity of the good such that a private producer is indifferent between producing and receiving a banknote and not producing.

When a private consumer meets a banker, we assume that the redemption price is determined exogenously. The fact that bankers are not anonymous and that their histories are public information means that bankers could potentially be punished for failing to redeem their notes at this exogenously determined price. We consider two cases. The first is par redemption, which we interpret as $Q_j = q_j$; that is, banks must redeem their notes at their issue value. During the period we are considering, the laws governing banking required that banks redeem their notes at par. Failure to do so would mean that except in extraordinary circumstances, a bank could face penalties and ultimately lose its banking privileges.

These extraordinary circumstances were generally understood to be times when banks in the whole country or large sections of the country suspended specie payments on notes due to exceptionally large drains on their specie reserves. Because of the widespread nature of these suspensions and fears of the consequences if banks were required to redeem notes at par, state banking authorities usually did not take actions against banks during these suspensions, although in many cases, states passed laws requiring banks to resume payments by a certain date or lose their banking privileges. To account for the fact that suspensions of specie payments on notes did occur, we consider the case in which bankers have suspended payments on notes, which we interpret as $Q_j < q_j$.

B. Private agents

We assume that private agents in both locations have a momentary utility function of the form

$$u(c) - y,$$

where c denotes consumption and y denotes production. The utility function has the properties $u(0) = 0, u' > 0, u'(0) = \infty, u'(\infty) = 0$, and $u'' < 0$. Private agents are assumed to maximize expected discounted lifetime utility, where β is the discount factor.

Let V_k and W_k, $k \in \{0, H, F\}$ denote the beginning-of-period expected value of having no banknotes ($k = 0$), holding a home banknote ($k = H$), and holding a foreign banknote ($k = F$) for a home agent and a foreign agent, respectively. Further, let $x_j \in \mathbb{R}^+$ and $z_j \in \mathbb{R}^+$, $j \in \{H, F\}$ denote the production of a home agent and a foreign agent, respectively, in a meeting with the holder of a home note ($j = H$) or a foreign note ($j = F$). Since we consider only steady-state equilibria, the value functions and the production quantities are independent over time. We say that foreign banknotes go at a discount in the local market if agents with foreign notes obtain fewer goods from local producers than agents with local notes. The discount on foreign notes in the home market is

$$d_H = 1 - \frac{x_F}{x_H}.$$

If we apply the same concept in the foreign market, the discount on home banknotes in the foreign market is

$$d_F = 1 - \frac{z_H}{z_F}.$$

If either discount is negative, then nonlocal notes are said to be going at a surplus.

Let $0 \leq m_{ji} \leq 1$ be the fraction of private agents from location j holding an i banknote and $0 \leq \lambda_{kj}^i \leq 1$ be the probability that a k (consumer) with an i banknote trades with a j (producer). Further, define Ω_{ki} to be the expected value for a k consumer with an i banknote of going to the private market. Specifically,XXX

$$\Omega_{Hi} = \max_{\lambda_{HH}^i,\lambda_{HF}^i} \pi\{P_H m_{H0} \lambda_{HH}^i \Lambda_H^0 \left[u\left(x_i\right) - \Delta_{Hi}\right] \tag{1}$$
$$+ P_F m_{F0} \lambda_{HF}^i \Lambda_F^0 \left[u\left(z_i\right) - \Delta_{Hi}\right]\}, i \in \{H, F\}$$

$$\Omega_{Fi} = \max_{\lambda_{FH}^i,\lambda_{FF}^i} \pi\{P_H m_{H0} \lambda_{FH}^i \Lambda_H^0 \left[u\left(x_i\right) - \Delta_{Fi}\right] \tag{2}$$
$$+ P_F m_{F0} \lambda_{FF}^i \Lambda_F^0 \left[u\left(z_i\right) - \Delta_{Fi}\right]\}, i \in \{H, F\},$$

where Λ_{jk}^0 denotes the probability that agents from location j without a banknote trade with an agent with a k note, $\Delta_{Hi} = \beta(V_i - V_0)$, and $\Delta_{Fi} = \beta(W_i - W_0)$. The first term on the right-hand side of (1) is the expected value of meeting another home agent without a banknote, and the second term is the expected value of meeting a foreign agent without a banknote. The sum of these terms is multiplied by the probability that the other agent in the meeting is a producer to obtain the expected value to a home consumer of going to the private market with an i banknote. The terms in (2) are similarly interpreted.

Let $0 \leq \gamma_j \leq 1$ be the probability that a private agent from location j trades with a banker. Then the flow Bellman equations for home agents with a banknote are

$$(1 - \beta)V_H = \pi\{\theta_H \max_{\gamma_H \in [0,1]} \gamma_H \left[u(Q_H) - \Delta_{HH}\right] + (1 - \theta_H)\Omega_{HH}\} \tag{3}$$

$$(1 - \beta)V_F = (1 - \theta_H)\pi\Omega_{HF} \tag{4}$$

and those for foreign agents with a banknote are

$$(1 - \beta)W_F = \pi\{\theta_F \max_{\gamma_F \in [0,1]} \gamma_F \left[u(Q_F) - \Delta_{FF}\right] + (1 - \theta_F)\Omega_{FF}\} \tag{5}$$

$$(1 - \beta)W_H = (1 - \theta_F)\pi\Omega_{FH} \tag{6}$$

The incentive compatibility conditions are

$$(7)\qquad \gamma_H \in \begin{cases} \{1\} \text{ if } u(Q_H) - \Delta_{HH} > 0 \\ \{\phi\} \text{ if } u(Q_H) - \Delta_{HH} = 0 \\ \{0\} \text{ if } u(Q_H) - \Delta_{HH} < 0 \end{cases} \qquad \gamma_F \in \begin{cases} \{1\} \text{ if } u(Q_F) - \Delta_{FF} > 0 \\ \{\phi\} \text{ if } u(Q_F) - \Delta_{FF} = 0 \\ \{0\} \text{ if } u(Q_F) - \Delta_{FF} < 0 \end{cases}$$

$$(8)\qquad \lambda^i_{HH} \in \begin{cases} \{1\} \text{ if } u(x_i) > x_i \\ \{\phi\} \text{ if } u(x_i) = x_i \\ \{0\} \text{ if } u(x_i) < x_i \end{cases} \qquad \lambda^i_{FH} \in \begin{cases} \{1\} \text{ if } u(x_i) > z_i \\ \{\phi\} \text{ if } u(x_i) = z_i \\ \{0\} \text{ if } u(x_i) < z_i \end{cases}$$

$$\lambda^i_{HF} \in \begin{cases} \{1\} \text{ if } u(z_i) > x_i \\ \{\phi\} \text{ if } u(z_i) = x_i \\ \{0\} \text{ if } u(z_i) < x_i \end{cases} \qquad \lambda^i_{FF} \in \begin{cases} \{1\} \text{ if } u(z_i) > z_i \\ \{\phi\} \text{ if } u(z_i) = z_i \\ \{0\} \text{ if } u(z_i) < z_i \end{cases}$$

where $\phi = [0, 1]$.

The Bellman equations for agents without a banknote are obtained in a similar manner. Define Ω_{k0} to be the expected value for a k producer with no banknote of going to the private market. Specifically,

$$\begin{aligned} (9)\qquad \Omega_{H0} &= \max_{\lambda^0_H, \lambda^0_F} [\pi(P_H m_{HH} \Lambda^H_{HH} + P_F m_{FH} \Lambda^H_{FH}) \lambda^0_H (\Delta_{HH} - x_H) \\ &\quad + (P_H m_{HF} \Lambda^F_{HH} + P_F m_{FF} \Lambda^F_{FH}) \lambda^0_F (\Delta_{HF} - x_F)] \end{aligned}$$

$$\begin{aligned} (10)\qquad \Omega_{F0} &= \max_{\lambda^0_H, \lambda^0_F} \pi[(P_H m_{HH} \Lambda^H_{HF} + P_F m_{FH} \Lambda^H_{FF}) \lambda^0_H (\Delta_{FH} - z_H) \\ &\quad + (P_H m_{HF} \Lambda^F_{HF} + P_F m_{FF} \Lambda^F_{FF}) \lambda^0_F (\Delta_{FF} - z_F)], \end{aligned}$$

where λ_j^0 and Λ_{kj}^i are defined analogously to Λ_j^0 and λ_{kj}^i above. The first term on the right-hand side of (9) is the expected value of meeting an agent with a home banknote, and the second term is the expected value of meeting an agent with a foreign banknote. The sum of these terms is multiplied by the probability that the other agent in the meeting is a consumer to obtain the expected value to a home producer of going to the private market with an i banknote. The terms in (10) are similarly interpreted. With this notation, the Bellman equations for agents without a banknote are

$$(11) \quad (1-\beta)V_0 = \pi\{\theta_H \max[\Delta_{HH} - q_H, 0] + (1-\theta_H)\Omega_{H0}\}$$

$$(12) \quad (1-\beta)W_0 = \pi\{\theta_F \max[\Delta_{FF} - q_F, 0] + (1-\theta_F)\Omega_{F0}\}$$

Bankers and private agents with banknotes are assumed to make TIOLI offers to private agents who are producers without banknotes. These offers extract all of the surplus from the trade. Note that the agent making the offer takes V_j and W_j as given. These offers will be

$$(13) \quad q_H = \beta(V_H - V_0)$$

$$(14) \quad q_F = \beta(W_F - W_0)$$

$$(15) \quad x_j = \beta(V_j - V_0), \qquad j \in \{H, F\}$$

$$(16) \quad z_j = \beta(W_j - W_0), \qquad j \in \{H, F\}$$

Substituting (13)–(16) into (11) and (12) yields $V_0 = W_0 = 0$.

C. Steady-state banknote holdings

Because we are focusing only on steady-state equilibria, we also require that the distribution of banknote holdings remain the same from period to period. Listing the outflows from a given banknote holding on the left-hand side and the inflow on the right-hand side, the conditions on the distribution of banknote holdings that have to be satisfied for a steady-state equilibrium are as follows.

For agents without a banknote:

$$(17) \qquad m_{i0}\left[\theta_i + (1-\theta_i)\pi P_j \sum_{k=H,F} m_{jk}\Lambda_{ji}^k\right]$$

$$= \quad m_{ii}[\Gamma_i\theta_i + (1-\theta_i)\pi P_j m_{j0}\Lambda_{ij}^i] + m_{ij}(1-\theta_i)\pi P_j m_{j0}\Lambda_{ij}^i, \qquad i,j \in \{H,F\}, i \neq j$$

For agents with a banknote of a bank in their location:

$$(18) \quad m_{ii}[\Gamma_i\theta_i + (1-\theta_i)\pi P_j m_{j0}\Lambda_{ij}^i] = m_{i0}[\theta_i + (1-\theta_i)\pi P_j m_{ji}\Lambda_{ji}^i], i,j \in \{H,F\}, i \neq j$$

For agents with a banknote of a bank in the other location:

$$(19) \quad m_{ij}m_{j0}\Lambda_{ij}^j = m_{i0}m_{ji}\Lambda_{ji}^j, \qquad i,j \in \{H,F\}, i \neq j$$

3. Equilibrium

We are now ready to define a symmetric steady-state equilibrium for this economy.

Definition: *Symmetric steady-state equilibrium (SSSE)*

Given Q_H and Q_F, an SSSE is a set

$$\Phi = \{q_i, x_i, z_i, V_i \geq 0, W_i \geq 0, m_{ji} \geq 0, \gamma_i, \Gamma_i, \lambda_{jk}^i, \Lambda_{jk}^i, \forall i,j,k \in \{H,F\}\}$$

that satisfies maximization with TIOLI offers, the steady-state banknote holding equations, $\lambda^i_{jk} = \Lambda^i_{jk}$, and $\gamma_i = \Gamma_i, \forall i, j, k \in \{H, F\}$.

Definition: *Nonmonetary SSSE*

A nonmonetary SSSE is the set

$$\Phi_N = \{\phi \in \Phi : \Lambda^i_{jk} = \Gamma_i = V_i = W_i = 0, \forall i, j, k \in \{H, F\}\}.$$

That is, in a nonmonetary equilibrium there is no trade and banknotes are not valued.

PROPOSITION 1. *A nonmonetary SSSE exists for this economy.*

Proof: Substitution of Φ_N into (3)–(8) and (11)–(16) shows that all conditions of an SSSE are satisfied and $x_H = x_F = z_H = z_F = 0$. ■

Definition: *Monetary SSSE*

A monetary SSSE is a set

$$\Phi_M = \{\phi \in \Phi : \Gamma_i, \Lambda^i_{ii}, > 0 \text{ for } i \in \{H, F\} \text{ and } V_H, W_F > 0\}.$$

This definition of a monetary SSSE is strong in that it requires that both types of banknotes are valued ($V_H, W_F > 0$) and circulate ($\Gamma_i > 0$). We adopt this definition because banknote discounts were quoted in terms of notes of local banknotes, which implies that more than one type of banknote was in circulation. Note, however, that our definition of a monetary SSSE does not require interlocation circulation of banknotes. A question that we explore below is, for what parameter values do there exist equilibria in which there is interlocation banknote circulation, that is, monetary equilibria in which either $\Lambda^i_{ij} > 0$ or $\Lambda^i_{ji} > 0$ for some $i, j \in \{H, F\}, i \neq j$.

LEMMA 1. *In a monetary SSSE with $\Gamma_i = 1$, $m_{i0} = m_{ii}, i \in \{H, F\}$.*

This result is proved in the Appendix.

Lemma 2. *In a monetary SSSE,* $\Gamma_H(x_H - q_H) = \Gamma_F(z_F - q_F) = 0$; *that is, banknotes circulate locally at par (at their issue value).*

This result follows from the assumption that bankers and nonbankers with notes get to make TIOLI offers to nonbankers without notes.

4. Par redemption equilibria

The first type of monetary SSSE we examine is one in which the issue, redemption, and circulation values of banknotes are the same.

Definition: *Par redemption SSSE*

A par redemption SSSE is a set

$$\Phi_P = \{\phi \in \Phi_M : Q_i - q_i = 0, i \in \{H, F\}\}.$$

In the Appendix we prove that a par redemption SSSE exists for small enough θ_H and θ_F when agents have utility functions that are homogeneous of degree $(0, 1)$. We are unable to prove existence for all values of $\theta_H \in [0, 1)$ and $\theta_F \in [0, 1)$.[3] Instead, we present results for numerical examples in which we assume that $u(c) = c^\alpha, \alpha = \frac{1}{2}, \rho = 0.01, \pi = 0.5$, and $\delta = 0.9$.[4] Initially, we first set $H/F = 1$, so that there is the same measure of private agents in both locations.

The properties of the par redemption SSSE for this numerical example are shown in Figure 7. In region 1, both home and foreign consumers play pure strategies and exchange

[3] No solution exists for $\theta_H = \theta_F = 1$. However, this case is uninteresting because in it private agents can meet and trade only with bankers and can never meet and trade with other private agents.

[4] We have also computed the SSSE for various $\alpha \in (0, 1)$. The qualitative results are the same as those presented below.

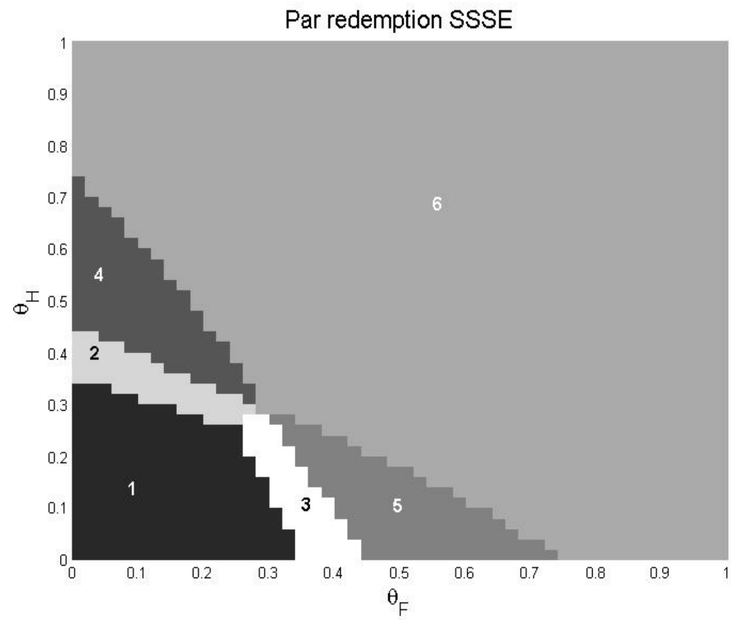

Figure 7: Par redemption SSSE

banknotes with both home and foreign producers with probability 1. In regions 2 and 3, one type of agent continues to play a pure strategy while the other plays a mixed strategy. Specifically, in region 2, foreign consumers play a pure strategy and trade with both types of producers with probability 1, while home consumers play a mixed strategy when trading with foreign producers, although they trade with home producers with probability 1. In region 3, the behavior of the two types of consumers is reversed. In regions 4 and 5, both consumers again play pure strategies. However, now consumers from one location—home consumers in the case of region 4 and foreign consumers in the case of region 5—do not trade with producers from the other location, while consumers from the other location continue to trade with all producers with probability 1. Finally, in region 6, consumers play pure strategies, but trade only with producers from their own location. That is, in region 6, the two locations are distinct; there is no interlocation trade.

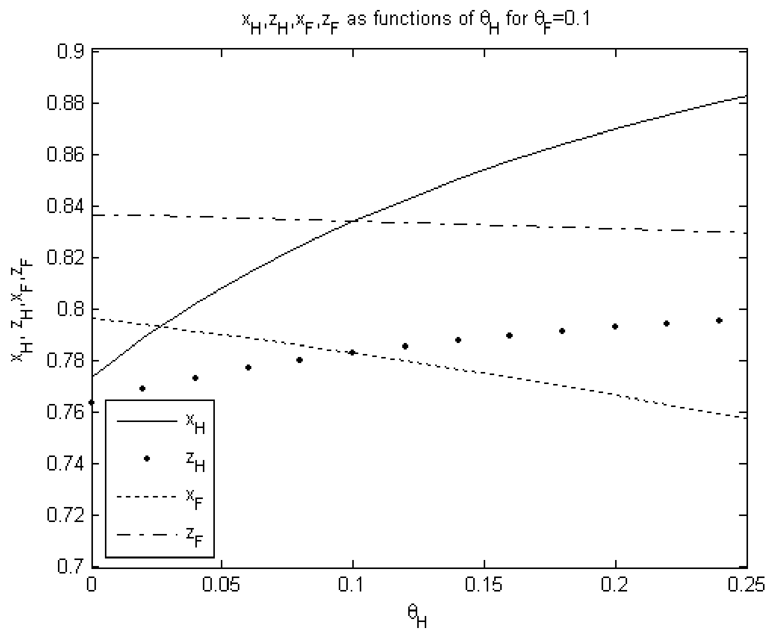

Figure 8: Effects of changes in θ_H on quantities of purchased goods

Because we are interested in the discounts on banknotes, the case of interest to us is region 1 where banknotes trade in all meetings between agents from both locations. Thus, from this point, we will restrict our attention to the region in which θ_H and θ_F are small. Because the probability of private agent–banker meetings is low in this case, focusing on this case is equivalent to making Cavalcanti and Wallace's (1999) assumption that the measure of bankers in the economy is small.

The effects of changes in θ_H on the quantities of the good that can be purchased with banknotes in the two locations are shown in Figure 8. The quantity of goods that can be obtained with a home banknote from both home producers (x_H) and foreign producers (z_H) increases with θ_H, and the quantity of goods that can be obtained with a foreign banknote from both home producers (x_F) and foreign producers (z_F) decreases with θ_H. Because of the symmetry of the model, increases in θ_F would have the opposite effects.

To see the intuition behind these results, recall that because note holders get to make TIOLI offers, the quantity of goods obtained in exchange for a banknote is proportional to the expected value of holding a banknote. Consider the case of a home agent holding a home banknote. The larger θ_H, the higher the probability of meeting a home banker and being able to redeem the note at par with certainty, as opposed to going to the private agent market where an agent can exchange a banknote for goods only in matches with an agent who happens to be a producer without a banknote. Thus, the larger θ_H, the higher the expected value of home banknotes to a home agent. Hence, x_H is larger, the larger θ_H. Further, the higher x_H increases the expected payoffs to foreign holders of home banknotes, so that z_H increases as well.

The intuition for the response of the quantities of goods that are obtained in exchange for foreign notes to changes in θ_H is similar. The larger θ_H, the higher the probability that a home agent holding a foreign note will not be able to trade because this agent now has a higher probability of meeting a home banker who will refuse to redeem the note and a lower probability of meeting another private agent who might potentially trade. Thus, foreign banknotes become less valuable to home agents, so that x_F is lower. In turn, the lower x_F reduces the payoff to foreign agents of holding a foreign banknote, although the effect is partially offset by the fact that a larger θ_H means that foreign agents have a lower probability of meeting home producers.

The effects of changes in θ_H on banknote discounts are shown in Figure 9. The discount on foreign notes in the home market (d_H) increases with θ_H. This occurs because x_H is increasing in θ_H, whereas x_F is decreasing as shown in figure 8. (Foreign notes are actually trading at a premium for low θ_H because the figure is drawn for $\theta_F = 0.1$, and this

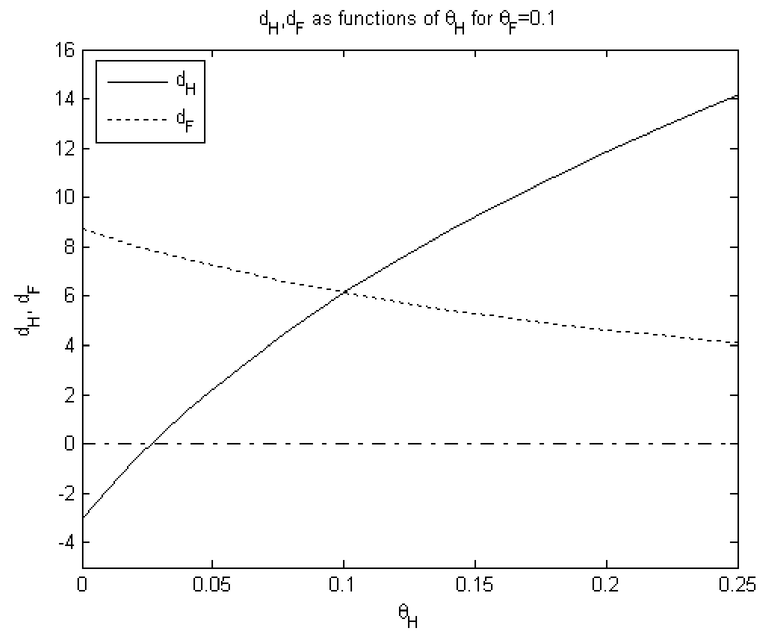

Figure 9: Effects of changes in θ_H on banknote discounts

difference between θ_H and θ_F makes foreign notes more valuable in the home market.) The discount on home notes in the foreign market (d_F) is decreasing in θ_H, however. This occurs because z_H is increasing in θ_H, whereas z_F is decreasing. Because of symmetry, the opposite effects occur with increasing θ_F.

Figure 9 shows that the predictions of the model are consistent with two of the facts about banknote discounts. First, it shows that nonlocal banknotes generally trade at a discount in the local market. Second, because discounts vary with the θs, it shows that discounts can change over time if there are changes in the accessibility of bankers. (Below we also show that discounts vary with the relative measure of agents in the two markets H/F.)

The figure also shows that the model's predictions are consistent with another fact about actual banknote discounts: these discounts were generally asymmetric, meaning that the discount on foreign notes in the home market differed from the discounts on home notes

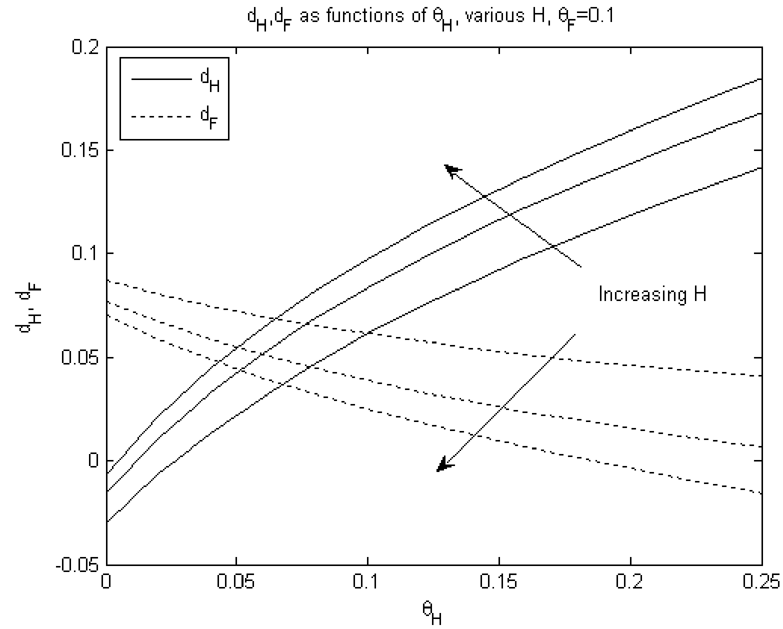

Figure 10: Effects of increasing H with $F = 1$

in the foreign market. The figure shows that discounts on foreign notes in the home market can be either larger or smaller than discounts on home notes in the foreign market, depending on the relative values of θ_H and θ_F, and that the difference $d_H - d_F$ increases with θ_H. In our example, discounts are equal only when $\theta_H = \theta_F$.

Next we determine the effects on x_H and z_H of changes in the measures of home and foreign agents. Specifically, in Figure 10, we show the effects of increasing H with $F = 1$. The figure shows that as H increases, discounts on foreign notes in the home market increase and discounts on home notes in the foreign market increase. Increasing H increases the probability that the holder of a banknote will meet a producer from the home location rather than a producer from the foreign location in the private market. Because a home producer is willing to exchange more goods for a home banknote than for a foreign banknote, this increases the value of the home banknote relative to a foreign banknote. The discount on foreign banknotes in the home market increases. And because a foreign producer is willing to produce more goods for a foreign banknote than for a home banknote, the lower probability of meeting such a producer reduces the value of foreign banknotes relative to home banknotes.

5. Suspension equilibria

We now turn to the case in which bankers in at least one location have suspended note redemption in the sense that the redemption value of their notes is less than the issue value. We discuss the predictions of the model and whether these predictions are consistent with the data.

Definition: *Suspension SSSE*

A suspension SSSE is a set

$$\Phi_S = \{\phi \in \Phi_M : Q_i - q_i < 0, \text{ for some } i \in \{H, F\}\}.$$

We present results for the same numerical example as for the case of par redemption SSSE. We model suspension by assuming that $Q_i = (1 - k_i)q_i, k_i \in [0, \bar{k}_i], i \in \{H, F\}$, where $\bar{k}_i$ is that value such that $u[(1 - \bar{k}_i)q_i] = q_i$. That is, $\bar{k}_i$ is the smallest redemption value of a banknote as a percentage of issue value that is consistent with the existence of a monetary SSSE.

We first consider the case in which $k_H = 0, k_F > 0$, that is, the case in which home bankers redeem their notes at par, but foreign bankers have suspended (in our sense) note redemption. Figure 11 shows that as k_F increases, discounts on foreign notes in the home market increase and discounts on home notes in the foreign market fall. Increasing k_F is similar to decreasing θ_F because both make the option of going to a banker less valuable to the holder of a note. Thus, one can follow the reasoning for the case of changes in θ_H

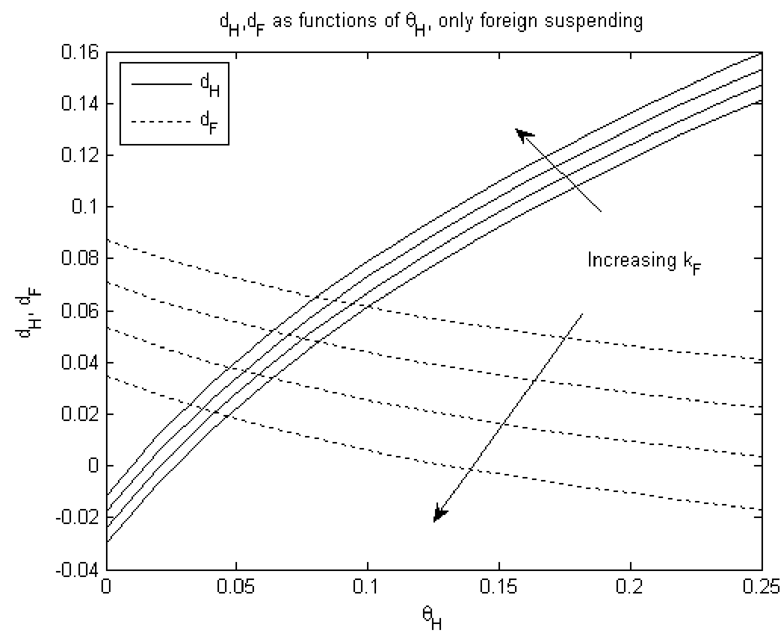

Figure 11: Effects of increasing k_F on discounts on home and foreign notes

to get the intuition for why changing k_F affects discounts as it does. These predictions are consistent with the data shown in Figures 3 and 4 since, except for a short period from May through August 1837, New York banks were redeeming their notes while Wilmington and Philadelphia banks were suspended.

Because virtually all banks in the United States were suspended from May through August 1837, and banks in Philadelphia and most of the southern and western part of the country were suspended from October 1839 through March 1842, we also examine the case in which all banks are suspended. Specifically, we examine the case in which $k_H = k_F = k > 0$; that is, all banks have suspended by the same percentage. Figure 12 shows that both d_H and d_F decline with k. The intuition for this result is that as k increases, the amount of goods that local agents can get from a local banker decreases. This reduces the wedge between the values of local and nonlocal notes to a local agent. The discount on nonlocal notes declines

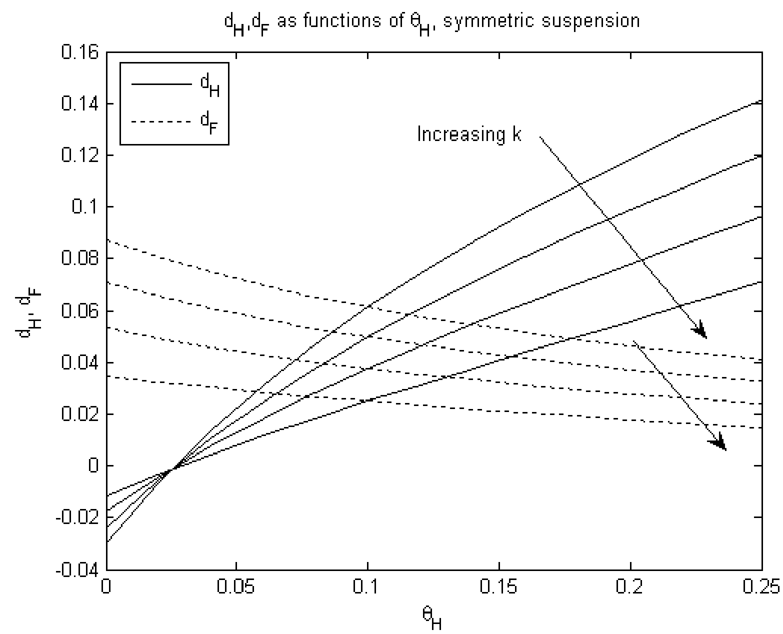

Figure 12: Effects of increasing k on d_H and d_F

as a result.

This prediction of the model is consistent with the data on the discounts on New York banknotes quoted in Philadelphia from May through August 1837, when banks in both New York and Philadelphia were suspended. However, it is inconsistent with the behavior of discounts quoted in both New York and Philadelphia on the notes of banks in all other locations when both that bank and New York or Philadelphia banks were also suspended. However, if we allow for asymmetric suspensions by New York and Philadelphia banks with respect to banks in other locations, then the predictions of the model would be consistent with the data if $k_F - k_H$ were large enough.

Our model also predicts that the value of banknotes declines when bankers suspend payment. This is shown in Figure 13 where we plot the quantity of goods that are exchanged for a home note (x_H) when home bankers are suspended and when they are redeeming at par.

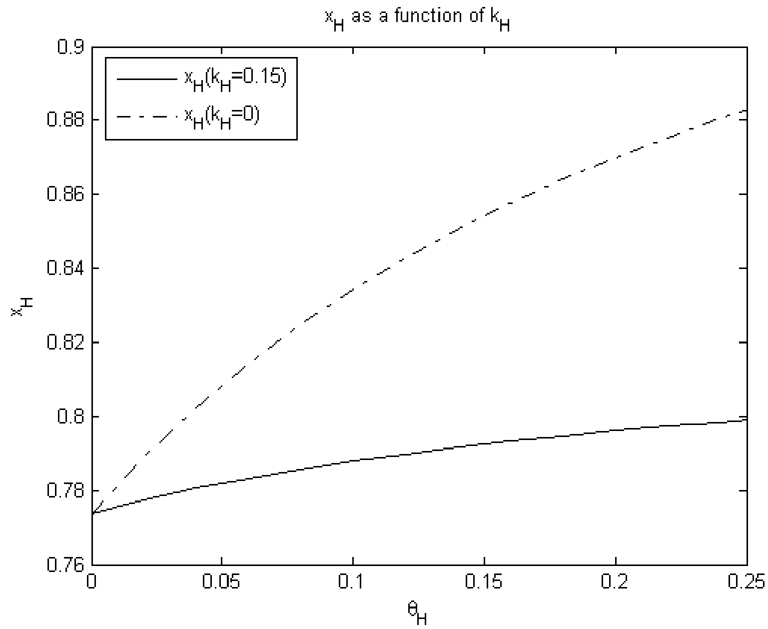

Figure 13: Quantity of goods exchanged for home note (x_H) when home bankers suspended or redeeming at par

Note that the difference increases with θ_H because as θ_H increases, the redemption option is a large fraction of the value of a banknote, and suspension reduces this option value. If we interpret the value of x_H when banks are redeeming at par as the value of specie, then this result can be interpreted as saying that banknotes go at a discount to specie when banks suspend. Such discounts are observed in the data and are one of the facts we want to explain.

6. Conclusion

In this paper we have constructed a model to qualitatively match some facts about the discounts on the notes—dollar-denominated promises to pay specie to the bearer on demand—issued by state banks in the United States prior to 1863. The model we construct builds on the basic search model of money by Shi (1995) and Trejos and Wright (1995). To attempt to account for locational differences in banknote discounts, our model has agents that come from two distinct locations. Each location also has bankers that can issue notes. We model the fact that banknotes had to be redeemed in specie on demand by adapting the Cavalcanti-Wallace (1999) innovation that banks are required to produce when a banknote is presented for redemption and their past actions are public information. This redemption possibility is one reason why banknotes are valued in our model. A second reason is that banknotes can also be used to purchase goods for other agents.

We present two versions of the model. In the first, banks are threatened with permanent autarky if they ever fail to produce the same quantity of goods when their notes are presented for payment as they obtain when issuing a note. We refer to this case as par redemption. In the second, banks are allowed to suspend payments on their notes by redeeming them below their issue value.

The specific goal of the paper was to explain five facts about the discounts on banknotes. One fact is that notes of local banks were always quoted at par. Our model is consistent with this fact, but trivially because we allow bankers and private agents with notes to make TIOLI offers. A second fact is that foreign banknotes usually were quoted at a discount to local banknotes, and this discount varied by the location of where this discount was being quoted. Our model is consistent with this fact. It is also consistent with a third fact: discounts were asymmetric across locations, meaning that the discounts quoted in location A on the notes of banks in location B generally differed from the discounts quoted in location B on the notes of banks in location A.

The last two facts involve banknote discounts when banks had suspended payment on their notes. One fact is that local banknotes were quoted at a discount to specie when local banks suspended. Even though specie does not explicitly appear in our model, it can be interpreted as being consistent with this fact if we interpret the value of specie in our model to be the value of banknotes when banks are redeeming. The second fact about the behavior of discounts when banks were suspended is that the discounts on foreign notes increased. Here the model is consistent with the data only under the assumption that local bankers have suspended to a much lesser degree than have nonlocal bankers. Otherwise, the model predicts that discounts will fall, not increase.

Thus, we think the overall performance of the model is quite good. It delivers predictions consistent with four of the facts we wanted to explain and consistent with the fifth fact in some special cases. Of course, we have only examined how well the model fits with the data in a qualitative sense. The next step, which we are beginning to pursue, is to see how the model's predictions match the actual quantitative discounts.

Appendix

A1. Proof of Lemma 1

Proof. A stationary distribution of banknote holdings must satisfy the following equations:

$$m_{H0}[\theta_H + (1-\theta_H)\pi P_F(m_{FH}\Lambda^H_{FH} + m_{FF}\Lambda^F_{FH})] = m_{HH}[\Gamma_H\theta_H + \tag{A-1}$$
$$+(1-\theta_H)\pi P_F m_{F0}\Lambda^H_{HF}] + (1-\theta_H)m_{HF}\pi P_F m_{F0}\Lambda^F_{HF}$$

$$m_{HH}[\Gamma_H\theta_H + (1-\theta_H)\pi P_F m_{F0}\Lambda^H_{HF}]$$
$$= m_{H0}[\theta_H + (1-\theta_H)\pi P_F m_{FH}\Lambda^H_{FH}] \tag{A-2}$$

$$m_{HF}m_{F0}\Lambda^F_{HF} = m_{H0}m_{FF}\Lambda^F_{FH} \tag{A-3}$$

$$m_{F0}[\theta_F + (1-\theta_F)\pi P_H(m_{HF}\Lambda^F_{HF} + m_{HH}\Lambda^H_{HF})] = m_{FF}[\Gamma_F\theta_F + \tag{A-4}$$
$$+(1-\theta_F)\pi P_H m_{H0}\Lambda^F_{FH}] + (1-\theta_F)m_{FH}\pi P_H m_{H0}\Lambda^H_{FH}$$

$$m_{FF}[\Gamma_F\theta_F + (1-\theta_F)\pi P_H m_{H0}\Lambda^F_{FH}]$$
$$= m_{F0}[\theta_F + (1-\theta_F)\pi P_H m_{HF}\Lambda^F_{HF}] \tag{A-5}$$

$$m_{FH}m_{H0}\Lambda^H_{FH} = m_{F0}m_{HH}\Lambda^H_{HF} \tag{A-6}$$

$$m_{H0} + m_{HF} + m_{HH} = 1 \tag{A-7}$$

$$m_{F0} + m_{FF} + m_{FH} = 1 \tag{A-8}$$

Equations (A-1) and (A-4) are redundant. Substituting (A-3) into (A-2) and (A-6) into (A-5), we obtain

$$m_{H0} = m_{HH}$$

$$m_{F0} = m_{FF}$$

■

A2. Existence of a Monetary SSSE

We show existence of a monetary SSSE for small values of θ_H and θ_F and a particular class of utility functions. The strategy is to guess that $\Lambda_{HF}^{H} = \Lambda_{FH}^{F} = \Lambda_{FH}^{H} = \Lambda_{HF}^{F} = 1$ are equilibrium strategies and then to verify under what conditions all the equilibrium conditions are satisfied. Note that this guess implies that under a par redemption equilibrium, $\Gamma_H = \Gamma_F = 1$.

LEMMA 3. If $\Lambda_{HF}^{H} = \Lambda_{FH}^{F} = \Lambda_{FH}^{H} = \Lambda_{HF}^{F} = 1$, then stationary note holdings are equal to $\frac{1}{3}$.

Proof. From Lemma 1 we have

(A-9) $m_{F0} = m_{FF}$

(A-10) $m_{H0} = m_{HH}$

Combining (A-9) with (A-3) yields

(A-11) $m_{HF} = m_{H0} = m_{HH}$

Combining (A-10) with (A-6) yields

(A-12) $m_{FH} = m_{F0} = m_{FF}$

Now combining (A-11) and (A-12) with (A-7) and (A-8), it follows that $m_{ij} = \frac{1}{3}, \forall i, j \in \{0, H, F\}$.

$\blacksquare$

We will focus only on the home market; for the foreign market the same arguments apply.

LEMMA 4. The quantities x_H and z_H are part of a monetary SSSE and $\Lambda^H_{HF} = \Lambda^F_{FH} = \Lambda^H_{FH} = \Lambda^F_{HF} = 1$ if they solve the following two equations and four incentive constraints.

(A-13) $\rho x_H = \pi\theta_H \left[u(x_H) - x_H\right] + (1-\theta_H)\frac{\pi^2}{3}\{P_H \left[u\left(x_H\right) - x_H\right] + P_F \left[u\left(z_H\right) - x_H\right]\}$

(A-14) $\rho z_H = (1-\theta_F)\frac{\pi^2}{3}\{P_H \left[u\left(x_H\right) - z_H\right] + P_F \left[u\left(z_H\right) - z_H\right]\}$

(A-15) $u(z_H) \geq x_H$

(A-16) $u(x_H) \geq z_H$

(A-17) $u(z_H) \geq z_H$

(A-18) $u(x_H) \geq x_H$

To show existence of equilibrium, we proceed in two steps. First we establish that a solution to (A-13) and (A-14) exists. Then we show that for small values of θ_H and θ_F, such a solution is incentive compatible.

LEMMA 5. Suppose $u(\cdot)$ is homogeneous of degree n, with $n \in (0, 1)$, $n = \frac{m}{k}$, where m and k are natural numbers. For any value of θ_H and θ_F in $[0, 1)$, there exists a solution (x_H, z_H) to equations (A-13) and (A-14).

Proof. Rewrite (A-13) as

$$\text{(A-19)}\left\{\rho+\pi\theta_H+(1-\theta_H)\frac{\pi^2}{3}\delta\right\}x_H=\left\{\pi\theta_H+\frac{\pi^2}{3}(1-\theta_H)P_H\right\}u(x_H)+\frac{\pi^2}{3}(1-\theta_H)P_F u\left(z_H\right)$$

and (A-14) as

$$\text{(A-20)}\left\{\rho+\frac{\pi^2}{3}\delta(1-\theta_F)\right\}z_H=\frac{\pi^2}{3}(1-\theta_F)P_H u(x_H)+\frac{\pi^2}{3}(1-\theta_F)P_F u(z_H).$$

The above system can be written as

$$\text{(A-21)}\, Ax=Bu(x)+Cu(z)$$

$$\text{(A-22)}\, Dz=Eu(x)+Fu(z)$$

with all six coefficients positive. Solving for (x,z) is equivalent to solving for (x,α) with $\alpha\in\mathbb{R}$, defined as $z=\alpha x$. We are looking for a pair (x,z) strictly positive, so we exclude the case $\alpha=0$. Substituting, we then have

$$\text{(A-23)}\, Ax=Bu(x)+C\alpha^n u(x)$$

$$\text{(A-24)}\, D\alpha x=Eu(x)+F\alpha^n u(x),$$

that is,

$$\text{(A-25)}\, x=\frac{B+C\alpha^n}{A}u(x)$$

$$\text{(A-26)}\, x=\frac{E+F\alpha^n}{D\alpha}u(x).$$

Given $x\neq 0$, the system becomes

$$\text{(A-27)}\, x=\Gamma u(x)$$

$$\text{(A-28)}\, \frac{B + C\alpha^n}{A} = \frac{E + F\alpha^n}{D\alpha} = \Gamma.$$

We know that (A-27) has a solution for any $\Gamma > 0$. We look at equation (A-28) and rewrite it as

$$\text{(A-29)}\, CD\alpha^{n+1} + BD\alpha - AF\alpha^n - AE = 0.$$

Since this is a continuous function of α, $p(\alpha)$, note that $p(0) = -AE < 0$ (so that $\alpha = 0$ is not a solution). Also, since $CD > 0$, $\lim_{\alpha\to\infty} p(\alpha) = \infty$. So a positive solution must exist. Moreover, since n is a rational number, we can change variables, setting $\hat{\alpha} = \alpha^{\frac{1}{k}}$, and (A-29) becomes

$$\text{(A-30)}\, CD\hat{\alpha}^{m+k} + BD\hat{\alpha}^k - AF\hat{\alpha}^m - AE = 0.$$

The above is now a polynomial of degree $m + k$ with the same zeros as (A-29). Also (A-30) has only one sign change, so by Descartes' sign rule, there can be only one positive solution. ■

Denoting the solution of (A-13) and (A-14) by $(\hat{x}(\theta), \hat{z}(\theta))$ with $\theta = (\theta_H, \theta_F)$, we now need to show for what parameter values such a solution is incentive compatible.

THEOREM 1. *If $\theta_H = \theta_F = 0$, a monetary SSSE exists and $\hat{x}(0) = \hat{z}(0) = \tilde{x}$, where*

$$\text{(A-31)}\, \left(\rho + \frac{\pi^2\delta}{3}\right)\tilde{x} = \frac{\pi^2\delta}{3}u(\tilde{x}).$$

Proof. When $\theta_H = \theta_F = 0$, the right-hand sides of (A-19) and (A-20) are equal. Given that $P_H + P_F = \delta$, then $\hat{x} = \hat{z} = \tilde{x}$. The existence of $\tilde{x} > 0$ can easily be proved by noting that the left-hand side and right-hand side of (A-31) are continuous, are equal at zero, and that u satisfies the Inada conditions. This is a standard argument in Trejos-Wright (1995) models. ■

Definition: Given the utility function $u(\cdot)$, define $\tilde{u} \in \mathbb{R}$ as the unique value such that $u(\tilde{u}) = \tilde{u}$.

Note that by (A-31), since $\left(\rho + \frac{\pi^2\delta}{3}\right) / \frac{\pi^2\delta}{3} > 1$, $\tilde{x} < \tilde{u}$. Define $\tilde{\omega} = u(\tilde{x}) - \tilde{x} > 0$. The strategy to show that $(\hat{x}(\theta), \hat{z}(\theta))$ are incentive compatible is as follows. We show that for small values of θ, $\hat{x}$ and $\hat{z}$ are arbitrarily close to each other and arbitrarily close to $\tilde{x}$, so that both are less than $\tilde{u}$. Finally, we show that arbitrarily close $\hat{x}$ and $\hat{z}$, together with being less than $\tilde{u}$, implies that the incentive constraints are satisfied.

The coefficients A, B, C, D, E, F are continuous in θ. The solution for α is the unique positive zero of (A-30), and since the zero of a polynomial changes continuously with its coefficients,[5] we conclude that $\hat{\alpha}$ changes continuously with θ. The solution $\hat{x}$ to (A-27) is given by

(A-32) $\hat{x} = (u(1)\Gamma)^{\frac{1}{1-n}}$,

which depends continuously on Γ, so $\hat{x}$ depends continuously on θ. There exists a unique solution for every $\theta \in [0, 1) \times [0, 1)$. We have that as $\theta \to 0$, (A-30) converges to

(A-33) $\hat{\alpha}^{m+k} + \hat{\alpha}^k - \hat{\alpha}^m - 1 = 0$,

so that $\hat{\alpha} = 1$ at $\theta = 0$, and for $\theta = 0$, $\hat{x} = \tilde{x}$. We then conclude that $(\hat{x}, \hat{\alpha}) \to (\tilde{x}, 1)$ as $\theta \to 0$, or in other words, $(\hat{x}, \hat{z}) \to (\tilde{x}, \tilde{x})$.

Define the distance in $\mathbb{R}^2$ between θ and 0 as $d(\theta, 0) = \|(\theta_H, \theta_F) - (0, 0)\|$.

[5] A statement of this theorem can be found in Beauzamy (1999).

Lemma 6. Let $\hat{x}$ and $\hat{z}$ be a solution to (A-13) and (A-14). Then for small $d(\theta, 0)$, $\hat{x}$ and $\hat{z}$ satisfy the incentive constraints (A-15) to (A-18).

Proof. Given $\tilde{x} < \tilde{u}$, and given $(\hat{x}, \hat{z}) \to (\tilde{x}, \tilde{x})$ for small values of d, we have that $\hat{x} < \tilde{u}$ and $\tilde{z} < \tilde{u}$ so that (A-17) and (A-18) hold. As before, let $\hat{z} = \hat{\alpha}\hat{x}$ so that we rewrite (A-15) and (A-16) as

$$\text{(A-34)}\; u(\hat{x}) \geq \frac{1}{\hat{\alpha}^n}\hat{x}, \qquad u(\hat{x}) \geq \hat{\alpha}\hat{x}.$$

But given $(\hat{x}, \hat{\alpha}) \to (\tilde{x}, 1)$ as $d \to 0$, the result follows for small enough d, given $\tilde{x} < \tilde{u}$. ■

Combining the previous results, we get the following theorem.

Theorem 2. *Suppose $u(\cdot)$ is homogeneous of degree n, with $n \in (0,1)$, $n = \frac{m}{k}$, where m and k are natural numbers. Then for small values of θ_H and θ_F, a monetary SSSE exists.*

References

Beauzamy, Bernard. 1999. How the roots of a polynomial vary with its coefficients: A local quantitative result. *Canadian Mathematical Bulletin* 42 (1): 3–12.

Cavalcanti, Ricardo and Wallace, Neil. 1999. A model of private bank-note issue. *Review of Economic Dynamics*, 2 (1): 104–36.

Gorton, Gary. 1999. Pricing free bank notes. *Journal of Monetary Economics*, 44 (1): 33–64.

Shi, Shouyong. 1995. Money and prices: A model of search and bargaining. *Journal of Economic Theory*, 67 (2): 467–96.

Trejos, Alberto and Wright, Randall. 1995. Search, bargaining, money, and prices. *Journal of Political Economy*, 103 (1): 118–41.