Production, Collateral and the Risk-Free Rate

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February 14, 2006

Abstract

In this paper, I examine the implications of collateral constraints in a production economy and demonstrate that collateral constraints may have a role to play in resolving two outstanding puzzles: the risk-free rate puzzle and the total factor productivity puzzle. The first puzzle, as noted by Mehra and Prescott (1985), Weil (1989) and others is simply that it is difficult to obtain plausible values of the risk-free real interest rate in production economies without assuming implausibly high values of risk-aversion. This paper demonstrates that the risk-free real interest is related to idiosyncratic productivity risk through the collateral constraint and that a low risk-free real interest rate can be obtained for small, and plausible, values of risk-aversion. The second puzzle is more recent - namely why has the risk-free real interest rate fallen while measured total factor productivity has risen during the 1990's in the United States? The argument put forth here is that the level and persistence of idiosyncratic productivity risk is related to measured aggregate total factor productivity and the risk-free real interest rate via the collateral constraint. Hence, increases in aggregate total factor productivity that occur in conjunction with decreases in the risk-free real interest rate may simply reflect unanticipated increases in the level (or persistence) of idiosyncratic productivity risk.

JEL classification: E43, E44, G21

Keywords: Collateral; Wealth Heterogeneity, Risk-Free Rate, Total Factor Productivity, Equity Premium

[1] I thank, without implicating, Gregor Smith, Allen Head, Sumon Majumdar, Jason Allen, Huw Lloyd-Ellis, J-F Houde, Thor Koeppl, participants at the 2004 MidWest Macro Meetings at Iowa State, participants at the 2004 Canadian Economic Association Meetings and PhD seminar participants at Queen's University. The author gratefully acknowledges financial support from SSHRC. Corresponding author: Geoffrey Dunbar, Department of Economics, Simon Fraser University, 8888 University Drive, Burnaby, BC, V5A 1S6, Phone: 604-291-5909, email: geoffrey_dunbar@sfu.ca

1 Introduction

That collateral constraints are complicit in the story of financial intermediation is, in most developed economies, a well-known fact. However, the role that collateral constraints play in the allocation of capital and wealth remains, for the most part, unclear. To the extent that wealth and capital are central ingredients to asset prices and production then perhaps the role of collateral constraints in the equilibrium distributions of wealth and capital deserve inspection.

Intuitively, collateral constraints have two key implications for the distribution of wealth: they offer insurance for both borrowers and lenders and they prevent borrowers with insufficient collateral resources from borrowing. Collateral insures borrowers *ex-post* by allowing them to exchange a risky asset for a risk-less obligation. In addition, collateral insures lenders *ex-ante* by providing a enforcement mechanism for the repayment of loans. Collateral constraints limit borrowing by restricting the maximum amount of debt to the value of collateralizable assets. To the extent that collateral insures lenders against loan default, collateral constraints will affect the interest rate on debt by obviating the need for a default premium. However, collateral constraints also affect the interest rate on debt by reducing the available amount of debt that can be exchanged whenever (potential) borrowers face binding collateral constraints. To the extent that binding constraints reduce the effective demand for loans, collateral constraints should also lower the interest rate. Hence, collateral constraints may help resolve the outstanding risk-free real interest rate puzzle, noted by Mehra and Prescott [34] and Weil [39], among others.

A second puzzle which may be addressed by collateral constraints is why the total factor productivity increases during the 1990's in the US were accompanied by a fall in the risk-free real interest rate. To the extent that the risk-free real rate is related to the marginal product of capital one might expect that total factor productivity increases should be associated with an increasing risk-free real interest rate. I refer to the negative relationship between total factor productivity increases and the risk-free real interest rate as a disconnect puzzle. To see how collateral constraints can address this second puzzle imagine a production economy with unverifiable, idiosyncratic, productivity shocks. In such an economy, collateral constraints can reduce the efficiency of the distribution of capital stocks. To the extent that collateral constraints limit borrowing, they can affect the risk-free real interest rate. Collateral constraints may also induce some agents, who would otherwise choose to save wealth through risk-free debt, into saving wealth through (risky) capital to smooth consumption intertemporally. As a result, collateral constraints can lead to inefficiently high levels of output. More subtly, when the idiosyncratic risk is persistent, those agents with high capital stocks will tend to have higher than average realisations of the productivity shock. A direct consequence is that an aggregate production function will weight idiosyncratic shocks by the size of the capital stock and an aggregate productivity measure will overstate the true mean level of productivity. Hence, the level of idiosyncratic risk may be positively related to aggregate total factor productivity while being negatively related to the risk-free real interest rate. As a result, collateral constraints and idiosyncratic productivity shocks may help explain the aggregate total factor productivity and the risk-free real interest rate disconnect puzzle.

Finally, to the extent that the risk-free real rate puzzle is often linked to the equity-premium puzzle, collateral constraints may also have a role to play in resolving the latter as well. It is plausible to think that collateral constraints can affect both the price and the dividend from equity. In a production economy, capital which is inefficiently allocated across producers will have an effect on the output, and hence dividend yield, of capital. Moreover, if producers cannot borrow as a result of binding collateral constraints then the price of capital should likewise be affected. The model presented in this paper is, unfortunately, ill-equiped to address the equity premium puzzle directly. However, the model does provide some evidence of collateral effects on the marginal and average real return to capital. To the extent that the average return to capital in the model is related to real-world average equity returns, then collateral constraints and idiosyncratic production risk can return low risk-free real interest rates and reasonable equity premiums for relatively modest levels of risk aversion.

The model examined in this paper is the following. I imagine a world populated by households who own a particular production technology which is identical across all households. Households in the model can be thought of as independent producers, *i.e.* households with a backyard technology. All households produce an identical consumption good using the production technology. In each period, each household suffers an idiosyncratic shock which affects the quantity of output she produces. The shock is intended to capture such vagaries as bad weather, crop disease, or illness. There is no aggregate uncertainty. The idiosyncratic shock provides households with an explicit consumption-smoothing motive. However, lenders require an enforcement technology to ensure that borrowers will repay their obligations. Collateral suffices to ensure that intermediary contracts are enforceable. Lending and borrowing contracts are therefore two dimensional objects: an amount borrowed (lent) and an amount of capital posted as collateral.

The paper is structured as follows. Section 2 describes the relation of this paper to the literature on incomplete markets and limited enforcement of contracts. Section 3 presents some recent evidence on the total factor productivity puzzle and the equity premium puzzle. Section 4 presents the model environment. Section 5 discusses the enforcement technology which arises in equilibrium. Section 6 defines the autarkic equilibrium of the model. Section 7 presents a benchmark social planning equilibrium of the model which is, naturally, the complete markets outcome. Section 8 describes the types of contracts which arise as equilibrium behavior. Section 9 defines a competitive equilibrium, where collateralized lending and borrowing contracts are traded in equilibrium. Section 10.1 discusses the solution concept applied to determine an (approximate) solution to the competitive equilibrium. Section 10 presents the results from different specifications of the model parameters. In particular, the sensitivity of the collateralized equilibrium is discussed. Section 11 concludes.

2 Research Context

This paper intersects three strands of the literature on financial intermediation. In particular, it examines the role of financial intermediation in a similar manner to models which focus on: incomplete markets and wealth inequality; endogeneous borrowing constraints; and limited contract enforcement mechanisms. The primary motivation of this paper is to build on Huggett [17] and Aiyagari [3] by developing a heterogeneous-agents model with collateralized borrowing/lending.

Several authors have constructed theoretical models where collateral arises as an optimal debt contract. Lacker [28] constructs a two-period model where output is only observable to the borrower and demonstrates that, when the borrower values collateral more than the lender, collateral ensures repayment. Kocherlakota [24] demonstrates that collateralized debt contracts are optimal in models where the ex-post value of collateral and the ex-post investment return are known only to the borrower. Geanakoplos and Zame [16] construct a two-period general equilibrium model where households can default at any time and show that financial assets are only traded when backed by collateral. Finally, Andolafatto and Nosal [5] construct a model where agents endogenously circulate claims which are implicitly backed by collateral. The model examined in this paper can be viewed as similar to the type of model examined theoretically by Araujo, Pascoa and Torres-Martinez [6], who demonstrate that collateral constraints rule out Ponzi schemes and hence by extension demonstrate the existence of stationary equilibria in models with collateral constraints.

This paper demonstrates that a collateral requirement restricts the amount of intermediation somewhat like endogenous solvency constraints. That is, the amount an household may borrow is limited by the assets she may post as collateral. In this respect, the present paper is related to Kehoe and Levine [20] and Alvarez and Jermann [4] who study the effects of solvency constraints. One difference of this paper from [20] and [4] is that I do not assume that households determine borrowing constraints to ensure that repayment is individually rational for the borrower.

Kiyotaki and Moore [21] , Kocherlakota [23], Lustig [30], Kubler and Schmedders [27] and Cordoba and Ripoll [11] examine the macroeconomic impacts of collateral constraints explicitly. Kiyotaki and Moore construct an infinite-horizon economy populated by two types of agents, farmers and gathers (where the nature of the economy specifies a fixed rate of interest) and demonstrate the presence of credit cycles resulting from the collateral constraints. Kocherlakota considers an open economy variant of a model with collateral constraints and shows that the amplification of shocks inherent in collateral constraints depends on the parametrisation of the economy. Cordoba and Ripoll examine the extent to which collateral acts as an amplification mechanism for exogenous shocks. They find that the amplification effects of collateral are typically small. My paper differs substantially from those mentioned above as I do not consider amplification effects or credit cycles. Lustig examines a model of bankruptcy where agents post shares in Lucas-trees as collateral for state-contingent loans. He demonstrates that the set of equilibria are constrained relative to those of Alvarez and Jermann. Kubler and Schmedders examine a similar model and characterize the wealth distribution when there are two agents. My paper extends this literature by examining the consequences of collateralized lending on the distribution of wealth and the risk-free real interest rate in a general equilibrium model with production with many households.

Bankruptcy rules have been examined in a number of papers, including Livshits, MacGee and Tertilt [29], Chatterjee, Corbae, Nakajima and Rios-Rull [9] and Zame [40] among others. Although bankruptcy is possible in my model, I abstract from modeling different bankruptcy rules and instead assume that assets offered as collateral are seized by creditors once repayment stops. Once collateral is seized any remaining debt is discharged by the lender.

Finally, this paper touches briefly on two widely examined, and often linked, puzzles in the literature: namely the risk-free rate puzzle and the equity premium puzzle. As Mehra and Prescott [34] famously noted, the historical average real rate of return on equities in the US is roughly 6 per cent above the historical average risk-free real interest rate. As they, and others, have demonstrated, it is typically difficult to reconcile this puzzle in a neoclassical framework without implausibly high degrees of risk aversion. The risk-free rate puzzle specifically was stressed by Weil [39] and has been explored further by numerous authors. For a survey of the literature concerning the equity-premium puzzle, and the risk-free rate puzzle, the interested reader is referred to a subsequent review article by Mehra and Prescott [35] who exhaustively account for most of the prevailing research.

3 Some Macro Evidence

In this section, I present some evidence on total factor productivity, the average risk-free real interest rate and the equity premium for the U.S. I begin with some empirical evidence on the second puzzle, namely why has the risk-free rate fallen why total factor productivity has risen? This second puzzle, the total factor productivity and risk-free real interest rate disconnect puzzle for the United States, is best illustrated in the following table, Table 1. Total factor productivity changes are measured as the average percentage increase in measured multifactor productivity for private non-farm businesses in the US for the periods considered.[1] The real interest rate is calculated as the average annual nominal interest rate on a US 90 day Treasury Bill minus the average annual CPI inflation for the same period.[2] As the table demonstrates, during the 1980s and 1990s increases (decreases) in multifactor productivity occurred while the risk-free rate fell (rose). To the extent that the risk-free rate is related to the marginal product of capital and by extension total factor productivity, these negatively associated movements constitute a puzzle. One might argue that other factors, such as temporary oil price shocks, explain the negative co-movements but this is perhaps harder to rationalise over the length of time period (decades) considered. Another factor which might explain a decrease in the marginal product of capital and hence the real interest rate in a typical neoclassical model is a rising capital-to-labour ratio, implying that the growth rate of capital exceeds the growth rate of labour. As the table illustrates, this explanation may be true but it is nevertheless still difficult to reconcile the changes in the real interest rate in 1980s and 1990s.[3] Moreover, the changes in the capital-labour growth rate have been modest and it seems unlikely that they could explain all the movement in the real interest rate over the periods considered.

The other puzzle tangentially addressed in this paper is the equity premium puzzle. The following table, Table 2, presents some recent data on those movements.[4] As the table illustrates,

[1]The data on multifactor productivity is from the Bureau of Labor Statistics. It calculates productivity as output per combined unit of labor and capital input. For data methodology see Trends in Multifactor Productivity, 1948-81, Bulletin 2178, September 1983. The series used in this chapter dates to 1961 which explains why only the period 1962-1969 was covered.

[2]The CPI index used is from the Bureau of Labor Statistics series . The methodology used to construct the series is from the BLS Handbook of Methods, Chapter 17. The treasury bill data was collected from the US Board of Governors.

[3]Data on the capital to labour ratio is from the Bureau of Labor Statistics series titled Capital Services per Hour [of Labor], series MPU750027.

[4]The equity data are the annual average rate of return on equity, including dividends, for the Standard and Poors

Table 1: Multifactor Productivity and the Real Interest Rate in the US

Decade	TFP	Real Interest Rate	K/L
1962-1969	2.2	1.72	2.43
1970-1979	1.1	-0.77	2.88
1980-1989	0.3	3.3	2.78
1990-1999	0.8	1.87	2.34

TFP refers to the average annual increase in measured multifactor productivity; Real Interest Rate refers to the period average of the average annual return on a 90 day US treasury bill minus the annual CPI inflation rate; K/L refers to the average annual change in the ratio of capital to labour.

between the 1960s and 1980s the return on equity and the real interest rate tended to move in the same direction. Higher real interest rates were associated with higher real returns-to-equity. This relationship tended to disappear during the 1990s as the real return on equity rose while the real interest rate fell. As the model presented in this paper will show, the relationship between the return-to-equity and the real interest rate observed between the 1960s and 1980s is consistent with idiosyncratic productivity risk and collateral constraints. The model will do less well at replicating the experience of the 1990s.

Table 2: Return-to-Equity and the Real Interest Rate in the US

Decade	Return on Equity	Real Interest Rate	Premium
1962-1969	4.2	1.72	2.5
1970-1979	0.3	-0.77	1.1
1980-1989	11.6	3.3	8.3
1990-1999	15.6	1.87	13.7

Return on Equity is the real return to the S&P 500 Index, including dividends, and deflated by the CPI; Real Interest Rate refers to the real interest rate on a risk-free bond; Premium is the excess real return to the S&P 500 Index.

4 General Model

In this section, a simple heterogeneous agent model is developed where collateralized lending arises in equilibrium. households are differentiated by a shock process in their production function. Apart from the shock, the production technology is identical across all households. As a result, considerations of human capital, monopolistic competition, or labor supply are absent.

500 Index. The data are from Robert Shiller's data on equity prices, available at www.econ.yale.edu/ shiller/data.htm, originally published in *Market Volatility*, University of Chicago Press, 1989 and since updated.

4.1 Preferences and Technology

There exists a continuum of households who live forever. The population is constant and normalized to unity. Households have the following preferences:

$$U^i = E \sum_{t=0}^{\infty} \beta^t u(c_t^i); \tag{1}$$

where U^i is the discounted lifetime utility of household i, $\beta \in (0,1)$ is the discount factor and $u(c_t^i)$ is the period utility of an household of type i in period t with consumption $c_t^i \in \mathbb{R}^+$. The function $u(c_t^i)$ is assumed to satisfy $u'(c_t^i) > 0$, $u''(c_t^i) < 0$ and $u'(0) = \infty$.

Each household is initially endowed with k_0^i units of the capital stock and a production technology:

$$y_t^i = \eta_t^i (k_t^i)^\alpha \tag{2}$$

where $y_t^i \in \mathbb{R}^+$ is the output of the consumption good by household i in period t, $k_t^i \in \mathbb{R}^+$ is the capital stock used in production by household i in period t, $\alpha \in (0,1)$ is the productivity of capital and $\eta_t^i \in N = \{\overline{\eta}, \underline{\eta}\}$; $\overline{\eta} > A > \underline{\eta}$, is a mean A, idiosyncratic, stochastic shock to household i in period t. The shock, η_t^i, follows a Markov process with stationary transition probabilities $\pi(\eta'|\eta) = Prob(\eta_{t+1}^i = \eta'|\eta_t^i = \eta) > 0$ for $\eta, \eta' \in N$.

The shock is intended to capture idiosyncratic elements of production, such as: drought, illness, median age, median experience, time constraints, *etc.*[5] The shock is assumed to be private information. Households realize their shock and output simultaneously. I assume the initial distribution of the shock is such that $\eta_0^i = \overline{\eta}$ for exactly half of the households, while the remaining households have $\eta_0^i = \underline{\eta}$. Finally, capital is assumed to depreciate by a fraction $\delta \in [0,1)$ each period.

4.2 Contracting

The idiosyncratic shock implies that intertemporal risk-sharing can be Pareto-improving; hence contracting between households is advantageous. The assumptions specified in this section are sufficient for collateralized lending and borrowing contracts to arise.

Assumption 1 An household's shock, η, is private information.

Assumption 2 Capital stock holdings, k_t^i, and preferences are common knowledge (or costlessly verifiable).

[5] In an infinite-horizon model, age and experience are clearly identical across agents. However, infinite-horizon agents may be considered also as dynastic households. Thus, the shock process also can represent the proportion of productive members of the household at any given point in time.

Assumption 3 Contracts which specify a transfer, b_t^i, from (to) an household i in period t at a cost $q_{t-1}^i b_t^i$ in period $t-1$ may be written costlessly. Contracts are assumed to be common knowledge and contracts may be breached.

Assumption 4 Households may contract once per period through an intermediary (henceforth referred to as the bank). There is assumed to be free entry to banking.

Assumption 1 implies that households cannot be differentiated by observation. In particular, households with a high shock may consume their additional output without public observation. Moreover, unlike a two-agent model, no household can infer the output (or shock) of any other household given the assumption of a continuum of households.Assumption 3 characterizes the nature of the bargaining problem. Assumption 4 rules out some contracting equilibria in that each agent can only bargain once per period.

5 Enforcement

In this section, I describe the enforcement mechanism for contracts. That an enforcement technology is required is straightforward to establish.

Any enforcement mechanism used by banks to enforce contracts must be binding in the sense that the enforcement mechanism must be robust to free-entry. That is, no borrower can avoid enforcement costs by banking with an entrant bank. One enforcement mechanism which is robust to free-entry is a collateral mechanism. I assume that a collateral mechanism is used by banks to enforcement contracts.

Assumption 5 Banks require collateral for loans where collateral is defined as the exchange of a claim which transfers ownership of an asset, $a_{t+1}^i \in [0, (1-\delta)k_t^i]$, from the borrower i to the lender conditional on the breach of a contract in period $t+1$. The collateral claim is assumed to be costless to write and the seizure of collateral, conditional on the breach of a contract, is assumed to be costless and immediate.

Together with Assumption 2, Assumption 5 implies collateral cannot be privately consumed by the borrower. However, collateral may be used by household i for production. In addition, only capital currently held by the household in period t can be pledged as collateral for period $t+1$. That is, agents cannot commit to capital investment. [6]

[6] The restriction that only capital currently held by the household can be pledged as collateral prevents a Ponzi

I note that collateral constraints are different from other constraints considered in the literature. In particular, [20] and [4] among others, have studied economies where the individual borrowing constraints are determined by individual rationality constraints which are, in effect, censure mechanisms. A censure mechanism in the context of this paper would be a mechanism which excludes a household from borrowing or lending for a period of time, ν. In relation to the model studied in this paper, censure and collateral have contrasting implications for borrowing.

First, no censure contract which specifies a time period of censure $\nu > 0$ can be constructed as a collateral contract. Clearly, for censure to impose an enforcement cost then the defaulting household must have lower expected utility when censured than it would otherwise. However, the household who invokes censure also suffers a cost in addition to the cost it suffers when the repayment of its loan is forfeit. The additional cost results from the inability to intermediate with the censured household. With collateral, the household does not incur the additional cost and, generally, reaps a positive return from the value of the collected collateral.

Second, censure mechanisms imply less (more) borrowing than collateral mechanisms for poor (rich) households. For example, imagine an household with very low capital holdings $k \to 0$ such that the costs of autarky are relatively large. Then, for this household, the value of intermediation is relatively high and thus the borrowing limit where it is indifferent between default and repayment is relatively high. However, under a collateral mechanism, it has very low access to intermediation because it has little capital to post as collateral. Hence, the distributional effects of collateral mechanisms will be quite different from those of censure mechanisms.

The final assumptions are largely self-explanatory.

Assumption 6 The bank proposes (the terms of) contracts.

Assumption 7 The bank can disburse received collateral.

Without Assumption 7, it may not be feasible for a bank to write collateral contracts since there may be a positive probability that a bank acquires collateral in a given period. Without a mechanism to disburse collateral, the bank is acquiring an asset to which it ascribes no value. Hence, for collateral contracts to be feasible for a bank, there must be a mechanism by which the bank can transform collateral to output or transfer collateral to creditors in lieu of output claims.

style game from arising. If households could borrow against future but unenforceable capital investment, households have an incentive to continually promise more capital investment (without necessarily following through) in order to pay-off current period debts while banks have a similar incentive to 'believe' such promises in order to remain solvent. Such an equilibrium would be, in effect, a type of Ponzi scheme.

5.1 Timing

The timing of events in each period is as follows:

1. Households produce using their available capital stocks;
2. Households learn their shock and receive their output;
3. Households contract with the bank, *i.e.* households settle or default on their previous-period contracts, if such contracts exist, and re-contract (or not) with a bank;
4. Banks redeem all contracts;
5. Households choose next-period capital holdings and consume;
6. Depreciation occurs.

Hence, contracts are offered after production has occurred so a household's prior expectation of the shock cannot be used to differentiate households.[7] Additionally, the contracting stage implies an implicit match which occurs between agents and banks within each period. Given that the goal of this chapter is to describe the effects of collateral as the enforcement mechanism for contracts, I abstract from formally modeling any matching frictions. That is, I assume that when $L \geq 2$ banks offer the same contract, agents are randomly allocated to each bank. A law of large numbers argument then applies and the resulting portfolio distribution of agents at each bank is assumed identical.

5.2 Bank's Problem

The profit, θ_t, of a bank in any period t is simply its flow of funds and may be written as:

$$\theta_t = -\sum_i (1-\tau_t^i) b_t^i + \sum_i q_t^i b_{t+1}^i + \sum_i \tau_t^i a_t^i \tag{3}$$

where θ_t is profit in period t, and a_t^i is the collateral seized from household i in period t in the event that household i defaults on its loan.

[7] One could imagine that contracts are signed prior to production in which case both soon-to-be high types and soon-to-be low types with identical histories prefer *ex-ante* the same contract. However, post-production both types will prefer separating contracts (different bond choices) and thus some additional form of commitment would be needed.

Free entry into banking implies that, in expectation, no bank makes positive profits. In addition, I impose a non-negative condition on bank profits:

$$\theta_t \geq 0. \tag{4}$$

The reason is straightforward. In this environment, a bank is simply a credit market. In any period where the flow of funds is negative, then the bank must be adding aggregate resources to the economy. Since the output good is non-storable this is impossible. Hence, in any period where the bank earns a negative return on its previous period contracts, *i.e.* $-\sum_i(1-\tau_t^i)b_t^i + \sum_i \tau_t^i a_t^i < 0$, then the bank must earn positive profit on its current sales $\sum_i(1+\tau_t^i)q_t^i b_{t+1}^i > 0$ to avoid bankruptcy.

One implication of free entry for banks is that no bank can earn a profit on bond sales, *i.e.* $\sum_i q_t^i b_{t+1}^i > 0$. The reason is immediate. Any entrant bank could offer a menu of contracts such that $\hat{q}_t^i \leq (\geq) q_t^i$ for all lenders (borrowers) i such that $\sum_i q_t^i b_{t+1}^i > \sum_i \hat{q}_t^i b_{t+1}^i > 0$ and the entrant bank would earn a profit. Moreover, all agents i would be strictly better off.[8] Hence, $\sum_i q_t^i b_{t+1}^i > 0$ will not be offered by a bank in a banking equilibrium.

In the event of bankruptcy, $\theta_t < 0$, the bank earns a payoff $\theta_t < 0$ and I assume the bank cannot re-enter the banking sector. In essence, I imagine the bank suffers a bad reputation effect and cannot service any contracts b_{t+1}^i. Further, I assume that banks are required to simply liquidate their assets (repaid loans and enforcement penalties) proportionately to all depositors (those agents with $b_t^i > 0$) in bankruptcy.

6 Autarky

Under autarky, agents face a trivial problem inasmuch as they cannot exchange and therefore they either consume their production or invest in capital every period. The period budget constraint for an agent of type i is:

$$c_t^i + \frac{k_{t+1}^i}{1-\delta} - k_t^i = \eta_t^i (k_t^i)^\alpha \tag{5}$$

Let Λ represents the state vector under autarky, $\Lambda = (k^i, \eta^i)$; Λ' and $k^{i'}$ represent the state vector and capital holdings in the $t+1^{th}$ period respectively. The agent's problem can be represented as (subscripts are omitted):

$$V(\Lambda) = \max_{c^i, k^{i'}} \; u^i(c^i) + \beta \sum_{\eta'} V(\Lambda')\pi(\eta'|\eta) \tag{6}$$

[8] Clearly, given a different menu of prices, agents will also be better off if the costs of borrowing fall since unconstrained agents can choose higher levels of borrowing or lending.

Let $\mu(\Lambda)$ represent the measure of agents in state Λ. A stationary autarkic equilibrium may be defined as follows:

A stationary recursive autarkic equilibrium is a set of functions $V(\Lambda)$, $c(\Lambda)$, $k(\Lambda)$ and $\mu(\Lambda)$ such that given δ, α, and $\pi(\eta'|\eta)$:

1. Agents in state Λ choose $c(\Lambda), k(\Lambda)$ to maximize $V(\Lambda)$.

2. There exists an invariant probability measure P^{Λ} defined over the ergodic set of equilibrium distributions, Λ.

7 Social Planner

Consider the social planner's problem. In the current environment, in every period a household may find itself in one of two possible realizations of the idiosyncratic shock, $\eta_t^i = \{\underline{\eta}, \bar{\eta}\}$. Since the social planner is constrained by *aggregate* resources then, clearly, the feasible set of autarkic allocations is a subset of the feasible set of social planning allocations. Let μ_1 and μ_2 refer to the measures of households receiving shocks $\underline{\eta}$ and $\bar{\eta}$, respectively, in a given period and let $\hat{K}_t$ be the aggregate capital stock in period t. The social planner can transfer capital and the consumable output to solve the following program:

$$\begin{aligned} W(\hat{\Lambda}) \quad &= \quad \max_{c^1, c^2} \ \lambda\mu_1\big(u^1(c_t^1)\big) + (1-\lambda)\mu_2\big(u^2(c_t^2)\big) + \beta W(\hat{\Lambda}') \\ s.t. \quad & \quad \sum_{i=1}^{2} \mu_i[c_t^i + \frac{k_{t+1}^i}{1-\delta} - k_t^i] = \sum_{i=1}^{2} \mu_i f(k_t^i) \\ & \quad c_t^1, c_t^2 \geq 0 \\ & \quad \sum_{i=1}^{2} \mu_i k_t^i = \hat{K}_t \end{aligned} \tag{7}$$

where $\hat{\Lambda} = (\hat{K}, \mu_1, \mu_2)$ refers to the state vector under the planner's problem and λ reflects the planner's weight on the utility of a household of type 1. The social planner will transfer capital to households who receive a good shock (to maximize the aggregate production) and subsequently distribute the consumable output. The planner's problem gives some intuition for the role of collateral. The planner redistributes capital to (efficiently) maximize total output. Hence, one possibly overlooked role for collateral contracts is simply that they allow the possibility of a redistribution of capital. Under highly persistent idiosyncratic shocks such a redistribution may be desirable.

8 Collateralized Contracts

In this section, I describe the collateral contract in detail.

8.1 Household's Problem

Where a_t^i is a collateral requirement leveled in period t by the bank for a loan b_t^i; τ_t^i is an indicator on default; q_t^i is the price of a period $t+1$ bond in period t; and $\underline{b}_{t+1}^i \leq 0$ is the borrowing constraint for household i in period t, then a household's problem in any period t can be written as:

$$V(k_t^i, b_t^i, \eta_t^i) = E_t \sum_{s=t}^{\infty} \beta^{s-t} \max_{c_s^i} u(c_s^i); \tag{8}$$

subject to:

$$c_t^i + \frac{k_{t+1}^i}{1-\delta} + (1+\tau_t^i) q_t^i b_{t+1}^i = \eta_t^i (k_t^i)^\alpha + (1-\tau_t^i) b_t^i + k_t^i - \tau_t^i a_t^i \tag{9}$$

$$b_{t+1}^i \geq \underline{b}_{t+1}^i \tag{10}$$

$$\tau_t^i = \{0, 1\} \tag{11}$$

The endogenous collateral constraint on borrowing, $\underline{b}_{t+1}^i$ and the savings technology which gives rise to q_t are discussed in Section 9. Under default households are assumed to forgo repayment of a debt b_t^i and forfeit their collateral. Hence the remaining capital stock of household i after default is simply the difference between the capital stock at the beginning of the period and the amount claimed by the bank.[9]

A household i's state can be described by the *tuple* $S^i = \{k^i, b^i, \eta^i\}$. Suppose that (as will be true in equilibrium): all households' capital holdings belong to a compact set, $k^i \in [\underline{k}, \bar{k}]$ and; all households' bond holdings belong to a compact set, $b^i \in [\underline{b}, \bar{b}]$. Define $\mathcal{S} : [\underline{k}, \bar{k}] \times [\underline{b}, \bar{b}] \times \mathcal{N}$ and $\mathcal{B}_{\mathcal{S}}$ as the Borel σ-algebra on $\mathcal{S}$. I define μ as the probability measure on $(\mathcal{S}, \mathcal{B}_{\mathcal{S}})$ with an assumed transition function $\mathcal{P} : \mathcal{S} \times \mathcal{B}_{\mathcal{S}}$. I note that $\mathcal{P}$ is known by all agents in the economy given the environment and a law of large numbers. In a slight abuse of notation, I write μ_t as the measure of households in time t and $\mathcal{P}_t$ as the transition function at time t such that $\mu_{t+1} = \int_{\mathcal{S}} \mathcal{P}_t d\mu_t$. Finally, let $\mu(S^i) \geq 0$, $\sum_i \mu(S^i) = 1$, be the measure of households in state S^i. Thus, the expectation E_t operator in Equation (8) is defined over the appropriate Borel set as $E_t : \mu_t \times \mathcal{P}_t \to \mathbb{R}$.

[9] Default in the model is a binary choice variable. That is, households cannot default on a fraction of their debt. This assumption is by construction restrictive but it serves the purpose of restricting renegotiation. However, the assumption of a binary choice over default seems justified since only one-period debt contracts are considered. Were this model extended to include multiple-period debt, then it would seem plausible to allow households to default on their debt at some maturities and not others. I leave this analysis to extensions of the current paper.

8.2 Collateral Mechanism

In order to describe the contracting mechanism, let $\Xi_t^i = A_t^i \times B_t^i$ represent the space of feasible collateral contracts in period t for agent i where $B_t^i = [\underline{b}_t^i, \overline{b}_t^i]$ is the feasible set for credit balances (*i.e.* borrowing and lending) and $A_t^i = [0, (1-\delta) \times k_t^i]$ is the feasible set for collateral. $\underline{b}_t^i = -(1-\delta) \times k_t^i$ is the lower bound on borrowing and $\overline{b}_t^i$ is the upper bound on lending which is given by the period t budget constraint of agent i. $b_{t+1}^i(a_{t+1}^i; q_t) \subseteq \Xi_t^i$; is a feasible enforceable collateral contract where $b_{t+1}^i \in B_{t+1}^i$ denotes a consumption good transfer (payment) in period $t+1$ to (by) agent i, at a net cost (revenue) of $q_t b_{t+1}^i$ in period t and $a_{t+1}^i \in A_{t+1}^i$ denotes a capital obligation contingent on b_{t+1}^i being forfeited in period $t+1$ by agent i.

Let $\tilde{c}_t^i$ represent the period t consumption of agent i in autarky and let $u_t^i(\tilde{c}_t^i - q_t b_{t+1}^i)$ represent the period t utility of an agent i with a prospective contract $b_{t+1}^i(a_{t+1}^i; q_t)$. For the remainder of the paper, I suppress the arguments of $b_{t+1}^i(a_{t+1}^i; q_t)$ and simply write b_{t+1}^i.[10] Further, let $V_{t+1}^i(k_{t+1}^i, b_{t+1}^i, \eta_{t+1}^i)$ be the discounted expected utility value at time $t+1$ for an agent of type i with capital stock k_{t+1}^i, shock η_{t+1}^i, and existing collateralized contract b_{t+1}^i. Finally, $V_t^{i,aut} = u_t^i(\tilde{c}_t^i) + \beta \sum_{\eta_{t+1}} V_{t+1}^i(k_{t+1}^i, 0, \eta_{t+1}^i)\pi(\eta_{t+1}|\eta_t)$, is the expected utility value to agent i of foregoing intermediation in period t. This case includes the possibility of postponing intermediation one period. Any contract $b_{t+1}^i(a_{t+1}^i; q_t)$, is individually rational for agent i at time t where:

$$u_t^i(\tilde{c}_t^i - q_t b_{t+1}^i) + \beta \sum_{\eta_{t+1}} V_{t+1}^i(k_{t+1}^i, b_{t+1}^i, \eta_{t+1}^i)\pi(\eta_{t+1}|\eta_t) \geq V_t^{i,aut} \tag{12}$$

Next, I note that the autarky problem at time t can be written, for an agent i, as:

$$\max_{c,b,k} u_t^i(\tilde{c}_t^i - q_t b_{t+1}^i) + \beta \sum_{\eta_{t+1}} V_{t+1}^i(k_{t+1}^i, b_{t+1}^i, \eta_{t+1}^i)\pi(\eta_{t+1}|\eta_t) \tag{13}$$

$$\text{subject to: } \Xi_t^i = \emptyset \tag{14}$$

Thus, whenever Ξ_t^i is non-empty then there exists at least one contract in Ξ_t^i which satisfies (12) for an household i (since a household could always choose no intermediation). However, typically there will be many contracts which satisfy (12) for a given household i. Moreover, by construction it must be true that $\bar{V}_t^i \geq \tilde{V}_t^i$ which then illustrates that a household can never be made worse off, *ex-ante*, by the availability of collateral contracts. A contract b_{t+1}^i does not, however, necessarily imply that loans will be repaid. (12) implies both of the following for a lending household i:

- agent i is weakly better off lending and receiving repayment than she would be under autarky

[10] Recall that throughout this paper, contracts b_{t+1}^i are assumed to be feasible to honor. That is, agent i has either (both) sufficient output or (and) sufficient capital to satisfy the terms of the contract b_{t+1}^i.

- agent i is weakly better off lending and receiving the collateral payment than she would be under autarky

For a borrowing agent i who has the default choice, it implies either:

- agent i is weakly better off borrowing and repaying than she would be under autarky
- agent i is weakly better off borrowing and defaulting the collateral obligation than she would be under autarky

The types of collateral contracts, b^i_{t+1}, which may be written in an economy are limited only by feasibility. Contracts are feasible **iff**:

$$\sum_i b^i_{t+1} = 0$$

and

$$a^i_{t+1} \leq (1-\delta)k^i_t \ \ \forall i.$$

The first condition is standard and the second implies that the amount pledged as collateral is less than or equal to the capital stock for each agent i.

The literature on incomplete markets and borrowing constraints typically focuses on imposing a solvency constraint where it is never individually rational for the borrower to default. This exposition makes the point that such contracts may be *overly tight* in the sense that they impose greater restrictions than may be socially optimal or even privately necessary in order to permit exchange. For instance, such contracts imply, by necessity, that the borrowing agent always prefers to repay. In their essence, collateral contracts permit a form of intertemporal trade which is conducted through the collateral constraint.[11]

A comment on the specific role of the bank in determining the collateral requirement is deserved. Equation 12 defines the bargaining problem implicit in collateralized lending and borrowing. Indeed, collateral contracts may be written where default is certain but which are incentive compatible for both the borrower and the lender. In such cases, collateral contracts act as mechanisms facilitating inter-temporal trade in the collateral good. As a consequence, the amount a required for a given loan b, is difficult to characterize in a decentralized environment where agents contract individually since collateral may act as either a enforcement mechanism or as a means to inter-temporal trade. The value of collateral is potentially different in each case. A borrower will default whenever $V(k-a,0,\eta) \geq V(k,b,\eta)$. A lender prefers default whenever $V(k+a,0,\eta) \geq V(k,b,\eta)$. Hence,

[11] Dubey, Geanakoplos and Shubik [14] make a similar point.

depending on the curvature of the value function, it can be that lenders will accept repayment of an amount b but would prefer sure default of an amount $a < b$ while a borrower would prefer to repay. As a result, most of the literature (c.f. [27] and [28]), assumes an exogenous menu of collateral requirements. In this paper, the bank obviates the need for an exogenous menu of collateral requirements by removing the curvature associated with the lender's payoff to received collateral. The bank has a constant marginal value to collateral and hence, the bargaining problem has an explicit solution.

9 Equilibrium

Any definition of equilibrium is complicated by the overlap of the household's problem and the bank's problem. Specifically, the household's problem (8) requires that the household form an expectation of the collateral level required by the bank over all possible future state realisations. The specific complication is that the level of the collateral requirement chosen by the bank may depend on the household's problem. Unless this expectation can be appropriately defined, no recursive equilibrium can be constructed. That this expectation can be defined follows from the separability of the household's problem and the bank's problem.

Proposition 1 *Collateral requirements, a_t^i, and bond prices, q_t^i, are insufficient to identify a household's shock, η_t^i.*

Proof: Consider the household's budget constraint,

$$c_t^i + \frac{k_{t+1}^i}{1-\delta} + q_t^i b_{t+1}^i = \eta_t^i (k_t^i)^\alpha + (1-\tau_t^i) b_t^i + k_t^i - \tau_t^i a_t^i. \tag{15}$$

Recall that $\eta_t^i (k_t^i)^\alpha$ is private information (or costlessly falsifiable). It is immediate that a_t^i cannot depend on η_t^i since a_t^i is agreed at time $t-1$. No household would voluntarily choose to pay more collateral for default in period t. Next, given two prices $q_t^{i,1}$ and $q_t^{i,2} > q_t^{i,1}$, all households i who borrow (lend) prefer to pay (receive) the highest (lowest) price, $q_t^{i,2}$ ($q_t^{i,1}$). Thus, since η is private information all households (who borrow or who lend) will prefer the same cost-minimizing price so the price cannot be state-dependent. *Q.E.D.*

The next proposition demonstrates that no bank will offer contracts where $a_t^i < -b_t^i$. Key to the proposition is that collateral requirements, a_t^i, map one-to-one into borrowing limits, b_t^i, for households. Also key is that the realisation of the shock during the period of repayment is not known with certainty during the period of bond sale. Moreover, the default decision of the household is independent of the price of the bond, q_t^i, since prices are determined at the time of sale, in period $t-1$.

Proposition 2 *Banks will offer only contracts such that $a_t^i = -b_t^i$.*

Proof: That $a_t^i \ngtr -b_t^i$ follows directly from free-entry into banking. That $a_t^i \nless -b_t^i$ is less straightforward. A profit-maximizing bank would set a_t^i such that $\tau_t^i = 0$ for any $a_t^i < -b_t^i$. The decision to default (or not) depends on an incentive compatibility constraint for the household in period t:

$$\max\left\{\max_{k_{t+1},b_{t+1}}\left\{u(\eta_t^i(k_t^i)^\alpha + b_t^i + k_t^i - \frac{k_{t+1}^i}{1-\delta} - q_t^i b_{t+1}^i) + \beta\sum_{\mathcal{N}}\pi(\eta_{t+1}|\eta)V(k_{t+1}^i, b_{t+1}^i, \eta_{t+1}^i)\right\},\right.$$
$$\left.\max_{k_{t+1},b_{t+1}}\left\{u(\eta_t^i(k_t^i)^\alpha + \hat{k}_t^i - \frac{k_{t+1}^i}{1-\delta} - q_t^i b_{t+1}^i) + \beta\sum_{\mathcal{N}}\pi(\eta_{t+1}|\eta)V(k_{t+1}^i, b_{t+1}^i, \eta_{t+1}^i)\right\}\right\} \quad (16)$$

subject to borrowing constraints, where $\hat{k}_t^i = (k_t^i - a_t^i)$. Thus the first element is the utility value of repayment and the second element is the utility value of default. Hence, a profit-maximizing entrant would determine a_t^i such the expected utility value of default was equal to the expected utility value of repayment.

Since a household gains wealth by default then it must lose some slackness in future borrowing after default to not choose to default. In order to ensure repayment of a collateralised debt in the period of repayment and thus avoid bankruptcy, the bank must set the collateral, a_t^i, and incentive compatible borrowing limits, $\hat{b}_{t+1}^i$ and $\underline{b}_{t+1}^i$, such that:

$$\eta_t^i(k_t^i)^\alpha + \hat{k}_t^i - \frac{k_{t+1}^i}{1-\delta} - q_t^i\hat{b}_{t+1}^i \le \eta_t^i(k_t^i)^\alpha + b_t^i + k_t^i - \frac{k_{t+1}^i}{1-\delta} - q_t^i\underline{b}_{t+1}^i \quad (17)$$

which implies:

$$0 \le \frac{-a_t^i - b_t^i}{q_t^i} \le \hat{b}_{t+1}^i - \underline{b}_{t+1}^i, \quad (18)$$

where the inequality follows because borrowing implies $b_t^i < 0$ and $b_{t+1}^i < 0$.[12] [13] So it must be that a household can borrow less as k_t^i falls after default. Crucially, this also implies, at the point

[12]Recall that collateral is required to enforce repayment of a debt and not required to enforce acceptance of such a repayment.

[13]From the structure of the incentive constraint (3.14) it appears as if there is a hold-up problem for the bank. That is, it appears that there is an assumption that all the punishment levied by the bank must be in the current period because otherwise it is insolvent and unable to punish in future periods. I argue here that this assumption is redundant. The reason is straightforward. If the inequality were reversed in (3.14) then a direct implication is that the consumption sequences under default must be strictly above the consumption sequences for repayments. Hence, the future expected utility value for default must also be higher than the expected utility value for repayment. Thus, there would be no punishment for default.

of indifference, a one-to-one mapping between the collateral requirement determined in period t, a_t^i, and the repayment period borrowing constraint after default determined in period $t+1$, $\hat{b}_{t+1}^i$. Thus, a collateral mechanism with $a_t^i < -b_t^i$ requires an implicit commitment mechanism to enforce repayment. This implies a contradiction between incumbent banks and entrant banks. An entrant bank can, and will, offer a contract where $0 \geq \hat{b}_{t+1}^i - \underline{b}_{t+1}^i$ since it does not care about the repayment of any existing period t debt. This implies that the borrowing limit $\hat{b}_{t+1}^i$ is at least as great as the borrowing limit $\underline{b}_{t+1}^i$ for an entrant which implies the contradiction.[14] Hence, free-entry rules out an incumbent writing contracts where $a_t^i < -b_t^i$. Banks thus set a borrowing limit such that they are indifferent between default and repayment or equivalently where there is no difference between the contract offered by an incumbent and the contract offered by the entrant. This directly implies $a_t^i = -b_t^i$. *Q.E.D.*

One direct implication of Proposition 2 is that, in equilibrium, all households pay the same price for contracts. This is intuitive since all households are purchasing the same good, that is, a risk-less claim on future output. Hence, Bertrand price competition applies.

Remark 3 *When contracts are fully collateralised, $a_{t+1}^i = b_{t+1}^i$, then in equilibrium all households pay the same price for bonds, $q_t^i = q_t$ for all i.*

9.1 Collateral Default

Recall that the price of capital is trivially determined in the model. The household's problem implies that the price of capital is simply the numeraire price, 1, $\forall t$. The bank is assumed to sell off the received collateral so that all households receive an identical share as long as they have sufficient resources to purchase capital.[15] However, in equilibrium, it must be that:

$$\sum_i \frac{k_{t+1}^i}{1-\delta} - (k_t^i - (1-\tau_t^i)a_t^i) \geq \sum_{\tau_t^i=0} a_t^i \tag{19}$$

If equation (19) is not satisfied then the total amount of collateral to be sold is greater than the total amount of net investment and hence the price of capital must adjust.

Proposition 4 *In a full-collateral equilibrium, no household has an incentive to default, i.e. $\tau^i = 1$ $\forall i$ when Equation (19) holds.*

[14] I note that the incentive constraint for repayment, (18), is similar to the borrowing constraints of Aiyagari with the exception that they are one-period constraints. When $a_t^i = 0$ and $\hat{b}_{t+1}^i = 0$ then the sequence of individual borrowing constraints such that $\frac{-b_t^i}{q_{t+1}^i} = -\underline{b}_{t+1}^i$ trace the frontier of incentive compatible no-default one-period borrowing limits.

[15] Alternative disbursement programs won't affect the aggregate amount of net investment and thus the sale format has no impact on the equilibrium in this case.

Proof: Since the price of capital and the face value price of bonds are identical, then there is no effect on an household's current period budget constraint of exchanging capital for bonds. By defaulting, the household gains the difference $a_t^i - b_t^i = 0$. Hence, no household can be made better off by defaulting. In addition, households who face borrowing constraints are strictly better off by having more capital in order to collateralize future borrowing. Hence, there is no incentive for default. *Q.E.D.*

Remark: For default to arise in equilibrium in models similar to the one presented in this chapter either (or both) of the following must be true:

1. The price of capital $\neq$ the realized face-value price of bonds (*e.g.* if bonds were state-contingent).

2. Capital investment is irreversible

In both instances, default acts as means of portfolio re-balancing which is otherwise not feasible for households to undertake.

Finally, the bank determines q_t in the model so that lending and borrowing are expected to be in zero net supply. In equilibrium it must be that:

$$\sum_i b_t^i = 0 \quad s.t.\ (10) \tag{20}$$

9.2 Equilibrium Definition

A stationary recursive competitive equilibrium in the model is defined as a:
set of functions, $V(S), b(S), k(S), c(S), f(S), \tau(S), \mu(S)$, *and a bond price* q, *such that given* u, δ, α, β, η *and* $\pi(\eta'|\eta)$*:*

1. The agent chooses $b(S), k(S), c(S), \tau(S)$ to maximize her dynamic problem $V(S)$ given by equation (8).

2. The agent's output is given by $f(S)$.

3. The bank determines q to satisfy equation (20).

4. Aggregates result from individual behavior, $K = \mu(S)k(S)$ and $A = \mu(S)a(S)$.

5. There exists an invariant probability measure P defined over the ergodic set of equilibrium distributions.

10 Results

In this section, I parametrise an infinite-horizon model and numerically solve for the stationary equilibrium. The results suggest that collateral borrowing constraints have a significant effect on the distribution of wealth in the economy. By limiting the opportunities for risk-sharing, collateral constraints distort the market clearing prices of bonds and capital which, in turn, tend to distort the distribution of wealth in the economy. In particular, under most parametrisations, the steady-state distributions of wealth are skewed towards the poor, with some parametrisations exhibiting bi-modal characteristics.

I use a CRRA utility function which lends itself, albeit imperfectly, to considerations of risk-aversion and time discounting. The specification of the utility function is:

$$U^i = \sum_{t=0}^{\infty} \beta^t \frac{(c_t^i)^{1-\gamma}}{1-\gamma} \tag{21}$$

where c_t^i is consumption at time t by household i, β is a discount factor and γ is a constant parameter of relative risk aversion. Also, $\gamma \equiv \omega^{-1}$ where ω measures an household's willingness to smooth consumption through time.

It is not immediately clear how to calibrate the model economy presented in this paper. However, the goal of this paper is to examine the effect of collateral constraints and idiosyncratic risk *persay* rather than to developed a calibrated model designed to quantitatively match empirical regularities. Thus, the particular parametrisations employed are less important than how the model responds to changes in particular parameters. In particular, I consider a base-case scenario characterized by:

- high risk: $\{\overline{\eta}, \underline{\eta}\} = \{1.25, 0.75\}$
- high persistence: $\pi(\eta'|\eta) = 0.9$

and $\beta = 0.96$, $\alpha = 0.4$, $\gamma = 2$ and $\delta = 0.1$. I set the mean of idiosyncratic productivity shocks to $A = 1$ for most scenarios considered. The base-case parametrisation is chosen to roughly match some typical parametrisations used for neoclassical models. The choice of the base case shock process is somewhat arbitrary. Intuitively, one would expect collateral mechanisms to cause distortions relative to the riskiness and persistence of the shock process. The logic is that the riskier the shock, the more likely households will borrow and lend to smooth the idiosyncratic uncertainty.

However, given that households face borrowing constraints, riskier shocks lead to a greater measure of the households becoming borrowing constrained. Interestingly, both effects impact the price of the risk-free bond in the same manner. As the shock becomes riskier, more households have an incentive to purchase the risk-less asset, thus increasing the price of the bond. As the shock becomes more persistent, fewer households are free to optimally smooth the risk and hence richer households lower the price at which they are willing to smooth uncertainty in order to attract more borrowers. Hence, one should expect that as the risk and persistence dimensions of the shock rise, the market clearing price of the risk-less bond should rise. Hence, although the base-case is somewhat arbitrary, it is chosen as it returns a risk-free real interest rate of roughly 2.5 percent which is around the level of rates which have been observed. I then consider variations of the base-case where I reduce persistence and/or risk and also variations where I change the level of risk-aversion and also depreciation.

10.1 Computational Strategy

In any stationary steady-state equilibrium it must be the case that the bank accrues zero profits. Moreover, this restriction also implies that q_t and the steady-state distribution of households $\mu(S)$ must satisfy equation (20) and that the amount of received collateral is not greater than the amount of capital demanded, equation (19). In a stationary equilibrium, the bank's program must be satisfied.

For the autarky model, a grid for capital is assumed and value-function iteration is used to determine household's policy functions. Natural cubic spline interpolation is used to determine choices that are not on the grid. A rough grid of 200 points is used for the value function iteration. A convergence tolerance of 1×10^{-5} is chosen for the norm of the distribution of households, μ.

For the social planning problem, a value-function collocation iteration method is used. The grid points for capital are the Cheybshev nodes. A grid of 28 points was chosen for the social planning problem as larger grids did not materially increase the accuracy of the interpolation.

For collateral, a state space grid for capital and bonds is assumed for the model. The collateral constraint on borrowing implies that the maximum borrowing is a fraction, $(1-\delta)$, of the maximum grid point of capital. The computational complexity of the problem is due largely to the need for a two dimensional grid. That is, capital is not a sufficient statistic to determine an household's borrowing decision. In equilibrium, an household's behavior is restricted by the amount of capital with which she enters the period. Her behavior is also affected by the amount of debt (savings) which she has upon entering the period. As a result, for capital to be a sufficient statistic, it must be that an household's capital stock at the beginning of a period uniquely identifies her debt (savings) at that point. However, it does not since collateral constraints imply that one household may have faced an unconstrained optimization in the previous period while the second household may have faced a constrained optimization. As a result, a second statistic - bond holdings - is necessary to identify an household's equilibrium behavior. The grid for capital is set, at a minimum, to be roughly 50 points, while the grid for bonds is set, at a minimum, at roughly 80 points. Hence the total number of possible state realizations for an household is approximately $Ne = 8000 - 10000$. The exact number depends on the parametrisation.

Value function iteration is used to determine households' policy function choices. The solution algorithm must iterate on the bond price q in order to clear the bond market. Since the bond price q is a function of the measure of households, μ then the solution algorithm must also update μ after solving for q and then check to see if the bond price q is consistent with the new distribution. The exact algorithm used is:

1. Choose a grid for capital and for bonds.

2. Choose an initial distribution of households μ. Typically, the autarkic distribution was chosen as the starting point. However, a robustness check to this choice was performed by considering uniform and other arbitrary distributions.

3. Iterate on the value function defined over the grid to determine the optimal policy functions for k, b, $g(k, b)$.

4. Solve for the market clearing bond price q using a bisection algorithm.

5. Update the distribution of households μ using a transition matrix, P, defined over the optimal policy functions. Note that P is a $Ne \times Ne$ matrix which is large. To circumvent memory restricitions, Kroenecker factor P into arbitrary n blocks and update blockwise.

6. Repeat steps 3-5 until the distribution of households μ converges to a stationary distribution, $\hat{\mu}$.

7. Conditioning on $\hat{\mu}$ iterate on the value function defined over the grid according to the following procedure:

    (a) Choose the optimal policy function given the grid values.

(b) Fix the choice of capital, k^*, and use a natural cubic spline to interpolate over choices of b. Choose the optimal b^{**} using a golden section bisection approach.

(c) Fix the choice of bonds, b^*, and use a natural cubic spline to interpolate over the choices of k. Choose the optimal k^{**} using a golden section bisection approach.

(d) Choose highest value among points $(k^*, b^*), (k^{**}, b^*), (k^*, b^{**})$.

8. Solve for the market clearing bond price q.

9. Update the distribution of households μ using a transition matrix, P, defined over the optimal policy functions. For choices k^{**} and b^{**} which are not on the grid, allocate a fraction, λ, of households to the nearest two grid points. Again, to circumvent memory restricitions, Kroenecker factor P into arbitrary n blocks and update blockwise.

10. Repeat steps 7(a) - 9 until the distribution of households converges.

Convergence tolerances for the collateral program are set as follows:

1. The convergence tolerances for the value function iteration and the bond price iteration are set at 1×10^{-5}.

2. The convergence tolerance for bond market clearing is required to be less than 1×10^{-3}.

3. The convergence tolerance for the distribution of households, μ is set to 2.5×10^{-3}. The measure of convergence chosen is $\|\mu_{new} - \mu_{old}\|_{\infty}$.

The convergence tolerance for the stationary distribution is, admittedly, not very strict in an absolute sense. However, given the grid size employed in the simulations it appears to reflect the a reasonable level of convergence with some trade-off for time. Moreover, experiments with tighter convergence criteria, 1×10^{-4}, did not qualitatively change the results. It should also be mentioned that the results do not appear to qualitatively change for small perturbations of the grid or for modest increases in the size of the grid. Experiments with 80 grid points for bonds did not qualitatively change the results and resulted in a significantly slower simulations.

10.2 Numerical Examples

The general flavor of the results is as follows. Collateral constraints typically cause bi-modal distributions of wealth as households transit at a relatively high rate through the middle wealth levels. There are two main reasons for the bimodal wealth distributions. First, binding collateral constraints prevent households which experience a switch from the *good* state to the *bad* state in terms of productivity from borrowing to invest to reap the benefits of higher productivity. Hence, households spend a disproportionate length of time at or near their constraint. Second, the middle wealth households are beneficiaries of the binding collateral constraints. Since wealthy households still have an incentive to smooth their income, they lend to the middle wealth households, albeit at a lower return than they could obtain from the poor.[16] Thus, the collateral constraints tend to depress the real interest rate which also means that households have a marginally greater incentive to accumulate capital than under models with a higher real interest rate. The net effect is that, at the margin, the middle class hold higher allocations of risky capital and thus face marginally riskier incomes hence increasing their rates of transition among wealth states.

An additional finding is that collateral constraints do not universally imply an over-accumulation of capital in the steady-state unlike other Bewley-type models. Capital over-accumulation does occur when shocks are both large and persistent, regardless of risk preferences or depreciation rates. However, for other parametrisations of the shocks, steady-state capital levels are virtually the same as in the social planning equilibrium. The reason is that when the idiosyncratic shock is large and persistent, the risk-free rate is lower than in the other parametrisations. Hence, the marginal borrowing agent has a greater incentive to borrow and invest in risky capital.

Third, the risk-free real interest rate, conditional on the coefficient of risk-aversion γ and the mean of the idiosyncratic productivity shock, proxies wealth inequality. Specifically, a higher risk-free real interest rate is indicative of relatively lower wealth inequality, while a lower risk-free real interest rate is indicative of relatively higher wealth inequality. The reason stems from the collateral constraint. As the idiosyncratic productivity shocks become more persistent or more risky, a greater segment of households have binding collateral constraints. As a consequence, wealthy households which wish to save must do so entirely through risky capital or else lower the price at which they are willing to lend in order to entice more households to borrow. Hence, the risk-free real interest rate proxies the measure of households which are borrowing constrained. To see why the risk-free real interest rate also proxies inequality in the upper tail of the distribution, consider that the marginal lending household will lend until the expected gain from capital is equal to the risk-free rate. As the risk-free rate falls to entice more borrowing, the marginal lending household will also

[16] One suspects therefore, that the rich have an incentive to create financial intermediation between themselves and the poor.

have more capital. Since higher capital stocks imply higher amounts of risky income, the wealth dispersion amongst the upper tail of the wealth distribution also increases.

The fourth result from the paper is that as the dispersion of household capital stocks increase, any aggregate production function applied to the economy will tend to overstate the level of productivity as measured by a Solow residual relative to the true mean of the idiosyncratic shocks. This result is true as long as the idiosyncratic productivity shocks exhibit some persistence, *i.e.* are not i.i.d. The reason is again intuitive. As the dispersion of capital stocks increase, those households with high productivity shocks are more likely to have *good* productivity shocks. A production function applied to aggregate capital stocks will weight the idiosyncratic productivity shocks by the size of the household's capital stock. Hence, a measure of aggregate productivity will tend to overstate the true mean of the idiosyncratic productivity shocks. As a consequence, using aggregate data, an increase in mean level of idiosyncratic productivity would be indistinguishable from an increase in the idiosyncratic productivity risk (or persistence). However, increasing risk (or persistence) also leads to a decrease in the risk-free real interest rate in the model. The conclusion, therefore, is that measured aggregate productivity increases that occur in conjunction with decreases in the risk-free real interest rate are likely due to increasing idiosyncratic risk (or persistence) and are unlikely to result from an increase in the true-level of productivity.

Finally, the model presented in this paper appears somewhat able to reconcile a low risk-free real interest rate with plausible levels of the equity premium.

The following subsections discuss some of the numerical examples specifically.

10.3 Base Case

As a base case, I examine the consequences of collateral constraints when the idiosyncratic shock exhibits high risk and high persistence. Table 3 presents the steady-state of the economy. As is evident from the table, collateral constraints on borrowing have a marked effect on the economy. The aggregate capital stock in the collateral equilibrium is significantly higher than in either a social planning equilibrium or an autarkic equilibrium. The reason is straightforward: collateral constraints on borrowing limit the risk-sharing opportunities of lenders in addition to borrowers. households who desire to smooth consumption and cannot do so using the bond market turn instead to capital. Hence, the overaccumulation of capital in the collateral steady-state is driven by saving motives. As a result of the overaccumulation of capital, output is higher in the collateral equilibrium than in either the social planning or autarky equilibria.

Table 3: Steady-State: Base Case

Model	Y	C	K	B	i	ROE
Autarky	1.98	1.36	5.43			4.11%
Collateral	2.04	1.39	5.68	1.61	2.55%	3.26%
Social Planner	1.92	1.38	4.88	1.37	4.17%	4.33%

Y refers to aggregate equilibrium output; C refers to aggregate equilibrium consumption; K is aggregate equilibrium capital stock; B is the face-value of all bonds traded; i is the interest rate on the risk free bond; and ROE refers to the average expected marginal return to capital.

Table 4 presents wealth and welfare comparisons of the collateral and autarkic equilibrium.[17] Wealth is measured as the net end of period holdings of capital and bonds by households.[18] Welfare is measured by the lump-sum consumption transfer to all households required each period such that the expected utility of a randomly chosen household is identical to that under the social planner's equilibrium. In particular, it is important to note that this measure of welfare assumes that consumption can not be redistributed across households. If consumption redistribution was possible then it may be that the social planning outcome could be achieved through an appropriate schedule of taxes and transfers since aggregate consumption in the collateral model is marginally higher than that chosen by the social planner. I leave consideration of the optimality of consumption taxes to future research.

The collateral equilibrium has a higher steady-state welfare than autarky although inequality is higher in the collateral equilibrium than in the autarky equilibrium. Moreover, roughly 14 per cent of households in the collateral economy face binding borrowing constraints in equilibrium. Figure 1 in Appendix **??** displays the distribution of portfolio holdings across households. A mass of borrowing constrained households is clearly evident in the spike in the lower left part of the graph. Figure 3 translates the portfolio holdings into a single measure of wealth and compares the distribution of wealth in the collateral model with than in the autarky model. The increase in inequality is evident as is the significant fraction of poor wealth households. Figure 2 then compares the distribution of consumption across households in the collateral model with the distribution in the autarky model. Again, consumption is also more unequally distributed in the collateral model than in the autarky model in the steady-state. Hence, the steady-state welfare benefits of collateralised intermediation are not universally shared.

Finally, the model returns an average rate of consumption growth for a household of approxi-

[17] The social planning equilibrium is, by definition, the first-best outcome and I assume the social planner weights all individuals equally, so all households, by construction, have the same welfare in the social planning equilibrium.

[18] I consider end of period holdings so that all households have, by construction, positive levels of wealth.

mately 0.7 per cent although aggregate consumption is stationary. The reasoning for the apparent disconnect is straightforward - the ratio of a sum is not the same as the sum of a ratio. Hence, average household consumption growth may be different than aggregate consumption growth.

Table 4: Wealth Inequality: Base Case

Model	Welfare	Gini	GE	Top 30	Constrained	Δc
Autarky	0.11	0.15	0.04	45.1		
Collateral	0.077	0.392	0.329	56.3	13.5	0.7%

Welfare is the lump-sum consumption transfer to households each period such that expected lifetime utility of a randomly chosen household is equal to that achieved by the social planner; Gini refers to the Gini coefficient on the distribution of wealth, where wealth is defined as the sum of capital and bonds held by an household; GE refers to the Generalized Entropy Measure on wealth using a weighting parameter of 0; Top 30 refers to the percentage of wealth held by the top 30 percent of households; and Constrained refers to the percentage of households who are at their borrowing constraint; Δc is the average percentage consumption change for a household.

10.4 Sensitivity to the Level of Risk

Next, I examine changes to the level of risk of the idiosyncratic shock to see how sensitive the wealth distribution is to the parametrisation of the shock. The results are reported in Table 5. As is apparent, as the level of risk increases (decreases) the collateral constraint has more (less) effect on the aggregate distributions of capital, output and consumption relative to autarky and the social planner's model. In particular, the level of the risk-free real interest rate can become negative in the presence of greater levels of risk. Somewhat surprising is that as the level of risk rises, then aggregate consumption in the collateral model falls below the aggregate consumption in the social planner's model. The reason is that when there is more risk, households insure against the risk to a greater degree since the costs of being constrained rise. Hence, households shift more consumption to savings to smooth the idiosyncratic risk.

Table 6 presents the wealth and welfare comparisons of the equilibria. As is clear from the table, wealth inequality generally tends to increase as the level of risk increases. As well, the level of consumption transfer required to bring a randomly chosen household in the collateral model to the same steady-state level of utility as in the social planner's model rises sharply (to almost 20% of average period consumption). Perhaps the only surprising result is that the fraction of total wealth possessed by the top 30 per cent of households is relatively static. The reason appears to be largely a result of the construction of the statistic. Since the fraction of wealth possessed by the top 30 per cent of households is sizeable, then those households also exert a disproportionate influence on aggregate wealth. Hence, as the total value of their wealth increases so to does the mean. Thus,

Table 5: Steady-State: Sensitivity to the Level of Risk

Less Risk, $\{\overline{\eta}, \underline{\eta}\} = \{1.1, 0.9\}$

Model	Y	C	K	B	i	ROE
Autarky	1.88	1.34	4.84			4.54%
Collateral	1.91	1.35	5.04	1.35	3.65%	4.09%
Social Planner	1.87	1.34	4.75	0.55	4.17%	4.60%

More Risk, $\{\overline{\eta}, \underline{\eta}\} = \{1.5, 0.5\}$

Model	Y	C	K	B	i	ROE
Autarky	2.31	1.42	7.74			2.59%
Collateral	2.49	1.51	8.44	2.16	-0.25%	0.85%
Social Planner	2.12	1.52	5.38	2.71	4.17%	3.46%

See Table 3 for definitions.

both the denominator and numerator of the fraction rise and it exhibits little variability.

Table 6: Wealth Sensitivity to the Level of Risk

Less Risk, $\{\overline{\eta}, \underline{\eta}\} = \{1.1, 0.9\}$

Model	Welfare	Gini	GE	Top 30	Constrained	Δc
Autarky	0.017	0.10	0.015	36.3		
Collateral	0.022	0.380	0.270	57.5	6.4	0.12%

More Risk, $\{\overline{\eta}, \underline{\eta}\} = \{1.5, 0.5\}$

Model	Welfare	Gini	GE	Top 30	Constrained	Δc
Autarky	0.22	0.376	0.286	56.0		
Collateral	0.29	0.418	0.454	57.9	16.6	1.26%

See Table 4 for definitions.

10.5 Less (and Less) Persistence

Since the idiosyncratic shock considered in the base case is relatively highly persistent, I examine the sensitivity of the collateral model to decreasing degrees of persistence of the shock. In particular, I examine two cases: (1) moderately persistent with $\pi = 0.75$ and (2) not persistent with $\pi = 0.5$. Not surprisingly, as the persistence of the productivity shock falls, there is less incentive for households to accumulate capital in the collateral model. Moreover, as the persistence falls households are less able to smooth their consumption using capital and hence households bid down the price of the risk-less bond thus increasing the risk-free rate. Another consequence is that as the persistence falls, aggregate consumption and output falls, reflecting the decrease in aggregate capital.

Table 7: Sensitivity to the Persistence of Risk

Moderate Persistence, $\pi = 0.75$

Model	Y	C	K	B	i	ROE
Autarky	1.93	1.35	5.14			4.36%
Collateral	1.96	1.35	5.23	1.37	3.16%	3.83%
Social Planner	1.92	1.38	4.88		4.17%	4.60%

No Persistence, $\pi = 0.5$

Model	Y	C	K	B	i	ROE
Autarky	1.34	1.90	4.91			4.64%
Collateral	1.88	1.34	4.85	1.29	3.80%	4.45%
Social Planner	1.92	1.38	4.88		4.17%	4.60%

See Table 3 for definitions.

Table 6 presents the wealth and welfare comparisons of the equilibria. The most interesting observation from this exercise is simply how little the persistence of idiosyncratic shocks seems to matter in terms of many inequality and welfare measures in the steady-state of the collateral model. The main movement is in terms of the measure of constrained agents and the variability in household consumption across the levels of persistence. One final result from this exercise is that the steady-state welfare costs of the collateral model seem to decrease non-monotonically with the decrease in the persistence. The steady-state welfare cost falls 0.01 consumption units as the persistence falls from $\pi = 0.9$ to $\pi = 0.75$ while there is no decrease from $\pi = 0.75$ to $\pi = 0.5$. The reason is intuitive - as the productivity shock becomes less persistent all households have a greater incentive to borrow and invest in risky capital since the expected returns to doing so rise for household which have a current poor realisation of the shock. Thus, some households substitute more out of current consumption than when the shock is more persistent and hence their consumptions fall.

10.6 Increase in the Mean of the Idiosyncratic Shock

In this section I consider $A = 2$ which I interpret as an exogenous increase in the mean level of the idiosyncratic shock relative to the base case. However, I hold constant the percentage of the level of risk such that $\{\overline{\eta}, \underline{\eta}\} = \{2.5, 1.5\}$. Thus, I imagine the increase in the mean of the shock to represent an exogenous total factor increase. It is interesting to note in Table 9 that there is little impact on the risk-free real interest rate. The size of the decrease in the risk-free real interest rate is marginal, roughly 0.1 percentage points, considering that the mean of the idiosyncratic shock

Table 8: Wealth Sensitivity to the Persistence of Risk

Moderate Persistence, $\pi = 0.75$

Model	Welfare	Gini	GE	Top 30	Constrained	Δc
Autarky	0.077	0.17	0.05	41.4		
Collateral	0.067	0.357	0.249	55.1	9.5	0.96%

No Persistence, $\pi = 0.5$

Model	Welfare	Gini	GE	Top 30	Constrained	Δc
Autarky	0.065	0.12	0.025	38.0		
Collateral	0.067	0.355	0.233	54.5	5.4	0.1%

See Table 4 for definitions.

has increased by 100 per cent. The reason is simple - the increase in the mean of the idiosyncratic shock increases the output produced by all households. Hence, average household consumption rises and there is less demand for consumption insurance. As a result, the risk-free real interest rate falls slightly to clear the market.

Table 9: Steady-State: Increase in Mean Productivity

Model	Y	C	K	B	i	ROE
Autarky	6.27	4.33	17.22			4.11%
Collateral	6.52	4.41	18.52	5.07	2.43%	3.01%
Social Planner	6.11	4.39	15.47	4.33	4.17%	4.35%

See Table 3 for definitions.

Table 10 presents the wealth and welfare comparisons. The increase in the mean level of the idiosyncratic productivity shock leads to lower measured inequality and by extension marginally lower steady-state welfare losses. The percentage of consumption transfer required is 5.4% of steady-state consumption as compared to 5.5% in the base case simulation. One perhaps comforting conclusion therefore is that the steady-state welfare costs of collateral constraints fall as the mean level of productivity rises, though admittedly the fall is negligble considering the size of the exogenous increase.

10.7 Constrained Households and the Risk-Free Rate

In this section, I highlight a particular result from the base case numerical examples: conditional on preferences and the mean level of the idiosyncratic productivity shock, the risk-free interest rate

Table 10: Wealth Inequality: Increase in Mean Productivity

Model	Welfare	Gini	GE	Top 30	Constrained	Δc
Autarky	0.33	0.227	0.086	45.1		
Collateral	0.239	0.380	0.297	55.7	12.3	0.9%

See Table 4 for definitions.

proxies the measure of borrowing constrained households. Table 11 illustrates this result.[19] As is evident, the level of inequality is also negatively correlated with the level of the real interest rate in the collateralized model across all parametrisations, although not perfectly. The results in this section highlight the effect of the borrowing constraint on the risk-free rate. As more households become constrained, the real interest rate must fall to clear the market for bonds. This result unambiguous across all examples.

The results also suggest that changes in either the level or persistence of risk map into the inequality measures, again though not perfectly. The obvious exception is the moderate persistence case which highlights the fact that the inequality statistics also depend on the distributions of the marginal products of capital across all models. The reason is that as the persistence falls, the expected marginal product of capital for poorer households rises while the expected marginal product of capital for richer households falls. Hence, in terms of wealth dynamics, the poorer households tend to transit up the distribution relatively more quickly while the richer households do the reverse. Whereas, a decrease in the level of the productivity risk in the examples considered here have less relative movements in the marginal products. Hence, one cannot directly compare movements in real interest rates as movements in broader measures of inequality *per say* without determining more precisely the underlying distributions of the idiosyncratic shocks.

10.8 The Aggregate Solow Residual

The results presented thusfar suggest that collateral constraints and idiosyncratic risk can have significant effects on the distributions of capital, bonds and by extension net wealth and incomes. Additionally, collateral constraints and idiosyncratic risk affect the aggregate levels of these variables. Hence, it is natural to ask whether collateral constraints affect measures based on macro-aggregates. One obvious question is to ask whether collateral constraints affect total factor productivity as measured by the Solow residual. In other words, does the total factor productivity and risk-free real

[19] Results from examples not explicitly considered in this paper have also been included. Results for those models are available on request from the author.

Table 11: Base Case: Wealth Inequality and the Real Interest Rate

Model	i	Gini	GE	Constrained
Higher Risk	-0.25%	0.418	0.454	16.6
Base Case	2.55%	0.392	0.329	13.5
Base Case but Medium Persistence	3.16%	0.357	0.249	9.5
Lower Risk	3.65%	0.380	0.270	6.4
No Persistence	3.80%	0.355	0.233	5.4
Very Low Risk and No Persistence	4.19%	0.07	0.004	0.00002

Gini refers to the Gini coefficient on the distribution of wealth, where wealth is defined as the sum of capital and bonds held by an agent; GE refers to the Generalized Entropy Measure on wealth using a weighting parameter of 2; Top 30 Wealth refers to the percentage of wealth held by the top 30 percent of agents; and Constrained refers to the percentage of agents who face a binding borrowing constraint.

interest rate disconnect puzzle relate to collateral constraints?

One can think of an aggregate production function, $Y = zK^{\alpha_K}$, for the economy presented in this chapter. However, if one imagines an econometrician being confronted with this specification then, given the stationarity of the environment, the unknowns z and α_K can not be identified without further restrictions at the aggregate level. For tractability, I assume that the econometrician has some information about the level of α_K. Specifically, I assume the econometrician sets $\alpha_K = \alpha$.[20] Given α_K, one can compute $z = Y/K^{\alpha_K}$.

Unless the distribution of capital stocks is identical across all agents or the distribution of shocks is i.i.d., z will not be the mean of the idiosyncratic shocks, which for (most of) the examples considered here is 1. In all versions of the model examined these restrictions are rarely true because of the distortions to capital holdings induced by the insurance motive for capital savings. Even in the social planner's problem, the planner allocates capital efficiently to equalise the marginal products of capital which also leads to asymmetric capital holdings by size of shock. The net result of the dispersion of capital holdings is that, in all versions of the model, those agents with high capital stocks are more likely to have high idiosyncratic productivity shocks - the exception is when shocks are i.i.d. At the aggregate level of capital, the productivity shocks are weighted by the size

[20] I could relax this assumption by assuming that the econometrician has access to a random sample of the capital stocks, output and idiosyncratic shock of individual firms. Imposing the, true, restriction that α_K is identical across all firms would also allow the econometrician to estimate the value of α_K.

of the capital stock and thus high idiosyncratic shocks receive a higher weight. Hence, average total factor productivity measured at the aggregate level will appear to be higher than the mean value of the shocks when capital holdings are dispersed and the shock is not i.i.d. I refer to the difference between z and the true mean of the idiosyncratic shocks as the **bias**.

The level of the computed z induced by the collateral constraints is related to the level of the risk-free rate. The reason is that both the risk-free rate and the level of computed z depend on the measure of constrained households. That the level of computed z also depends, to some degree, on inequality stems from the fact that in the collateral model those households with high capital stocks tend to be more likely to have a high idiosyncratic shock than households with lower capital stocks. As a result, the measured aggregate z will rise. It is important to note that the increase in z is true for all versions of the model, including the social planner's problem.

The parameter(s) of the model which are most relevant to the results in this section are the risk and persistence of the idiosyncratic shock. Table 12 presents the computed z compared to the mean of the distribution of the idiosyncratic shocks. As is evident from the table, as the risk of the idiosyncratic shocks rise the level of the computed z rise as well. In the final row of the table I present the results from a model where I set the mean level of the idiosyncratic shocks to be 2 but where I otherwise retain the parameters from the base case. This exercise illustrates that the bias to z from the collateral constraint persists as the other components of total factor productivity rise.

Table 12: Total Factor Productivity and the Real Interest Rate in the Collateral Model

Parametrisation	TFP	Bias	i	GE Inequality
Higher Risk ($\eta = \{1.5, 0.5\}$)	1.056	5.6	-0.0025	0.5313
Base Case	1.014	1.4	2.55	0.4311
Medium Persistence ($\pi = 0.75$)	1.011	1.1	3.16	0.249
Medium Risk ($\eta = \{1.1, 0.9\}$)	1.002	0.2	3.65	0.3635
No Persistence ($\pi = 0.5$)	1.0	0	3.80	0.233
Lower Risk and Lower Persistence	1.0	0	4.19	0.0290
Base Case, mean shock 2	2.028	1.4	2.43	0.3822

TFP refers to the aggregate Solow residual, z; Bias refers to the percentage distortion in the aggregate Solow residual from the mean of the idiosyncratic shocks; Real Interest Rate refers to the real interest rate on a risk-free bond; and GE Inequality is the generalised entropy measure of inequality with weighting parameter 0 which implicitly weights poor household more than rich.

A related point to make is that the bias in measured total factor productivity covaries with the level of the risk-free real interest rate across the collateral models examined here. The same is not true of the bias in measured total factor productivity and the risk-free real interest rate (which is constant) across the social planner's models. A further significant point to make is that an increase in the true mean level of total factor productivity has less effect on the risk-free real interest rate than an increase in the bias, which can be much smaller in magnitude. Hence, measured total factor productivity increases that occur in conjunction with decreases in the risk-free real rate may suggest increasing idiosyncratic productivity risk rather than increases in the mean level of productivity. This is suggestive for a new avenue of research concerning measured total factor increases and the risk-free rate. It also suggests one possible method of testing for the presence of collateral constraints in an economy though clearly a dynamic model would be required before trying to estimate any links in the data.

10.9 The Return-to-Capital

In this section, I present some evidence which suggests that the equity premium puzzle is consistent with, in part, collateral constraints. As has been presented to this point, collateral constraints and idiosyncratic risk yield reasonable levels of the risk-free real interest rate. This section argues that collateral constraints and idiosyncratic risk yield reasonable albeit high values of the equity premium. However, the collateral model presented in this paper does less well at capturing the apparently negative correlation in the data between the risk-free real rate and the equity premium during the 1980s and 1990s.

One difficulty in calculating the equity premium for the model is that both the price of equity and dividend from equity is difficult to measure in the current environment. Hence, I present two statistics, the marginal expected return-to-capital investment and the realised total return-to-capital investment. To the extent that a share of equity is, in the data, a share of the capital stock of a firm, then capital in my model is a household's net equity holding. The marginal return-to-capital measures the average expected marginal real return of capital across the households in the economy. The realised total return-to-capital investment is calculated in an effort to compute the return-to-equity used in studies of the equity premium for the S&P 500, for instance in Mehra and Prescott. The return-to-equity is typically calculated as $(P_{t+1} + D_t - P_t)/P_t$ where P_t was the real price of equity at time t and D_t was the real dividend yield of a share purchased at time t. In my model, the price of capital is fixed in both periods which makes it difficult to compute the return-to-equity in the typical manner. Instead, I note that the return to equity measures the consumption return a household realises from equity investment. For example,by foregoing P_t units of consumption in time t a household can consume $P_{t+1} + D_t$ in time $t+1$. If $P_{t+1} + D_t > P_t$ then a household realises a positive consumption return in period $t+1$. Hence I measure the return-to-capital investment for a household as the consumption dividend a household takes from its capital in a given period. Specifically, I calculate the average return-to-capital (AER) as: $AER = \int_{\mu} (\eta_t^i (k_t^i)^{\alpha} + k_t^i - k_{t+1}^i/(1-\delta))/(k_t^i/(1-\delta))di$. Another justification for this approach is that firms in the S&P 500 pay dividends after retained earnings have been deducted - which has an impact on the price of equity but which does not have an analagous price effect in my model (rather they have a consumption dividend effect). One outcome of this formulation is that some households in my model earn negative average equity returns. I note that a negative average equity return is akin to share dilution in the sense that an increase in outstanding equity by a firm (through, for example a share issue) is a dilution of existing equity holdings.

Using these characterisations of the marginal and total returns I find that the marginal return-to-capital investment is only slightly above the risk-free rate. There is some distortion owing to borrowing constrained households but the distortion is, on net, small. However, in terms of average total returns I find that capital investment earns, roughly, an $10-20$ per cent excess return depending on the parametrisations. It is also interesting to note that the excess average return-to-capital in my model falls as the idiosyncratic shock becomes more risky. The reason for the significant fall in excess returns is due to capital investment swings when households receive an unexpected positive, and persistent, idiosyncratic productivity shock. In addition, the average return-to-capital falls as a household's stock of capital rises (due to the falling marginal product of capital). Thus, greater dispersion in capital holdings across households can also lead to a lower average return-to-capital.

Another conclusion seems to be that, in the collateral model, there is a premium in terms of average real returns but a neglible premium in marginal expected real returns to capital. One suggestion, therefore, is that in the empirical data equity is priced at the average return. In many respects this is reasonable since equity markets typically do not distinguish between the marginal and average investor. Consequently, the reason that more real investment in equities does not occur is simply that, for the most part, firms do not want more real equity (and perhaps those that do face borrowing constraints or unverifiable idiosyncratic risk). Table 13 presents some of the results from the collateral model in more detail.

One might be concerned that the average excess return-to-capital is actually too high relative to that measured in the data. There are, at least, two reasons why this should not be too concerning.

Table 13: Average and Marginal Return-to-Capital and the Real Interest Rate in the Collateral Model

Parametrisation	MER	AER	AEP	Real Interest Rate
Higher Risk ($\eta = \{1.5, 0.5\}$)	0.85	10.15	10.39	-0.0025
Base Case	3.26	20.6	18.07	2.55
Medium Risk ($\eta = \{1.1, 0.9\}$)	4.09	23.9	20.22	3.65
Lower Risk and Lower Persistence	4.66	25.5	21.29	4.19
Base Case, mean shock 2	3.01	20.0	17.55	2.43

MER refers to the average marginal expected real return-to-capital investment for households; AER refers to the average realised average real return-to-capital investment by households; AEP is the average excess return-to-capital over the risk-free rate; and the Real Interest Rate is the real interest rate on a risk-free bond.

First, there are no taxes, financing costs or other transactions costs in the model which might appear in the data. To the extent that all would reduce the dividend returns or lead to a relative increase in P_t then the model should over-estimate the premium. Second, the equity premium measured in the data varies substantially across the time-period used to measured, and during some time-periods, the realised returns are indeed in the order of 20 per cent. It is thus highly likely that measured equity returns do not satisfy the stationarity assumptions used in my model and that aggregate uncertainty may play a significant role. Clearly, the incentives for increasing the retained earnings are sensitive to expectations of aggregate future returns which are held constant in my model. In particular, casual empiricism suggests this might have a role to play in explaining the excess returns during the 1990s. The collateral model presented in this paper would suggest that if an unanticipated increase in idiosyncratic risk were responsible for the falling real interest rate and rising multifactor productivity then the excess return on equity should have fallen. Given the lack of an aggregate stochastic shock, arguments relating to expectations of future prices (for example *irrational exuberence*) cannot be explored here. The results presented simply suggest that collateral constraints and idiosyncratic productivity risk may have a part to play in reconciling the level of the equity premiums observed in the data.

Finally, the excess return-to-capital results actually suggest that perhaps the coefficient of risk aversion is actually too high in the parametrisations studied. A lower coefficient of risk aversion, perhaps closer to 1, would reduce the demand for consumption insurance and thus increase the risk-free real interest rate, *ceteris paribus*. In addition, a lower coefficient of risk aversion would tend to increase the level of the household capital stock (in essence reducing the gap in excess marginal returns). One consequence is that the average return would also fall, thus lowering the premium to capital.

11 Conclusion

This paper began by posing the question: what is the impact of collateralized lending on wealth inequality and how does this impact affect macrovariables? The model developed in this paper has gone some way to answer this question. Collateralized lending tends to polarize wealth and tends to increase the rate of transition from the middle class in economies where shocks are large or persistent. Collateralized intermediation does not benefit the poor as they do not possess enough collateral to borrow efficiently. Indeed, the steady-state welfare in the collateral model is lower than in the autarky model for some parametrisations of the shock process. However, there is some evidence that increases in the mean level of the idiosyncratic productivity shock have a mitigating effect on the the steady-state welfare costs of the collateral.

Another finding is that wealth inequality is related to the risk-free interest rate. In particular, lower risk-free rates arise in steady-state equilibrium which exhibit higher levels of inequality. The reason is that as more poor agents face binding borrowing constraints, the risk-free rate must fall in order to clear the market.

Perhaps more relevant is the finding that the collateral model presented here performs reasonably in terms of matching the recent movements of total factor productivity and the real interest rate. Looking at the data for total factor productivity and the risk-free real interest rate for the U.S. during the 1980s and 1990s, it is surprising that the real interest rate fell as total factor productivity increased. The collateral model replicates such negative co-movements when the level of the idiosyncratic increases.

Another surprising finding is that the collateral model also yielded somewhat plausible levels of the equity premium, as measured by the excess consumption dividend earned by investing in capital over the risk-free real interest rate. The collateral model suggested that the average marginal excess returns to capital were slight but that the average returns could be in the neighbourhood of $10-20$ per cent. Unlike other models, only a modest level of risk aversion on the part of households was required. Indeed, there is some evidence that the risk aversion parameter used here could be lowered and return more plausible excess returns to capital.

Finally, the model presented in this paper suggests several areas of future research. Firstly, taxes, in particular capital taxes, can have unintended consequences - they can make it more costly for poor agents to save capital in order to relax their borrowing constraints. Hence, capital taxes may increase inequality. Secondly, it would be interesting to augment the model to include a laborleisure decision to see if and how the distortions caused by collateral constraints persist in such an environment.

References

[1] Abreu, D. and H. Matsushima (1994), "Exact Implementation," *Journal of Economic Theory* 64, 1-19.

[2] Abreu, D., D. Pearce and E. Stachetti (1990), "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica* 58, 1041-1063.

[3] Aiyagari, R. (1994). "Uninsured idiosyncratic risk and aggregate saving," *Journal of Economics* 109, 659-684.

[4] Alvarez F. and U. Jermann (2000), "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica* 68, 775-798.

[5] Andolfatto, D. and E. Nosal (2002), "A Theory of Money and Banking," *mimeo*, Simon Fraser University.

[6] Araujo, A.P., M.R. Pascoa and J.P Torres-Martinez (2002), "Collateral Avoids Ponzi Schemes in Incomplete Markets," *Econometrica* 70, 1613-1638.

[7] Banerjee, P. (2001). "Community Enforcement with Endogenous Information," *mimeo.*

[8] Bewley, T. (1980), "The optimum quantity of money," in J. Kareken and N. Wallace, eds., *Models of Monetary Economies*, Federal Reserve Bank of Minneapolis, Minneapolis.

[9] Chatterjee, S., D. Corbae, M. Nakajima and J.-V. Rios-Rull (2001), "A Quantitative Theory of Unsecured Consumer Credit with Risk of Default," *University of Pennsylvania Working Paper.*

[10] Chen, N-K. (2001), "Bank Net Worth, Asset Prices and Economic Activity," *Journal of Monetary Economics*, 48,415-436.

[11] Cordoba, J. and M. Ripoll (2002). "Credit Cycles Redux," *mimeo*, Rice University.

[12] Cowell, F. (1995), *Measuring Inequality*, Prentice Hall, London.

[13] De Soto, H. (2000), *The Mystery of Capital*, Basic Books, New York.

[14] Dubey,P., J. Geanakoplos and M. Shubik (2004), "Default and Punishment in General Equilibrium," *Econometrica*, forthcoming.

[15] Fernandez-Villaverde, J. and D. Krueger (2001), "Consumption and Saving over the Life-Cycle: How Important are Consumer Durables?" *Federal Reserve Bank of Minneapolis.*

[16] Geanakoplos, J. and W. Zame (2002), "Collateral and the Enforcement of Intertemporal Contracts," *Yale University mimeo.*

[17] Huggett, M (1993). "The risk-free rate in heterogeneous- agents, incomplete-insurance economies." *Journal of Economic Dynamics and Control* 17, 953-969.

[18] Kandori, M. (1992). "Social Norms and Community Enforcement," *Review of Economic Studies*, 59, 63-80.

[19] Kandori, M. and H. Matsushima (1998). "Private Observation, Communication and Collusion," *Econometrica*, 66, 627-52.

[20] Kehoe, T., and D.K. Levine (1993), "Debt-constrained Asset Markets," *Review of economic studies*, 60, 595-609.

[21] Kiyotaki, N, and J. Moore (1997), "Credit Cycles," *Journal of Political Economy*, 105, 211-248.

[22] Kocherlakota, N.R. (1996), "Implications of Efficient Risk Sharing without Commitment," *Review of Economic Studies* 63, 595-609.

[23] Kocherlakota, N.R. (1996), "Consumption, Commitment and Cycles," *Journal of Monetary Economicss* 37, 461-474.

[24] Kocherlakota, N.R. (2000), "Risky Collateral and Deposit Insurance," *Federal Reserve Bank of Minneapolis Staff Report.*

[25] Kocherlakota, N.R. (2000), "Money as Memory," *Journal of Economic Theory*, 81, 232-251.

[26] Krishnamurthy, A. (2000a), "Collateral Constraints and the Amplification Mechanism," *forthcoming Journal of Economic Theory.*

[27] Kubler,F. and K. Schmedders (2003), "Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral," *Econometrica*, 71, 1767-1793.

[28] Lacker, J.M. (2001), "Collateralized Debt as the Optimal Contract," *Review of Economic Dynamics*,4,842-859.

[29] Livshits, I., J. MacGee and M. Tertlit (2001), "Consumer Bankruptcy: A Fresh Start," *Federal Reserve Bank of Minneapolis Working Paper.*

[30] Lustig, H. (2001), "Bankruptcy and Asset Prices" *Stanford manuscript.*

[31] Miranda, M. and P. Fackler (2002), *Applied Computational Economics and Finance*, MIT Press, Cambridge.

[32] Rothschild, M. and J. Stiglitz (1976), "Equilibrium in Competitive Insurance Markets: An Essay on the Economics of Imperfect Information," *Quarterly Journal of Economics*, 90(4), 629-649.

[33] Rutherford, S. (2000), *The Poor and Their Money*, Oxford University Press, Oxford.

[34] Mehra, R. and E. Prescott (1985), "The Equity Premium: A Puzzle...," *Journal of Monetary Economics*, 15, 145-61.

[35] Mehra, R. and E. Prescott (2003), 'The Equity Premium in Retrospect. ", *NBER Working Paper 9525*

[36] Shiller, R. (1989), *Market Volatility*, University of Chicago Press, Chicago.

[37] Stokey, N. and R. Lucas (1989), *Recursive Methods in Economic Dynamics*, Harvard University Press, Boston.

[38] Taylor, J. and H. Uhlig (1990), "Solving nonlinear stochastic growth models: a comparison of alternative solution methods," *Journal of Business and Economic Statistics*, 8, 1-19.

[39] Weil, P. (1989), "The Equity Premium Puzzle and the Risk-Free Rate Puzzle," *Journal of Monetary Economics*, 24, 401-21.

[40] Zame, W.R. (1993). "Efficiency and the Role of Default when Securities Markets are Incomplete," *American Economic Review* 83, 1142-1164.

12 Appendix A

Figure 1: Distribution of Capital and Bond Holdings: Base Case

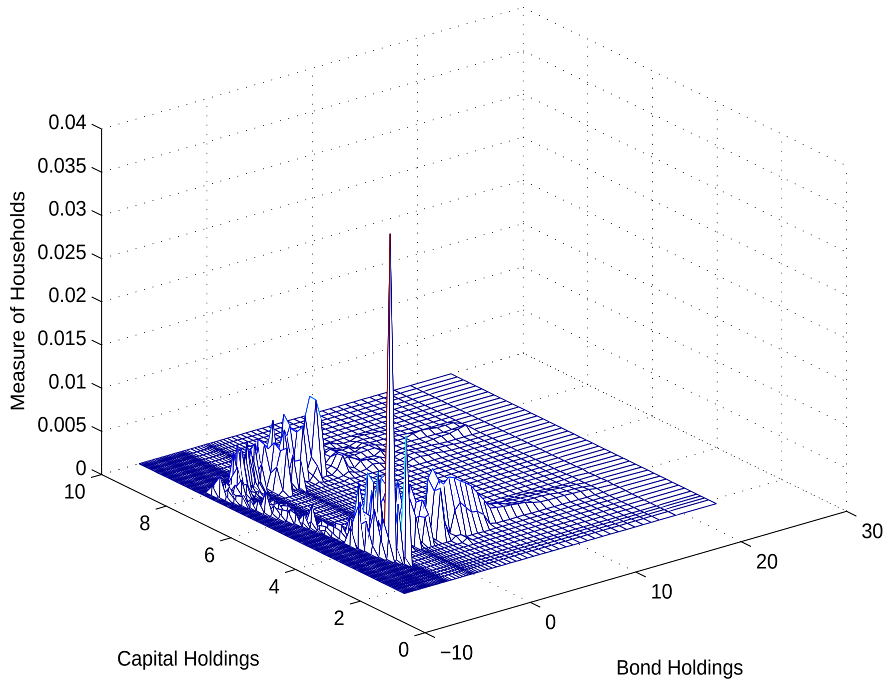

Figure 2: Distribution of Consumption: Base Case

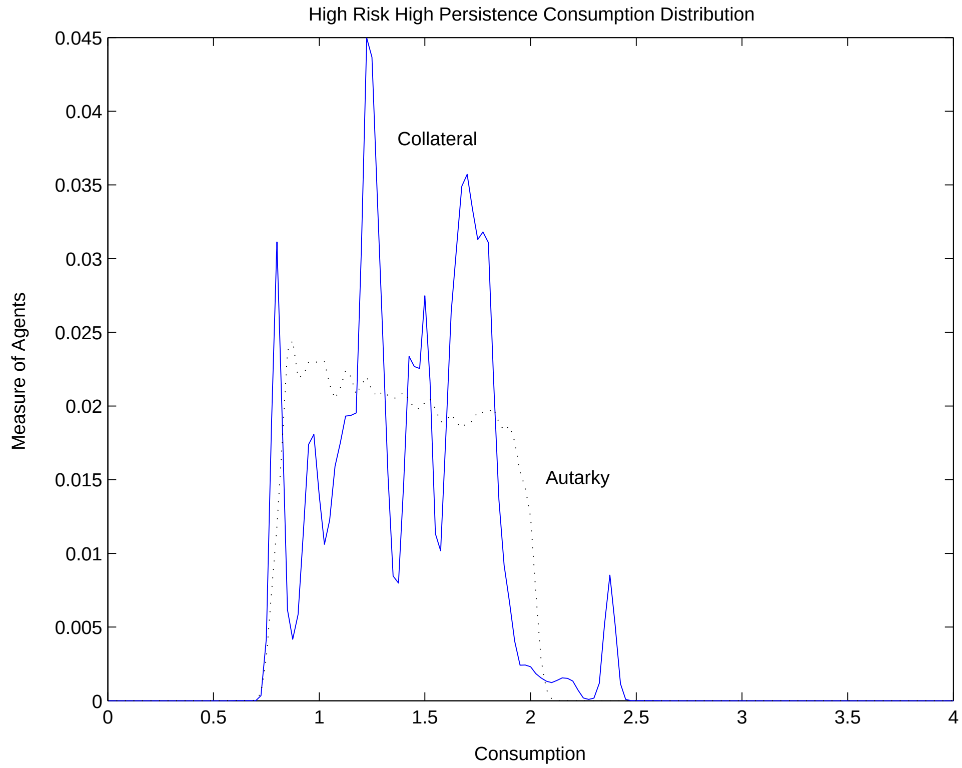

Figure 3: Distribution of Wealth: Base Case