Strategic Trading with Market Closures*

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Abstract

This paper analyzes the equilibrium trading strategies of informed traders in the presence of market closures defined as periodic predictable stops of trading. We construct a dynamic auction model based on rational strategic behavior with asymmetric information across the agents. Empirical evidence indicates that market closures have important impact on the information structure of financial markets, in particular the private information flow. In our model, the insiders repeatedly increase their informational advantage over other agents by receiving private signals about fundamentals when the market is closed. In a continuous-time setting, we solve a dynamic programming problem and derive closed-form solutions for optimal intertemporal strategies of both insiders and the market maker. The key feature of insiders' optimal strategy is that they act strategically by anticipating future market closures. Because of this, even though market closures are periodic, the intertemporal pattern of optimal trading strategies is *not* periodic. This aperiodicity of trading is quite important since while it is a definitive feature of the data, it has been missing from the existing theoretical literature on market closures. In agreement with broad empirical evidence, we obtain a U-shaped pattern of trading volume during the periods when the market is open, superimposed on a U-shaped pattern during the lifetime of the economy, before all information about the asset is revealed.

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1 Introduction

There exists strong empirical evidence that market closures have an important impact on the information flow in financial markets (see e.g. French and Roll (1986), and Ito, Lyons and Melvin (1998)). In particular, private information obtained when the market is closed plays an important role in the trading process. This paper analyzes the consequences of the information structure imposed by market closures on several characteristics of the market, such as the aggregate trading intensity, trading volume, and market uncertainty.

Specifically, we construct a dynamic auction model based on rational strategic behavior with asymmetric information across different informed agents. There are three agent types in our model: the risk-neutral market maker (MM), insiders (informed traders), and liquidity (noise) traders. The market maker observes aggregate demand and sets price in a regret-free way. The informed traders strategically compete with each other to maximize their total profits. They do not know the fundamentals exactly, but learn about them through private signals. Price conveys information about the fundamentals, complementary to each of the informed traders' private information. For this reason, the informed traders' optimal strategies are conditioned on both private signals and the price.

We assume that when the market is closed, insiders keep receiving information about the fundamentals via their private signals, while the information set of the MM does not change. This allows informed traders to repeatedly increase their informational advantage over the MM. The effect of increasing information asymmetry during market closures is consistent with empirical evidence.[1] To emphasize this effect, we further assume that the insiders receive their private signals only during the periods when the market is closed.

To fully characterize the information and trading dynamics in such a setting, one must have a suitable technical understanding of how the informed traders compete when they are differentially endowed with private information which they repetitively upgrade over time. They must learn from past and current prices, fully recognizing that those prices reflected

[1]The evidence comes from the introduction of trading in Tokyo over the lunch hour. Ito, Lyons and Melvine (1998) find that lunch-return variance doubles with the introduction of trading, which cannot be due to public information since the flow of the public information did not change with the trading rules. They try to discriminate between the two alternatives: mispricing and private information and show that their evidence is broadly in favor of the private information inflow.

both the information of other traders and their attempts to learn from others. The informed traders must literally learn how to 'forecast the forecasts of others'. As the history grows, the nature of this forecast problem becomes overwhelming, seemingly requiring an infinite dimensional belief space. This problem alone has long proven a major roadblock in solving intertemporal trading models with differential information. On top of that, informed traders in our model receive multiple private signals and we must further determine how intensively they should trade on each private signal and the price. In this paper we develop a method to solve the above problem in our setting. Our method is quite effective, and allows to characterize the problem of optimal private information aggregation analytically. It also leads to a very intuitive solution for equilibrium, with full analytical solution available in the case of two informed traders.

The new information arrival and the strategic competition among informed traders are most crucial components of our model. When combined together, they give rise to several important results. First, the anticipated arrival of new information causes the strategic informed traders to use the old information more freely and, therefore, to compete more fiercely against each other than in the case of no market closures (see Back, Cao and Willard (2000), henceforth BCW, for comparison). For example, in contrast to the results of BCW, we find that in an economy with closures, two informed traders with uncorrelated signals typically trade *more* intensively than a monopolist trader with the same aggregate information.

Second, by increasing the competition among insiders, the dynamic injection of new information facilitates more efficient information transmission into prices. As a result, the model predicts a reduction in return volatility as compared to when market closures are absent. This is consistent with the empirical observations of French and Roll (1986) and Ito, Lyons and Melvin (1998).

Finally, our model predicts a rich aperiodic temporal U-shaped pattern in trading volume between each opening and closure.[2] This is one of the empirical stylized facts associated with market closures.[3] This aperiodicity of trading is quite important since while it is a definitive

[2]Our model is flexible enough to allow for combined daily and weekly closure intervals.
[3]See, for example, Chan et. al. (1996) and Jain and Joh (1988).

feature of the data, it has been missing from the existing theoretical literature on market closures. It arises quite naturally in our model due to the fact that informed traders act strategically by anticipating the future information inflows in their strategies. The intuition is quite transparent. Every time the market reopens, it briefly enters the so-called "rat race" phase.[4] During this phase, imperfectly informed insiders, who have not yet learned much from the price process about other insiders' information, compete and trade very aggressively, trying to quickly make use of their new private information. At later times, the price reflects more of the informed traders' private information. Each informed trader can make a better inference of the other private signals. At this stage, the informed traders have an incentive to trade less, in order not to reveal their own information. Therefore, trading intensity subsides during this phase. However, as the market approaches the time of closure, insiders once more have an incentive to increase their trading intensity to make use of their current private information. Altogether these leads to the U-shaped trading volume and since informed traders act strategically, the actual shape of this U-shape varies across openings.

Our paper essentially contributes to two areas of the market microstructure literature: trading in the presence of market closures and imperfect competition among informed traders.[5] There exists a rich theoretical literature that uses both strategic[6] and competitive settings to study the effect of market closures on trading and prices. Foster and Viswanathan (1990) examine a variation of a Kyle (1985) model with market closures. In their model, a closure allows private information to accumulate, altering the amount of adverse selection in the market and, thus, the equilibrium price process. While our model shares this feature with Foster and Viswanathan (1990), the crucial difference in our model is that we have multiple informed traders competing for profits, while Foster and Viswanathan consider a monopolist informed trader. In order to generate a time-varying trade pattern, they assume that there is a noisy public signal at the end of each trading period that reduces the information advantage

[4]See Foster and Viswanathan (1996) and BCW.

[5]The dynamic arrival of the new private information is what separates our paper from the work Holden and Subrahmanyam (1992), Foster and Viswanathan (1996), and BCW on the imperfect competition among insiders.

[6]This literature has been pioneered by Admati and Pfleiderer (1988, 1989) who, while not directly addressing market closures, analyze how investors' discretion in timing their liquidity trade can lead to the endogenous concentration of trades and price changes.

of the informed trader and makes him trade more aggressively than in Kyle (1985). The downside of such a setting is that while private information is long-lived, it gets attenuated quickly. In fact, an informed trader uses only the next day when choosing how much of her private information to carry forward in maximizing profits. Monday trades have no impact on Thursday trades, for instance. In our model, the private information is long-lived, in that each informed trader maximizes her *expected* profits over *all* trading periods until the public announcement date. Therefore, trading activity around market openings and closings is related in our model to both past and future openings and closings of the market. Moreover, our model allows for a closed form solution, while in Foster and Viswanathan (1990) only a numerical solution is possible.

Hong and Wang (2000) study trading patterns implied by periodic market closures in the economy with heterogeneously informed agents. In their model the better informed investors also receive additional information during market closures. Consequently, investors optimally adjust their trading strategies during the trading period in anticipation of the following market closures, which gives rise to time variation in equilibrium returns. However, Hong and Wong (2000) and others[7] consider competitive market setting. That is, informed agents are individually small price takers who ignore the price impact of their trades. Assuming that agents are informationally small circumvents individual strategic behavior, as agents do not have to trade off profit-taking against information release. This is the key difference between these models and ours, as we focus on the effect of the strategic competition among informed traders in the presence of market closures. The strategic behavior plays a crucial role in our model in generating aperiodic trading patterns, even for periodic market closures. That is, while the optimal trading strategies remain U-shaped in every period when the market is open, the quantitative structure of this U-shape changes systematically across periods. This finding is, for instance, in sharp contrast with Hong and Wang (2000) who obtain strictly periodic solution when market closures are periodic.

Methodologically, our paper is related to Taub, Bernhardt, and Seiler (2005), henceforth TBS. They also develop a method for analyzing informed trading when agents possess

[7] In the partial equilibrium model Brock and Kleidon (1992) point out the link between time-variation in market activity and closures. Slezak (1994) examines the impact of closure on equilibrium returns using a noisy rational expectation equilibrium setting.

long-lived private information about a firm's value and have repeated access to new information over time. In their model, every time period insiders receive heterogeneous private information through infinite sums of AR(1) processes. The firm may be liquidated at any time with a given probability, but the firm is liquidated at the prevailing market price and, therefore, the true value of the firm is never revealed. Such a *stationary* structure of the economy allows TBS to map informed agents optimization problems into the frequency domain. In the frequency domain they can use standard variational methods to find the optimal policy function. Essentially, they convert their problem, that has to be solved using conditional information, into an unconditional optimization problem. The whole problem then reduces to an Euler equation which takes the form of a Wiener-Hopf equation, which TBS solve to obtain optimal trading strategies.

There are several key differences between our model and that of TBS. First, we consider market closures and their effect on strategies of informed traders and TBS do not. Second, in the model of TBS informed traders both receive new information and trade every period. In our model, informed traders can trade continuously between signals which allows for much more of the informed traders' private information to be incorporated in the price between signals. As a result, our model features the "waiting game" phase of the market, when informed trading temporarily subsides.[8] This feature of the informed trading is missing from the model of TBS. Third, in our model, the value of the asset is fully revealed at some pre-specified future date, thus rendering all past private information useless. As a result, we have to solve the *non-stationary* optimization problem. As a consequence, the solution method developed in TBS is not applicable in our case since the Wiener-Hopf equation can be derived only for stationary problems. Thus, we develop an alternative approach for analyzing informed trade with dynamic information acquisition. We formulate the problem of insiders recursively. We then show that the optimal trading strategy for each informed trader should involve the linear combination of the current and past signals up until the current time period. In equilibrium, the weights of this linear combination are chosen by the informed traders in order to optimize their expected payoffs and can be defined explicitly.

The rest of the paper is organized as follows. The set-up of the model and the derivation

[8] See Foster and Viswanathan (1996) for more details.

of optimal trading strategies are presented in Section 2. The optimization problem of informed traders with market closures is formulated and solved in Section 3. The comparative dynamics and the implications of the model on the intensity of trading and on the distribution of payoffs across the informed traders are considered in Section 4. We conclude in Section 5.

2 The model

We consider an economy in which a single risky asset is traded in the financial market over the time period $[0, T)$. There are three types of risk-neutral agents in the economy: $N \geq 1$ informed traders, the market maker (MM), and a number of uninformed liquidity ("noise") traders. Informed traders trade continuously when the market is open. MM observes the aggregate order of informed and liquidity traders and sets the asset price, $P(t)$. The cumulative order of the liquidity traders per one insider is equal to $u(t) = \sigma_u Z(t)$, where $Z(t)$ is a standard Wiener process. The risk-free rate is set to zero.

We assume the financial market is closed periodically, and that there are a total of *K-1* closures and K openings before the true value of the asset is revealed at $t = T$. We will use t_k^o with $k = 0, ..., K-1$ to indicate the time of k-th opening and let $\mathbf{O} = \overset{K-1}{\underset{k=0}{\cup}} \mathcal{O}^k$, where $\mathcal{O}^k = \{t : t \in [t_k^o, t_k^c)\}$, denote the set of all times when market is open. Analogously, we will use t_k^c with $k = 0, ..., K-2$ to indicate the time of k-th closure and let $\mathbf{C} = \overset{K-2}{\underset{k=0}{\cup}} \mathcal{C}^k$, where $\mathcal{C}^k = \left\{t : t \in [t_k^c, t_{k+1}^o)\right\}$, denote the set of all times when market is closed. We will also use a subscript k throughout the paper to indicate that a parameter has a value specific to the k-th opening and closure. Note that our model is not limited to this "equidistant" choice of trading and non-trading intervals. Since we are focusing on the trading process, we will consider all quantities as a function of the "event time" when the non-trading periods are omitted. However, we will take into account the changes of the informed traders' information sets during the non-trading periods. This does not lead to any loss of generality and is merely a convenient notation.

We construct the private information acquisition of informed traders as follows. At $t = 0$ as well as during each period when the market is closed, $t \in \mathcal{C}^k$, each informed trader i receives a private signal, S_i^k. Informed traders receive no private signals when the market is open. The true value of the asset is revealed at T. By using this structure, we capture the important fact that there exists a potentially large inflow of information independent of the organizational structure of the financial market: investors keep receiving private information about assets when the market is closed and all trading activity is suspended. This complete separation of periods of private information acquisition from trading periods enables us to investigate the essential effect of market closures on trading activity while allowing for closed-form analytic solutions of the optimal strategies.[9] Moreover, these assumptions together with the fact that the liquidation value of the asset is fully revealed at T imply that, contrary to the case considered by TBS, we are dealing with a non-stationary problem.

The next set of assumptions highlights the informational structure of our model. At initial moment $t = 0$, there is a random draw of the fundamental value V of the asset from the normal distribution with mean zero and variance σ_s^2. We assume that the private signals at time t_k $(k = 0, ..., \ K-1)$ are given by

$$S_i^k = V + \sigma_s \left(\varepsilon_i^k - \frac{1}{N} \sum_{j=1}^{N} \varepsilon_j^k \right), \tag{1}$$

with $\varepsilon_i^k \sim \mathcal{N}(0,1)$ being uncorrelated for different time periods, and correlated in the cross-section

$$\begin{aligned} \operatorname{corr}(\varepsilon_i^k, \varepsilon_j^k) &= \rho, \quad (\forall k = 0, ..., \ K-1, \ i \neq j), \\ \operatorname{corr}(\varepsilon_i^k, \varepsilon_j^{k'}) &= 0, \quad (\forall i, j, \ \forall k = 0, ..., \ K-1, \ k' = 0, ..., \ K-1, \ k \neq k'). \end{aligned} \tag{2}$$

Assumption (1) implies, in the spirit of Kyle (1985) and analogous to BCW, that the arithmetic mean of the signals obtained by all insiders during every closure period, does not change with time and equals the fundamental value

$$V = \frac{1}{N} \sum_{j=1}^{N} S_j^k, \ k = 0, ..., \ K-1, \tag{3}$$

and that the relation (3) is satisfied for each closure period. To make the information structure analogous to BCW, we also assume that the informed traders' as well as the market maker's valuations *before* the private signals at $t = 0$ are observed is given by

$$V \sim \mathcal{N}\left(0, \ \phi\sigma_s^2\right), \tag{4}$$

[9] When this assumption is relaxed, the model can still be solved numerically. However, this would not change the main conclusions.

where

$$\phi = \frac{1 + (N-1)\,\rho}{N}. \tag{5}$$

From the definition (1), it follows that the conditional marginal distribution of the private signals obtained at the period k is given by

$$S_i^k|V \sim \mathcal{N}\left(V, (1-\phi)\,\sigma_s^2\right),\ \ k = 0, ...,\ K-1, \tag{6}$$

with

$$\begin{aligned} \mathrm{cov}(S_{i\neq j}^k, S_j^k|V) &= (\rho - \phi)\,\sigma_s^2, \\ \mathrm{var}(S_i^k|V) &= (1-\phi)\,\sigma_s^2, \end{aligned} \tag{7}$$

as well as that, conditioned on V, private signals are i.i.d. across time. Combining (4) and (7), we obtain for the unconditional correlations of the private signals

$$\mathrm{corr}(S_{i\neq j}^k, S_j^k) = \rho, \tag{8}$$

which is also analogous to the BCW setting. The details of the calculation are provided in Appendix A.

The prior distribution (4) is a common knowledge, and it is, therefore, known to the MM. Distributions (6) and (4) determine the dynamics of the Bayesian updating process of informed traders in our model. Appendix A provides full details of this Bayesian updating process.

At each trading period, the MM sets $P(t)$ to be regret-free based on her current beliefs and the observation of the total order flow which is a function of the aggregate private signal. As a result, informed traders may learn about the aggregate private signal from the price while not being able to observe it directly. Therefore, in contrast to BCW, informed traders in our model update their information set over time using two *independent* but *complementary* sources: by learning about the aggregate signal from the price $P(t)$ during periods when the market is open, and also by learning from their own private signals during the market closure periods.[10] This is an important feature of our model since it makes the

[10]The first feature is essentially present in the standard Kyle (1985) model. However, in Kyle (1985) and further work based on it, (e.g. Holden and Subrahmanyam (1992) and BCW), the combined impact of both sources of learning has not been addressed.

learning process of insiders more efficient than that of the MM, who can only update her beliefs at times when informed trade occurs. The more efficient information updating by insiders causes their superior informativeness relative to that of the MM to be comparatively long-lived in our model, which, in turn, has important pricing implications.

2.1 Equilibrium Strategies

To fully capture the effect of competition among informed traders on trading as well as to obtain the results in a compact and parsimonious form, we cast the problem in a continuous time setting. According to the above discussion, we look for the dynamic equilibrium strategies when the i-th informed trader's incremental demand, $dx_i(t)$, during the time interval dt depends on her signal and the instantaneous price, $P(t)$. As it is analyzed in Foster and Viswanathan (1996) , such a specification allows us to avoid the problem of "forecasting of the forecasts" which occurs when the informed traders' strategies are based on their valuations. On the other hand, the optimal informed traders' strategies are expressed in terms of the instantaneous valuations and the price at equilibrium.

Following Foster and Viswanathan (1996) and BCW, the dynamic trading strategy of the i-th informed trader when her incremental demand $dx_i(t)$ is linear in her private signal as well as the instantaneous price, $P(t)$, and is given by

$$dx_i(t) = \beta(t)\left[\widehat{S}_i(t) - P(t)\right] dt, \tag{9}$$

where the effective private signal, $\widehat{S}_i(t)$, is a noisy estimate of the fundamental, based on the private signals available at the time t. In contrast to Foster and Viswanathan (1996) and BCW, informed traders in our model receive multiple private signals at times when the market is closed. Since taking into account both past and current signals increases the precision of the effective signal, the optimal trading strategy for each informed trader should involve an optimally weighted combination of the current and past signals. By assumption, the arrival of the private signals is completely separated from the trading process. For this reason, each insider constructs his effective signal independently of the trading process. Therefore, the effective signals change during the market closure periods, and remain constant when the market is open. Note that this is in a sharp contrast with the results of TBS, where the fundamentals receive shocks every period, leading to constant updating of the informed traders' information sets.

For that reason, the effective signal at period $t \in \mathcal{O}^k$, is given by

$$\widehat{S}_i(t) = \sum_{l=0}^{k} a_{l,k} S_i^l, \tag{10}$$

where the weights $\{a_{l,k},\ l = 0, ...k\}$, are chosen by informed traders to maximize the precision of their effective signals and will be derived in equilibrium in the next section. It will also be shown that these weights satisfy the normalization condition $\sum_{l=0}^{k} a_{l,k} = 1$. Note that at each closure period $k = 0, ..., K$, the informed traders update all the weights $\{a_{l,k},\ l = 0, ...k\}$. Therefore, the weights are different during different periods when the market is open.

The MM observes the total incremental demand per one informed trader

$$dy(t) = \frac{1}{N} \sum_{i=1}^{N} dx_i(t) + du = \beta(t) [V - P(t)]\, dt + \sigma_u dZ, \tag{11}$$

where we have used that

$$\frac{1}{N} \sum_{i=1}^{N} \widehat{S}_i(t) = V \sum_{l=0}^{k} a_{l,k} = V. \tag{12}$$

Let $\mathbf{F} \equiv \{\mathcal{F}(t) : t \in \mathbf{O}\}$ denote the filtration generated by $y(t)$ and set $P(t) = \mathrm{E}[V \mid \mathcal{F}(t)]$. We interpret $\mathbf{F}$ as the MM's information structure. Analogous to BCW, the market maker's uncertainty $\Sigma_M(t)$ and the price are revised according to

$$d\left(\frac{1}{\Sigma_M(t)}\right) = \frac{\beta(t)^2}{\sigma_u^2} dt, \tag{13}$$

and

$$dP(t) = \lambda(t)\, dy(t), \tag{14}$$

where the inverse market depth parameter is given by

$$\lambda(t) = \frac{\beta(t)}{\sigma_u^2} \Sigma_M(t). \tag{15}$$

At time $t = 0$, the market maker starts with

$$\Sigma_M(0) = \phi \sigma_s^2,$$

$$P(0) = 0.$$

Next we need to make a similar analysis of the information structure of informed traders. Let $P(t)$ denote the solution (assumed to exist) of equation (14). The informed trader i observes $P(t)$ and her effective signal $\widehat{S}_i(t)$. Let $\mathbf{F}_i \equiv \{\mathcal{F}_i(t) : t \in \mathbf{O} \cup \mathbf{C}\}$ denote the filtration generated by $P(t)$ and $\widehat{S}_i(t)$, and set $V_i(t) = \mathrm{E}[V \mid \mathcal{F}_i(t)]$. This is the informed trader's information structure. Analogous to BCW, we obtain that each insider's residual uncertainty $\Sigma_i(t)$ and valuation $V_i(t)$ are described by the processes

$$d\left(\frac{1}{\Sigma_i(t)}\right) = \frac{\beta(t)^2}{\sigma_u^2} dt, \tag{16}$$

and

$$dV_i = \frac{\beta(t)\Sigma_i\left[\frac{1}{\lambda(t)}dP(t) + \beta(t)(P(t) - V_i(t))\,dt\right]}{\sigma_u^2}. \tag{17}$$

At time $t = 0$ informed trader i starts with

$$\Sigma_i(0) = (1-\phi)\,\phi\sigma_s^2 = (1-\phi)\,\Sigma_M(0), \tag{18}$$
$$V_i(0) = \phi S_i^0.$$

The revision processes for both MM's and the insiders' information sets follow immediately from an application of the Kalman-Bucy filter (see Lipster and Shiryaev (2000)).

Note that (18) describes the i-th informed trader's best estimate of the aggregate signal, as has been pointed out by Foster and Viswanathan (1996) and BCW. If the informed trader is perfectly informed, i.e. $\phi = 1$, then $\Sigma_i(0) = 0$. The factor ϕ measures, therefore, the initial informativeness of the informed trader relative to the MM. Making use of the characterizations for the informed trader's strategy and the price process given above, we obtain

$$dP(t) = \lambda(t)\left\{\beta(t)\left[V - P(t)\right]dt + \sigma_u dZ(t)\right\}, \tag{19}$$

and

$$dV_i(t) = \left[1 - \delta(t)\right]\lambda(t)\left\{\beta(t)\left[V - V_i(t)\right]dt + \sigma_u dZ(t)\right\}, \tag{20}$$

where the information spread parameter $\delta(t) = 1 - \frac{\Sigma_i(t)}{\Sigma_M(t)}$ characterizes the information asymmetry between the MM and the informed traders. From (18), we obtain

$$\delta(0) = \phi, \tag{21}$$

analogous to BCW. It follows from (13) and (16) that both price and the informed trader's valuation approach the true value V as time increases. This reflects the fact that the informed traders and the MM learn about the fundamentals over time. As we will see below, the individual rates of learning may depend on time in a complex way.

2.2 The Effective Signals

Comparing equations (13) and (16), one can see that the MM's and the informed trader's learning processes are equally efficient when the market is open and the informed traders do not receive any private signals. However, the MM's beliefs have less precision at the market re-opening periods, since the MM observes the aggregate demand but does not receive a private signal. The MM's information set depends on the liquidity parameter $\beta(t)$, which has to be defined in a self-consistent way by optimizing the informed trader's expected payoffs. Since, by assumption, the MM does not learn during the closure periods, the market maker's uncertainty $\Sigma_M(t)$ is continuous across the closures

$$\Sigma_M(t_k^c) = \Sigma_M\left(t_{k+1}^o\right), \qquad k = 0, ..., \ K-2. \tag{22}$$

Since the informed trader is learning from both the price process and the private signals, one should expect that her valuation can be expressed in terms of these two quantities. These relations are formalized in Proposition 1 below.

Proposition 1 *The proposition consists of three main results.*

1. If we define the relative valuation of the i-th informed trader as

$$\widehat{V}_i(t) \equiv V_i(t) - (1-\delta) P(t), \tag{23}$$

then

$$d\widehat{V}_i(t) = -k(t)\, \widehat{V}_i(t) dt, \tag{24}$$

with $k(t) = [1-\delta(t)]\, \lambda(t)\, \beta(t)$.

2. The informed trader's valuation $V_i(t)$ *after the market reopens at* $t = t_k^o$ *is related to the price and the effective private signal by*

$$V_i(t) = [1-\delta(t)]\, P(t) + \delta(t)\, \widehat{S}_i(t_k^o), \ t \in \mathcal{O}^k, \tag{25}$$

where the effective signal is defined recursively by

$$\widehat{S}_i\left(t_{k+1}^o\right) = \mu_{k+1}\widehat{S}_i\left(t_k^o\right) + \left(1-\mu_{k+1}\right)S_i^{k+1}, \tag{26}$$

with

$$\mu_{k+1} = \frac{1-\delta\left(t_{k+1}^o\right)}{1-\delta\left(t_k^c\right)}\frac{\delta\left(t_k^c\right)}{\delta\left(t_{k+1}^o\right)}, \tag{27}$$

and the initial condition

$$\mu_0 = 0. \tag{28}$$

3. The dynamics of $\delta\left(t\right)$ *are described by the ODEs*

$$\begin{aligned} \frac{d}{dt}\delta\left(t\right) &= -\delta\left(t\right)\left[1-\delta\left(t\right)\right]\lambda\left(t\right)\beta\left(t\right), \quad t\in\mathcal{O}^k,\ \forall k, \\ \frac{1}{1-\delta\left(t_{k+1}^o\right)} &= \frac{1}{1-\delta\left(t_k^c\right)} + \frac{\Sigma_M\left(t_k^c\right)}{\left(1-\phi\right)\sigma_s^2}, \quad \forall k. \end{aligned} \tag{29}$$

Proof: See Appendix B.

Note that the second equation in (29) describes the dynamics of $\delta(t)$ due to the private learning process. If the private learning is absent, the dynamics of $\delta(t)$ are fully described by the first ODE in (29), and the information asymmetry is a monotonically decreasing function of time. In the presence of private learning, $\delta(t)$ increases during the market closure periods due to the private learning and thus is no longer a monotonically decreasing function of time. This makes the temporal pattern of $\delta(t)$ non-monotonic in our model. Therefore, the information asymmetry decreases when the market is opened and increases during the market closure periods. This result has a simple intuition. Namely, the informed trader's valuation is based on both the price process and her private signal. From equation (25), it is clear that the information asymmetry parameter $\delta(t)$ characterizes the relative weights of each of these two factors. During the periods when the market is open, the price process acquires more weight in the informed traders' information sets and therefore the asymmetry $\delta(t)$ should decrease. By the same argument, $\delta(t)$ should increase during the market closure periods, when the informed traders only learn from the private signals. As we will see below, this is the key reason for the increasing trading activity after the market re-openings.

After the market reopens at t_k^o, the trading strategies for the period $\mathcal{O}^k$ are characterized by

Corollary 1 *The trading strategy of the i-th informed trader during the k-th market opening is*

$$\theta_i(t) = \beta(t)\left[\widehat{S}_i(t_k^o) - P(t)\right] = \frac{\beta(t)}{\delta(t)}\left[V_i(t) - P(t)\right]. \tag{30}$$

Proof. Immediately follows from (25).

The dynamics of the asymmetry parameter $\delta(t)$ are characterized by the following proposition.

Proposition 2 *The information asymmetry $\delta(t)$ satisfies*

$$\frac{1}{\delta(t)} = 1 + \frac{(1-\phi)\sigma_s^2}{(k+1)\Sigma_M(t)},\ t \in \mathcal{O}^k,\ k = 0, ..., K-1. \tag{31}$$

Proof: See Appendix B.

With the help of Proposition 2 we can now fully characterize the weights of the effective signal, $a_{l,k}$.

Proposition 3 *The weights of the effective signal are given by*

$$a_{l,k} = \frac{1}{k+1}, \qquad l = 0, ...k. \tag{32}$$

Proof: See Appendix B.

Equation (32) implies that all optimal weights for the signals obtained at different times enter with *equal weights* into the effective signal. This is a consequence of the fact that all private signals obtained by each insider are drawn from the same distribution, and is consistent with the result that the equally-weighted average is the optimal linear unbiased estimator of the fundamental. At the same time, as it follows from (25), the insider's valuation also involves the factor related to the information asymmetry which exhibits non-monotonic and non-periodic dynamics over the different market closure periods. This is in sharp contrast to the model of TBS, where a stationary time structure of the information inflow has been assumed.

Now, we are in a position to formulate and solve the dynamic optimization problem for the informed traders.

3 Dynamic Optimization and Equilibrium

We start by defining the set of feasible trading strategies for informed traders. We define a trading strategy θ_i to be feasible for trader i if there exists a unique solution $P(t)$ to the stochastic differential equation (19) for the given λ and for the given β that characterize the other traders' strategies and if

$$\lim_{t\to T} P(t) \text{ exists a.s.,} \tag{33}$$

$$\int_0^T \theta_i(t)dt \text{ exists a.s.,} \tag{34}$$

and

$$\mathrm{E}\left[\int_0^T P^2(t)dt|V\right] < \infty. \tag{35}$$

These limits define, respectively, the price and number of shares held by trader i just before the announcement. Condition (35) is the "no doubling strategies" condition introduced in Back (1992).

Making use of Propositions 1 and 2, we can analyze the optimal trading intensity for the informed traders which characterize their optimal strategies. Proposition 4 below summarizes these results.

Proposition 4 *The Bellman equation for the informed trader's value function*

$$\pi(P,t) = \max_{\theta(t)} \left\{ \mathrm{E}_t \left[\int_t^T dt' \theta(t') \left[V_i(t') - P(t')\right] \right] \right\}, \tag{36}$$

is given by

$$\max_{\theta} \left\{ \left(\frac{\partial}{\partial t} + L_D^{\theta} \right) \pi(P,t) + \theta \left[V_i - P\right] \right\} = 0, \tag{37}$$

where the optimization is over the set of feasible strategies and the Dynkin operator is defined by

$$L_D^{\theta} = \left[\lambda\beta \left(V - \frac{1}{N}\widehat{S}_i - \frac{N-1}{N}P \right) + \frac{1}{N}\lambda\theta \right] \frac{\partial}{\partial P} + \frac{1}{2}\lambda^2\sigma_u^2 \frac{\partial^2}{\partial P^2}.$$

Proof: See Appendix B.

Analogous to BCW, the equilibrium solution exists if

$$\frac{1}{N}\frac{\partial \pi}{\partial P} = \frac{\left[P - V_i\right]}{\lambda(t)}. \tag{38}$$

Making use of (38) and partially differentiating (37) with respect to P yields

$$(P - V_i)\,\eta + \beta\left(V_i - \frac{1}{N}\widehat{S}_i - \frac{N-1}{N}P\right) - \beta\left(\frac{N-1}{N}\right)(P - V_i) = 0, \tag{39}$$

where $\eta(t) = \frac{d}{dt}\left(\frac{1}{\lambda(t)}\right)$. Combining (39) and (25), we finally obtain the dynamics of the inverse market depth parameter $\lambda(t)$ summarized by Proposition 5 below.

Proposition 5 *When the optimal strategies are exercised, the dynamics of the market depth parameter are described by the ODE*

$$\frac{d}{dt}\left(\frac{1}{\lambda(t)}\right) = \left(2N - 1 - \frac{1}{\delta(t)}\right)\frac{\beta(t)}{N}, \tag{40}$$

and

$$\lim_{t\to T}\Sigma_M(t) = 0. \tag{41}$$

Proof: See Appendix B.

An equilibrium with market closures is defined as follows.

Definition 1 *An equilibrium with market closures is a set of* $\{\beta_k(t), \lambda_k(t)\}_{k=0}^{K-1}$ *that are continuous on* $t \in [0, T)$ *and continuously differentiable on* $t \in \mathbf{O}$*, with* β_k *and* λ_k *positive* $\forall k$*, and which satisfy (i)* $P(t) = V_M(t)$ *for all* t*, and (ii) the trading strategy* (30) *is feasible and maximizes the expected profits of the informed trader given by* (36) *over the set of feasible strategies.*

We would like to note that the continuity of $\{\beta_k(t)\}_{k=0}^{K-1}$ and consequently the inverse market depth $\{\lambda_k(t)\}_{k=0}^{K-1}$ across the closures is the dominating strategy for the informed traders.[11] The intuition behind this is simple. According to (47) the inverse depth of the market depends on the amount of information in the order flow, which depends on the amount of information outstanding measured by Σ_M and the trading intensity of the informed traders measured by β. Suppose that the informed traders increase their trading intensity β_k after they receive new signals over the market closure period $\mathcal{C}^k$, $\beta_k(t_k^o) > \beta_k(t_{k-1}^c)$. This would increase λ_k and, consequently, the informed traders' expected payoffs in a short run. However, the greater intensity of trading would lead to greater learning by the market and faster declining Σ_M. Consequently, the expected profits of the informed traders would decline more in a long run than in the case of continuous β_k. Overall, the extra long-run

[11]The proof is available from the authors upon request.

expected profit losses from increasing β_k during the market closure outweigh the short-run expected profit gains from this strategy. Therefore, the informed traders are better off by not "tipping" their hands early and keeping β_k continuous over the market closures.

We put some extra structure on market closures by denoting the durations of the periods when the market is open or closed as τ and $\Delta\tau$, respectively (see Figure 1), so that

$$t_k^o = k(\tau + \Delta\tau), \qquad (42)$$
$$t_k^c = t_k^o + \tau = k(\tau + \Delta\tau) + \tau.$$

As we have discussed above, we use the event time, which formally leads to $\Delta\tau = 0$. Our solution is obtained by solving the ODE (40) with the help of Propositions 1, 2, and 4.

Theorem 1 *If there are multiple informed traders ($N > 1$), there is a unique linear equilibrium. Set $\Sigma_M(0) = \phi\sigma_s^2$ and define the constants $\{C_k\}_{k=0}^{K-1}$ as follows*

$$C_k = \frac{1}{\tau}\int_{r_k}^{r_{k+1}} x^{2(N-2)/N} \exp\left[-\frac{2x(1-\phi)}{N\phi(k+1)}\right] dx, \qquad k = 1, ..., K-2, \qquad (43)$$

with

$$C_0 = \frac{1}{\tau}\int_{1}^{r_1} x^{2(N-2)/N} \exp\left[-\frac{2x(1-\phi)}{N\phi}\right] dx, \qquad (44)$$

and

$$C_{K-1} = \frac{1}{\tau}\int_{r_{K-1}}^{\infty} x^{2(N-2)/N} \exp\left[-\frac{2x(1-\phi)}{N\phi K}\right] dx, \qquad (45)$$

where $r_k = \Sigma_M(0)/\Sigma_M(t_k^o)$. The constants $\{C_k\}_{k=1}^{K-1}$ and $\{r_k\}_{k=0}^{K-1}$ are defined by the recursion relations

$$C_k = C_{k-1}\exp\left[\frac{2r_k(1-\phi)}{N\phi}\left(\frac{1}{k} - \frac{1}{k+1}\right)\right].$$

For each $t \in \mathcal{O}^k$, $k = 0, ..., K-1$, define $\Sigma_M(t)$ as

$$\int_{r_k}^{r(t)} x^{2(N-2)/N} \exp\left[-\frac{2x(1-\phi)}{N\phi(k+1)}\right] dx = (t - t_k^o)C_k, \qquad (46)$$

with $r(t) \equiv \Sigma_M(0)/\Sigma_M(t)$, and define $\Sigma_M(t) = \Sigma_M(t_k^c)$ for each $t \in \mathcal{C}^k$, $k = 0, ..., K-1$. The equilibrium is $\{\beta_k(t), \lambda_k(t)\}_{k=0}^{K-1}$, where

$$\beta_k(t) = \sigma_u (C_k)^{1/2}\left(\frac{\Sigma_M(t)}{\Sigma_M(0)}\right)^{(N-2)/N} \exp\left[\frac{(1-\phi)}{N\phi(k+1)}\frac{\Sigma_M(0)}{\Sigma_M(t)}\right], \qquad (47)$$
$$\lambda_k(t) = \beta_k(t)\frac{\Sigma_M(t)}{\sigma_u^2},$$

for $t \in \mathcal{O}^k$. Finally, $\{C_k, \Sigma_M(t_k^c)\}_{k=0}^{K-1}$ is fully determined by the Proposition 5, system (46)*, and the continuity of $\Sigma_M(t)$ across market closures*

$$\Sigma_M\left(t_{k+1}^o\right) = \Sigma_M\left(t_k^c\right), \; k = 0, ..., K-2. \tag{48}$$

Proof: See Appendix B.

In the case of a monopolist insider, we recover the results of Kyle (1985). The intuition is that, in the monopolist case, the insider has the exact information of the fundamentals at the very beginning of the trading process when $t = 0$, and therefore the additional signals obtained during the closure periods are irrelevant. Therefore, the problem becomes identical to the one considered in BCW with $N = 1$, which reduces to the case of Kyle (1985). This is summarized in the corollary below.

Corollary 2 (Monopolist informed trader) *Consider $N = 1$ and $K - 1$ closures. The unique equilibrium for the monopolist is given by $\{\beta_k(t), \lambda_k(t)\}_{k=0}^{K-1}$, where*

$$\begin{aligned} \beta_k(t) &= \frac{\sigma_u}{\sigma_s}\frac{1}{\sqrt{T}}\frac{1}{1-\frac{t}{T}}, \\ \lambda_k(t) &= \frac{\sigma_s}{\sigma_u}\frac{1}{\sqrt{T}}, \qquad \forall k = 0, ..., K-1. \end{aligned} \tag{49}$$

The market maker's variance for each $k = 0, ..., K$, is

$$\Sigma_M(t) = \Sigma_M(0)\left(1 - \frac{t}{T}\right), \tag{50}$$

where $T = K\tau$.

Proof: See Appendix B.

Analogous to BCW, the case of two informed traders in our model allows for an explicit analytical solution. The main reason for this is that for $N = 2$, the integrals (46) can be evaluated explicitly. The results for the duopoly case are presented in the corollary below.

Corollary 3 *Consider $N = 2$. Define constants $\{A_k\}_{k=0}^{K}$ and $\{r_k\}_{k=0}^{K}$ as follows*

$$\begin{aligned} A_K &= +\infty, \\ A_k &= \left(\sum_{n=k+1}^{K}\frac{1}{n}\right)^{-1}, \qquad k = 0, ..., K-1, \end{aligned} \tag{51}$$

and

$$r_0 = 1, \qquad r_K = +\infty, \tag{52}$$
$$r_k = 1 + \sum_{n=1}^{k} \frac{n\phi}{1-\phi} \ln\left(1 + \frac{A_n}{n}\right), \qquad k = 0, ..., K-1.$$

The unique equilibrium is given by $\{\beta_k(t), \lambda_k(t)\}_{k=0}^{K-1}$, *where, in each of the intervals* $t \in \mathcal{O}^k$, *the coefficients* $\{\beta_k(t), \lambda_k(t)\}_{k=0}^{K-1}$ *are defined by* (47), *and the market maker's variance for each* $k = 0, ..., K-1$, *is*

$$\Sigma_M(t) = \Sigma_M(0) \left\{ r_k + \frac{(k+1)\phi}{1-\phi} \ln\left[\left(1 - \frac{(t - t_k^o) A_k}{k+1}\right)^{-1}\right]\right\}^{-1}, \ t \in \mathcal{O}^k. \tag{53}$$

Proof: See Appendix B.

4 Comparative Dynamics

The result (40) looks similar to the one from BCW. As we have discussed above, the key difference is that in our case the dynamics of the asymmetry parameter $\delta(t)$ are different due to the private signals received during market closures. In particular, $\delta(t)$ undergoes jumps after the market re-openings due to the informed traders' private learning over the market closure periods. These jumps cause jumps in the aggregate trading volume.

The contribution to the trading volume of the i-th informed trader can be measured by unconditional variance

$$TV_i(t) = \operatorname{var}\left[\frac{dx_i(t)}{dt}\right] = \beta^2(t) \operatorname{var}\left[\widehat{S}_i(t) - P(t)\right], \tag{54}$$

where $dx_i(t)$ is the incremental demand during the time interval dt. Since the price increments in (19) depend on the aggregate signal $\frac{1}{N}\sum_{j=1}^{N} \widehat{S}_j(t) = V$, and the liquidity trades, they are unconditionally correlated with all individual signals $\left\{\widehat{S}_j(t)\right\}$. Therefore, (54) yields

$$TV_i(t) = \beta^2(t) \left(\operatorname{var}\left[\widehat{S}_i(t)\right] + \operatorname{var}\left[P(t)\right] - 2\operatorname{cov}\left[\widehat{S}_i(t), P(t)\right]\right). \tag{55}$$

In order to evaluate (55), we will prove the following result analogous to the one obtained by BCW.

Corollary 4 *At each time* $t \in [0, T)$, *the price process* (19) *takes the form*

$$P(t) = \Sigma_M(t) \int_0^t \left\{ \frac{1}{N} \sum_{j=1}^{N} \widehat{S}_j(t') d\left(\frac{1}{\Sigma_M(t')} \right) + \frac{\lambda(t')}{\Sigma_M(t')} \sigma_u dZ(t') \right\}. \tag{56}$$

Proof: See Appendix B.

Note that (56) describes how the price accumulates the aggregate private signals of the informed traders and the liquidity trades over time. Clearly, the private signals enter the price linearly, but with the time-dependent weights, and the price movements are unconditionally correlated. Since the valuation of each informed trader is based on both the effective private signal and on the current price by (25), the current valuations can also be represented as a linear functional of all the available private signals with the time-dependent weights.

Define an integer function

$$n(t) = \left[\frac{t}{\tau} \right] + 1, \tag{57}$$

where $[\cdot]$ stands for the integer part of the argument. Clearly, $n(t)$ corresponds to the number of private signals obtained by each informed trader up until the time t. With this definition and making use of the above corollary, we obtain the following

Corollary 5 *The trading volume defined by* (54) *is given by*

$$\begin{aligned} TV_i(t) &= T_1(t) + T_2(t) + T_3(t), \\ T_1(t) &= \beta^2(t) \sigma_s^2 \frac{\phi k + 1}{k + 1}, \\ T_2(t) &= -2\phi\sigma_s^2 \frac{\beta^2(t)}{n(t)} \left(1 - \frac{\Sigma_M(t)}{\Sigma_M(0)} \right), \end{aligned} \tag{58}$$

with

$$\begin{aligned} T_3(t) &= 2\phi\sigma_s^2 \beta^2(t) \Sigma_M^2(t) \Pi(t), \\ \Pi(t) &= \frac{1}{n(t)} \left(\frac{1}{\Sigma_M(t)} - \frac{1}{\Sigma_M((n(t) - 1)\tau)} \right) \times \\ &\quad \left[\frac{1}{2} \left(\frac{1}{\Sigma_M(t)} + \frac{1}{\Sigma_M((n(t) - 1)\tau)} \right) - \frac{1}{\Sigma_M(0)} \right] \\ &\quad + \sum_{k=1}^{n(t)} \frac{1}{k} \left(\frac{1}{\Sigma_M(k\tau)} - \frac{1}{\Sigma_M((k - 1)\tau)} \right) \times \\ &\quad \left[\frac{1}{2} \left(\frac{1}{\Sigma_M(k\tau)} + \frac{1}{\Sigma_M((k - 1)\tau)} \right) - \frac{1}{\Sigma_M(0)} \right]. \end{aligned} \tag{59}$$

Proof: See Appendix B.

The trading volume (58) has three different contributions. The first term in the brackets on the right hand side of (58) reflects the amount of the trading activity when the informed traders take opposite sides of the market. By analogy with Evans and Lyons (2002), it may be referred to as the "inter-informed" trading volume component. Clearly, the "inter-informed" component originates from the heterogeneity of signals across the informed traders. The second and the third terms in the brackets on the right-hand side of (55) correspond to the amount of trading when informed traders are mostly on the same side of the market ("market-informed" component) and are proportional to the market uncertainty. In order to construct a measure of the total informed trading volume using (55), one should take into account that the "market-informed" contribution to the total trading volume from (55) has to be multiplied by the number of informed traders N, whereas the "inter-informed" component should not have such a factor. This is because the "market-informed" component corresponds to the case when most the informed traders stay on the same side of the market, as opposed to the "inter-informed" one. Therefore, the total informed trading volume is given by

$$TV(t) = N\left(\frac{1}{N}T_1(t) + T_2(t) + T_3(t)\right). \qquad (60)$$

The measure of the trading volume (60) is consistent with the results of Evans and Lyons (2002), where the *signed* order flow (the sign reflects the direction of the trade) can be considered an appropriate measure of trading activity. This implies that the aggregate order flow may be low just because different agents take opposite sides of the market, whereas the trading activity may be quite substantial (due to the heterogeneous information across the agents). Therefore, the sum of the variances of the incremental demands provides a better measure of trading activity. Analogous to (55), the total informed trading volume (60) has the "inter-informed" and the "market-informed" components.

To analyze the time profiles of all market parameters, we have to simultaneously solve the ODEs (29), (31), (13) and (40) taking into account the boundary condition (41). In particular, we obtain a closed-form analytical characterization of the equilibrium. The results of the numerical solution are presented in Figures 2 through 9.

Figure 2 presents the dynamics of the information asymmetry parameter $\delta(t)$ as a function of time for the case of five informed traders $N = 5$ and a correlation coefficient between the private signals ρ equal to 0.5. We assume that there are four trading periods and, therefore, three market closures in the economy. Clearly, the information asymmetry increases at the beginning of each trading period, since the informed traders learn from their private signals over the market closure periods. However, the magnitude of the change for the asymmetry parameter is different across the closure periods and tends to decrease towards the last trading period. The intuition is simple: the role of private learning decreases as the price becomes more informative towards the announcement date T.

In Figure 3, we plot the dynamics of the optimal informed trading parameters $\beta(t)$ for the case of the three closures described above as a function of time and compare to the optimal $\beta(t)$ in the absence of market closures. Note that the time profile of the trading parameter in the presence of market closures has a typical U-shape over the trading period which is consistent with the results of BCW. As it was pointed out by Foster and Viswanathan (1996), in the oligopolistic economy with heterogeneous signals across informed traders, the time profiles of the optimal trading strategies can be characterized by two different phases. For sufficiently small time intervals, the informed traders do not learn much from the price process and are therefore trading very aggressively in an attempt to quickly make use of their initial private information. This is called the "rat race" phase. At later times, the informed traders learn more from the price and can make better inferences of the others' private signals, and therefore have less incentive to trade in order not to reveal their information. The second stage is called the "waiting phase." In contrast to Foster and Viswanathan (1996) and BCW, the information inflow in our model happens at each market closure and is therefore periodic. For this reason, one should expect that such a "two-stage" pattern occurs during each of the trading periods. However, one should also take into account that the informed traders strategically anticipate the future information inflows in their strategies and therefore the U-shaped pattern described above may have significant non-periodic distortions. This is what can be seen in Figure 3.

In Figure 4, we plot the dynamics of the trading volume defined by (60) as a function of time for the case of five informed traders with the initial correlation coefficient between the private signals $\rho = 0.5$, and compare to the optimal trading volume in the absence of market closures. From (60), it follows that the dynamics of the trading volume are directly related to the dynamics of the informed trading parameter $\beta(t)$. The trading volume in the presence of the market closures exceeds the volume obtained in the case of no closures for the whole time interval presented in Figure 4. Analogous to the dynamics of $\beta(t)$, the time profile of the trading volume in the presence of market closures has a typical U-shape over the trading period, following the "rat race" and the "waiting" phases of the optimal trading strategies described above. Also, the U-shape of the time profile of the trading volume has significant distortions since the informed traders strategically anticipate the future information inflows in their strategies.

In Figure 5, we plot the dynamics of relative trading intensity as a function of time for the case of five informed traders and three closures described above and compare to the relative trading intensity with no closures. Following BCW, the relative trading intensity is defined as the ratio of the oligopolistic trading intensity defined in Corollary 1 as $\beta(t)/\delta(t)$, to the optimal trading intensity of the monopolist from Kyle (1985). As one can see from Figure 5, the relative trading intensity in presence of market closures exceeds the one with no closures. Note that the trading intensity drops at the beginning of each trading period, which is consistent with the results of Foster and Viswanathan (1996). The trading intensity follows monotonically increasing patterns during each trading period, but aperiodic across the different trading periods. Analogous to the above discussion, this non-periodicity arises due to the strategic nature of trading combined with the decreasing role of the private signals during the trading periods.

In Figure 6, we compare the market's residual uncertainty, showing the ratio of $\Sigma_M(t)$ for the oligopolistic and monopolistic cases, as a function of time, for the case of five informed traders and three closures described above, and comparing to the otherwise identical case with no closures. The relative residual uncertainty has a typical U-shape during the whole time interval consistent with the results of BCW. As one can see from Figure 6, the relative residual uncertainty in the absence of market closures exceeds the one with closures. This implies that the periodic inflow of information in the heterogenous oligopolistic economy is informationally more efficient than the "one-shot" information inflow considered in BCW.

Figure 7 presents the dynamics of the relative market depth parameter $1/\lambda(t)$ as a function of time, for the same case of five informed traders and three market closures, compared to the case with no closures. As before, the relative market depth is defined as a ratio of the oligopolistic and the monopolistic market depth parameters. The relative market depth increases at the beginning of each trading period and decreases towards the time horizon T. The reason for the relative market depth increasing is exactly analogous to the one described in BCW. From Figure 7, one can see that the inverse market depth only slightly increases during the last trading period, implying that the liquidity problems in our model are less severe than in the case when the information inflow is non-periodic. The intuition is simple: since the informed traders strategically anticipate the future information inflows and have less incentive to wait, the adverse selection problem is less important.

In Figure 8, we plot the dynamics of the trading intensity as a function of time for the case of two informed traders and different initial correlations between the private signals and compare to the optimal trading intensity of a monopolist (Kyle (1985)). The oligopolistic trading intensity in our model exceeds that of a monopolist over the whole range of the initial correlations presented in Figure 8. For sufficiently short time intervals (around the first trading period), the trading intensity is quite sensitive to the correlations, but it becomes less sensitive after several closure periods. The intuition is that the role of initial correlations diminishes since the informed traders learn more from the price process after several trading periods.

Figure 9 presents the dynamics of the market depth parameter $1/\lambda(t)$ as a function of time for the same case of two informed traders and different signal correlations. One should note that the market depth decreases in signal correlations for short times and increases for sufficiently long time intervals, which corresponds to the "rat race" and the "waiting" phases of the optimal strategies, respectively.

As we discussed early, an important result of our model is that contrary to the results of BCW, the trading intensity with oligopolistic heterogeneously informed traders is typically higher than the one of the monopolist. This situation is illustrated in Figure 8, where we present the ratio of the trading intensities of the two informed traders and the monopolist. The reason for this is the same mechanism of the "adverse selection problem suppression" due to the strategic anticipation of the future signals described above (see Figure 6).

Finally, we study the distribution of payoffs across all agents which is important from the perspective of welfare analysis. As usual for Kyle (1985) type models, the informed traders are winning at the expense of the uninformed ones. The payoff distribution across the informed traders can be analyzed analogous to BCW. For the informed trader i, the payoff follows the process

$$d\pi_i(t) = [V - P(t)]\, dx_i, \tag{61}$$

where the incremental demand $dx_i(t)$ during the time interval dt is given by (9). Making use of equation (25) and equation (9), we obtain

$$E[d\pi_i(t)] = \beta(t) \operatorname{var}[V - P(t)|V]\, dt = \lambda(t)\, \sigma_u^2 dt. \tag{62}$$

In order to analyze the dependence of the expected payoffs on the number of informed traders, we have to re-scale $\beta(t) \to \beta(t)/N$, according to (11). Taking this into account, we obtain from (62) the following result.

Corollary 4 *In equilibrium, the expected profit of each informed trader is*

$$\frac{\sigma_u^2}{N} \sum_{k=0}^{K} \int_{t_k^o}^{t_k^c} \lambda_k(t) dt = \frac{\sigma_u^2}{N} \int_0^T \lambda(t) dt. \tag{63}$$

Proof: Follows immediately from (62).

The result (63) has the same form as the one of BCW, because the inverse depth $\lambda(t)$ is continuous across the market closures. One should note that the total expected gain of informed traders is the sum of the expected profits earned during the periods when the market is open.

5 Conclusions

This paper constructs a dynamic auction model based on rational strategic trading with asymmetric information across agents and *dynamic* information acquisition. The key feature of the model is that informed traders do not know the fundamentals exactly but repeatedly receive imperfect private signals about the fundamentals. Learning from prices also enables each informed trader to improve her estimation of fundamentals. Therefore, to solve for equilibrium outcomes in such setting, one must first determine how informed traders combine current and past signals together with the information in current and past prices, then determine how much they trade at each time, and finally put this all together and solve for the associated equilibrium pricing. This is a very challenging problem which we successfully solve in this paper.

In our model agents trade continuously and informed traders receive private information during times when the market is closed. We characterize analytically how information received during one market closure interacts with information from other closures. We show that it is optimal for informed traders to use a linear combination of all of their private signals obtained at different times as an effective signal in their optimal trading strategy. We demonstrate that optimal weights for the signals obtained at different times enter with *equal weights* into the effective signal. This is a consequence of the fact that in our model all private signals obtained by each insider are drawn from the same distribution, and is consistent with the result that the equally-weighted average is the optimal linear unbiased estimator of the fundamental. At the same time, we show that the informed traders' valuations also involve the factor related to the information asymmetry which exhibits non-monotonic and non-periodic dynamics over the different market closure periods. Our results indicate that the use of private information and its revelation through price never ends: each new realization of private information leads agents to re-interpret the history of private information as well as prices.

While the model only analyzes the role of market closures from a purely information perspective, it is consistent with a wealth of empirical evidence. It predicts a U-shaped patterns for trading activity, during the periods when the market is open. We also obtain rich aperiodic patterns for the trading volume over the whole time period, which is changing across different periods when the market is open. This aperiodicity property of trading activity is quite important and arises because the informed traders strategically trade during different periods when the market is open anticipating the final announcement about the fundamentals.

Appendix A: Bayesian Updating

First, lets derive the distribution in (6). By assumption, the private signals are given by (1) from which it immediately follows that $\left\{S_i^k\right\}$ are normally distributed, with

$$E\left[S_i^k|V\right] = V, \quad (\forall k = 0, ..., \ K-1)\,. \tag{A1}$$

The conditional variance for $\forall k = 0, ..., \ K-1$ is given by

$$\begin{aligned} Var\left[S_i^k|V\right] &= \sigma_s^2 E\left[\left(\varepsilon_i^k - \frac{1}{N}\sum_{j=1}^N \varepsilon_j^k\right)^2\right] \\ &= \sigma_s^2\left(1 - \frac{2}{N}\left[1 + (N-1)\,\rho\right] + \frac{1}{N^2}\left[N + N\,(N-1)\,\rho\right]\right) \\ &= \sigma_s^2\,(1-\phi)\,, \end{aligned} \tag{A2}$$

and (6) follows immediately. Analogously, we obtain for the covariances

$$\begin{aligned} Cov\left[S_i^k, S_{j\neq i}^k|V\right] &= \sigma_s^2 E\left[\left(\varepsilon_i^k - \frac{1}{N}\sum_{j=1}^N \varepsilon_j^k\right)\left(\varepsilon_j^k - \frac{1}{N}\sum_{l=1}^N \varepsilon_l^k\right)\right] \\ &= \sigma_s^2\left(\rho - \frac{2}{N}\left[1 + (N-1)\,\rho\right] + \frac{1}{N^2}\left[N + N\,(N-1)\,\rho\right]\right) \\ &= \sigma_s^2\,(\rho-\phi)\,. \end{aligned} \tag{A3}$$

Analogously, the unconditional variances and covariances are

$$\begin{aligned} Var\left[S_i^k\right] &= E\left[\left(V + \sigma_s\left(\varepsilon_i^k - \frac{1}{N}\sum_{j=1}^N \varepsilon_j^k\right)\right)^2\right] \\ &= Var\left[V\right] + \sigma_s^2\left(1 - \frac{2}{N}\left[1 + (N-1)\,\rho\right] + \frac{1}{N^2}\left[N + N\,(N-1)\,\rho\right]\right) \\ &= \sigma_s^2\phi + \sigma_s^2\,(1-\phi) = \sigma_s^2, \end{aligned} \tag{A4}$$

and

$$\begin{aligned} Cov\left[S_i^k, S_{j\neq i}^k\right] &= E\left[\left(V + \sigma_s\left(\varepsilon_i^k - \frac{1}{N}\sum_{j=1}^N \varepsilon_j^k\right)\right)\left(V + \sigma_s\left(\varepsilon_j^k - \frac{1}{N}\sum_{l=1}^N \varepsilon_l^k\right)\right)\right] \\ &= Var\left[V\right] + \sigma_s^2\left(\rho - \frac{2}{N}\left[1 + (N-1)\,\rho\right] + \frac{1}{N^2}\left[N + N\,(N-1)\,\rho\right]\right) \\ &= \sigma_s^2\phi + \sigma_s^2\,(\rho-\phi) = \sigma_s^2\rho, \end{aligned} \tag{A5}$$

respectively. Combining (A4) and (A5), we obtain (8).

Next, we derive the Bayesian updating procedure for our model. Making use of the Bayes rule, the conditional p.d.f. of V, $p_{k+1}\left(V|\mathcal{F}_i(t^o_{k+1})\right)$, is given by recursion

$$p_{k+1}\left(V|\mathcal{F}_i(t^o_{k+1})\right) = \frac{p_k\left(V|\mathcal{F}_i(t^c_k)\right)\exp\left[-\frac{\left(S_i^{k+1}-V\right)^2}{2(1-\phi)\sigma_s^2}\right]}{\int_{-\infty}^{+\infty} dx p_k\left(x|\mathcal{F}_i(t^c_k)\right)\exp\left[-\frac{\left(S_i^{k+1}-x\right)^2}{2(1-\phi)\sigma_s^2}\right]}, \tag{A6}$$

where S_i^{k+1} is the private signal received by the informed trader i during the k-th closure period. Taking into account the prior distribution (4) and her own signal (6), the informed trader i's posterior p.d.f. of V at $t=0$ *after* she observes her private signal S_i^0 becomes

$$\begin{aligned} p_0\left(V|S_i^0\right) &= \frac{\exp\left[-\frac{V^2}{2\phi\sigma_s^2}\right]\exp\left[-\frac{\left(S_i^0-V\right)^2}{2(1-\phi)\sigma_s^2}\right]}{\int_{-\infty}^{+\infty} dx\exp\left[-\frac{x^2}{2\phi\sigma_s^2}\right]\exp\left[-\frac{\left(S_i^0-x\right)^2}{2(1-\phi)\sigma_s^2}\right]} \\ &= \frac{1}{\sqrt{2\pi\phi\left(1-\phi\right)\sigma_s^2}}\exp\left[-\frac{\left(V-\phi S_i^0\right)^2}{2\phi\left(1-\phi\right)\sigma_s^2}\right]. \end{aligned} \tag{A7}$$

Equation (A7) can be used as an initial condition for the recursion (A6). Now, we apply the updating rule (A6) to obtain (A12) and (A13). Since all the conditional distributions $\{p_k\left(V|\mathcal{F}_i(t^o_k)\right),\ k=0,...K-1\}$ are normal, we have

$$p_k\left(V|\mathcal{F}_i(t^c_k)\right) = \frac{1}{\sqrt{2\pi\Sigma_i\left(t^c_k\right)}}\exp\left[-\frac{\left(V_i\left(t^c_k\right)-V\right)^2}{2\Sigma_i\left(t^c_k\right)}\right]. \tag{A8}$$

Note that all informed traders receive private signals drawn from the same distribution, and therefore have the same updating rules. The substitution of (A8) into (A6) yields the updating rule (A12) and (A13). To see that we first calculate the integral in the denominator of (A6)

$$\int_{-\infty}^{+\infty} dx p_k\left(x|\mathcal{F}_i(t^c_k)\right)\exp\left[-\frac{\left(S_i^{k+1}-x\right)^2}{2\left(1-\phi\right)\sigma_s^2}\right] = \tag{A9}$$

$$= \sqrt{\frac{\widehat{\Sigma}_i\left(t^c_k\right)}{\Sigma_i\left(t^c_k\right)}}\exp\left\{-\left(\frac{V_i\left(t^c_k\right)^2}{2\Sigma_i\left(t^c_k\right)}+\frac{\left(S_i^{k+1}\right)^2}{2\left(1-\phi\right)\sigma_s^2}\right)+\frac{\widehat{\Sigma}_i\left(t^c_k\right)}{2}\left(\frac{V_i\left(t^c_k\right)}{\Sigma_i\left(t^c_k\right)}+\frac{S_i^{k+1}}{\left(1-\phi\right)\sigma_s^2}\right)^2\right\},$$

where

$$\widehat{\Sigma}_i\left(t^c_k\right) \equiv \frac{1}{\frac{1}{\Sigma_i\left(t^c_k\right)}+\frac{1}{(1-\phi)\sigma_s^2}}. \tag{A10}$$

Substituting (A9) and (A8) into (A6) yields

$$p_{k+1}\left(V|\mathcal{F}_i(t^o_{k+1})\right) = \frac{1}{\sqrt{2\pi\widehat{\Sigma}_i\left(t^c_k\right)}}\exp\left\{-\frac{\left[\widehat{\Sigma}_i\left(t^c_k\right)\left(\frac{V_i\left(t^c_k\right)}{\Sigma_i\left(t^c_k\right)}+\frac{S^{k+1}_i}{(1-\phi)\sigma^2_s}\right)-V\right]^2}{2\widehat{\Sigma}_i\left(t^c_k\right)}\right\} = \frac{1}{\sqrt{2\pi\Sigma_i\left(t^o_{k+1}\right)}}\exp\left[-\frac{\left(V_i\left(t^o_{k+1}\right)-V\right)^2}{2\Sigma_i\left(t^o_{k+1}\right)}\right]. \quad \text{(A11)}$$

Comparing two distributions in (A11) gives the necessary updating rules for the informed trader's precision $\Sigma_i\left(t\right)$

$$\frac{1}{\Sigma_i\left(t^o_{k+1}\right)} = \frac{1}{\Sigma_i\left(t^c_k\right)} + \frac{1}{\left(1-\phi\right)\sigma^2_s}, \quad \text{(A12)}$$

and valuation $V_i\left(t\right)$

$$\frac{V_i\left(t^o_{k+1}\right)}{\Sigma_i\left(t^o_{k+1}\right)} = \frac{V_i\left(t^c_k\right)}{\Sigma_i\left(t^c_k\right)} + \frac{S^{k+1}_i}{\left(1-\phi\right)\sigma^2_s}. \quad \text{(A13)}$$

Appendix B: Proofs

Proof of Proposition 1

1. Making use of (13), (16) and the definition $\delta(t) = 1 - \Sigma_i(t)/\Sigma_M(t)$, we immediately obtain

$$\frac{d}{dt}\delta = -\delta(1-\delta)\lambda\beta. \tag{B1}$$

Combining (B1) and (23) yields (24) with $k(t) = [1-\delta(t)]\lambda(t)\beta(t)$. Taking into account (B1), we obtain that $k(t) = -\frac{d}{dt}[\ln\delta(t)]$.

2 and 3. Let us prove equation (25) first. After market reopens at $t = t_k^o$ informed trader i starts with the valuation $V_i(t_k^o)$. We now have to solve an ODE for $\widehat{V}_i(t) \equiv V_i(t) - (1-\delta(t))P(t)$ that directly follows from (24)

$$\frac{d\widehat{V}_i(t)}{dt} = -k(t)\widehat{V}_i(t),$$

with initial condition $\widehat{V}_i(t_k^o)$ which will be determined later. The solution is given by

$$\widehat{V}_i(t) = \widehat{V}_i(t_k^o)\exp\left[-\int_{t_k^o}^{t} k(s)ds\right] = \widehat{V}_i(t_k^o)\frac{\delta(t)}{\delta(t_k^o)}, \tag{B2}$$

where we made use of the relation $k(t) = -\frac{d}{dt}[\ln\delta(t)]$, and (25) follows after we introduce the effective private signal

$$\widehat{S}_i(t_k^o) = \frac{1}{\delta(t_k^o)}\widehat{V}_i(t_k^o). \tag{B3}$$

Now we can prove (26). It is shown in Appendix A that after each of the market re-openings, the informed trader's precision $\Sigma_i(t)$ and valuation $V_i(t)$ are given by the updating rule (A12) and (A13) respectively. Combining (A12) and (A13), we immediately obtain

$$\frac{V_i\left(t_{k+1}^o\right)}{\Sigma_i\left(t_{k+1}^o\right)} = \frac{V_i(t_k^c)}{\Sigma_i(t_k^c)} + \left[\frac{1}{\Sigma_i\left(t_{k+1}^o\right)} - \frac{1}{\Sigma_i(t_k^c)}\right]S_i^{k+1}. \tag{B4}$$

Using the definition of $\delta(t)$ and (22), we can rewrite (B4) as

$$\frac{\widehat{V}_i\left(t_{k+1}^o\right)}{1-\delta\left(t_{k+1}^o\right)} = \frac{\widehat{V}_i(t_k^c)}{1-\delta(t_k^c)} + \left[\frac{1}{1-\delta\left(t_{k+1}^o\right)} - \frac{1}{1-\delta(t_k^c)}\right]S_i^{k+1}. \tag{B5}$$

In deriving the relation (B5) we have also used the definition of $\widehat{V}_i(t)$, namely

$$\frac{V_i\left(t_{k+1}^o\right)}{1-\delta\left(t_{k+1}^o\right)} = \frac{\widehat{V}_i\left(t_{k+1}^o\right)}{1-\delta\left(t_{k+1}^o\right)} + P\left(t_{k+1}^o\right), \tag{B6}$$

and the fact that the price does not change during closure periods, $P\left(t_{k+1}^{o}\right) = P\left(t_{k}^{c}\right)$. Next, we multiply both sides of (B5) by $1-\delta\left(t_{k+1}^{o}\right)$ and use the identity $\widehat{V}_i\left(t_k^c\right) = \widehat{V}_i\left(t_k^o\right)\left[\delta\left(t_k^c\right)/\delta\left(t_k^o\right)\right]$ to obtain

$$\widehat{S}_i\left(t_{k+1}^{o}\right) = \mu_{k+1}\widehat{S}_i\left(t_k^c\right) + \frac{1}{\delta\left(t_{k+1}^{o}\right)}\left[1-\frac{1-\delta\left(t_{k+1}^{o}\right)}{1-\delta\left(t_k^c\right)}\right]S_i^{k+1}, \tag{B7}$$

with

$$\mu_{k+1} = \frac{1-\delta\left(t_{k+1}^{o}\right)}{1-\delta\left(t_k^c\right)}\frac{\delta\left(t_k^c\right)}{\delta\left(t_{k+1}^{o}\right)}. \tag{B8}$$

Finally, in order to establish (26) we need to show that the coefficient in front of the second term in (B7) is equal to $1-\mu_{k+1}$. A simple algebra exercise shows that

$$\begin{aligned} 1-\mu_{k+1} &= 1-\frac{1-\delta\left(t_{k+1}^{o}\right)}{1-\delta\left(t_k^c\right)}\frac{\delta\left(t_k^c\right)-1+1}{\delta\left(t_{k+1}^{o}\right)} = \\ &= 1+\frac{1-\delta\left(t_{k+1}^{o}\right)}{\delta\left(t_{k+1}^{o}\right)}-\frac{1}{\delta\left(t_{k+1}^{o}\right)}\frac{1-\delta\left(t_{k+1}^{o}\right)}{1-\delta\left(t_k^c\right)} = \\ &= \frac{1}{\delta\left(t_{k+1}^{o}\right)}\left[1-\frac{1-\delta\left(t_{k+1}^{o}\right)}{1-\delta\left(t_k^c\right)}\right]. \end{aligned} \tag{B9}$$

The recursion (B7) has initial condition

$$\widehat{S}_i\left(t_0^o\right) = \widehat{S}_i(0) = \frac{\widehat{V}_i(0)}{\delta(0)} = \frac{V_i(0)}{\delta(0)} = S_i^0, \tag{B10}$$

and can be solved forward. This is equivalent to the condition

$$\mu_0 = 0. \tag{B11}$$

Using the definition of $\delta(t)$ and (22), we can rewrite (A12) as

$$\frac{1}{1-\delta\left(t_{k+1}^{o}\right)} = \frac{1}{1-\delta\left(t_k^c\right)} + \frac{\Sigma_M\left(t_{k+1}^{o}\right)}{(1-\phi)\,\sigma_s^2}, \tag{B12}$$

which is the second of (29). Finally, the ODEs for the information asymmetry parameter follow immediately from (B1). *Q.E.D.*

Proof of Proposition 2

Combining the results of the Proposition 1, we obtain

$$\frac{d\delta(t)}{\delta(t)\,(1-\delta(t))} = -\Sigma_M(t)\,d\left(\frac{1}{\Sigma_M(t)}\right). \tag{B13}$$

Integration on the interval $t \in \mathcal{O}^k$ with the initial conditions at $t = t_k^o$ yields

$$\delta(t) = \frac{\delta(t_k^o)\,\Sigma_M^{-1}(t_k^o)}{(1-\delta(t_k^o))\,\Sigma_M^{-1}(t) + \delta(t_k^o)\,\Sigma_M^{-1}(t_k^o)}, \tag{B14}$$

and therefore

$$\frac{1}{1-\delta(t_k^c)} = \frac{1}{1-\delta(t_k^o)}\left(\frac{\Sigma_M(t_k^c)}{\Sigma_M(t_k^o)}\right) + \left(1 - \frac{\Sigma_M(t_k^c)}{\Sigma_M(t_k^o)}\right). \tag{B15}$$

Substituting (B15) into the second equation of (29) and taking into account that $\Sigma_M(t_k^c) = \Sigma_M\left(t_{k+1}^o\right)$, we obtain

$$\frac{1}{1-\delta\left(t_{k+1}^o\right)} = \frac{1}{1-\delta(t_k^o)}\left(\frac{\Sigma_M(t_k^c)}{\Sigma_M(t_k^o)}\right) + \left(1 - \frac{\Sigma_M\left(t_{k+1}^o\right)}{\Sigma_M(t_k^o)} + \frac{\Sigma_M\left(t_{k+1}^o\right)}{(1-\phi)\,\sigma_s^2}\right). \tag{B16}$$

Iterating the recursion (B16) with the initial conditions $\delta(0) = \phi$, we finally have

$$\frac{1}{1-\delta\left(t_{k+1}^o\right)} = 1 + \frac{(k+1)\,\Sigma_M\left(t_{k+1}^o\right)}{(1-\phi)\,\sigma_s^2}, \tag{B17}$$

with $k = 0, ...K-1$. Combining (B14) and (B17), we observe that

$$\frac{1}{\delta(t)} = 1 + \frac{(1-\phi)\,\sigma_s^2}{(k+1)\,\Sigma_M(t)}, \tag{B18}$$

where $t \in \mathcal{O}^k$, $\forall k$, which is the result of Proposition 2. The result (B18) will be useful for the solution of the dynamic optimization considered below. *Q.E.D.*

Proof of Proposition 3

Combining (26) and the definition of the effective signal (10), we obtain the recursion relation for the weights of the effective signal

$$\begin{aligned} a_{k,k} &= 1 - \mu_k, \\ a_{l,k+1} &= \mu_{k+1} a_{l,k}, \qquad l = 0, ..., k. \end{aligned} \tag{B19}$$

Solving the recursion (B19) with the initial condition $a_{0,0} = 1$ yields

$$a_{l,k} = (1-\mu_l)\prod_{m=l+1}^{k} \mu_m. \tag{B20}$$

From (B20), it follows that $\sum_{l=0}^{k} a_{l,k} = 1 - \prod_{m=0}^{k} \mu_m$. Taking into account (28), we observe that the weights satisfy the normalization condition $\sum_{l=0}^{k} a_{l,k} = 1$. According to the above discussion, the effective signal (10) reflects the informed traders' updating of the new signals. From Proposition 2 we have

$$\frac{1}{\delta\left(t_k^o\right)} = 1 + \frac{(1-\phi)\,\sigma_s^2}{(k+1)\,\Sigma_M\left(t_k^o\right)}, \qquad \text{(B21)}$$
$$\frac{1}{\delta\left(t_k^c\right)} = 1 + \frac{(1-\phi)\,\sigma_s^2}{(k+1)\,\Sigma_M\left(t_k^c\right)},\ \ k=0,...,K-1.$$

Making use of (B21) and that the MM's residual uncertainty is continuous over the closure periods, we immediately obtain

$$\frac{1-\delta\left(t_{k+1}^o\right)}{\delta\left(t_{k+1}^o\right)}(k+2) = \frac{1-\delta\left(t_k^o\right)}{\delta\left(t_k^o\right)}(k+1),\ \ k=0,...,K-1, \qquad \text{(B22)}$$

and therefore

$$\mu_{k+1} = \frac{1-\delta\left(t_{k+1}^o\right)}{1-\delta\left(t_k^c\right)}\frac{\delta\left(t_k^c\right)}{\delta\left(t_{k+1}^o\right)} = \frac{k+1}{k+2}, \qquad \text{(B23)}$$
$$\mu_k = \frac{k}{k+1},$$

which also satisfies the initial condition (28) for $k=0$. Combining (B23) and (B20) yields (32). *Q.E.D.*

Proof of Proposition 4

According to (19), we have that

$$dP\left(t\right) = \lambda\left(t\right)\left[\sum_{i=1}^{N}\theta_i\left(t\right)dt + \sigma_u dZ\left(t\right)\right]. \qquad \text{(B24)}$$

Assuming that all informed traders except for the i-th informed trader are following the linear strategies and taking expectations, we have

$$dP = \lambda\left(t\right)\left[\theta\left(t\right) + \beta\left(t\right)N\left(V - \frac{1}{N}\widehat{S}_i - \frac{N-1}{N}P\right)\right]dt + \lambda\left(t\right)\sigma_u dZ\left(t\right), \qquad \text{(B25)}$$

Making use of (B25), we immediately obtain the result of Proposition 3. *Q.E.D.*

Proof of Proposition 5

Follows directly from (39) and (25). *Q.E.D.*

Proof of Theorem 1

From (13), and (16), it follows that the equilibrium is given by β_k and λ_k satisfying the following conditions

$$\beta_k(t) = \sigma_u \left(\frac{d\left(\Sigma_M^{-1}(t)\right)}{dt} \right)^{1/2},$$
$$\lambda_k(t) = \frac{\beta_k(t)\,\Sigma_M(t)}{\sigma_u^2},$$

during the kth period when market is open.

In order to derive the equilibrium, one has to derive a solution of the PDE (40) with the limit condition (41), with continuity of market maker beliefs across market closures, and which is such that the informed trader's strategy (9) is feasible for each informed trader. Ignoring the feasibility condition, β_k and λ_k are uniquely defined by $\Sigma_M(t)$, which is known at $t = 0$ and satisfies the limit condition (41). Therefore, if we solve for $\Sigma_M(t)$ for all time periods, we would be able to completely characterize the solution for the whole problem. Define $R(t) \equiv \Sigma_M(t)^{-1}$, and $R'(t) \equiv \frac{d}{dt}R(t)$. The necessary and sufficient conditions for the equilibrium can be written as

$$\beta_k(t) = \sigma_u \sqrt{R'(t)}, \tag{B26}$$

$$\lambda_k(t) = \frac{1}{\sigma_u} \frac{\sqrt{R'(t)}}{R(t)}, \tag{B27}$$

$$R''(t) + \left[\frac{2N-4}{N}\right] \frac{R'(t)^2}{R(t)} - \frac{2(1-\phi)}{N\phi\,(k+1)} \frac{R'(t)^2}{R(0)} = 0,\ t \in \mathcal{O}^k,\ \forall k, \tag{B28}$$

$$R(t) = R(t_k^c),\ t \in \mathcal{C}^k,\ \forall k, \tag{B29}$$

and

$$\lim_{t\to T} R(t) = +\infty. \tag{B30}$$

In order to obtain equation (B28) note that $\delta(t)$ satisfies (B14) for $t \in \mathcal{O}^k$, from which it follows that

$$\delta(t)^{-1} = 1 + \frac{(1-\delta(t_k^o))}{\delta(t_k^o)} \frac{R(t)}{R(t_k^o)} = 1 + \frac{(1-\phi)}{(k+1)\,\phi} \frac{R(t)}{R(0)},$$

where the last equality is obtained from (B17). We can use the expression (B27) to compute

$$\frac{d}{dt}\left(\frac{1}{\lambda_k(t)}\right) = \frac{\sigma_u}{\sqrt{R'(t)}} \left[R'(t) - \frac{R(t)R''(t)}{2R'(t)}\right] = \tag{B31}$$
$$= \left(\frac{2N-2}{N} - \frac{(1-\phi)}{N\,(k+1)\,\phi} \frac{R(t)}{R(0)}\right) \sigma_u \sqrt{R'(t)}.$$

A simple rearrangement of terms in (B31) immediately yields (B28). Equation (B28) is fully integrable. To see that divide it by $R'(t)$ to obtain

$$\frac{d}{dt}\left[R'(t)R(t)^{\frac{2N-4}{N}}\exp\left(-\frac{2(1-\phi)}{N(k+1)\phi}\frac{R(t)}{R(0)}\right)\right]=0,\ t\in\mathcal{O}^k,\ \forall k, \tag{B32}$$

We can now integrate (B32) twice to get

$$\int_{t_k^o}^{t} R'(s)R(s)^{\frac{2N-4}{N}}\exp\left(-\frac{2(1-\phi)}{N(k+1)\phi}\frac{R(t)}{R(0)}\right)ds=(t-t_k^o)C_k,\ t\in\mathcal{O}^k,\ \forall k,$$

and, finally, the change of variables $x = R(s)/R(0)$ yields expression (46) with

$$r_k = \Sigma_M(0)/\Sigma_M(t_k^o). \tag{B33}$$

Since $R(T)=\infty$ and $R(0)=(\Sigma_M(0))^{-1}$ we have

$$\begin{aligned} r_K &= \infty, \\ r_0 &= 1. \end{aligned} \tag{B34}$$

From (46), it immediately follows that

$$\begin{aligned} C_k &= \frac{1}{\tau}\int_{r_k}^{r_{k+1}} x^{2(N-2)/N}\exp\left(-\frac{2(1-\phi)}{N(k+1)\phi}x\right)dx, \qquad k=1,...,K-2, \\ C_0 &= \frac{1}{\tau}\int_{1}^{r_1} x^{2(N-2)/N}\exp\left(-\frac{2(1-\phi)}{N\phi}x\right)dx, \\ C_{K-1} &= \frac{1}{\tau}\int_{r_{K-1}}^{\infty} x^{2(N-2)/N}\exp\left(-\frac{2(1-\phi)}{NK\phi}x\right)dx. \end{aligned} \tag{B35}$$

To complete the solution for $R(t)$ we have to evaluate the constants $\{C_k\}_{k=0}^{K-1}$ and $\{r_k\}_{k=0}^{K}$. To accomplish that, consider a kth closure at $t=t_k^c$ with a corresponding opening at $t=t_{k+1}^o$. We have the following continuity conditions

$$\begin{aligned} R'(t_{k+1}^o) &= R'(t_k^c), \\ R(t_{k+1}^o) &= R(t_k^c), \end{aligned} \tag{B36}$$

which follow from the continuity of the market uncertainty $\Sigma_M(t)$ and the inverse market depth $\lambda(t)$ over the closure periods. Making use of (B36) yields

$$C_k = C_{k-1}\exp\left[\frac{2r_k(1-\phi)}{N\phi}\left(\frac{1}{k}-\frac{1}{k+1}\right)\right], \tag{B37}$$

which is a recursion condition for the constants $\{C_k\}$. The conditions (B36) combined with (B36), form a complete set of equations defining the constants $\{C_k\}_{k=0}^{K-1}$ and $\{r_k\}_{k=0}^{K}$.

Define the constant

$$\psi = \frac{2}{N}\left(\frac{1-\phi}{\phi}\right). \tag{B38}$$

For the practical calculations, it is convenient to introduce the new variables

$$B_k = C_k \frac{1}{\tau}\left[\psi \Sigma_M(0)\right]^{(4-3N)/N}, \tag{B39}$$

and

$$x_k = \psi r_k, \tag{B40}$$

which is simply a change of scale. In terms of the new variables, the recursion takes the form

$$\begin{aligned} B_k &= \int_{x_k}^{x_{k+1}} x^{2(N-2)/N} \exp\left(-\frac{x}{k+1}\right) dx, \\ B_k &= B_{k-1} \exp\left[x_k \left(\frac{1}{k} - \frac{1}{k+1}\right)\right], \end{aligned} \tag{B41}$$

for $k = 1, ..., K-1$, with the boundary conditions

$$\begin{aligned} B_0 &= \int_{\psi}^{x_1} x^{2(N-2)/N} \exp(-x)\, dx, \\ B_{K-1} &= \int_{x_{K-1}}^{\infty} x^{2(N-2)/N} \exp\left(-\frac{x}{K}\right) dx. \end{aligned} \tag{B42}$$

Finally, we have to prove the feasibility of the equation (9). It is feasible for all time intervals when market is open by the same argument as in BCW, and, since price does not change during the market closures, it is feasible at any $t \in [0, T]$. *Q.E.D.*

Proof of Corollary 2

Consider $N = 1$ and $K - 1$ closures. In this case, $\phi = 1$, and (B28) yields

$$R''(t) - 2\frac{R'(t)^2}{R(t)} = \frac{d}{dt}\left(\frac{R'(t)}{R^2(t)}\right) = 0, \tag{B43}$$

and therefore

$$-\frac{R'(t)}{R^2(t)} = \frac{d}{dt}\Sigma_M(t) = -\gamma \equiv const, \tag{B44}$$

during each period when the market is open, $t \in \mathcal{O}^k$, $\forall k$. Making use of (B44), the boundary conditions

$$\Sigma_M(0) = \phi\sigma_s^2, \qquad \Sigma_M(T) = 0, \tag{B45}$$

and the continuity condition (48), we obtain the unique solution for the residual market uncertainty in the form

$$\Sigma_M(t) = \phi\sigma_s^2\left(1 - \frac{t}{T}\right), \tag{B46}$$

and therefore the unique equilibrium for the monopolist case is given by $\{\beta_k(t), \lambda_k(t)\}_{k=0}^{K-1}$, where

$$\begin{aligned} \beta_k(t) &= \frac{\sigma_u}{\sigma_s}\frac{1}{\sqrt{T}}\frac{1}{1-\frac{t}{T}}, \\ \lambda_k(t) &= \frac{\sigma_s}{\sigma_u}\frac{1}{\sqrt{T}}, \qquad \forall k = 0, ..., K. \end{aligned} \tag{B47}$$

The market maker's variance for each $k = 0, ..., K$, is

$$\Sigma_M(t) = \Sigma_M(0)\left(1 - \frac{t}{T}\right), \tag{B48}$$

where $T = K\tau$. *Q.E.D.*

Proof of Corollary 3

Setting $N = 2$ in (B41), and (B42) immediately leads to

$$\begin{aligned} B_k &= (k+1)\left\{\exp\left(-\frac{x_k}{k+1}\right) - \exp\left(-\frac{x_{k+1}}{k+1}\right)\right\}, \\ B_{k-1} &= B_k \exp\left[-x_k\left(\frac{1}{k} - \frac{1}{k+1}\right)\right], \end{aligned} \tag{B49}$$

for $k = 1, ..., K-1$, with the boundary conditions

$$\begin{aligned} B_0 &= \exp(-\psi) - \exp(-x_1), \\ B_{K-1} &= K\exp\left(-\frac{x_{K-1}}{K}\right). \end{aligned} \tag{B50}$$

Now introduce the new variables A_k's

$$B_k = A_k \exp\left(-\frac{x_k}{k+1}\right), \tag{B51}$$

for $k = 0, ..., K$. A substitution of (B51) into (B49) yields a recursion for the A_k's, in the form

$$\begin{aligned} A_{k-1} &= A_k \frac{k}{k + A_k}, \\ A_{K-1} &= K. \end{aligned} \tag{B52}$$

Inverting both sides of (B52), we obtain

$$\begin{aligned} (A_{k-1})^{-1} &= (A_k)^{-1} + \frac{1}{k}, \quad k = 0, ...K - 1, \\ A_{K-1} &= K. \end{aligned} \tag{B53}$$

The recursion (B52) has the following solution

$$(A_k)^{-1} = \sum_{n=k+1}^{K} \frac{1}{n}, \tag{B54}$$

or

$$A_k = \left(\sum_{n=k+1}^{K} \frac{1}{n} \right)^{-1}. \tag{B55}$$

Making use of (B55), we obtain from (B51) and (B49)

$$x_k = \psi + \sum_{n=1}^{k} n \ln \left(1 + \frac{A_n}{n} \right), \tag{B56}$$

for $k = 0, ..., K - 1$. Substituting into (B40), and taking into account that for $N = 2$, $\psi = \left(\frac{1-\phi}{\phi}\right)$, we obtain from (B56)

$$r_k = 1 + \left(\frac{\phi}{1 - \phi} \right) \sum_{n=1}^{k} n \ln \left(1 + \frac{A_n}{n} \right), \ k = 0, ..., K - 1. \tag{B57}$$

We can also define $A_K = +\infty$, to make it consistent with the condition $r_K = +\infty$. Setting $N = 2$ in (46), we obtain

$$r(t) = -(k + 1) \ln \left[\exp \left(-\frac{r_k}{k + 1} \right) - B_k \frac{t - t_k^o}{k + 1} \right],$$

or, making use of (B51)

$$r(t) = r_k + \frac{(k + 1) \phi}{1 - \phi} \ln \left[\left(1 - A_k \frac{t - t_k^o}{k + 1} \right)^{-1} \right],$$

which is equivalent to

$$\Sigma_M(t) = \Sigma_M(0)\left\{r_k + \frac{(k+1)\phi}{1-\phi}\ln\left[\left(1 - \frac{(t-t_k^o)A_k}{k+1}\right)^{-1}\right]\right\}^{-1}, \ t \in \mathcal{O}^k. \tag{B58}$$

Q.E.D.

Proof of Corollary 4

From (19), it follows that

$$\begin{aligned} P(t) &= \int_0^t \exp\left[-\int_{t'}^t d\tau \lambda(\tau)\beta(\tau)\right] \times \\ &\left\{\lambda(t')\beta(t')\frac{1}{N}\sum_{j=1}^N \widehat{S}_j(t')dt' + \lambda(t')\sigma_u dZ(t')\right\}. \end{aligned} \tag{B59}$$

Taking into account (13) and (15), we observe that

$$\lambda(t)\beta(t) = \Sigma_M(t)\frac{d}{dt}\left(\frac{1}{\Sigma_M(t)}\right) = \frac{d}{dt}\left(\log\left[\frac{1}{\Sigma_M(t)}\right]\right). \tag{B60}$$

Combining (B59) and (B60), we obtain (56). *Q.E.D.*

Proof of Corollary 5

From the distributions (6), (4) and (32), it follows that the unconditional distribution of the effective private signals is given by

$$\widehat{S}_i^k \sim \mathcal{N}\left(0, \frac{\phi k+1}{k+1}\sigma_s^2\right), \ k = 0, ..., \ K-1. \tag{B61}$$

Making use of (56), we obtain from (55)

$$\begin{aligned} TV_i(t) &= T_1(t) + T_2(t) + T_3(t), \\ T_1(t) &= \beta^2(t)\,\mathrm{var}\left[\widehat{S}_i(t)\right] = \beta^2(t)\,\sigma_s^2\frac{\phi k+1}{k+1}, \\ T_2(t) &= -2\beta^2(t)\,\mathrm{cov}\left[\widehat{S}_i(t), P(t)\right], \\ T_3(t) &= \beta^2(t)\,\mathrm{var}\left[P(t)\right], \end{aligned} \tag{B62}$$

with

$$T_2(t) = -2\beta^2(t)\,\Sigma_M(t)\int_0^t d\left(\frac{1}{\Sigma_M(t')}\right)\frac{1}{N}\sum_{j=1}^N E\left[\widehat{S}_i(t), \widehat{S}_j(t')\right], \tag{B63}$$

and

$$T_3(t) = \beta^2(t)\,\Sigma_M^2(t)\int_0^t d\left(\frac{1}{\Sigma_M(t')}\right)\int_0^t d\left(\frac{1}{\Sigma_M(t'')}\right)\times \frac{1}{N^2}\sum_{j=1}^{N}\sum_{i=1}^{N} E\left[\widehat{S}_i(t'),\widehat{S}_j(t'')\right]. \tag{B64}$$

The definition of the effective signal and (32) yield

$$\frac{1}{N}\sum_{j=1}^{N} E\left[\widehat{S}_i(t),\widehat{S}_j(t')\right] = \phi\sigma_s^2\frac{\min\{n(t),n(t')\}}{n(t)\,n(t')}, \tag{B65}$$

with

$$n(t) = \left[\frac{t}{\tau}\right]+1, \qquad n(t') = \left[\frac{t'}{\tau}\right]+1. \tag{B66}$$

Substituting (B65) into (B63) and (B64), we obtain

$$\begin{aligned} T_2(t) &= -2\phi\sigma_s^2\beta^2(t)\,\frac{\Sigma_M(t)}{n(t)}\int_0^t d\left(\frac{1}{\Sigma_M(t')}\right) \\ &= -2\phi\sigma_s^2\frac{\beta^2(t)}{n(t)}\left(1-\frac{\Sigma_M(t)}{\Sigma_M(0)}\right), \end{aligned} \tag{B67}$$

and

$$\begin{aligned} T_3(t) &= \phi\sigma_s^2\beta^2(t)\,\Sigma_M^2(t)\int_0^t d\left(\frac{1}{\Sigma_M(t')}\right)\int_0^t d\left(\frac{1}{\Sigma_M(t'')}\right)\frac{\min\{n(t),n(t')\}}{n(t)\,n(t')} \\ &= 2\phi\sigma_s^2\beta^2(t)\,\Sigma_M^2(t)\int_0^t d\left(\frac{1}{\Sigma_M(t')}\right)\frac{1}{n(t')}\left(\frac{1}{\Sigma_M(t')}-\frac{1}{\Sigma_M(0)}\right). \end{aligned} \tag{B68}$$

Evaluating (B68), we obtain

$$\begin{aligned} T_3(t) &= 2\phi\sigma_s^2\beta^2(t)\,\Sigma_M^2(t)\,\Pi(t), \\ \Pi(t) &= \frac{1}{n(t)}\left(\frac{1}{\Sigma_M(t)}-\frac{1}{\Sigma_M((n(t)-1)\tau)}\right)\times \\ &\quad \left[\frac{1}{2}\left(\frac{1}{\Sigma_M(t)}+\frac{1}{\Sigma_M((n(t)-1)\tau)}\right)-\frac{1}{\Sigma_M(0)}\right] \\ &\quad +\sum_{k=1}^{n(t)}\frac{1}{k}\left(\frac{1}{\Sigma_M(k\tau)}-\frac{1}{\Sigma_M((k-1)\tau)}\right)\times \\ &\quad \left[\frac{1}{2}\left(\frac{1}{\Sigma_M(k\tau)}+\frac{1}{\Sigma_M((k-1)\tau)}\right)-\frac{1}{\Sigma_M(0)}\right]. \end{aligned} \tag{B69}$$

Combining (B62), (B67) and (B69) yields (58). *Q.E.D.*

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Figure 1: **Timing of Events**

This Figure illustrates the timing of events. Market closures are periodic so that the durations each trading and non-trading periods are equal to τ and $\Delta\tau$ respectively.

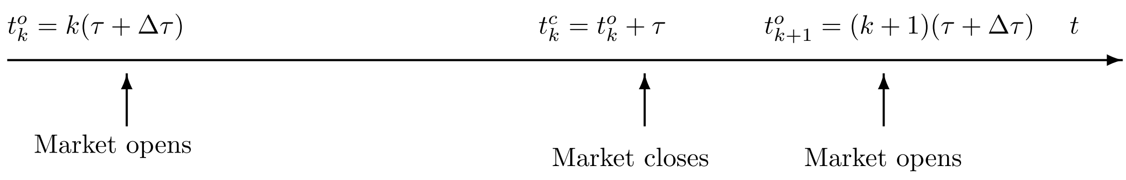

Figure 2: **Dynamics of** $\delta(t)$

The dynamics of the information asymmetry parameter $\delta(t)$ is shown as a function of time for the case of three market closures. There are five informed traders ($N = 5$) in the economy and the initial correlation coefficient between the private signals, ρ, equal to 0.5.

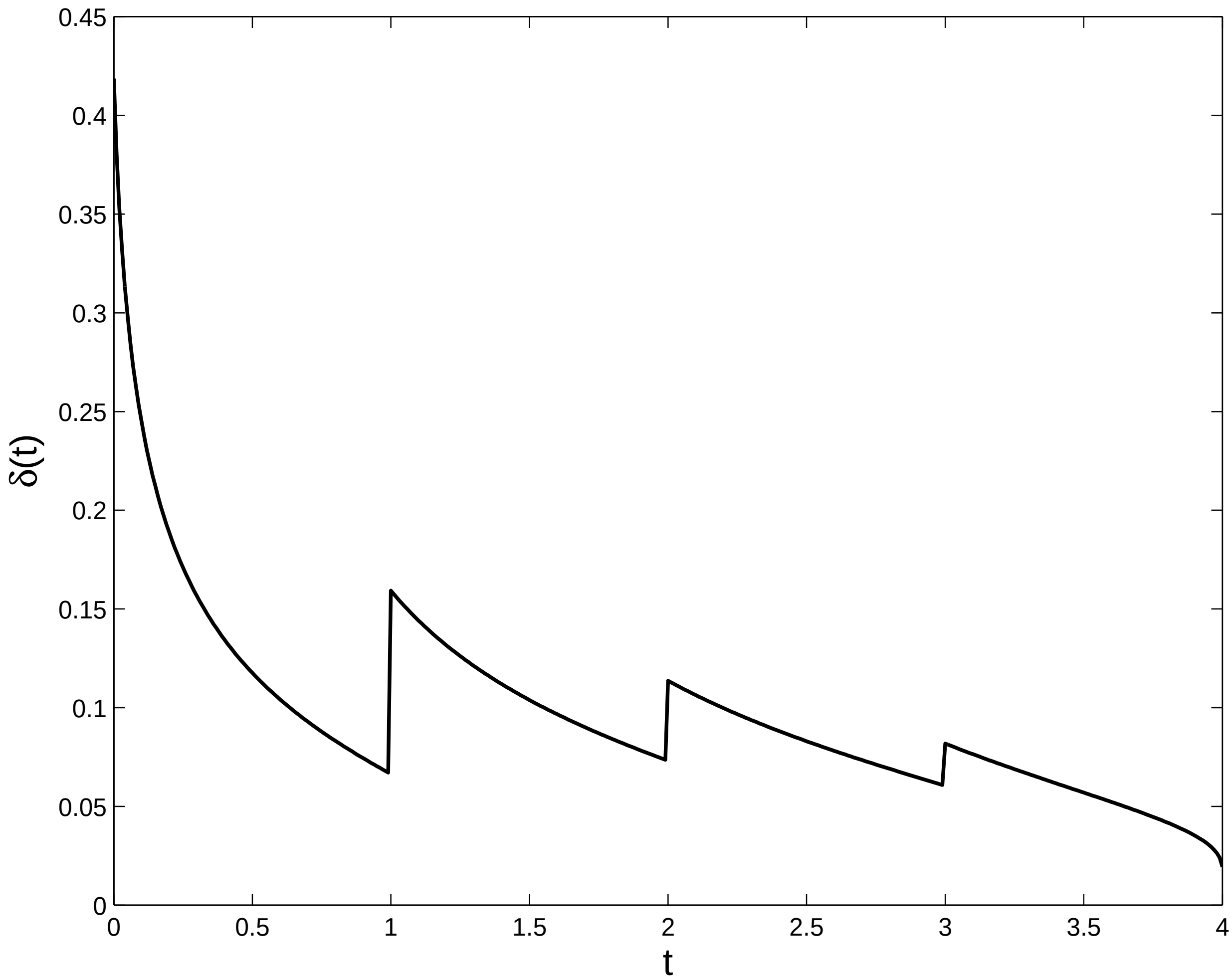

Figure 3: **Dynamics of $\beta(t)$**

The dynamics of the optimal informed trading parameter $\beta(t)$ is shown as a function of time for the case of the three market closures. It is compared to the intertemporal profile of the optimal $\beta(t)$ in the absence of market closures. There are five informed traders ($N = 5$) in the economy and the initial correlation coefficients between the private signals, ρ, equal to 0.5.

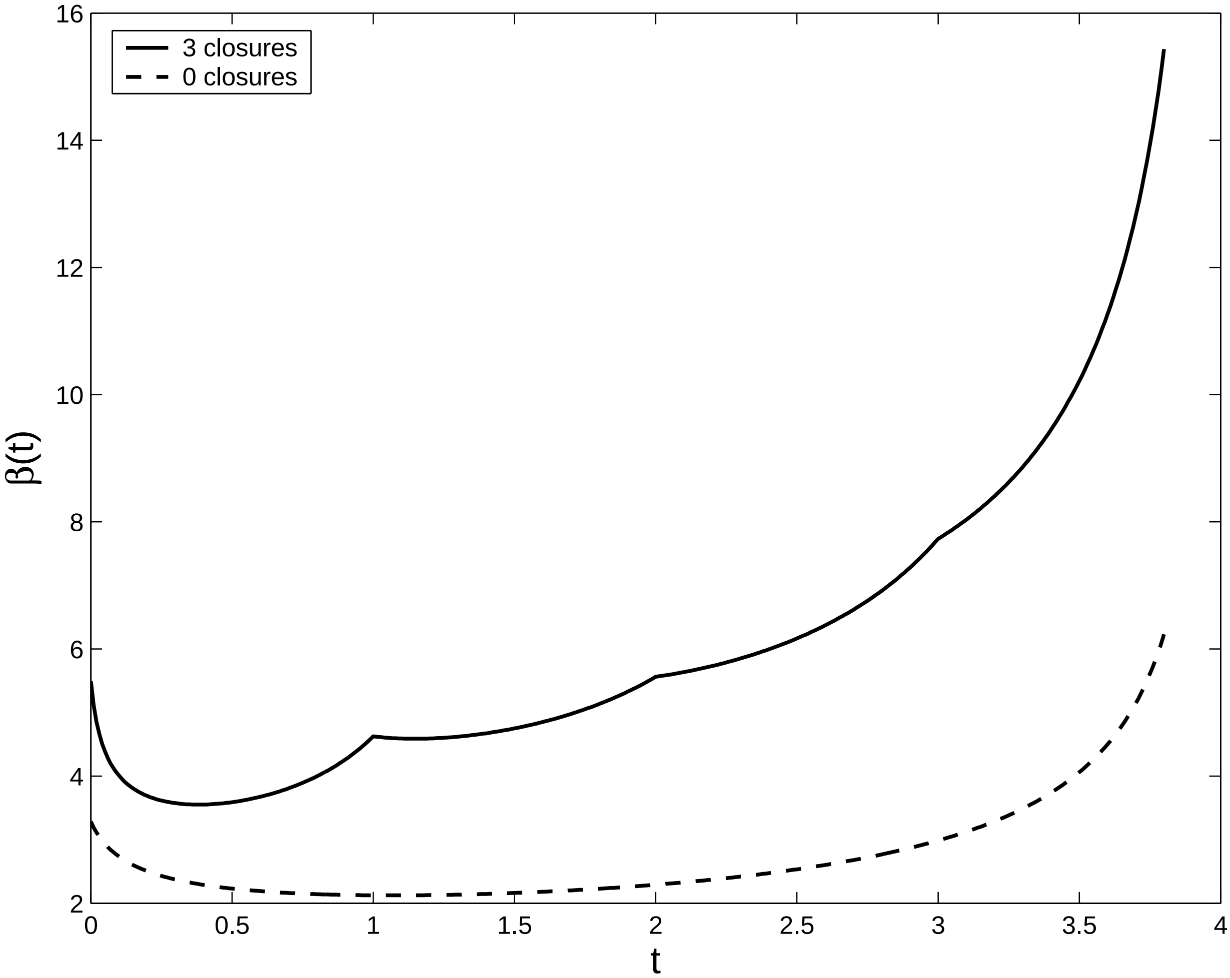

Figure 4: **Dynamics of Informed Trading Volume**

The dynamics of the informed trading volume, $TV(t)$, is shown as a function of time. It is compared to the optimal informed trading volume in the absence of market closures. There are five informed traders ($N = 5$) in the economy and the initial correlation coefficients between the private signals, ρ, equal to 0.5.

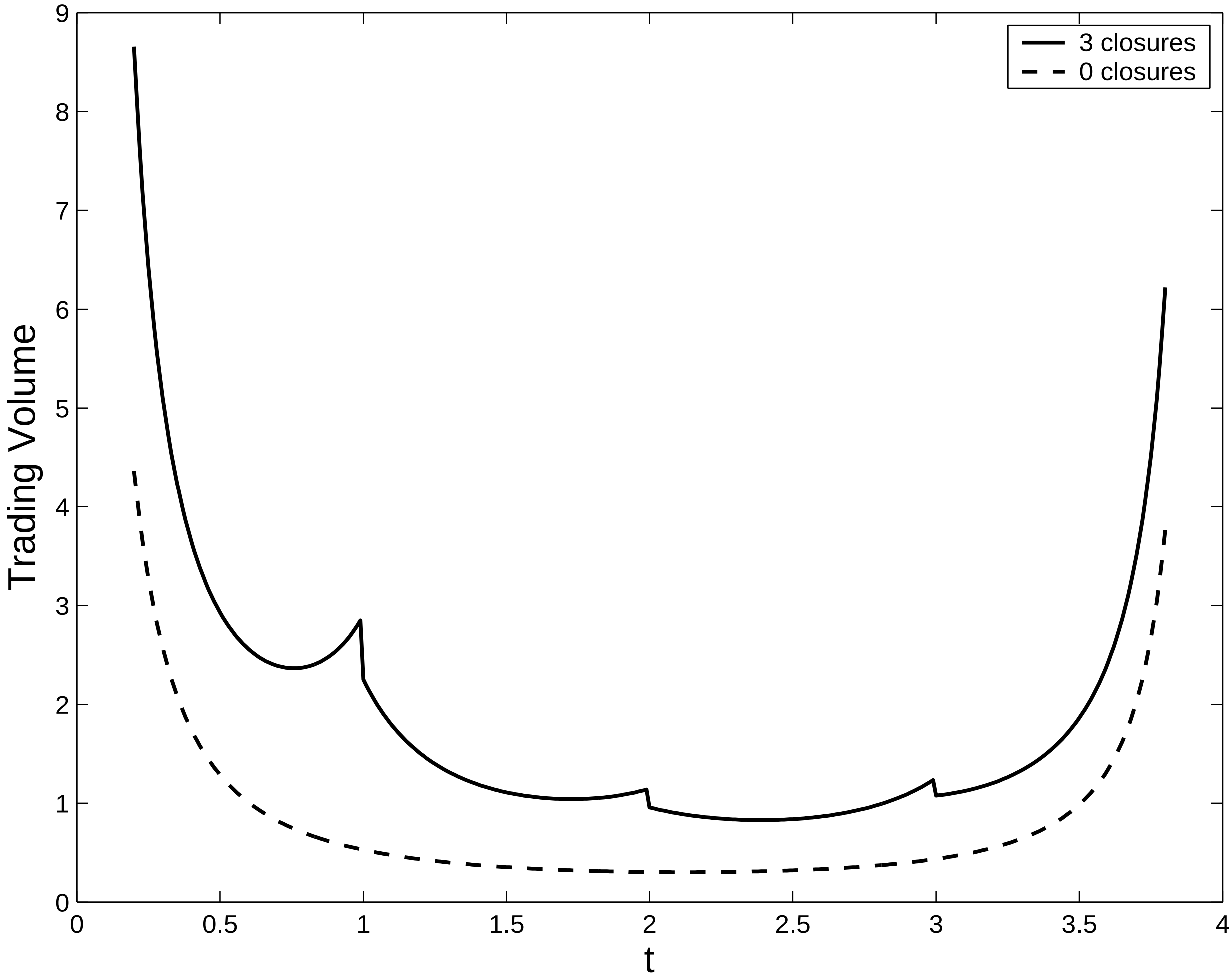

Figure 5: **Dynamics of Relative Trading Intensity**

The dynamics of the relative trading intensity is shown as a function of time for the case of the three market closures. It is compared the relative trading intensity in the absence of market closures. Following BCW, the relative trading intensity is defined as the ratio of the oligopolistic trading intensity defined in Corollary 1 as $\beta(t)/\delta$, to the optimal trading intensity of the monopolist informed trader from Kyle (1985). There are five informed traders ($N = 5$) in the economy and the initial correlation coefficients between the private signals, ρ, equal to 0.5.

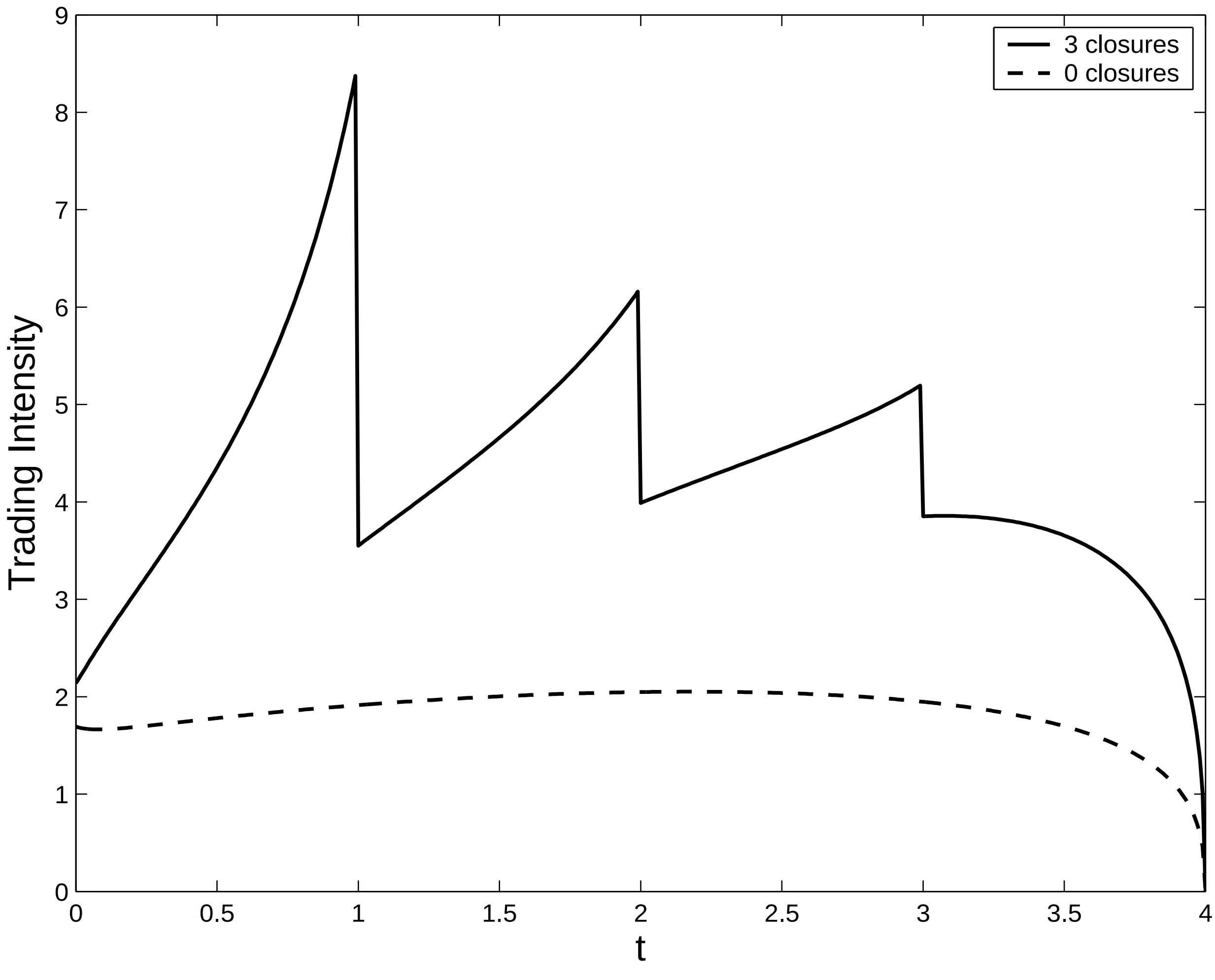

Figure 6: **Dynamics of Relative Market's Residual Uncertainty**

The relative market's residual uncertainty defined as the ratio of $\Sigma_M(t)$ for the oligopolistic and monopolistic cases is shown as a function of time for the case of three market closures. It is compared to the relative market's residual uncertainty in the absence of market closures. There are five informed traders ($N = 5$) in the economy and the initial correlation coefficients between the private signals, ρ, equal to 0.5.

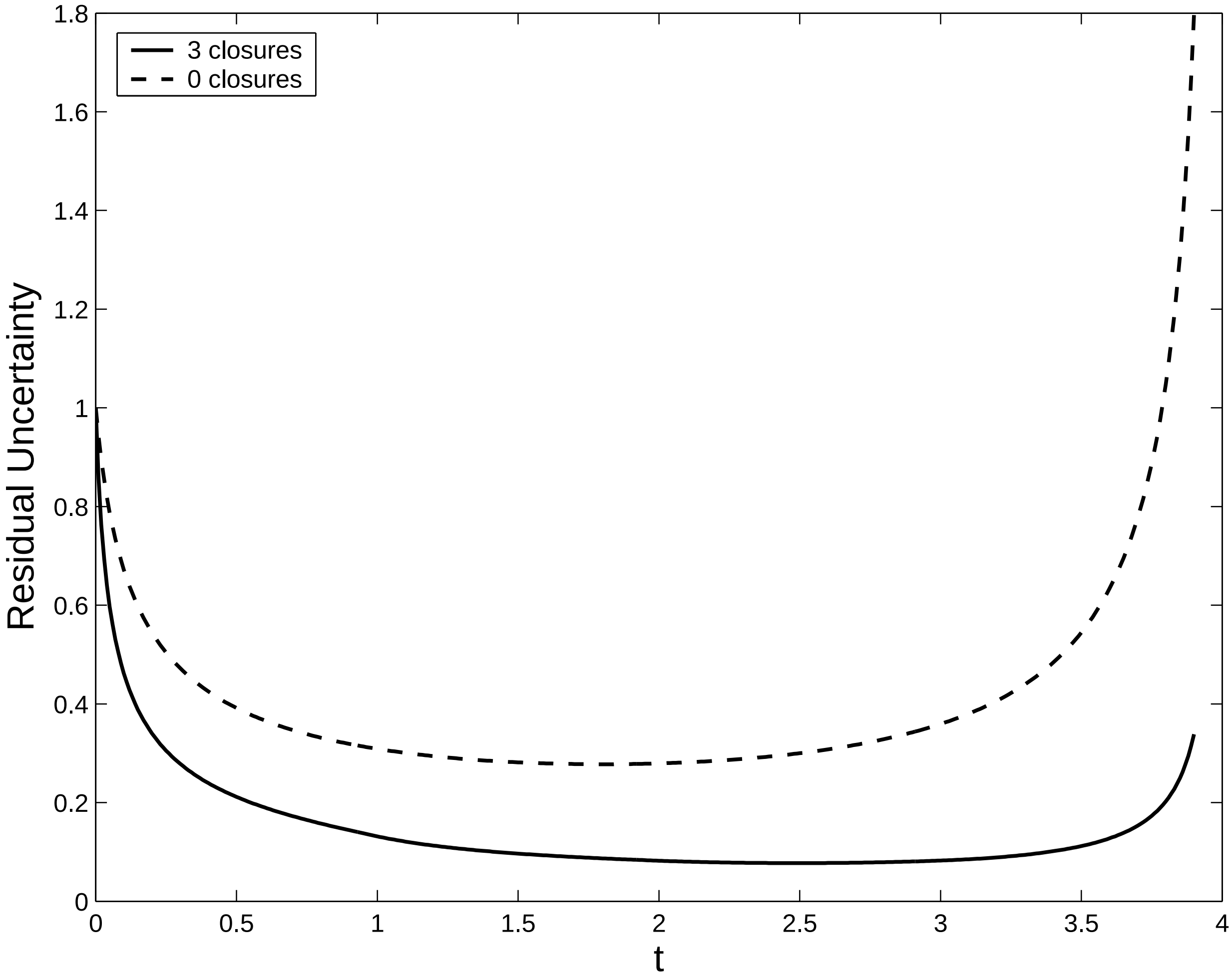

Figure 7: **Dynamics of Relative Market Depth**

The dynamics of the relative market depth parameter $1/\lambda(t)$ is shown as a function of time for the case of three market closures. It is compared to the relative market depth in the absence of market closures. The relative market depth is defined as a ratio of the oligopolistic and the monopolistic market depth parameters. There are five informed traders ($N = 5$) in the economy and the initial correlation coefficients between the private signals, ρ, equal to 0.5.

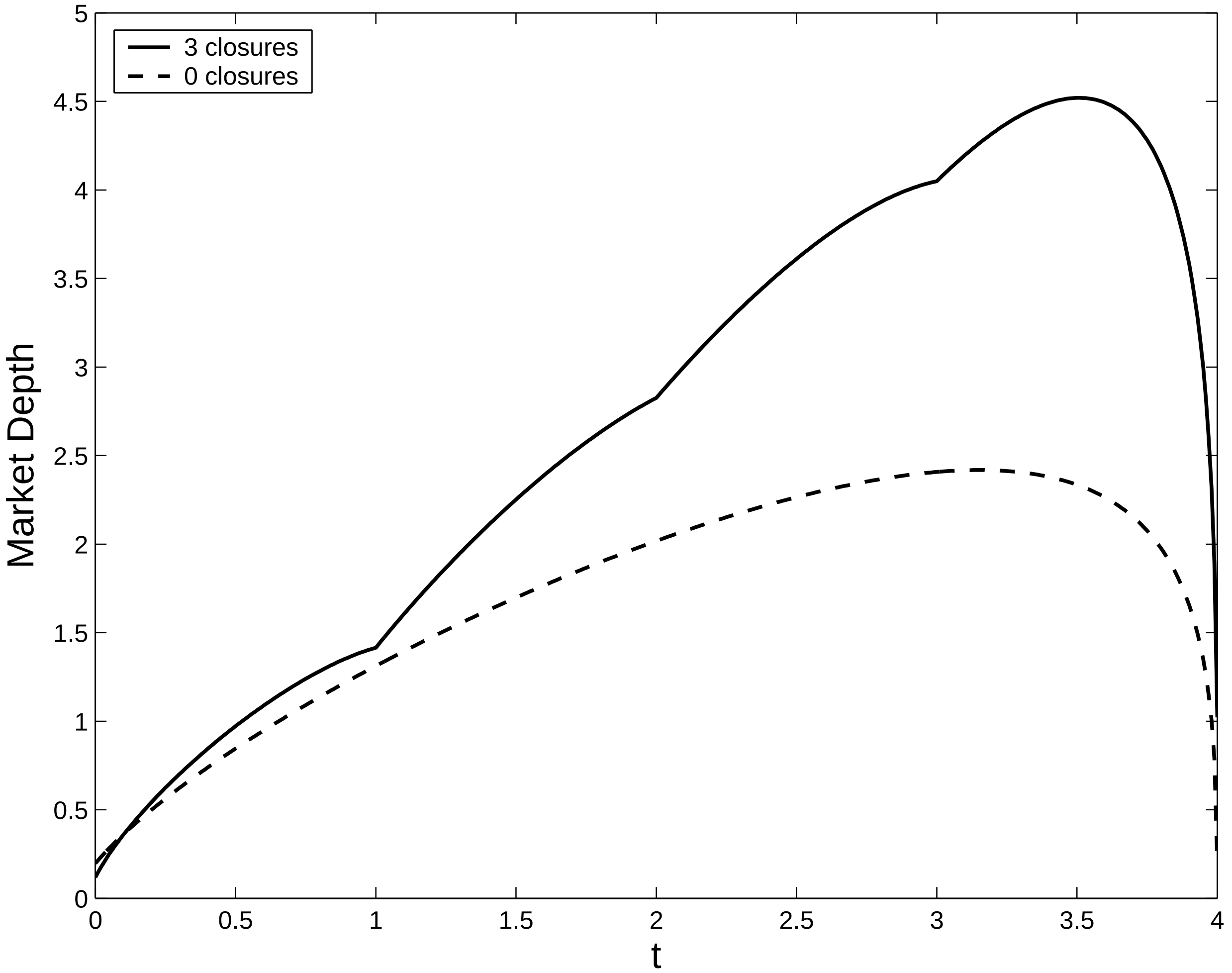

Figure 8: **Dynamics of Relative Trading Intensity for Different Values of ρ**

The dynamics of the relative trading intensity is shown as a function of time for several different values of the initial correlations between the private signals, ρ. There are three market closures and two informed traders ($N = 2$) in the economy.

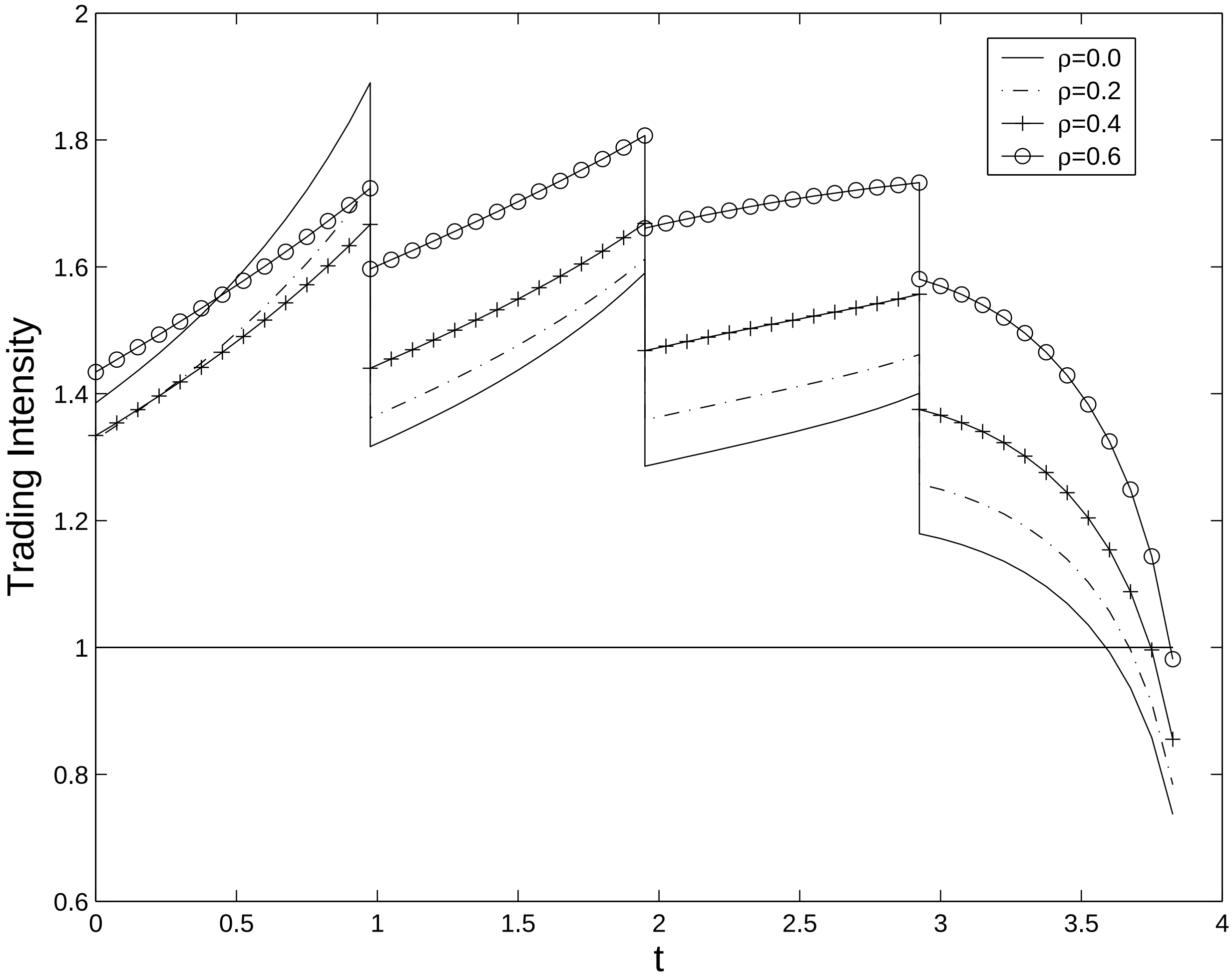

Figure 9: **Dynamics of Relative Market Depth for Different Values of ρ**

The dynamics of the relative market depth is shown as a function of time for several different values of the initial correlations between the private signals, ρ. There are three market closures and two informed traders ($N = 2$) in the economy.