Allocation of Individual Risks in a Market Economy

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The pricing kernel - the stochastic process that assigns value to state-contingent payoffs - has been the object of an enormous amount of study, both empirically and theoretically. Associated with the pricing kernel is the stochastic discount factor (SDF), which can be linked with the intertemporal marginal rate of substitution of consumption (IMRS) in consumption-based asset pricing models. Representative agent models, which are based on frictionless capital markets with heterogeneous agents, have fared poorly in empirical testing and efforts to improve the empirical results have moved in many directions. One direction is to introduce both consumer heterogeneity and incomplete consumption insurance to see if the asset pricing implications are improved. So far, the performance of the models have not improved significantly. Constantinides and Duffie [1996] argue that the reason is the way in which idiosyncratic risk is introduced, namely that agents are *ex ante* identical, which has important implications for opportunities to share idiosyncratic risk even in restricted asset markets. In this paper, I construct a simple model in which agents face endowment distribution risk which is not fully insurable because consumption insurance markets are not actuarially fair.

The ability to insure against idiosyncratic risk depends on market conditions and the availability of consumption insurance, which in turn depend on the risks faced by other types of agents. At one extreme - pure autarky - there is no market in which consumption insurance can be purchased and all risk is borne by the individual. When all agents are identical and face the same risk,

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which is generally referred to as aggregate risk, there are no opportunities to diversify and no consumption insurance is available. In an endowment economy with complete markets where all agents are identical *ex ante* and there is a countable infinity of agents, full consumption insurance is available because the cross sectional distribution of the endowment is identical to the endowment distribution an individual faces over his lifetime. As a result, consumption insurance is actuarially fair since all agents face the identical frequency of being in a particular state. This framework - idiosyncratic risk and *ex ante* identical agents - has been studied extensively in the literature. If markets are complete, then full consumption risk-sharing can be achieved and a representative agent constructed. If markets are incomplete, then depending on the source of the incompleteness, full consumption insurance typically can't be achieved. Typical market frictions that have been studied include borrowing constraints (Aiyagari, Hugget), endogenous borrowing constraints (Alvarez and Jermann), one-sided commitment (Kocherlakota), liquidity constraints (Kehoe and Levine), and private information. This research was motivated by the failure of the complete markets set-up with heterogeneous agents to explain asset-pricing behavior, sensitivity of income to income shocks over time, or many other features of economic time series has been well documented.

Constantinides and Duffie [1996] have criticized models in which agents are *ex ante* identical on the grounds that the consumption insurance opportunities at each point in time (cross sectionally) exactly match the kind of consumption insurance that agents require over their lifetime to achieve full insurance. They develop a model with a countable infinity of *ex ante* heterogeneous agents and study the asset pricing implications for a particular restriction on the types of securities that are traded. While the results of the model are promising, the question arises whether the exogenously imposed security market structure is the key determinant in the results or whether the *ex ante* heterogeneity is the key factor.

In this paper I explore the asset pricing and consumption/income relationship in a simple model in which agents are *ex ante* heterogeneous because they face different income distribution risk. This results in a very simple and plausible market friction: agents can purchase insurance against endowment realizations but this consumption insurance is not actuarially fair because the market is not indexed for endowment distribution risk. Markets fail to separate because agents cannot be prevented from participating in a market in which they can fulfill all contracts they voluntarily enter because of their risk class or individual-specific attributes. Essentially *ex ante* heterogeneous agents can achieve full risk-sharing within a risk class if they participate in segregated markets, where markets are segregated by risk class. But an individual agent may be better off to buy or sell claims in another market. Since market participation cannot be limited only to agents from the same risk class, the prices for a contingent claim conditional on a particular endowment realization will be the same across risk classes. Agents are then able to insure against income shocks but not against differences in the distribution of income by risk class.

1 Basic Model

Let $\theta \in \Theta$, where $\Theta \equiv \{\theta_1, \ldots, \theta_M\}$, be a discrete random variable such that $\theta_1 \geq 0$ and θ_M is finite. Let $\eta \in N \equiv \{\eta_1, \ldots, \eta_N\}$ be a discrete random variable and let $f(\eta)$ denote the measure over N. Assume there is a countable infinity of agents indexed by $I = \{1, 2, \ldots\}$. At the beginning of time, each agent draws an η, which is fixed for rest of his lifetime.

The distribution of θ is parameterized by η so that, for a particular $\eta \in N$, the distribution of θ is $g(\theta \mid \eta)$. For example, if $\eta_2 > \eta_1$, $g(\theta \mid \eta_2)$ may be a mean-preserving spread of $g(\theta \mid \eta_1)$. More generally, η should be thought of as a vector of attributes, such as race, gender, religion, inherent abilities, and so on. The key idea is that the distribution over the endowment sample space Θ will differ across agents, with a fraction $f(\eta)$ facing distribution $g(\theta \mid \eta)$. Because the realization of η determines the endowment distribution, an agent with η_i and an agent with η_j such that $\eta_i \neq \eta_j$

are said to be in different income distribution risk categories.

Since there is a countable infinity of agents, the sample realization over the population is a pair $\{\theta_i, \eta_i\}$ where $i = 1, \ldots, \infty$. The frequency of observing the pair θ_i, η_j where $\theta_i \in \Theta$ and $\eta_j \in N$ is $g(\theta_i \mid \eta_j) f(\eta_j)$.

Initially, there is no aggregate uncertainty. Total output is

$$y = \sum_{\eta \in N} \sum_{\theta \in \Theta} f(\eta) \theta g(\theta \mid \eta). \tag{1}$$

The distribution of θ over all agents is a *mixture*

$$\phi(\theta) = \sum_{\eta \in N} f(\eta) g(\theta \mid \eta). \tag{2}$$

Define the mean of the endowment for type η as

$$\theta_m(\eta) = \sum_{\Theta} g(\theta \mid \eta) \theta.$$

2 One-Period Model

Before there is any trading, agents observe their realization of η and enter into contingent contracts. A type-η agent has preferences

$$\sum_{\theta \in \Theta} U(c) g(\theta \mid \eta). \tag{3}$$

The function U is continuous and twice continuously differentiable, strictly increasing and strictly concave. Also, as $c \to 0$, $U'(c) \to \infty$.

2.1 Full Insurance in Separated Markets

At time 0, agents observe their η realization and they enter into contingent claims contracts. Let $z(\theta, \eta)$ be the amount of a contingent claim contract held by type η if θ occurs. Assume that contingent claims pricing is a nonzero pricing function defined on $\Re_+^{M\times N}$, so that prices for a unit of consumption in state $\theta \in \Theta$ are allowed to differ across agent types η_i, $i = 1, \ldots, N$; let $q(\theta, \eta)$ denote the price of one unit of consumption in state θ for type η. In this section, it is assumed that agents participate only in the η market for claims. In particular, a type η_i is not allowed to purchase claims issued in the η_j market where $i \neq j$. An agent's type is public information so that the assumption that agents enter into contingent contracts only with their own type is an assumption about the enforceability of excluding different agents from the market.

At time 0, all agents of type η are identical. A particular agent, agent $i \in I$, has a random endowment $\theta(i)$. Observe that after the endowment is realized, agents that are the same type are different *ex post*. Consumption choices for a type η agent are made over the commodity space $\Re_+^M$. A type-η has budget constraints

$$
\begin{aligned}
0 &\leq \sum_{\Theta} q(\theta, \eta) z(\theta, \eta) &(4)\\
0 &\leq \theta + z(\theta, \eta) - c &(5)
\end{aligned}
$$

Let $\bar{\mu}(\eta)$ denote the Lagrange multiplier for the first constraint and let $\mu(\eta, \theta)$ denote the multiplier for the second; observe the second constraint must hold for each θ and η. The first-order conditions with respect to $c, z(\theta, \eta)$ are

$$
\begin{aligned}
g(\theta \mid \eta) U_1(c) &= \mu(\theta, \eta), &(6)\\
\mu(\theta, \eta) &= \bar{\mu}(\eta) q(\theta, \eta). &(7)
\end{aligned}
$$

Eliminate $\mu(\theta, \eta)$ and solve for the price,

$$
q(\theta, \eta) = \frac{g(\theta \mid \eta) U_1(c)}{\bar{\mu}(\eta)}. \tag{8}
$$

Full consumption insurance is achieved when $c(\theta, \eta)$ is invariant with respect to θ; let $\bar{c}(\eta)$ denote such a solution. For constant consumption over θ to be optimal, it follows from (8), that q and g

are proportional

$$g(\theta \mid \eta) = Kq(\theta, \eta),$$

where K is a positive constant. Solve (5) for $z(\theta, \eta)$ and substitute into (4) and then eliminate q using (8) to obtain

$$0 = \sum_{\Theta} \left[\frac{g(\theta \mid \eta) U_1(\bar{c}(\eta))}{\bar{\mu}(\eta)} \right] [\theta - \bar{c}(\eta)] \tag{9}$$

Since $\bar{c}(\eta), \bar{\mu}(\eta)$ are invariant with respect to θ, these terms can be moved out of the summation and then eliminated since the left side is zero. It follows that $\bar{\theta}(\eta) = \bar{c}(\eta)$ is the solution.

Since there is only one time period and hence no borrowing or lending in the model, agents are able to achieve full risk sharing within an income distribution risk class η by trading in the N separate markets. In the example above, I assume that agents only consider trades in claims indexed by income distribution risk. Agents are not allowed to purchase claims in other markets. Whether or not this restricts the opportunity set of the agent is discussed in the section below.

Although well known, it worthwhile to mention two properties of the solution. First, the price of the claim depends entirely on the frequency of being in a particular state and not on the size of the endowment realization. Hence a high endowment state $\theta_M \geq \theta$ that occurs with a low frequency has low value for securing consumption insurance and conversely for states that occur with a higher probability. Second, complete risk sharing is achieved for each η type by trading only with other agents of the same type. Observe that the market-clearing condition is satisfied, where the market-clearing condition is

$$0 = \sum_{N} \sum_{\Theta} f(\eta) g(\theta \mid \eta) [\theta - \bar{c}(\eta)]. \tag{10}$$

In the standard set-up, with idiosyncratic risk and *ex ante* identical agents, $g(\theta \mid \eta_i) = g(\theta \mid \eta_j)$ for all i, j. In that case the frequency of being in a specific state $g(\theta)$ just equals the price of a contingent claim for consumption insurance.

The central planning problem is

$$\max\left\{\sum_{i=1}^{N}\psi(\eta_i)\sum_{j=1}^{M}g(\theta_j,\eta_i)U(c(\theta_j,\eta_i))\right\}$$

subject to the resource constraint, where $\psi(\eta)$ is the Pareto-weight attached to type η agents. Let λ_p denote the Lagrange multiplier for the resource constraint. The first-order condition is

$$\frac{\psi(\eta_i)}{f(\eta_i)}g(\theta_j,\eta_i)U_1(c(\theta_j,\eta_i)) = \lambda_p g(\theta_j,\eta_i)$$

Observe that $g(\theta,\eta)$ cancels out so the optimal consumption is invariant with respect to θ. In the competitive equilibrium described above, where $c(\eta)=\theta_m(\eta)$, the Pareto weights satisfy

$$\frac{\psi(\eta_i)}{\psi(\eta_j)} = \frac{f(\eta_i)}{f(\eta_j)}\frac{U_1(\theta_m(\eta_j))}{U_1(\theta_m(\eta_i))}.$$

2.2 Arbitrage

In the discussion above, agents of type η are assumed to trade contingent claims with agents of the same type. This restriction results in full risk-sharing for each η-type. Suppose now that agents are allowed to buy and sell contingent claims in all other markets. Recall that at time 0, all agents of the same type are identical and will enter into identical contracts. Agents now have the option of selling the contingent claims in other markets. Let $z(\theta_j,\eta_i,\eta_h)$ denote the claims issued by a type η_h contingent on θ_j, that are sold in market η_i. As before, if $z(\theta_j,\eta_i,\eta_h)>0$, the holder of the claim (which was purchased in market i) will pay agent η_h in the event θ_j occurs.

Agents continue to maximize (3) subject to the budget constraints, which are now modified,

$$0 = \sum_{i=1}^{N}\sum_{j=1}^{M}q(\theta_j,\eta_i)z(\theta_j,\eta_i,\eta_h) \tag{11}$$

$$0 = \theta + \sum_{i=1}^{N}z(\theta,\eta_i,\eta_h) - c(\theta,\eta_h) \tag{12}$$

Let $\bar{\mu}(\eta_h)$ denote the multiplier for the first constraint and $\mu(\eta_h, \theta)$ denote the multiplier for the second. The first-order conditions are

$$\bar{\mu}(\eta_h)q(\theta, \eta_i) \geq \mu(\eta_h, \theta) \tag{13}$$

$$g(\theta, \eta_h)U_1(c(\theta, \eta_h)) = \mu(\eta_h, \theta) \tag{14}$$

Rewrite (13) to show that for each $i \in N$

$$q(\theta, \eta_i) \geq \frac{\mu(\eta_h, \theta)}{\bar{\mu}(\eta_h)}.$$

Hence if agents are to hold contingent claims issued by all agent types, then the price must be equal for each type or $q(\theta, \eta_i) = q(\theta, \eta_j)$; otherwise agents will sell claims in only one market for each θ. The intuition is that the agent will seek the contingent claims price that maximizes the value of his portfolio. To see this, let $\theta - c(\theta, \eta_h)$ be given for each $\theta \in \Theta$. The portfolio maximization problem is

$$\max \sum_N \sum_\Theta q(\theta, \eta) z(\theta, \eta, \eta_h)$$

subject to

$$\theta - c(\theta, \eta_h) \geq \sum_N z(\theta, \eta, \eta_h)$$

Let $\lambda(\theta)$ denote the multiplier for the constraint. The first-order condition is

$$q(\theta, \eta) \geq \lambda(\theta).$$

There will be only one market η such that this holds with equality. The implication of integrated contingent claims markets is explored in the next section.

2.3 Integrated Markets

The dimensionality of the contingent claims price function, which is $\Re_+^{M \times N}$ in the separated markets economy, is now restricted to $\Re_+^M$. Essentially, all agents participate in the same contingent claims market trading contracts that are short or long in the endowment realization θ. When markets are integrated, agents can still enter into state contingent contracts, but the contract is contingent only on the endowment realization and not on the parameter η, which indexes the frequency of a particular type of agent being in a particular state. A type-η agent maximizes (3) subject to

$$0 \leq \sum_{\Theta} q(\theta)[\theta(\eta) - c]. \tag{15}$$

Let $\mu(\eta)$ denote the Lagrange multiplier. The first-order condition is

$$g(\theta \mid \eta) U_1(c) = \mu(\eta) q(\theta). \tag{16}$$

In general, $g(\theta \mid \eta)$ is not proportional to $q(\theta)$. Solve for μ and show, for $\theta_i, \theta_j \in \Theta$,

$$\mu(\eta) = \frac{g(\theta_i \mid \eta) U_1(c_i)}{q(\theta_i)} = \frac{g(\theta_j \mid \eta) U_1(c_j)}{q(\theta_j)}. \tag{17}$$

Based on the intuition from the separated markets example, conjecture that the contingent claims price for θ takes the form

$$q^{\star}(\theta) = \sum_{N} f(\eta) g(\theta \mid \eta), \tag{18}$$

which is the (unconditional) probability of θ in the general population. If this is the price, observe that (17) can be rewritten as

$$1 = \left(\frac{\Lambda(\theta_i, \eta)}{\Lambda(\theta_j, \eta)} \right) \left[\frac{U_1(c(\theta_i, \eta))}{U_1(c(\theta_j, \eta))} \right] \tag{19}$$

where

$$\Lambda(\theta, \eta) \equiv \left[\frac{g(\theta \mid \eta) f(\eta)}{\sum_N f(\eta) g(\theta \mid \eta)} \right].$$

This states that the weighted marginal rate of substitution across states is equal to unity, where the weights equal the ratio of the conditional probability $\Lambda(\theta, \eta)$ that an agent is a type η given he has endowment θ.

Let μ be given and use the price $q^\star$ to define a function $\hat{c}$ as

$$\hat{c}(\theta, \mu, \eta) = (U_1)^{-1} \left(\frac{\mu \sum_N f(\eta) g(\theta \mid \eta)}{g(\theta, \eta)} \right). \tag{20}$$

Observe that the function $\hat{c}$ is strictly decreasing in μ. Substitute the function $\hat{c}$ into the budget constraint using $q^\star(\theta)$,

$$0 = \sum_{\Theta} \sum_{N} f(\eta) g(\theta \mid \eta)[\theta - \hat{c}(\theta, \mu, \eta)] \tag{21}$$

Since $\hat{c}$ is decreasing in μ, the right side is strictly increasing in μ; Let $\mu(\eta)$ denote the unique solution, conditional on the price taking the form in (18). Define

$$c^\star(\theta, \eta) = \hat{c}(\theta, \mu(\eta), \eta).$$

The function $c^\star$ is the optimal consumption for a type η agent with endowment θ, conditional on the conjectured price. The next main issue is whether the conjectured price clears the market.

2.3.1 Equilibrium Conditions

The market-clearing condition is

$$\sum_{\eta \in N} \sum_{\theta \in \Theta} f(\eta) g(\theta \mid \eta)[\theta - c(\theta, \eta)] \tag{22}$$

To determine if this holds with the conjectured price so that consumption is $c^\star(\theta, \eta)$, first observe that the unconditional probability that any agent has endowment θ is $\sum_N f(\eta) \sum_\Theta g(\theta \mid \eta)$. The conditional probability that an agent with endowment θ is a type η is $\Lambda(\theta, \eta)$, defined earlier, which is also the frequency that an agent with endowment θ is a type η. Summing over the agents' budget constraints, observe the proportion of agents with endowment θ that are type η with excess demand $q(\theta)[\theta - c(\theta - \eta)]$ is

$$\Lambda(\theta, \eta) q(\theta)[\theta - c^\star(\theta, \eta)]. \tag{23}$$

Summing this across all $\eta \in N$ results in

$$\sum_{\eta \in N} \Lambda(\theta, \eta) q(\theta)[\theta - c^{\star}(\theta, \eta)] \tag{24}$$

which is the net excess supply or demand for claims conditional on θ. Summing this over all θ, and substituting in the conjectured price used in deriving the function $c^{\star}$,

$$\begin{aligned} 0 &= \sum_{\eta \in N} \sum_{\theta \in \Theta} \{\Lambda(\theta, \eta) q^{\star}(\theta)[\theta - c^{\star}(\theta, \eta)]\} && (25) \\ 0 &= \sum_{\eta \in N} \sum_{\theta \in \Theta} \{g(\theta \mid \eta) f(\eta)[\theta - c^{\star}(\theta, \eta)]\} && (26) \end{aligned}$$

where the last equation is the market-clearing condition. Hence the conjectured price used in deriving the policy function $c^{\star}$ is the price at which the market clearing condition holds.

A key property of the solution is that full consumption insurance is not achieved. In particular, given the price $q^{\star}(\theta)$, we know in general $\frac{q^{\star}(\theta)}{g(\theta|\eta)}$ will vary with η and θ. Hence, the solution $c^{\star}(\theta, \eta)$ varies with θ. The marginal rate of substitution is not equalized across states, although the weighted marginal rate of substitution is equal to unity, where the weights equal the conditional probability of η, conditional on θ. This is a natural outcome of the main feature of the model that consumption insurance is not actuarially fair.

To see the sense in which the consumption insurance is actuarially unfair, consider the following two-state example. Let $\theta_2 > \theta_1$, so that agents wish to insure against an endowment of θ_1. Define h as the amount of insurance an agent purchases at price ρ, so that the agent's consumption if $\theta = \theta_2$ is

$$\theta_2 - \rho h = c_2$$

and his consumption if θ_1 is realized is

$$c_1 = \theta_1 + h.$$

The agent solves

$$\max_h g_1 U(\theta_1 + h) + g_2 U(\theta_2 - \rho h)$$

The first order condition is

$$g_1 U_1(\theta_1 + h) = g_2(\theta_2 - \rho h)\rho.$$

Obviously, if

$$\frac{g_1}{g_2} = \rho$$

then the agent buys full insurance otherwise achieves only partial insurance. In the application here, notice that the first order condition can be rewritten as

$$\frac{g_1}{g_2} U'(\theta_1 + h) = \frac{q(\theta_1)}{q(\theta_2)} U'(\theta_2 - \frac{q(\theta_1)}{q(\theta_2)} h). \tag{27}$$

The amount of insurance h equals the consumption that the agent would like to shift to the low endowment state from the high endowment state. The premium per unit of consumption paid out of income in state 2 is ρ while the expected benefit in state 1 is $\frac{g(\theta_1)}{g(\theta_2)}$. If, as assumed, insurance is not actuarially fair, the premium doesn't equal the expected claim, so the agent doesn't fully insure. A wedge is created in the sense that the personal tradeoff between marginal utilities in states one and two, which is $\frac{g_{1,i}}{g_{2,i}}$, differs from the terms of trade over states offered in the market. The agent's budget constraint is

$$\theta_1 + \frac{\theta_2}{\rho} = c_1 + \frac{c_2}{\rho}.$$

2.3.2 A Simple Example

Suppose that there are only two types of agents η_1, η_2 with probabilities f_1, f_2 and only two states θ_1, θ_2. Let $g_{i,j} = g(\theta_i, \eta_j)$. Assume that $\theta_1 < \theta_2$ and $g_{11} > g_{12}$, $g_{22} > g_{21}$, so that agent 1 has a higher probability of the low state and agent 2 has a higher probability of realizing the high state.

The MRS condition for agent η_i is

$$1 = \frac{\Lambda_{1,i}}{\Lambda_{2,i}} \frac{U_1(c_{1,i})}{U_1(c_{2,i})} \tag{28}$$

Notice if the conditional probabilities where the same across agents, so that $\Lambda(j,1) = \Lambda(j,2)$, then consumption for agent i would be constant across states. Suppose that $f_1 = f_2$. Then, under the assumptions on g_{11}, g_{21}, observe that $\frac{\Lambda_{1,1}}{\Lambda_{2,1}} > 1$ from which it follows that $c_{11} > c_{21}$ because U is strictly concave. Hence a type-1 agent overinsures against a low shock θ_1 and underinsures against θ_2. The converse is true for the type-2 agent for whom $\frac{\Lambda_{1,2}}{\Lambda_{2,2}} < 1$. This outcome is the result of consumption insurance that is not actuarially fair.

Assume that preferences are logarithmic, so that a closed-form solution can be derived. The first-order condition for a type 1 is

$$\frac{g_{11}}{c_{11}q_1} = \frac{g_{21}}{c_{21}q_2} \tag{29}$$

so that

$$c_{21} = \frac{g_{21}}{g_{11}} \frac{q_1}{q_2} c_{11}$$

The type 1 has a budget constraint

$$\begin{aligned} 0 &= q_1[\theta_1 - c_{11}] + q_2[\theta_2 - c_{21}] \\ &= q_1[\theta_1 - c_{11}] + q_2[\theta_2 - \frac{g_{21}}{g_{11}} \frac{q_1}{q_2} c_{11}] \\ &= q_1\theta_1 + q_2\theta_2 - \frac{q_1 c_{11}}{g_{11}}[g_{11} + g_{21}] \end{aligned}$$

Solving for c_{11} and c_{21},

$$\begin{aligned} c_{11} &= \frac{g_{11}}{q_1}[q_1\theta_1 + q_2\theta_2] \\ c_{21} &= \frac{g_{21}}{q_2}[q_1\theta_1 + q_2\theta_2] \end{aligned}$$

The steps above can be repeated for a type 2 agent, yielding

$$\begin{aligned} c_{12} &= \frac{g_{12}}{q_1}\left[q_1\theta_1 + q_2\theta_2\right] \\ c_{22} &= \frac{g_{22}}{q_2}\left[q_1\theta_1 + q_2\theta_2\right] \end{aligned}$$

Using the intuition from the complete contingent claims market, suppose that the contingent claims prices are

$$q_1 = f_1 g_{11} + f_2 g_{12}, \tag{30}$$

$$q_2 = f_1 g_{21} + f_2 g_{22}. \tag{31}$$

Total endowment is

$$y \equiv [f_1 g_{11} + f_2 g_{12}]\theta_1 + [f_1 g_{12} + f_2 g_{22}]\theta_2.$$

For each agent, observe that

$$q_1 c_{11} + q_2 c_{21} = q_1\theta_1 + q_2\theta_2$$

Sum the budget constraints across agents using the conjectured prices to show that

$$[f_1 g_{11} + f_2 g_{12}]\theta_1 + [f_1 g_{12} + f_2 g_{22}]\theta_2 = f_1 g_{11} c_{11} + f_2 g_{12} c_{12} + f_1 g_{21} c_{21} + f_2 g_{22} c_{22} \tag{32}$$

Suppose that $\theta_2 > \theta_1$ and $g_{11} > g_{12}$ (so that $g_{21} < g_{22}$). Assume that $f_1 = f_2$. Then (28) is

$$1 = \frac{\Lambda_{1,i}}{\Lambda_{2,i}}\frac{c_{2,1}}{c_{1,1}}$$

for a type η_1 agent.

The implication for consumption-income correlations is this: for two agents from different risk categories and identical realizations θ, it is clear that

$$\frac{g(\theta \mid \eta)U'(c(\theta,\eta))}{g(\theta \mid \hat{\eta})U'(c(\theta,\hat{\eta}))} = \frac{\mu_\eta}{\mu_{\hat{\eta}}}$$

for each $\theta \in \Theta$ and $\eta, \hat{\eta} \in N$. As long as $g(\theta \mid \eta) \neq g(\theta \mid \hat{\eta})$ and assuming that $g(\theta)$ is not uniformly distributed and constant over the states, the marginal rate of substitution will vary over states. This occurs because individual agents have an incentive to trade outside of their income risk category, even though, as a group, agents are better off in expected utility terms by staying in the appropriate market for their income risk.

Next, given the ratio $\frac{\mu_\eta}{\mu_{\hat{\eta}}}$, if θ is a low probability state for type η, compared to $\hat{\eta}$, so that $g(\theta \mid \eta) < g(\theta \mid \hat{\eta})$ then $U'(c(\theta, \eta)) > U'(c(\theta, \hat{\eta}))$ so that $c(\theta, \eta) < c(\theta, \hat{\eta}))$. Remember that this is a one-period model and that complete risk sharing is feasible by income category. Complete risk sharing means that the agent's consumption doesn't fluctuate with respect to the endowment realization (high or low) or with the probability of a particular state.

3 Infinite-Horizon Markets

Assume now that an agent has an infinite time horizon. As before, at time 0, an agent observes the realization of a random variable η that affects the distribution of his endowment for all future periods. Let $\theta^t(i) = \{\theta_1(i), \ldots, \theta_t(i)\}$ denote the history of endowment realizations for a particular agent $i \in I$. Let $g_t(\theta^t(i) \mid \eta)$ denote the probability of the history $\theta^t(i)$ under the assumption that agent i is a type η agent. For the first period, the allocations across different η agents with different endowments θ lie in $N \times M$. The history of allocations for agents from periods 1 to t is an element of $(N \times M)^t$. There is no aggregate uncertainty and the history of the system is the entire space $(N \times M)^t$ at time t and hence is deterministic. Initially prices are allowed to be functions of history θ^t, since the history of endowments is observable for any agent and, moreover, it is possible for θ^t to be identical for two agents of different types $\eta_i \neq \eta_j$.

A stationary solution is an allocation in $N \times M$. Under the assumption that only stationary solutions are considered, the history of the system does not affect the price of a contingent claim in market θ. As before, there is no aggregate uncertainty. The distribution of the endowment remains unchanged so that the probability that a random agent has realization θ is

$$\sum_{N} f(\eta) g(\theta \mid \eta).$$

3.1 Separated Markets

I start with separated markets. At time 0, the type η agent solves

$$\max \sum_{t=0}^{\infty} \sum_{\theta^t} g_t(\theta^t \mid \eta) \beta^t U(c_t(\theta^t, \eta)) \tag{33}$$

subject to

$$0 \leq \sum_{t=0}^{\infty} \sum_{\theta^t} p_t(\theta^t \mid \eta)[\theta_t - c_t(\theta^t, \eta)] \tag{34}$$

Let $\hat{\mu}_\eta$ denote the Lagrange multiplier. The first-order condition is

$$\sum_{\theta^{t-1}} g_t(\theta^t \mid \eta) \beta^t U'(c_t(\theta^t, \eta)) = \hat{\mu}_\eta \sum_{\theta^{t-1}} p_t(\theta^t \mid \eta) \tag{35}$$

Notice that the summation over θ^{t-1} is assumed because it is assumed that, regardless of history θ^{t-1} or agent type η, agents can enter any $\theta \in \Theta$ market. If prices depended on θ^{t-1}, so an agent's specific history mattered, it would imply that two agents of the same type η but different histories would be unable to enter into contingent contracts.

Solve this for the price and substitute into the budget constraint

$$0 \leq \sum_{t=0}^{\infty} \sum_{\theta_t} \left[\frac{\sum_{\theta^{t-1}} g_t(\theta^t \mid \eta) \beta^t U'(c_t(\theta^t, \eta))}{\hat{\mu}_\eta} \right] [\theta_t - c_t(\theta^t, \eta)] \tag{36}$$

Notice that since $\hat{\mu}_\eta$ is constant it can be eliminated from the equation above. Under full consumption insurance, consumption is a constant across realizations, so that (39) can be written

$$0 = U_1(\bar{c}(\eta)) \left\{ \sum_{t=0}^{\infty} \beta^t \left[\sum_{\theta} g(\theta_t, \eta)\theta_t - \sum_{t=0}^{\infty} \beta^t \bar{c}(\eta) \right] \right\} \tag{37}$$

or

$$\frac{1}{1-\beta}\bar{c}(\eta) = \theta_m(\eta)[\frac{1}{1-\beta}].$$

Observe that, as in the one-period model, preferences don't matter nor does the wealth distribution over time. Since there is no aggregate uncertainty, total endowment of type η agents is

$$f(\eta)\left[\sum_{\Theta}\theta g(\theta \mid \eta)\right].$$

Hence the infinite-horizon model is just a repetition of the solution in the one-period model. Agents maintain a constant consumption over their lifetime, although this consumption varies by type (realization of η). Also notice that the price

$$\sum_{\theta^{t-1}} p_t(\theta^t \mid \eta) = \beta^t q(\theta, \eta)$$

where q was defined above.

3.2 Integrated Markets in the Infinite Horizon

As before, under the assumption that prices are stationary, agents will have an incentive to borrow and lend in other η-agent markets. In that case, where markets cannot be segregated, agents solve

$$\max \sum_{t=0}^{\infty}\sum_{\theta^t} g_t(\theta^t \mid \eta)\beta^t U(c_t(\theta^t, \eta)) \tag{38}$$

subject to

$$0 \leq \sum_{t=0}^{\infty}\sum_{\theta^t} p_t(\theta_t)[\theta_t - c_t(\theta^t, \eta)] \tag{39}$$

Let $\hat{\mu}_\eta$ denote the Lagrange multiplier. The first-order condition is

$$\sum_{\theta^{t-1}} g_t(\theta^t \mid \eta)\beta^t U'(c_t(\theta^t, \eta)) = \hat{\mu}_\eta \sum_{\theta^{t-1}} p_t(\theta_t) = \hat{\mu}_\eta p_t(\theta_t) \tag{40}$$

Consider only stationary solutions. Let

$$q(\theta_t) = \frac{p_t(\theta_t)}{\beta^t}$$

Observe that, if we assume only stationary solutions are to be considered so $c(\theta, \eta) = c_t(\theta^t, \eta)$, then

$$\sum_{\theta^{t-1}} g_t(\theta^t \mid \eta) = g(\theta_t \mid \eta)$$

The first-order condition becomes

$$\frac{U_1(c(\theta, \eta))g(\theta \mid \eta)}{\hat{\mu}(\eta)} = q(\theta) \tag{41}$$

Let

$$q(\theta) = \sum_{\eta} f(\eta)g(\theta \mid \eta)$$

and let

$$\hat{c}(\theta, \hat{\mu}(\eta), \eta)$$

be the solution to (41). Substitute $\hat{c}$ into the budget constraint

$$0 \leq \sum_{t=0}^{\infty} \beta^t \left[\sum_{\eta} f(\eta)g(\theta \mid \eta) \right] [\theta_t - \hat{c}(\theta, \hat{\mu}(\eta), \eta)] \tag{42}$$

and solve for $\hat{\mu}(\eta)$. The market-clearing condition is identical to (10) above, so that this is an equilibrium. Just as in the separated market, the integrated market solution has the property that the stationary consumption allocation in the intertemporal model is identical with the allocation in the one-period model. Since the endowment is nonstorable and there are no capital market frictions such as borrowing constraints, this is not surprising. The main friction in this model is the ability to shift consumption over states at a point in time, but as will be demonstrated, this has strong implications for the stochastic discount factor which is central to asset pricing.

The infinite horizon problem can also be expressed as a sequential dynamic programming problem and doing so provides some valuable insight into how the integration of markets affects agents facing uninsurable income distribution risk. The key is to note that the conditional probability of moving to a state $\theta' \in \Theta$ for agent η with endowment θ differs across different η-types. As mentioned earlier, the property that consumption insurance isn't actuarially fair creates a wedge between the trade-off over states offered by the market and the marginal rate of transformation of marginal utility across states for the agent. Let $\hat{q}(\theta_{t+1}, \theta_t, \eta)$ denote the effective price of a unit of consumption in state θ_{t+1} for a type η agent with current endowment θ_t. The Bellman equation is

$$V(z, \theta_t; \eta) = \max \left[U(c) + \beta \sum_{\Theta} g(\theta_{t+1} \mid \eta) V(z(\theta_{t+1}, \theta_{t+1}; \eta) \right] \tag{43}$$

subject to

$$\theta + z \geq c + \sum_{\Theta} \hat{q}(\theta_{t+1}, \theta_t, \eta) z(\theta_{t+1}, \theta_t, \eta) \tag{44}$$

Let $\lambda(\theta, \eta)$ denote the Lagrange multiplier. The first-order conditions and envelope condition are

$$U'(c) = \lambda(\theta, \eta) \tag{45}$$

$$\hat{q}(\theta_{t+1}, \theta_t, \eta)\lambda(\theta, \eta) = \beta g(\theta_{t+1}, \eta) V_z \tag{46}$$

$$V_z = \lambda(\theta, \eta) \tag{47}$$

These conditions can be simplified as

$$U'(c)\hat{q}(\theta_{t+1}, \theta_t, \eta) = \beta g(\theta_{t+1}, \eta) U'(c_{t+1}) \tag{48}$$

Recall that θ', θ are related according to (17) so that

$$U'(c(\theta_t, \eta)) = \frac{\mu(\eta) q(\theta_t)}{g(\theta_t, \eta)} \tag{49}$$

Update the time subscript in (49) by one and substitute into (48) to obtain

$$0 = \left[\frac{\mu(\eta) q(\theta_t)}{g(\theta_t, \eta)} \right] \hat{q}(\theta_{t+1}, \theta_t, \eta) - \beta g(\theta_{t+1}, \eta) \left[\frac{\mu(\eta) q(\theta_{t+1})}{g(\theta_{t+1}, \eta)} \right] \tag{50}$$

so that

$$\hat{q}(\theta_{t+1}, \theta_t, \eta) = \beta \left[\frac{q(\theta_{t+1})g(\theta_t, \eta)}{q(\theta_t)}\right] \tag{51}$$

The price $\hat{q}$ measures the intertemporal terms of trade for a type η agent with current endowment θ_t. Notice that (50) also implies

$$U'(c(\theta_t, \eta)\frac{g(\theta_t, \eta)}{q(\theta_t)} = U'(c(\theta_{t+1}, \eta)\frac{g(\theta_{t+1}, \eta)}{q(\theta_{t+1})}.$$

Notice the similarity between this equation and (27); the price $\hat{q}$ takes into account the wedge created by the property that consumption insurance is actuarially unfair.

Observe that by averaging over the different agent types

$$\begin{aligned} \sum_{\eta} \hat{q}(\theta_{t+1}, \theta_t, \eta) &= \beta \sum_{\eta} \left[\frac{q(\theta_{t+1})g(\theta_t, \eta)}{q(\theta_t)}\right] && (52) \\ &= \beta \left[\frac{q(\theta_{t+1})}{q(\theta_t)}\right] && (53) \end{aligned}$$

As a result, the intertemporal marginal rate of substitution for an individual type η agent is

$$m(\theta_{t+1}, \theta_t, \eta) \equiv \beta \frac{U'(c(\theta_{t+1}, \eta))}{U'(c(\theta_t, \eta))} = \beta \frac{\Lambda(\theta_t, \eta)}{\Lambda(\theta_{t+1}, \eta)} \tag{54}$$

For any θ', θ pair in Θ, observe that there is a distribution of IMRS $m(\theta', \theta, \eta)$.

To understand the implications for the agent's budget constraint, solve (44) for $z(\theta_t, \eta)$,

$$z(\theta_t, \eta) = c(\theta_t, \eta) - \theta_t + \sum_{\Theta} \hat{q}(\theta_{t+1}, \theta_t, \eta) z(\theta_{t+1}, \eta)$$

Define

$$q_c(\theta_{t+1}, \theta_t) \equiv \frac{q(\theta_{t+1})}{q(\theta_t)}.$$

Make the substitution for $\hat{q}$ and solve recursively forward to obtain

$$z(\theta_t, \eta) \quad = \quad c(\theta_t, \eta) - \theta_t + \beta \sum_{\theta_{t+1}} \left[q_c(\theta_{t+1}, \theta_t) g(\theta_t, \eta) [c(\theta_{t+1}, \eta) - \theta_{t+1} + \sum_{\theta_{t+2}} \hat{q}(\theta_{t+2}, \theta_{t+1}, \eta) z(\theta_{t+2}, \eta) \right]$$

$$= \quad c(\theta_t, \eta) - \theta_t + \sum_{j=1}^{\infty} \beta^j \left[\sum_{\theta_{t+j} \in \Theta} \left(\prod_{i=0}^{j-1} q_c(\theta_{t+1}, \theta_t) g(\theta_{t+i}, \eta) \right) [c(\theta_{t+j}, \eta) - \theta_{t+j}] \right]$$

4 Aggregate Risk

The model so far has deterministic output; there is idiosyncratic risk for an individual agent, and because the idiosyncratic endowment distribution risk is not insurable at a actuarially fair price, agents experience consumption fluctuations that average out in the aggregate. The next step is to introduce aggregate uncertainty in a tractable way.

Let $S = \{s_1, \ldots, s_S\}$ be a discrete random variable forming a Markov chain with

$$\pi(s_{t+1}, s_t)$$

denoting the one-step ahead probability of the state next period equaling s_{t+1} conditional on starting in state s_t. Now assume that the individual agent's probability of an endowment realization θ conditional on the aggregate state s and type η is

$$g(\theta \mid \eta, s)$$

For example, there may be some type η agents that have an endowment distribution that is unaffected by an aggregate shock while other type η agents have endowment distributions that are affected by the aggregate shock. If $s_2 > s_1$ then an example would be $g(\theta \mid \eta, s_2)$ is a mean-preserving spread of $g(\theta \mid \eta, s_1)$. Let $s_0, s_1, \ldots, s_t$ be a history of realizations of the aggregate shock. Denote the probability of the history as

$$\pi_t(s^t)$$

The type η agent solves

$$\max \left[\sum_{t=0}^{\infty} \sum_{s^t} \sum_{\theta^t} \beta^t \pi_t(s^t) g_t(\theta^t \mid s^t, \eta) U(c_t(\theta^t, s^t, \eta)) \right] \tag{55}$$

subject to

$$0 = \sum_t \sum_{s^t} \sum_{\theta_t} p_t(\theta, s^t)[\theta_t - c_t(\theta_t, s^t, \eta)] \tag{56}$$

The first-order condition is

$$\beta^t \pi_t(s^t) \sum_{\theta^{t-1}} g_t(\theta^t \mid s^t, \eta) U'(c_t(\theta^t, s^t, \eta) = \mu(\eta) \sum_{\theta^{t-1}} p_t(\theta^t, s^t) \tag{57}$$

Only stationary equilibria will be examined. Let $c(\theta_t, s_t, \eta)$ be a solution. Define

$$p(s_t, \theta_t) \equiv \frac{\sum_{\theta^{t-1}} p_t(\theta^t, s^t)}{\beta^t \pi_t(s^t)}$$

so that the first-order condition can be expressed as

$$\pi(s_t) g(\theta_t \mid s_t, \eta) U'(c(\theta_t, s_t, \eta)) = \mu(\eta) p(s_t, \theta_t) \tag{58}$$

For any two aggregate states $\hat{s}, \bar{s}$, observe that

$$\mu(\eta) = \frac{\pi(\hat{s}) g(\theta \mid \hat{s}, \eta) U'(c(\theta, \hat{s}, \eta))}{p(\hat{s}, \theta)} = \frac{\pi(\bar{s}) g(\theta \mid \bar{s}, \eta) U'(c(\theta, \bar{s}, \eta))}{p(\bar{s}, \theta)}$$

The marginal rate of substitution across aggregate states, given θ,

$$\frac{U'(c(\theta, \hat{s}, \eta))}{U'(c(\theta, \bar{s}, \eta))}$$

is not equalized across states since the marginal utility is weighted by the individual's conditional probability of θ, s.

Consider a price function of the form

$$p(s, \theta) = \pi(s) \sum_\eta f(\eta) g(\theta \mid s, \eta) \tag{59}$$

so that the price equals the probability that a random agent has endowment θ, conditional on the aggregate state s. Then given an aggregate state s, the consumption allocation across Θ satisfies

$$\frac{p(s, \hat{\theta})}{p(s, \bar{\theta})} = \frac{g(\hat{\theta} \mid s, \eta) U'(c(\hat{\theta}, s, \eta))}{g(\bar{\theta} \mid s, \eta) U'(c(\bar{\theta}, s, \eta))} \tag{60}$$

The agent's intertemporal marginal rate of substitution is

$$m(\theta_{t+1}, \theta_t, s_{t+1}, s_t, eta) \equiv \beta \frac{g(\theta_{t+1} \mid s_{t+1}, \eta) U'(c(\theta_{t+1}, s_{t+1}, \eta))}{U'(c(\theta_t, s_t, \eta))} \tag{61}$$

Observe that given $\theta_{t+1}, \theta_t, s_{t+1}, s_t$, the IMRS is a distribution over N.

5 Conclusion

Under full consumption insurance, heterogeneous agents are able to equalize their marginal rate of substitution (MRS) state by state. Hence the associated pricing kernel and stochastic discount factor for the heterogeneous agent model are identical with the representative agent model. Since the representative agent model has performed poorly in empirical testing, there have been many efforts to enrich the model by introducing uninsurable idiosyncratic risk. In many instances, agents are modeled as being identical *ex ante* and become differentiated only over sample paths. If there is a countable infinity of agents, the model has the property that the cross sectional distribution of idiosyncratic risk is identical to the endowment risk an agent faces over his lifetime. In the absence of capital market frictions, this means that the consumption insurance offered by a standard Arrow-Debreu contingent claims market allows complete smoothing of idiosyncratic risk. Restrictions on the asset market structure, such as simple borrowing constraints, have not substantially improved the empirical performance of the model. By introducing *ex ante* heterogeneous income distribution risk that is only partially insurable because the insurance is actuarially unfair, the asset market implications of the model are significantly altered. The stochastic discount factor is now a distribution over different categories of income distribution risk and the pricing kernel is a weighted average of the individuals' pricing kernels.

References

Aiyagari and Peled [1991]. "Dominant Root Characterization of Pareto Optimality and the Existence of Optimal Equilibria in Stochastic Overlapping Generations Models," *Journal of Economic Theory*, 54, 69–83.

Aiyagari, S.R. [1994]. "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, 109, no. 3, 659–684.

Alvarez F. and U. Jermann [2000]. "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica*, 68, 775–797.

Alvarez F. and U. Jermann [2005]. "Using Asset Prices to Measure the Persistence of the Marginal Utility of Wealth," *Econometrica*, 73, no. 6, 1977–2016.

Constantinides and Duffie [1994] *Journal of Political Economy*

Huggett, M. [1993]. "The Risk Free Rate in Heterogeneous-Agents Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 17, 953–969.

Hansen, L. and R. Jagannathan [1991]. "Implications of Security Market Date for Models of Dynamic Economies," *Journal of Political Economy*, 99, 225-262.

Kehoe, T. and D. Levine [1993]. "Debt-Constrained Asset Markets," *Review of Economic Studies*, 60, no. 4, 865–888.

Kehoe, T. and D. Levine [2001]. "Liquidity-Constrained versus Debt Constrained Markets," *Econometrica*, 69, no. 3, 575–598.

Kocherlakota, N. [1996]. "Implications of Efficient Risk Sharing without Commitment," *Review of Economic Studies*, 63, 595–609.