Payment networks in a search model of money

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Abstract

In a simple search model of money, we study a special kind of memory which gives rise to an arrangement resembling a payment network. Specifically, we assume that agents can choose to have access to a central data base which keeps track of payments made and received. We show that multiple equilibria can arise because of a network effect and we study policies that can help eliminate the equilibrium with low access. We also study policies that can loosen the participation constraint. Finally, we compare our model with the model of Cavalcanti and Wallace (1999 a and b).

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1 Introduction

Kocherlakota (1998) shows that memory plays a crucial role in achieving desirable allocations in economies where commitment is not possible. In particular, he shows that money can be thought of as a mnemonic device in a variety of environments. This has lead some authors to study alternative forms of memories and their interactions with money. Kocherlakota and Wallace (1998) study an economy in which there is money and a public record of all past actions that is updated with a lag. Cavalcanti, Erosa, and Temzelides (1999) consider an environment in which agents can issue notes which are redeemed at a central location. Cavalcanti and Wallace (1999 a and b) (CW) assume that some agents have public histories while other agents do not. They show that agents with public histories can issue notes that circulate and identify such agents with early banks. Corbae and Ritter (2004) assume that agents can remain in a long-term relationship as long as it is in their self-interest.

In this paper we study another form of memory which we believe shares some features with payment networks such as credit cards. We extend the standard divisible good model of Shi (1995) and Trejos and Wright (1995) and assume that agents can pay an entry cost, once and for all, to access a central data base (CDB) which can record the history of those agents' trades. When two agents who have access to this data base meet, they can transact without money. We compare equilibrium allocations with the ex-ante efficient allocation.

We show that agents holding money derive less benefit from having access to the CDB than agents who do not hold money. Since a participation constraint must ensure that an agent prefers to produce for another rather than loose access to the CDB, it is more difficult to convince the former type of agents to trade using the CDB. One way to loosen the participation constraint faced by agents holding money is to impose that sellers cannot require to be paid with money if the CDB can also be used. We also show that decreasing the quantity of goods exchanged for a unit of money (increasing the price paid when using money) loosens the participation constraint and the entry condition. These results suggest that there could be benefits from the 'no surcharge rule' of credit cards. More generally, our paper emphasizes the fact that both entry to the CDB and continued participation in the network are important and that the incentives for each may be different.

Because the benefit from having access to the CDB increases with the number of other agents who have access to it, there is a network effect. We show that in equilibrium the number of agents who access the CDB may be sub-optimally low. We also consider policies that can affect loosen the entry constraint without tightening the participation constraint. One policy is to impose a utility cost, which we interpret as a tax, another is to increase the supply of money, a third is to increase the price of goods purchased with money. We show that if the efficient allocation calls for every agent to access the CDB, then this allocation can be achieved by imposing a sufficiently high tax on agents who do not obtain access. We also show that it is not always possible to so the same by changing only the money supply. Moreover, we show that if the efficient allocation is such that not all agents access the CDB, then it is preferable to impose a tax than to increase the money supply. In turn, it is preferable to increase the money supply rather than increase the price of goods purchased with money.

The result that a change in the money supply can provide incentives for agents to access the CDB is similar to a result obtained by Corbae and Ritter (2004). They show that introducing money in their economy can weaken the incentive to produce in a credit relationship and thus weaken credit partnerships. The intuition is that money is an outside option for the parties of such partnerships and as the benefits of money increase, credit arrangement become relatively less attractive and thus more difficult to enforce. The same idea applies in our case, except that we consider a multilateral credit arrangement rather than bilateral arrangements. Studying a multilateral credit arrangement also allows us to show the importance of the size of the such an arrangement. We show that if agents who do not produce when they are supposed to can only be punished by loosing their access to the CDB, then small multilateral arrangements cannot be sustained.

Our model also emphasizes the role of both the entry condition and the participation constraint. There is anecdotal evidence that credit cards companies worry both about getting consumers and merchants to sign up and use credit cards, but also to keep them in the network once they have signed up. A typical incentive to sign up for consumers is a low interest rate for some period of time. Similarly, merchants may benefit from low 'introductory' interchange fees. On the other hand, consumers who want to cancel their credit card may be offered a rate reduction or additional benefits in order to keep the card and stay in the network. Similarly, some large merchants benefit from special conditions when they threaten to stop accepting cards.

We also compare allocations of our economy with the no-gift allocation considered in Cavalcanti and Wallace (1999 a). We show that the benefit from the kind of memory considered by CW is at least as great as the benefit from the kind of memory we consider. Hence, at equal costs, the benefits from a few 'banks' of the type considered by CW is greater then the benefit of a small payment network of the type we consider. In contrast, the benefit from all agents having access to the CDB is the same as the benefit from all agents having public histories. When either is beneficial, then a payment network of the type we consider would be adopted if it is only slightly less expensive.

The remainder of the paper proceeds as follows: Section 2 describes the model. Section 3 characterizes the ex-ante efficient allocation. Section 4 shows that there can be multiple equilibria. Section 5 studies policies that can loosen the entry condition and the participation constraint. Section 6 compares our model with CW. Section 7 concludes.

2 The model

Time is discrete and denoted by $t = 1, 2, ...$ There is a mass 1 of infinitely lived agents. There are $k > 2$ types of agents who are randomly matched in pairs in every period. There are also k types of perishable consumption goods in every period. Each agent is specialized in production and consumption. Agents of type i get period utility $u(c) > 0$ from consuming c units of good i. Agents of type i can only produce good $i + 1$, modulo k, and incur a cost $c > 0$ when producing an amount c of goods. Hence there can be no double coincidence of wants. Agents discount period utility with $\beta < 1$.

There is also a mass M of perfectly durable objects called money. Agents derive no utility from consuming money. Money comes in indivisible units and we assume that there is a storage constrain that prevents agents from holding more than one unit of money.

All agents can choose to pay an entry cost to access a central data base (CDB). The CDB is a central record keeping device. It can keep track of meetings between two agents who both have access to the CDB and whether an agent produces goods for, or receives goods from, another agent. The CDB is unable to directly monitor the behavior of agents and must rely on the reports of agents who have access. The possibility that agents do not report their actions limits the possible use of the CDB. For example, in a meeting between an agent who has access to the CDB and an agent who does not have access, the former agent would always have an incentive to not report the meeting and the latter agent would be unable to communicate with the CDB.

When two agents who have access to the CDB, a trade will be recorded by the CDB if both agents send consistent messages. We assume that agents who have access to the CDB receive a number–an infinite sequence of 0 and 1–that uniquely identify them. In a single coincidence meeting between two agents who have access to the CDB, each agent can send a verifiable message identifying her trading partner and whether that trading partner produces goods for her. We assume that agents who have access to the CDB are identifiable to each other so it is not possible to pretend not to have access. Agents who have access to the CDB but do not accept to produce goods when they meet another agent with access to the CDB are punished with loosing their access to the CDB.

The access cost to the CDB is paid once and for all at the beginning of the economy, before agents learn whether or not they will be money holders.[2] Assume that agents are indexed by $i \in [0, 1]$ and that the cost an agent must pay is given by $\kappa_i \geq 0$, where $\kappa_i \leq \kappa_j$ and $\kappa_{i+\Delta} - \kappa_i \leq \kappa_{j+\Delta} - \kappa_j$ if $i < j$, $\Delta > 0$; i.e., the costs are (weakly) increasing at a (weakly) increasing rate.[3] We also assume that κ_i is continuous in i.[4]

The CDB is a form of memory, as in Kocherlakota (1998) or Cavalcanti-Wallace

[2]While this cost is measured in terms of utility, we could assume that at the beginning of the economy agents are endowed with a nonstorable consumption good which must be consumed before any meeting with other agents. The access cost to the CDB could be expressed in terms of this good.

[3]Allowing some agents to receive a utility benefit from accessing the CDB ($\kappa < 0$) would not modify our results.

[4]Allowing for discontinuities complicates the exposition without providing additional insights.

(1999 a and b). The main difference is that in these authors' work, the history of some agents in publicly known. In contrast, in this paper the history of an agent who has access to the CDB is only available to agents who also have access to the CDB. It is thus a limited access memory as opposed to a full, or public, access memory. Corbae and Ritter (2004) also study a type of limited access memory. They assume that agents can form bilateral credit partnerships and that the partners can remember each other's histories as long as the partnership survives. Our CDB can be thought of as a multilateral partnership which resembles in some respect the partnerships in Corbae and Ritter's model. However, in contrast to their model, we assume that there is a form of public record keeping of some agents' histories but that access to this record is limited.

Note that in our model an agent's balance with the CDB does not matter. This is also the case for agents with public histories in CW. Because we assume enough enforcement, if agents with access to the CDB are willing to produce goods for other agents with access, this will be true for any history of past trade. The fact that an agent's balance with the CDB is not a state variable greatly simplifies the analysis.

2.1 The value functions

We can write the value functions for agents depending on whether or not they hold one unit of money and whether or not they have access to the CDB. Let $m_0 \equiv \frac{1-M}{k}$ denote the probability of a single-coincidence meeting in which the agent who can produce the good desired by the other agent does not hold a unit of money. Similarly, let $m_1 \equiv \frac{M}{k}$ denote the probability of a single-coincidence meeting in which the agent who likes to consume the good produced by the other agent holds a unit of money. We denote by V_0 and V_1 the value functions of agents who do not have access to the CDB and have no unit of money or one unit of money, respectively. Also, we denote by c_m the amount of goods exchange for a unit of money.

$$V_0 = m_1 [\beta V_1 - c_m] + (1 - m_1)\beta V_0, \quad (1)$$

$$V_1 = m_0 [\beta V_0 + u(c_m)] + (1 - m_0)\beta V_1. \quad (2)$$

An agent with no money will meet an agent with money of the right type with probability m_1. In such a meeting, the agent produces, and suffers the cost c_m, in exchange for a unit of money. With this unit of money, the agent will have value V_1

in the next period. In all other meetings, no trade can take place. It can be shown that

$$
\begin{aligned}
V_0 &= \frac{\beta m_0 m_1 \left[u(c_m) - c_m\right] - (1-\beta) m_1 c_m}{(1-\beta)\left[1 - \beta + \frac{\beta}{k}\right]}, && (3)\\
V_1 &= V_0 + \frac{m_0 u(c_m) + m_1 c_m}{\left[1 - \beta + \frac{\beta}{k}\right]} > V_0. && (4)
\end{aligned}
$$

We denote by V_0^a and V_1^a the value functions of agents who have access to the CDB and have no unit of money or one unit of money, respectively. We let c_{DB} denote the amount of good produced, and the cost of producing those goods, when the CDB is used.

$$
\begin{aligned}
V_0^a &= (1-\lambda) m_1 \left[\beta V_1^a - c_m\right] \\
&\quad + \lambda m_0 \left[u(c_{DB}) - c_{DB}\right] + \lambda m_1 \left[u(c_{DB}) - \theta c_{DB} - (1-\theta)(c_m - \beta V_1^a)\right] \\
&\quad + \left[1 - (1-\lambda) m_1 - \lambda(1-\theta) m_1\right] \beta V_0^a && (5)\\
&= (1-\lambda\theta) m_1 \left[\beta V_1^a - c_m\right] \\
&\quad + \lambda(m_0 + m_1) u(c_{DB}) - \lambda(m_0 + \theta m_1) c_{DB} + \left[1 - m_1(1-\theta\lambda)\right] \beta V_0^a, && (6)
\end{aligned}
$$

$$
\begin{aligned}
V_1^a &= (1-\lambda) m_0 \left[\beta V_0^a + u(c_m)\right] \\
&\quad + \lambda m_1 \left[u(c_{DB}) - c_{DB}\right] + \lambda m_0 \left[\theta u(c_{DB}) + (1-\theta)\left[u(c_m) + \beta V_0\right] - c_{DB}\right] \\
&\quad + \left[1 - (1-\lambda) m_0 - \lambda(1-\theta) m_0\right] \beta V_1^a && (7)\\
&= (1-\theta\lambda) m_0 \left[\beta V_0^a + u(c_m)\right] \\
&\quad + \lambda(m_1 + \theta m_0) u(c_{DB}) - \lambda(m_1 + m_0) c_{DB} + \left[1 - m_0(1-\theta\lambda)\right] \beta V_1^a. && (8)
\end{aligned}
$$

We let $\theta \in [0, 1]$ denote the probability that the CDB is used in a transaction that can take place both with money or using the CDB. We do not specify how θ is determined but consider the impact of different values of θ. An agent who has access to the CDB but does not carry one unit of money meets, with probability $1-\lambda$, an agent who does not have access to the CDB. In that case, a trade will take place only if the meeting partner wants the good produced by the agent and has a unit of money (probability m_1). With probability λ, the meeting partner has access to the CDB. If there is a single coincidence of wants but the meeting partner does not have a unit of money (probability m_0), then an exchange can only occur through the CDB. However, if there is a single coincidence of wants and the meeting partner has a unit of money (probability m_1), then an exchange can occur using money (probability θ) or the CDB.[5] In all other meetings, no exchange can occur. Similar reasoning applies for the case of an agent who has access to the CDB and holds one unit of money. These expressions can be rewritten as

$$
\begin{aligned}
V_0^a &= \frac{\lambda}{k}\frac{u(c_{DB})-c_{DB}}{1-\beta}+(1-\theta\lambda)\left[\frac{(1-\lambda\theta)\,\beta m_0 m_1\left[u(c_m)-c_m\right]-(1-\beta)m_1c_m}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]}\right] \\
&\quad -\lambda\,(1-\theta)\left[\frac{(1-\lambda\theta)\,\beta m_0 m_1\left[u(c_{DB})-c_{DB}\right]-(1-\beta)m_1c_{DB}}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]}\right] \qquad (9) \\
V_1^a &= V_0^a+(1-\theta\lambda)\,\frac{m_0u(c_m)+m_1c_m}{\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]}-\lambda\,(1-\theta)\,\frac{m_0u(c_{DB})+m_1c_{DB}}{\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]}. \qquad (10)
\end{aligned}
$$

From equation 10, it appears that V_1^a could be smaller than V_0^a; for example, if c_m is sufficiently small and $\theta < 1$. The intuition is that since c_m is very small, agents who have access to the CDB would prefer to use the CDB rather than money in a single coincidence meeting. However, since $\theta < 1$ money holders must use money in some cases. We can rule out the cases where $V_1^a < V_0^a$ by assuming that there is free disposal of money.

2.2 Participation constraints

Agents who have access to the CDB may have an incentive to renege on their obligation to produce in a meeting with an agent who also has access to the CDB. This is because such agents have to pay the immediate cost of production but only receive the potential benefits from access to the CDB later. Agent who refuses to produce in a single coincidence meeting looses access to the CDB and becomes indistinguishable from agents who chose not to access the CDB in the first place.[6] The participation constraint that must be verified is

$$\beta\left(V_i^a - V_i\right) \geq c_{DB}, i = 0, 1. \qquad (11)$$

[5]Note that two kinds of single coincidence meeting can occur: Either the agent considered want to consume the good produced by her meeting partner or she produces the good consumed by her meeting partner.

[6]It would be easier to sustain an arrangement such as the CDB if we assumed more severe punishments. For example, defecting agents could be punished with autarky. However, more severe punishments are harder to implement as they require more monitoring of agents' behavior.

As is standard, β cannot be too small, or c_{DB} too large relative to $u(c_{DB})$, if agents are not to defect. The expression for $V_1^a - V_1$ is given by

$$V_1^a - V_1 = V_0^a - V_0 - \lambda(1-\theta)\frac{m_0 u(c_{DB}) + m_1 c_{DB}}{\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]} - \frac{\theta\lambda(1-\beta)\left[m_0 u(c_m) + m_1 c_m\right]}{\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]\left[1-\beta+\frac{\beta}{k}\right]} \tag{12}$$

$$= \frac{\lambda}{k}\frac{u(c_{DB}) - c_{DB}}{1-\beta} - \lambda(1-\theta)\left[\frac{(1-\lambda\theta)\beta m_0 m_1\left[u(c_{DB}) - c_{DB}\right] + (1-\beta)m_0 u(c_{DB})}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]}\right]$$
$$-\theta\lambda\frac{\left[\beta m_0 m_1\left[u(c_m) - c_m\right]\left[(1-\beta)(2-\theta\lambda) + \beta\frac{1-\theta\lambda}{k}\right] + (1-\beta)^2 m_0 u(c_m)\right]}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]\left[1-\beta+\frac{\beta}{k}\right]}. \tag{13}$$

Lemma 1 *$V_1^a - V_1$ and $V_0^a - V_0$ display the follow properties:*

1. *$V_1^a - V_1 \leq V_0^a - V_0$.*

2. *$V_1^a - V_1$ is convex in M and reaches a minimum over $[0,1]$ at $M_{min} \leq \frac{1}{2}$, and a maximum at $M = 1$.*

The proof is provided in the appendix. The first result states that the benefit from having access to the CDB is greater for agents who do not hold money than it is for agents who do hold money.

Lemma 1 implies that we only need to be concerned about the incentive constraint for agents who are holding a until of money. Inspection of equation 13 reveals that if $\lambda \to 0$, then $V_1^a - V_1 \to 0$. Hence, it is not possible to sustain very small networks as the benefits from having access to the network are not large enough to provide incentives to agents to produce when they should.

2.3 Access decision

We assume that agents must decide whether or not to access the CDB at the very beginning of the economy, before they learn whether or not they will be money holders at date 1. Under this assumptions, all agents are identical when they make their access decision, with the possible exception of their access cost. In order to derive the expressions for expected utility, we must first know the mass of each type of agents in the economy in steady-state. Let N_0^a and N_1^a denote the steady-state mass of agents who have access to the CDB and carry zero or one unit of money, respectively. Similarly, N_0 and N_1 denote the steady-state mass of agents who do not have access to the network and carry zero or one unit of money. In the appendix we show that $N_0^a = \lambda(1-M)$, $N_1^a = \lambda M$, $N_0 = (1-\lambda)(1-M)$, and $N_1 = (1-\lambda)M$.

The expected utility, net of potential access cost, associated with having access to the CDB is given by $W^a = (1-M)V_0^a + MV_1^a$, which can be written as

$$\begin{aligned} W^a &= \frac{1}{k(1-\beta)}\left[u(c_{DB}) - c_{DB}\right]\lambda\left[1-(1-\theta)M(1-M)\right] \\ &\quad + \frac{1}{k(1-\beta)}\left[u(c_m) - c_m\right](1-\theta\lambda)M(1-M). \end{aligned} \tag{14}$$

The expected utility associated with not having access to the CDB is given by $W = (1-M)V_0 + MV_1$, which can be written as

$$W = \frac{u(c_m) - c_m}{k(1-\beta)}M(1-M). \tag{15}$$

Hence, the expected welfare benefit from having access to the CDB is

$$\begin{aligned} W^a - W &= \frac{\lambda}{k(1-\beta)}\left[u(c_{DB}) - c_{DB}\right] \\ &\quad - \frac{\lambda M(1-M)}{k(1-\beta)}\left\{(1-\theta)\left[u(c_{DB}) - c_{DB}\right] + \theta\left[u(c_m) - c_m\right]\right\}. \end{aligned} \tag{16}$$

From equation 16, it appears that $W^a - W$ could be negative; for example if $\theta > 0$ and c_{DB} is sufficiently small. The intuition is that since c_{DB} is very small, the benefit from using the CDB is very small. Moreover, since $\theta > 0$, agents who have access to the CDB must use it in some single coincidence meeting when they would prefer to use money. We focus on cases where $W^a - W \geq 0$.

In order to make her access decision, an agent forms beliefs about the mass λ of agents who obtain access. Based on that belief, agent i compares the benefit of having access to the CDB, $W^a(\lambda) - W$, with the cost, κ_i. Also, for the expected value of λ, the participation constraint 11 must hold. In a rational expectation equilibrium, the belief of the agent is verified.

2.4 Welfare

First, we study welfare assuming that neither the access condition nor the participation constraint binds. Then, we consider how changes in the parameters affect these constraints.

Lemma 2 *The welfare functions display the following properties:*

1. W^a *reaches a maximum at* $u'(c_{DB}) = 1$ *and* $u'(c_m) = 1$.

2. $\frac{\partial W^a}{\partial \theta} > 0$ *if and only if* $u(c_{DB}) - c_{DB} > u(c_m) - c_m$.

3. $\frac{\partial W^a}{\partial \lambda} > 0$ *if and only if*

$$[u(c_{DB}) - c_{DB}]\,[1 - (1-\theta)M(1-M)] > [u(c_m) - c_m]\,\theta M(1-M).$$

4. W^a *is convex in* M *and reaches a maximum at* $M = 1/2$ *if*

$$(1-\lambda\theta)\,[u(c_m) - c_m] > (1-\theta)\lambda\,[u(c_{DB}) - c_{DB}]\,,$$

while W^a *is concave in* M *and reaches a minimum at* $M = 1/2$ *if*

$$(1-\lambda\theta)\,[u(c_m) - c_m] < (1-\theta)\lambda\,[u(c_{DB}) - c_{DB}]\,.$$

5. W *reaches a maximum at* $u'(c_m) = 1$.

6. W *is concave in* M *and reaches a maximum at* $M = 1/2$.

The proof is provided in the appendix. 1) and 5), state the conditions that maximize the surplus from a trade. 2) says that if the surplus from using the CDB is greater then the surplus from using money, then the CDB should be chosen when both can be used. 3) notes that if the surplus from using the CDB is not too small, then welfare increases when more agents have access to the CDB. Intuitively, there is a network effect so that the value of having access to the CDB increases with the number of agents who have access to it. 4) states that if the surplus from using money is not to small, then the money supply should be neither too small nor too big. Note that if $\theta = 0$, so that the CDB is always used when both money and the CDB can be used, or if $u(c_{DB}) - c_{DB} > u(c_m) - c_m$, then W^a cannot be convex in M. W^a is convex in M only if agents are forced to use money is situations where they would prefer to use the CDB. 4) does not take into account the fact that λ might depend on M, as we will see below. The unconstrained maximum for W^a is reached at $u'(c_{DB}) = u'(c_m) = 1$, $M = 1/2$, and $\lambda = 1$. Note that if $c_{CB} = c_m$, welfare is independent of θ.

In the remainder of this section, we consider how changes in some parameters can affect the entry and the participation constraints. Recall that the entry constraint for agent i is given by

$$W^a - W \geq \kappa_i, \tag{17}$$

and the participation constraint is given by

$$\beta(V_1^a - V_1) \geq c_{DB}. \tag{18}$$

Lemma 3 *$W^a - W$ displays the following properties*

1. *$W^a - W$ reaches a maximum at $u'(c_{DB}) = 1$ and a minimum at $u'(c_m) = 1$.*

2. *$W^a - W$ is concave in M and reaches a minimum at $M = 1/2$.*

3. *$\frac{\partial W^a - W}{\partial \theta} > 0$ if and only if $u(c_{DB}) - c_{DB} > u(c_m) - c_m$.*

Lemma 4 *$V_1^a - V_1$ displays the following properties*

1. *$\left.\frac{\partial V_1^a - V_1}{\partial c_i}\right|_{u'(c_i)=1} < 0, i = m, DB$.*

2. *$\left.\frac{\partial V_1^a - V_1}{\partial M}\right|_{M=1/2} > 0$.*

3. *$\left.\frac{\partial V_1^a - V_1}{\partial \theta}\right|_{c_m = c_{DB}} > 0$.*

The proofs are provided in the appendix. From lemmas 3 and 4, it can be seen that both constraints can be loosened by making money less desirable. Any change in c_m away from $u'(c_m) = 1$ loosens the entry constraint as does any chance in M away from $M = 1/2$. However, only an decrease in c_m away from $u'(c_m) = 1$ and a decrease in M away from $M = 1/2$ will loosen the participation constraint. Intuitively, deviating from the level of consumption that maximizes the surplus from trades involving money affects agents who do not have access to the CDB more than those who do, and thus loosen the entry constraint. Decreasing c_m, which corresponds to increasing the price of goods purchased with money, makes it less attractive to loose access to the CDB. Similarly, any deviation from $M = 1/2$ affects agents who do not have access to the CDB more than those who do. Increasing M from $M = 1/2$ hurts money holders and thus makes it more costly to loose access to the CDB.

If $c_m = c_{DB}$, then $W^a - W$ is independent of θ. In that case, the participation constraint can be loosened by increasing θ because money holders prefer to hold on to their unit of money if they can use the CDB instead. The intuition is that buyers should be able to choose whether to use money or the CDB because this makes it easier to provide incentives for agents who have access to the CDB and use money to produce when they should.

These results suggest that the 'no surcharge' rule imposed by credit card companies may have some benefits. This rule states that merchants may not charge higher prices for purchases made with credit cards rather then money. Indeed, if $c_m = c_{DB}$, sellers prefer to be paid with money, than with the CDB, provided they do not already hold a unit of money. They thus might have an incentive to set a 'surcharge' for using the CDB and increase c_m relative to c_{DB}. Such an increase both reduces welfare and tightens the participation constraint.

3 The ex-ante efficient allocation

Agents in the economy are identical, except possibly for their cost of access to the CDB. If all agents could meet before the beginning of the economy, at a time where they are ignorant of the cost that any specific individual will face, but knowing the distribution of these costs, then they would all agree on the allocation that they prefer. We define this allocation as the ex-ante efficient allocation.

The benefit from letting a mass λ of agents have access to the CDB is given by $\lambda\left(W^a(\lambda) - W\right)$. Since agents may have different access costs, the lowest possible cost to let a mass λ of agents have access to the CDB is given by $\int_0^\lambda \kappa_i di$. Let SW denote the social welfare function.

$$SW \equiv \lambda W^a(\lambda) + (1-\lambda)W - \int_0^\lambda \kappa_i di = W + \lambda\left[W^a(\lambda) - W\right] - \int_0^\lambda \kappa_i di. \tag{19}$$

The value of λ which characterizes the ex-ante efficient allocation solves $\max_\lambda SW$.

Note that $W^a(\lambda) - W$ is a convex function of λ, so taking the first derivative of SW and setting it equal to zero may not provide a maximum. Whether or not it does depends on the particular shape of $\int_0^\lambda \kappa_i di$. Since we assumed that κ_i is weakly increasing at a weakly increasing rate, $\int_0^\lambda \kappa_i di$ is itself a convex function of λ. If it is sufficiently convex, then SW will be concave.

This can be illustrated by an example: Consider a particular functional form for $\int_0^\lambda \kappa_i di$,

$$\int_0^\lambda \kappa_i di = \alpha\lambda^n, \tag{20}$$

where $\alpha > 0$ and n is a positive integer, and assume that $c_m = c_{DB} = c$. If $n = 2$, then

$$SW = W + \lambda^2 \left[\frac{u(c) - c}{k(1-\beta)}\left[1 - M(1-M)\right] - \alpha\right]. \tag{21}$$

If $\alpha > \frac{u(c)-c}{k(1-\beta)}\left[1 - M(1-M)\right]$, then the term in brackets is negative and the efficient allocation is such that $\lambda = 0$. In contrast, if $\alpha < \frac{u(c)-c}{k(1-\beta)}\left[1 - M(1-M)\right]$ then the efficient allocation is such that $\lambda = 1$. If, instead, $n = 3$, then

$$SW = W + \lambda^2 \left[\frac{u(c) - c}{k(1-\beta)}\left[1 - M(1-M)\right] - \alpha\lambda\right]. \tag{22}$$

For λ sufficiently small, the term in brackets is positive, so that an increase in λ increases SW. However, if $\alpha > \frac{u(c)-c}{k(1-\beta)}\left[1 - M(1-M)\right]$ then the term in brackets is negative for sufficiently high λ so that a decrease in λ increases SW. In such a case, the solution to $\max_\lambda SW$ is interior.

Abstracting from the cost of access, we can make some observations about the benefit of increasing the mass of agents having access to the CDB at the margin. This benefit is given by

$$\left[W^a(\lambda) - W\right] + \lambda W^{a\prime}(\lambda). \tag{23}$$

The first element is the benefit received by the marginal agent who obtains access to the CDB when the mass of agents having access is λ. The second element is the benefit the mass λ or agents who already have access to the CDB receive from the addition of the marginal agent. Note that

$$\begin{aligned}\left[W^a(\lambda) - W\right] &= \lambda W^{a\prime}(\lambda) = \frac{\lambda}{k(1-\beta)}\left[u(c_{DB}) - c_{DB}\right] \\ &\quad - \frac{\lambda M(1-M)}{k(1-\beta)}\left\{(1-\theta)\left[u(c_{DB}) - c_{DB}\right] + \theta\left[u(c_m) - c_m\right]\right\}. \end{aligned} \tag{24}$$

In this economy, the benefit received by the marginal agent is exactly equal to the benefits received by all agents who have access to the CDB from the addition of the marginal agent. From lemma 2, we also know that regardless of λ the efficient allocation should set $M = 1/2$.

4 Equilibria

In this section, we describe equilibrium allocations. In equilibrium, some agents who should have access to the CDB under the ex-ante efficient allocation may prefer not to do so. The reason is that individual agents, when choosing to access the CDB, compare their private benefit to their cost but do not take into account the network effect.

The analysis in this section is conducted assuming that the participation constraint does not bind. Any candidate equilibrium with $\lambda > 0$ is not an equilibrium if the participation constraint binds.

Agent i, when considering whether or not to access the CDB, compares the cost of doing so, κ_i, with the benefit $b(\lambda) \equiv W^a(\lambda) - W$. We assume that agents who are indifferent choose to access the CDB. Equation (24) shows that the benefit from accessing the CDB is a linear function of λ. Since we assumed that the distribution of entry cost is continuous, weakly increasing, and weakly concave, we can consider three cases: The graph of κ, as a function of λ, may intersect the graph of $b(\lambda)$ either 0, 1, or 2 times.

4.1 No intersections

There are two cases to consider: Either $\kappa_i > b(i)$, for all i or $\kappa_i < b(i)$, for all $i > 0$.[7]

Proposition 1 *If $\kappa_i > b(i)$, for all i, then no agent accesses the CBD.*

Proof. Assume, to establish a contradiction, that a mass $\lambda > 0$ of agents choose to access the CDB. Since κ_i is continuous, then for ε sufficiently small, $\kappa_{\lambda-\varepsilon} > b(\lambda)$. Also $b(\lambda - \varepsilon) < b(\lambda)$ so all agents in the interval $[\lambda - \varepsilon, \lambda]$ prefer not to have access to the CDB. Since this will be true for any $\lambda > 0$, no agent access the CDB. ■

Proposition 2 *If $\kappa_i < b(i)$, for all $i > 0$, then there can be two equilibria: Either all agents access the CDB or no agents access the CDB.*

Proof. It is an equilibrium for all agents to access the CDB, since $b(1) > \kappa_1 \geq \kappa_i$ for all i.

[7]Since we restrict $\kappa_i \geq 0$, then it cannot be the case that $b(0) > \kappa_0$.

If the mass of agents with cost $\kappa_i = 0$ is zero, then it is an equilibrium for no agent to access the CDB (except for a set of measure zero) since agents face cost $\kappa_i > b(0) = 0$, all $i \in (0, 1]$. If instead, the mass of agents with cost $\kappa_i = 0$ is positive, then it is not an equilibrium for no agent to access the CDB since we assume that agents who are indifferent choose to access the CDB. ■

We can define a notion of stability of equilibria with respect to small deviations of beliefs about λ. Let $\eta \in [0, 1]$ denote the mass of agents who access the CDB if all agents believe that a mass λ_η of agents access the CDB.

Definition 3 *An equilibrium λ is unstable if, $\forall \varepsilon > 0$, $|\lambda_\eta - \lambda| > \varepsilon \Rightarrow \eta \neq \lambda$.*

The equilibrium characterized by $\lambda = 0$ in the above proposition is unstable when it exists. All other equilibria considered so far are stable.

4.2 One intersection

There are two cases to consider: First, $\kappa_i > b(i)$ for $i \in [0, \bar{\lambda})$ and $\kappa_i < b(i)$ for $i \in (\bar{\lambda}, 1]$, $0 \leq \bar{\lambda} \leq 1$. Second, $\kappa_i < b(i)$ for $i \in (0, \bar{\lambda})$ and $\kappa_i > b(i)$ for $i \in (\bar{\lambda}, 1]$, $0 \leq \bar{\lambda} \leq 1$. In the first case, the slope of the graph of κ is flatter than the slope of the graph of $b(\lambda)$ at the point at which they intersect, while the opposite is true in the second case.

Proposition 4 *If $\kappa_i > b(i)$ for $i \in [0, \bar{\lambda})$ and $\kappa_i < b(i)$ for $i \in (\bar{\lambda}, 1]$, $0 \leq \bar{\lambda} \leq 1$, then there are two stable equilibria: Either all agents access the CDB, or no agent accesses the CDB (except for sets of measure zero). There also an unstable equilibrium such that $\bar{\lambda}$ agents access the CDB.*

Proof. If all agents believe nobody accesses the CDB, then no agent chooses to access the CDB since $\kappa_i > b(0) \geq 0$, for all i. If all agents believe that everybody accesses the CDB, then all agents choose to access the CDB since $b(1) > \kappa_1 \geq \kappa_i$, for all i.

Now assume all agents believe that exactly $\bar{\lambda}$ agents will access the CDB. There is a mass $\bar{\lambda}$ of agents with cost $\kappa_i < \kappa_{\bar{\lambda}}$. They prefer to access the CD since $\kappa_{\bar{\lambda}} = b(\bar{\lambda})$. For all other agents, $\kappa_i > \kappa_{\bar{\lambda}}$ and they prefer not to access the CDB.

No other belief can be supported as an equilibrium. To see this, first consider any $\lambda \in (0, \bar{\lambda})$. By assumption, $\kappa_\lambda > b(\lambda)$ and by continuity, the same must be true in a neighborhood of λ. Hence, agents with a cost close to but smaller than λ choose not to access the CDB. Next, consider any $\lambda \in (\bar{\lambda}, 1)$. By assumption, $\kappa_\lambda < b(\lambda)$ and by continuity, the same must be true in a neighborhood of λ. Hence, agents with a cost close to but higher than λ choose to access the CDB. ■

Proposition 5 *If $\kappa_i < b(i)$ for $i \in [0, \bar{\lambda})$ and $\kappa_i > b(i)$ for $i \in (\bar{\lambda}, 1]$, $0 \leq \bar{\lambda} \leq 1$, then there is a unique stable equilibrium such that a mass $\bar{\lambda}$ of agents access the CDB. If the mass of agents with cost κ_i is equal to zero, then there is also an unstable equilibrium such that no agent accesses the CDB.*

Proof. The proof that it is an equilibrium for a mass $\bar{\lambda}$ of agents to access the CDB is the same as in the previous proposition. The proof that this equilibrium is stable is omitted. The proof of existence (or lack thereof) of the no access equilibrium is the same as in the case where $\kappa_i < b(i)$ for all $i > 0$.

Now I show that no other belief can be supported as an equilibrium. Suppose all agents believe that a mass $\lambda \in (0, \bar{\lambda})$ agents access the CDB. In that case $\kappa_\lambda < b(\lambda)$ and by continuity, the same must hold true in a neighborhood of λ. Hence, agents with a cost close to but higher than κ_λ choose to access the CDB. Suppose all agents believe that a mass $\lambda \in (\bar{\lambda}, 1]$ agents access the CDB. In that case $\kappa_\lambda > b(\lambda)$ and by continuity, the same must hold true in a neighborhood of λ. Hence, agents with a cost close to but lower than κ_λ choose not to access the CDB. ■

4.3 Two intersections

There is one case to consider: $\kappa_i > b(i)$ for $i \in [0, \bar{\lambda}_1) \cup (\bar{\lambda}_2, 1]$ and $\kappa_i < b(i)$ for $i \in (\bar{\lambda}_1, \bar{\lambda}_2)$, where $0 \leq \bar{\lambda}_1 < \bar{\lambda}_2 \leq 1$. The graph of κ_i is flatter then the graph of $b(i)$ at $\bar{\lambda}_1$, but steeper at $\bar{\lambda}_2$.

Proposition 6 *If $\kappa_i > b(i)$ for $i \in [0, \bar{\lambda}_1) \cup (\bar{\lambda}_2, 1]$ and $\kappa_i < b(i)$ for $i \in (\bar{\lambda}_1, \bar{\lambda}_2)$, where $0 \leq \bar{\lambda}_1 < \bar{\lambda}_2 \leq 1$, then there are three equilibria: Either nobody accesses the CDB, or a mass $\bar{\lambda}_1$, or a mass $\bar{\lambda}_2$ of agents accesses the CDB.*

The proof of this proposition is omitted as it follows the same logic as the proofs of previous propositions. Note that the equilibrium with a mass $\bar{\lambda}_1$ of agents accessing the CDB is unstable while the other equilibria are stable.

5 Policies

In this section we consider policies that can influence the mass of agent who choose to access the CDB. We restrict our attention to policies that either loosen or do not affect the participation constraint. The analysis is conducted assuming that the participation constraint does not bind and, thus, the policies we consider will not make the constraint bind. Note that any candidate equilibrium with $\lambda > 0$ is not an equilibrium if the participation constraint binds.

We consider three policies: First, a utility cost, τ, which we associate with a tax, can be imposed on agents in the economy.[8] The tax works as follows: Agents are taxed if they choose not to access the CDB but are not taxed if they access the CDB. Second, the money supply M can be chosen so loosen the entry condition. Third, the amount of goods exchanged for a unit of money, c_m, can be fixed at some level. These policies are evaluated on their ability to improve social welfare.

5.1 If $\lambda = 1$ is optimal

First, we consider the case where $\lambda = 1$ is optimal. As the following proposition shows, the use of taxes is particularly effective when the objective to achieve universal access to the CDB. Recall that κ_1 denotes the cost of access for the agent facing the highest cost.

Proposition 7 *Assume that the efficient allocation is such that $\lambda = 1$. If $\kappa_1 \in [W^a(1) - W, 2(W^a(1) - W)]$, then the optimal allocation cannot be achieved absent policy intervention. The efficient allocation can be achieved by setting a high enough tax for agents who do not access the CDB.*

Proof. Since $\kappa_1 > W^a(1) - W$, then some agents prefer not to get access to the CDB since their access cost is greater than their private benefit. However, since $\kappa_1 < 2(W^a(1) - W)$ it is desirable from the perspective of social welfare that all agents have access to the CDB. If follows that the efficient allocation cannot be achieved without some policy intervention.

[8] As in the case of the access cost, we could assume that at the beginning of the economy agents are endowed with a nonstorable consumption good which must be consumed before any meeting with other agents. The government could tax this good.

Assume that a tax greater than or equal to κ_1 is imposed on any agent who chooses not to gain access to the CDB faces. Under this threat, it is individually rational for all agents to obtain access to the CDB, regardless of what other agents do. ■

In equilibrium, of course, all agents obtain access to the CDB and no tax is paid. It is the case that agents whose cost of access is higher than $\frac{u-c}{k(1-\beta)}\left[1-M(1-M)\right]$ are made worse off by gaining access to the CDB than they would have been if they did not gain access and did not have to face the tax.

The key to the result is the assumption that $\kappa_1 \leq 2\frac{u-c}{k(1-\beta)}\left[1-M(1-M)\right]$ so that all agents should have access to the CDB in order to achieve the efficient allocation. The use of taxes is not as effective when $\kappa_1 > 2\frac{u-c}{k(1-\beta)}\left[1-M(1-M)\right]$, since in that case it is not desirable that all agents have access.

It may not be possible to obtain the efficient allocation by changing the money supply or the amount of goods exchanged for a unit of money. Indeed, the marginal social benefit of adding the last agent to the CDB, is always greater than the private benefit given by $W^a(1) - W$, for any values of M or c_m. This is shown formally in the following proposition.

Proposition 8 *If* $\kappa_1 \in (\frac{u(c_{DB})-c_{DB}}{k(1-\beta)}, 2\frac{u(c_{DB})-c_{DB}}{k(1-\beta)}]$*, then the ex-ante allocation is such that* $\lambda = 1$*, however it is not possible to achieve* $\lambda = 1$ *only by changing* M *or* c_m *(or both).*

Proof. From section 3, we know that the efficient allocation calls for $\lambda = 1$ if $\kappa_1 < 2\frac{u-c}{k(1-\beta)}$. Assume all agents join the CDB and consider the agent with the highest cost. This agent will choose to access the CDB if $[W^a(1) - W] \geq \kappa_1$. From equation 3, $[W^a(1) - W]$ can be no greater than

$$\frac{1}{k(1-\beta)}\left[u(c_{DB}) - c_{DB}\right]. \tag{25}$$

This maximum is reached if $M = 1$ (and, by symmetry, if $M = 0$) or if $c_m = 0$ and $\theta = 1$. It follows that if $\kappa_1 > \frac{u(c_{DB})-c_{DB}}{k(1-\beta)}$ then the agent with the highest cost will choose not to access the CDB. By continuity of the access cost schedule, the mass of agents joining the CDB is strictly less than 1. In this case, choosing the money supply does not make it possible to achieve the ex-ante efficient allocation. ■

Finally, note also that since $W^a(0) - W = 0$, it is always an equilibrium for no agents to access the CDB. Indeed, much as with money, if nobody expect the CDB to be used, then nobody has an incentive to use it. Changing M or c_m does not affect $W^a(0) - W$ and thus does not impact the no access equilibrium. In contrast, proposition 7 shows that a high enough tax will eliminate that equilibrium.

5.2 If $\lambda < 1$ is optimal

In this section, we focus on the case where the ex-ante efficient allocation is such that $\lambda < 1$. The problem is to choose τ, M, and c_m to maximize the social welfare function

$$SW = \lambda W^a + (1-\lambda)(W-\tau) - \int_0^\lambda \kappa_i di, \tag{26}$$

taking into account the fact that λ solves

$$\begin{aligned} \kappa_\lambda - \tau &= W^a(\lambda) - W \qquad (27) \\ &= \frac{\lambda}{k(1-\beta)}\left[u(c_{DB}) - c_{DB}\right] \\ &\quad - \frac{\lambda M(1-M)}{k(1-\beta)}\left\{(1-\theta)\left[u(c_{DB}) - c_{DB}\right] + \theta\left[u(c_m) - c_m\right]\right\}. \qquad (28) \end{aligned}$$

The question we ask is: What is the least costly way to provide incentives for $\tilde{\lambda}$ agents to access the CDB? Notice that the access cost must be the same regardless of the policy chosen, since

$$\int_0^{\tilde{\lambda}} \kappa_i di$$

is independent of the policy choice. Also, the social welfare function can be rewritten

$$SW = W - \tau + \tilde{\lambda}\left[W^a(\tilde{\lambda}) - (W-\tau)\right] - \int_0^{\tilde{\lambda}} \kappa_i di. \tag{29}$$

By equation 27, it must be the case that $W^a(\tilde{\lambda}) - (W-\tau) = \kappa_{\tilde{\lambda}}$ regardless of which policy is chosen. Hence, when comparing two policies, (τ, M, c_m) and (τ', M', c_m'), it is enough to compare $W(\tau, M, c_m) - \tau$ with $W(\tau', M', c_m') - \tau'$, subject to constraint 27.

We want to compare three sets of policies:

- $(\tau = \tilde{\tau} > 0, M = 1/2, c_m = c^*)$,

- $(\tau = 0, M = \tilde{M} \neq 1/2, c_m = c^*)$,

- $(\tau = 0, M = 1/2, c_m = \tilde{c}_m \neq c^*)$,

where $c*$ is defined by $u'(c^*) = 1$. We also assume throughout this section that $c_{DB} = c^*$.

The following proposition states that it is preferable to tax agents who do not access the CDB rather than change the money supply. In turn, changing M is preferable to changing c^*. Note that a linear combination of the policies consider cannot be better than the policy of only taxing agents who do not access the CDB.

Proposition 9 $W(\tilde{\tau}, 1/2, c^*) \geq W(0, \tilde{M}, c^*) \geq W(0, 1/2, \tilde{c}_m)$.

Proof. First, note that these expressions are given by

$$W(\tilde{\tau}, 1/2, c^*) = \frac{u(c^*) - c^*}{4k(1-\beta)} - \tilde{\tau}, \tag{30}$$

$$W(0, \tilde{M}, c^*) = \frac{u(c^*) - c^*}{k(1-\beta)} \tilde{M}(1 - \tilde{M}), \tag{31}$$

$$W(0, 1/2, \tilde{c}_m) = \frac{u(\tilde{c}_m) - \tilde{c}_m}{4k(1-\beta)}. \tag{32}$$

We can use equation 27 to obtain

$$\tilde{\tau} = \kappa_{\tilde{\lambda}} - \tilde{\lambda} \frac{3\left[u(c^*) - c^*\right]}{4k(1-\beta)}. \tag{33}$$

and

$$\frac{u(c^*) - c^*}{k(1-\beta)} \tilde{M}(1 - \tilde{M}) = \frac{u(c^*) - c^*}{k(1-\beta)} - \frac{\kappa_{\tilde{\lambda}}}{\tilde{\lambda}}. \tag{34}$$

We can show that $W(\tilde{\tau}, 1/2, c^*) \geq W(0, \tilde{M}, c^*)$ since

$$W(\tilde{\tau}, 1/2, c^*) - W(0, \tilde{M}, c^*) = \frac{1 - \tilde{\lambda}}{\tilde{\lambda}} \left[\frac{3\tilde{\lambda}\left[u(c^*) - c^*\right]}{4k(1-\beta)} - \kappa_{\tilde{\lambda}} \right] = \frac{1 - \tilde{\lambda}}{\tilde{\lambda}} \tilde{\tau} \geq 0 \tag{35}$$

Now we want to show that $W(0, \tilde{M}, c^*) \geq W(0, 1/2, \tilde{c}_m)$. From equation 27 we can get

$$\frac{\kappa_{\tilde{\lambda}}}{\tilde{\lambda}} k(1-\beta) = \left[u(c^*) - c^*\right] \left[1 - \tilde{M}(1 - \tilde{M})\right], \tag{36}$$

from policy $(0, \tilde{M}, c^*)$ and

$$\frac{\kappa_{\tilde{\lambda}}}{\tilde{\lambda}} k(1-\beta) = \left[u(c^*) - c^*\right] - \frac{1}{4} \left[(1-\theta)\left(u(c^*) - c^*\right) + \theta\left(u(\tilde{c}_m) - \tilde{c}_m\right)\right], \tag{37}$$

from policy $(0, 1/2, \tilde{c}_m)$. Combining these two expressions we get

$$[u(c^*) - c^*]\,\tilde{M}(1-\tilde{M}) = \frac{1}{4}\left[(1-\theta)\left(u(c^*) - c^*\right) + \theta\left(u(\tilde{c}_m) - \tilde{c}_m\right)\right] \geq \frac{1}{4}\left(u(\tilde{c}_m) - \tilde{c}_m\right). \tag{38}$$

This completes the proof. ■

The intuition for this result is that the wedge that must be created to provide incentives for the marginal agent to obtain access to the CDB is the same whether a tax is imposed or whether M or c_m is modified. In the case of the tax, however, only agents who do not access the CDB pay the cost associated with this wedge. In the case of a change in M or c_m, all agents must pay that cost. Also, note that $W(0, \tilde{M}, c^*) = W(0, 1/2, \tilde{c}_m)$ if $\theta = 1$. If the CDB is always used when money is also available, agents who have access to the CDB are not affected by the change in c_m unless money is the only payment method available.

One important caveat to this result is that it assumes the participation constraint 11 holds. If the participation constraint does not hold, then there might be a role for choosing $M > 1/2$. The key idea is that agents must have incentives to both access the CDB and produce under the CDB arrangement. Proposition 9 concerns the access decision, assuming agents are willing to produce.

If agents can be taxed when they decide not to produce, then it is optimal to set $M = 1/2$ and $c_m = c^*$ and use taxes to ensure that the participation constraint holds. However, if one assumes that agents cannot be taxed if they refuse to produce, then the only way to loosen the participation constraint may be to change the money supply or the amount of goods exchanged for a unit of money. The general point here is that changing the money supply is an easy way to affect all agents even if it is difficult to keep track of them while using taxes requires an ability to keep track of agents.

6 Comparison with Cavalcanti-Wallace

In this section, we compare some allocations of economies with limited memory studied in the previous sections with the 'no-gift' allocations studied in Cavalcanti-Wallace (1999). To facilitate the comparison, we assume that both economies share the environment described in section 2, except that in one case agents may have access to a CDB while in the other they can make their histories public information.

Also, we assume that the participation constraint is not binding in the two environments and that the amount of goods exchanged is the same in all meetings, denoted by c. Note that an important difference between the two economies is that in the case of a CW economy, the money supply is endogenous, while in the economies considered in this paper, it is exogenously given.

In a no-gift allocation, agents whose histories are public may issue notes that are used as a medium of exchange. Production always occurs in a single-coincidence-of-want meeting between two agents whose histories are public. In a meeting between two agents whose histories are private, production occurs if the buyer holds a note and the seller does not. In a meeting between an agent whose history is private and an agent whose history is public, production occurs if the agent whose history is private either wants to consume and holds a note or can produce and does not hold a note. Let λ^p denote the fraction of agents whose histories are public information. We can write the value functions as

$$\begin{aligned} V_0^{CW} &= (1-\lambda^p)\left[m_1\left[\beta V_1^{CW}-c\right]+(1-m_1)\beta V_0^{CW}\right] \\ &\quad +\lambda^p\left[\frac{1}{k}\left[\beta V_1^{CW}-c\right]+(1-\frac{1}{k})\beta V_0^{CW}\right], \qquad (39)\\ V_1^{CW} &= (1-\lambda^p)\left[m_0\left[\beta V_0^{CW}+u(c)\right]+(1-m_0)\beta V_1^{CW}\right] \\ &\quad +\lambda^p\left[\frac{1}{k}\left[\beta V_0^{CW}-c\right]+(1-\frac{1}{k})\beta V_1^{CW}\right]. \qquad (40)\end{aligned}$$

Taking into account the fact that $m_1 = m_0 = 1/2k$, it can be shown that

$$\begin{aligned} V_0^{CW} &= \frac{\frac{1+\lambda^p}{2k}\left[\frac{1+\lambda^p}{2k}\beta(u(c)-c)-(1-\beta)c\right]}{(1-\beta)\left[1-\beta+\beta\frac{1+\lambda^p}{k}\right]}, \qquad (41)\\ V_1^{CW} &= V_0+\frac{\frac{1+\lambda^p}{2k}(u(c)+c)}{\left[1-\beta+\beta\frac{1+\lambda^p}{k}\right]} > V_0. \qquad (42)\end{aligned}$$

The welfare of these agents, denoted by W_{CW} is

$$W_{CW} = \frac{1+\lambda}{4}\frac{u(c)-c}{k(1-\beta)}. \qquad (43)$$

There is no individual state variable for agents whose histories are public. These agents welfare, denoted by W_{CW}^p, is given by

$$W_{CW}^p = \beta W_{CW}^p + (1-\lambda^p)\left[m_1 u(c) - m_0 c\right] + \lambda^p\frac{1}{k}(u(c)-c) = 2W_{CW}. \qquad (44)$$

To compare social welfare in both economies, we assume that the distributions of costs are identical. Since we have no good guide to inform us about how these costs might differ, this assumption allows us to only change the type of memory available in each environment.

The social welfare function in a CW economy can thus be verified to be

$$SW_{CW} = \lambda^p W_{CW}^p + (1-\lambda^p)W_{CW} - \int_0^{\lambda^p} \kappa_i di = \frac{(1+\lambda^p)^2}{4}\frac{u(c)-c}{k(1-\beta)} - \int_0^{\lambda^p} \kappa_i di. \quad (45)$$

For a given λ^a, the social welfare function in the economies studied in the paper is maximized at $M = 1/2$. For such M, it is given by

$$SW_{CDB} = \lambda^a W^a + (1-\lambda^a)W - \int_0^{\lambda} \kappa_i di = \frac{1+3(\lambda^a)^2}{4}\frac{u(c)-c}{k(1-\beta)} - \int_0^{\lambda^a} \kappa_i di. \quad (46)$$

It can be seen that $SW_{CW} \geq SW_{CDB}$ if and only if $1 \geq \lambda \geq 0$. Hence, for all λ, welfare in a CW economy is at least as high as in an economy with a CDB. In fact, it is strictly greater if $\lambda \in (0,1)$. If $\lambda = 0$, both economies are identical and all trades require money. If $\lambda = 1$, then both economies are also identical, but in that case money is unnecessary.

We can summarize this result the following proposition.

Proposition 10 *If $\lambda^a = \lambda^p = \lambda$, then $SW_{CW} \geq SW_{CDB}$ for all λ.*

One might think that proposition 10 implies that there will always be at least as large a fraction of agents with public histories in a CW economy than agents with access to the CDB in an economy of the type we study; i.e., $\lambda^p \geq \lambda^a$. This turns out not to be the case. The condition for the above conjecture to hold is $W_{CW}^p - W_{CW} \geq W^a - W$ or, equivalently, $\lambda \geq [3 - 4M(1-M)]^{-1}$. If $M = 1/2$, this condition is verified for all values of λ. However, if $M \neq 1/2$, then the condition may not hold for large values of λ.

In the economies we study, if the participation constraint does not bind, then $M = 1/2$ and $\lambda^p \geq \lambda^a$ must hold. However, if the participation constraint binds, then it may be desirable to choose $M > 1/2$ in which case $\lambda^p < \lambda^a$ could occur. Note that $SW_{CW} \geq SW_{CDB}$ would still hold.

An interesting difference between the two economies is that small but positive values of λ^a cannot be supported as an equilibrium in the economies studied in this paper, while small positive values of λ^p can be supported as an equilibrium of a CW economy. This can be seen by looking at the participation constraint.

For the economies studied in this paper, we have already pointed out that $V_1^a - V_1 \to 0$ as $\lambda \to 0$, so that the participation constraint cannot hold. In a CW economy, we assume that agents with public histories become indistinguishable from agents whose histories are not public, if they refuse to produce when they should. Under this assumption, the participation constraint is given by $\beta\left(W_{CW}^p - W_{CW}\right) \geq c$. Since

$$W_{CW}^p - W_{CW} = \frac{u(c) - c}{k(1-\beta)} \frac{1+\lambda}{4}$$

the participation constraint can hold in a CW economy even for very small values of λ^p.

CW interpret agents with public histories as playing the role of early banks. We interpret the CDB as having some features of credit card networks. The results presented in this section suggest that in economies in which the cost of memory is high there are more benefits to be gained from having a few banks than from having a limited network resembling credit cards. Indeed, such a network may not be sustainable. On the other hand, in economies where the cost of memory is not too high, the benefits from having many agents with public information is not much greater than having many agents with access to the CDB. Hence, if the cost of the latter kind of memory is even slightly smaller than the cost of the former, it might be beneficial to adopt something resembling a credit card network

7 Conclusion

This paper considers an economy where agents can pay a cost to access a central data base. This CDB is a form of memory that keeps track of individual histories and allows agents who have access to it to engage in transactions that would otherwise not be possible without money. This form of memory has features that resemble those of some payment networks such as credit cards.

We show that agents holding money derive less benefit from having access to the CDB than agents who do not hold money. Thus it is more difficult to convince the former type of agents to trade using the CDB. One way to loosen the participation constraint faced by agents holding money is to impose that sellers cannot require to be paid with money if the CDB can also be used. Another way is to reduce the amount of goods exchanged for money (increase the price of goods purchased with money). This suggests that the 'no surcharge rule' of credit card may have benefits. More generally, our paper emphasizes the fact that both access to the CDB and continued participation in the network are important and that the incentives for each may be different.

We show that a network effect is present since the benefits of having access to the CDB is greater when more agents have access to it. Because of the network effect, fewer agents may access the CDB in equilibrium than would be efficient. We consider policies that can affect the entry condition: Imposing a utility cost, which can be interpreted as a tax, increasing the money supply, or decreasing the amount of goods exchanged for money. We show that if it is efficient for all agents to access the CDB, then imposing a high enough tax on agents who do not obtain access can achieve the efficient allocation. This is cannot be done by changing only the money supply.

We also compare our model with that of Cavalcanti and Wallace (1999) who consider an economy in which some agents have public histories. The type of memory that these authors consider provides greater benefits that the memory we study. This is particularly so when comparing an economy with few agents who have public histories with an economy with few agents having access to the CDB. However, if all agents have access to the CDB the benefits from that type of memory is the same as when all agents have public histories.

8 Appendix

Proof of lemma 1

The result that $V_1^a - V_1 \leq V_0^a - V_0$ follows directly from equation 12.

Note that

$$\begin{aligned}\frac{\partial(V_1^a - V_1)}{\partial M} &= -\lambda\left(1-\theta\right)\left[\frac{\left(1-\lambda\theta\right)\frac{\beta}{k^2}(1-2M)\left[u(c_{DB})-c_{DB}\right]-(1-\beta)\frac{1}{k}u(c_{DB})}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]}\right] \\ &\quad -\theta\lambda\frac{\left[\frac{\beta}{k^2}(1-2M)\left[u(c_m)-c_m\right]\left[(1-\beta)(2-\theta\lambda)+\beta\frac{1-\theta\lambda}{k}\right]-(1-\beta)^2\frac{1}{k}u(c_m)\right]}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]\left[1-\beta+\frac{\beta}{k}\right]}\end{aligned}$$

and

$$\frac{\partial^2 (V_1^a - V_1)}{\partial M^2} = \lambda (1-\theta) \frac{(1-\lambda\theta)\frac{2\beta}{k^2}\left[u(c_{DB}) - c_{DB}\right]}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]} + \theta\lambda \frac{\frac{2\beta}{k^2}\left[u(c_m) - c_m\right]\left[(1-\beta)(2-\theta\lambda)+\beta\frac{1-\theta\lambda}{k}\right]}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]\left[1-\beta+\frac{\beta}{k}\right]} > 0.$$

At $M = 1/2$, $\partial(V_1^a - V_1)/\partial M \geq 0$, so $M_{min} \leq 1/2$.

We show that $N_0^a = \lambda(1-M)$, $N_1^a = \lambda M$, $N_0 = (1-\lambda)(1-M)$, and $N_1 = (1-\lambda)M$:

First, note that by definition,

$$N_0^a + N_1^a = \lambda. \tag{47}$$

$$N_0 + N_1 = 1-\lambda. \tag{48}$$

$$N_0^a + N_0 = 1-M. \tag{49}$$

$$N_1^a + N_1 = M. \tag{50}$$

Now we need the transition probabilities between different types. First note that having access to the CDB is a permanent, once and for all decision. Hence we can consider agents having access separately from those who do not have access to the CDB. We start with the latter type.

An agent who does not have access to the CDB and is not holding a unit of money today could have been either an agent who was holding a unit of money yesterday and spent it (probability $(1-M)/k$), or an agent who was not holding a unit of money yesterday and did not acquire money (probability $1-(M/k)$). Thus we can write

$$N_{0,t} = N_{1,t-1}\frac{1-M}{k} + N_{0,t-1}\left(1-\frac{M}{k}\right). \tag{51}$$

Similarly, an agent who does not have access to the CDB and is holding a unit of money today could have been either an agent who did not have a unit of money yesterday but acquired one (probability M/k), or an agent who did have a unit of money yesterday but was unable to buys goods (probability $1-\left[(1-M)/k\right]$). Thus we can write

$$N_{1,t} = N_{0,t-1}\frac{M}{k} + N_{1,t-1}\left(1-\frac{1-M}{k}\right). \tag{52}$$

In steady state, either of these equations yields $N_0 M = N_1(1-M)$. This, combined with $N_0 + N_1 = 1-\lambda$ implies $N_0 = (1-\lambda)(1-M)$, and $N_1 = (1-\lambda)M$.

Now consider agents who have access to the CDB. An agent not holding money today could have been either an agent not holding money yesterday who did not acquire money (probability $1 - \left[\left(N_{1,t-1} + (1-\theta)N^a_{1,t-1}\right)/k\right]$) or an agent who did hold a unit of money yesterday but spent it (probability $\left(N_{0,t-1} + (1-\theta)N^a_{0,t-1}\right)/k$). Thus we can write

$$N^a_{0,t} = N^a_{0,t-1}\left(1 - \frac{N_{1,t-1} + (1-\theta)N^a_{1,t-1}}{k}\right) + N^a_{1,t-1}\frac{N_{0,t-1} + (1-\theta)N^a_{0,t-1}}{k}. \quad (53)$$

An agent holding a unit of money today could have been an agent not holding a unit money yesterday and who acquired it (probability $\left(N_{1,t-1} + (1-\theta)N^a_{1,t-1}\right)/k$) or an agent who was holding a unit of money yesterday and could not buy goods (probability $1 - \left[\left(N_{0,t-1} + (1-\theta)N^a_{0,t-1}\right)/k\right]$). Thus we can write

$$N^a_{1,t} = N^a_{0,t-1}\frac{N_{1,t-1} + (1-\theta)N^a_{1,t-1}}{k} + N^a_{1,t-1}\left(1 - \frac{N_{0,t-1} + (1-\theta)N^a_{0,t-1}}{k}\right). \quad (54)$$

In steady state, either of these equations yields $N^a_0 N_1 = N^a_1 N_0$ or, using the expressions for N_0 and N_1, $N^a_0 M = N^a_1(1-M)$. This, with $N^a_0 + N^a_1 = \lambda$ implies $N^a_0 = \lambda(1-M)$, and $N^a_1 = \lambda M$.

Proof of lemma 2

Item 1 follows from

$$\frac{\partial W^a}{\partial c_{DB}} = \frac{1}{k(1-\beta)}\left[u'(c_{DB}) - 1\right]\lambda\left[1 - (1-\theta)M(1-M)\right] \quad (55)$$

and

$$\frac{\partial W^a}{\partial c_m} = \frac{1}{k(1-\beta)}\left[u'(c_m) - 1\right](1-\theta\lambda)M(1-M). \quad (56)$$

Item 2 follows from

$$\frac{\partial W^a}{\partial \lambda} = \frac{\left[u(c_{DB}) - c_{DB}\right]\left[1 - (1-\theta)M(1-M)\right] - \left[u(c_m) - c_m\right]\theta M(1-M)}{k(1-\beta)}. \quad (57)$$

Item 3 follows from

$$\frac{\partial W^a}{\partial \theta} = \frac{1}{k(1-\beta)}\lambda M(1-M)\left\{\left[u(c_{DB}) - c_{DB}\right] - \left[u(c_m) - c_m\right]\right\}. \quad (58)$$

Item 4 follows from

$$\frac{\partial W^a}{\partial M} = \frac{(1-2M)}{k(1-\beta)} \left\{(1-\theta\lambda)\left[u(c_m)-c_m\right] - (1-\theta)\lambda\left[u(c_{DB})-c_{DB}\right]\right\}. \tag{59}$$

Item 5 follows from

$$\frac{\partial W}{\partial c_m} = \frac{1}{k(1-\beta)}\left[u'(c_m)-1\right]\lambda M(1-M). \tag{60}$$

Item 6 is a consequence of

$$\frac{\partial W}{\partial M} = \frac{(1-2M)}{k(1-\beta)}\left[u(c_m)-c_m\right]. \tag{61}$$

Proof of lemma 3

Item 1 follows from

$$\frac{\partial (W^a - W)}{\partial c_{DB}} = \frac{\partial W^a}{\partial c_{DB}} \tag{62}$$

and

$$\frac{\partial (W^a - W)}{\partial c_m} = \frac{-\theta\lambda M}{k(1-\beta)}\left[u'(c_m)-1\right](1-M). \tag{63}$$

Item 2 follows from

$$\frac{\partial (W^a - W)}{\partial M} = -\frac{\lambda(1-2M)}{k(1-\beta)}\left\{(1-\theta)\lambda\left[u(c_{DB})-c_{DB}\right] + \theta\left[u(c_m)-c_m\right]\right\}. \tag{64}$$

Item 3 follows from

$$\frac{\partial (W^a - W)}{\partial \theta} = \frac{\partial W^a}{\partial \theta}. \tag{65}$$

Proof of lemma 4

Item 1 follows from

$$\begin{aligned}\frac{\partial (V_1^a - V_1)}{\partial c_{DB}} \quad = \quad & \frac{\lambda\left[u'(c_{DB})-1\right]}{k(1-\beta)} - \frac{\lambda(1-\theta)(1-\theta\lambda)\beta m_0 m_1\left[u'(c_{DB})-1\right]}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]} \\ & - \frac{\lambda(1-\theta)(1-\beta)m_0 u'(c_{DB})}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]},\end{aligned}$$

so that

$$\left.\frac{\partial (V_1^a - V_1)}{\partial c_{DB}}\right|_{u'(c_{DB})=1} = -\frac{\lambda(1-\theta)(1-\beta)m_0 u'(c_{DB})}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]} < 0. \tag{66}$$

Also,

$$\frac{\partial(V_1^a - V_1)}{\partial c_m} = -\frac{\theta\lambda\beta m_0 m_1 \left[u'(c_m) - 1\right]\left[(1-\beta)(2-\theta\lambda) + \beta\frac{1-\theta\lambda}{k}\right]}{(1-\beta)\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]} - \frac{\theta\lambda(1-\beta)^2 m_0 u'(c_m)}{(1-\beta)\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]},$$

so that

$$\left.\frac{\partial(V_1^a - V_1)}{\partial c_m}\right|_{u'(c_m)=1} = -\frac{\theta\lambda(1-\beta)^2 m_0 u'(c_m)}{(1-\beta)\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]} < 0. \tag{67}$$

Item 2 follows from

$$\frac{\partial(V_1^a - V_1)}{\partial M} = -\lambda(1-\theta)\frac{(1-\theta\lambda)\frac{\beta}{k^2}(1-2M)\left[u(c_{DB}) - c_{DB}\right] - (1-\beta)\frac{1}{k}u(c_{DB})}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]} - \theta\lambda\frac{\frac{\beta}{k^2}(1-2M)\left[u(c_m) - c_m\right]\left[(1-\beta)(2-\theta\lambda) + \beta\frac{1-\theta\lambda}{k}\right] - (1-\beta)^2\frac{1}{k}u(c_m)}{(1-\beta)\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]},$$

so that

$$\left.\frac{\partial(V_1^a - V_1)}{\partial M}\right|_{M=\frac{1}{2}} = \frac{\lambda(1-\theta)(1-\beta)\frac{1}{k}u(c_{DB})}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]} + \frac{\theta\lambda(1-\beta)^2\frac{1}{k}u(c_m)}{(1-\beta)\left[1-\beta+\frac{\beta}{k}\right]\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]} > 0. \tag{68}$$

Finally, for item 3, if $c_{DB} = c_m = c$, then

$$V_1^a - V_1 = \frac{\lambda}{k}\frac{u(c) - c}{1-\beta} + (1-\lambda)\left[\frac{(1-\lambda\theta)\,\beta m_0 m_1(u(c) - c) + (1-\beta)m_0 u(c)}{(1-\beta)\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]}\right] - \frac{\beta m_0 m_1(u(c) - c) + (1-\beta)m_0 u(c)}{(1-\beta)\left[1-\beta+\frac{\beta}{k}\right]}. \tag{69}$$

so that, we a little algebra, we get

$$\left.\frac{\partial(V_1^a - V_1)}{\partial\theta}\right|_{c_{DB}=c_m} = \frac{(1-\lambda)}{k}\beta\lambda m_0\frac{(1-M)u(c) + Mc}{\left[1-\beta+\beta\frac{1-\theta\lambda}{k}\right]^2} > 0. \tag{70}$$

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