Public vs. Private Offers in the Market for Lemons

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Abstract

We analyze a version of Akerlof's market for lemons in which a sequence of buyers make offers to a long-lived seller endowed with a single unit for sale. We consider both the case in which previous offers are observable and the case in which they are not. When offers are observable, trade may only occur in the first period, so that the resulting inefficiency may be worse than in the static model. In the unobservable case, trade occurs with probability one eventually.

JEL codes: C72, D82, D83

1 Introduction

In this paper, we examine the relationship between the outcome of traded with asymmetric information and the observability of past offers. Search models typically assume that successive potential buyers never learn anything over time about the offering of the seller, so that the distribution of offers faced by the seller is stationary. On the other hand, bargaining models usually assume that potential buyers observe the entire public history, including past offers that were rejected. This affects the buyers' beliefs about the quality of the good that is being put on sale, and therefore the offers that are submitted.

While most markets characterized by adverse selection fall between those two extremes, they widely vary in this respect. In the housing market, potential buyers typically know the time on market, as well as the list price that is quoted by the seller. Buyers of second-hand cars do not usually have any reliable information regarding what offers and how many offers have been turned down by the seller. Employers may obtain verifiable information about the duration of unemployment of potential employees, but much less evidence regarding offers they may have rejected in the meantime. In yet other markets, it is up to the seller to decide ex ante whether his decisions will be public or private.

We consider two variants of Akerlof's market for lemons. In both variants, a sufficiently patient single seller with private information faces a sequence of (short-run) potential buyers who submit offers. Buyers know how long the item has been put on sale. In the first variant, buyers also know the offers that were rejected, while in the second those offers remain unobservable. All that is known by a potential buyer is that all the previous offers must have been turned down.

Results contrast sharply. In the case of public offers, only the first potential buyer submits a serious offer. That is, only the first offer is accepted with positive probability. In case this offer is turned down by the seller, as does occur if his valuation is high enough, all later offers are losing offers: from that point on, the seller rejects all equilibrium offers. Therefore, merely allowing for trade over time does not solve the moral hazard problem. In fact, it may worsen it. Not so, however, in the case of private offers. In every equilibrium, trade occurs with probability one eventually, and there always exist equilibria in which trade occurs in finite time.

This striking contrast can be understood as follows. Consider the case of private offers. Suppose that an out-of-equilibrium high offer is submitted. Because turning it down is interpreted as very favorable news by the next buyer, whose offer following such a deviation will be serious, the seller accepts this high offer only if his valuation is especially low. That is, the perspective of selling at a higher price tomorrow exacerbates the adverse selection problem today, making the high offer unattractive to the buyer. While this intuition help explain why it is equilibrium behavior for all potential buyers but the first one to submit losing offers, it is worth pointing out that we show that this is the *unique* equilibrium behavior. In the case of private offers, the equilibrium need not be unique. For the sake of contradiction, suppose that from some point on, all equilibrium offers are losing. Because offers are unobservable, the behavior of future bidders is not affected by a deviation by the current bidder. Therefore, a potential buyer would strictly prefer an offer slightly below his worst-case valuation for the good to his equilibrium offer, as the former always implies a strictly positive profit conditional on trade, and it is necessarily accepted by the seller with some positive probability.

In terms of actual payoffs, comparisons are less clear-cut. For instance, the dynamic game with public offers may actually be more or less efficient than the static game, depending on the parameters. As we argue, the low probability of trade in the dynamic model is driven by competition among potential buyers. Somewhat paradoxically, if there is a unique, long-lived and equally patient potential buyer, the good is traded with probability one in the unique equilibrium outcome. To shed more light on the relationship between the static game and the infinite-horizon game, we provide a detailed analysis of the game with finite, but arbitrary, horizon.

While it is possible to explicitly solve for equilibrium strategies in the case of private offers, the case of public offers is more complex, and we provide only a partial characterization of the equilibrium strategies. We show by means of specific example that equilibrium multiplicity can occur, and we prove that, quite generally, all potential buyers but the first and the last ones must use mixed strategies. Explicit solutions are available in some special cases: (i) the case of two periods, (ii) the case of two values, (iii) some particular ranges of parameters.

1.1 Related Literature

Our contribution is related to three strands of literature. First, several authors have already considered dynamic versions of Akerlof's model. Second, our model shares many features with the bargaining literature. Finally, a pair of papers have investigated the difference between public and private offers in the framework of Spence's educational signaling model.

Janssen and Roy (2002) consider a dynamic, competitive durable good setting, with a fixed set of sellers. They prove that all trade for *all* qualities of the good occurs in finite time. While there are several inessential differences between their model and ours, the critical difference lies in the market mechanism. In their model, the price in every period must clear the market. That is, by definition, the market price must be at least as large as the good's expected value to the buyer conditional on trade, with equality if trade occurs with positive probability (this is condition (ii) of their equilibrium definition).[1] This expected value is derived from the equilibrium strategies when such trade occurs with positive probability, and is assumed to be at least as large as the lowest unsold value even when no trade occurs in a given period (this is condition (iv) of their definition). This immediately implies that the price exceeds the valuation to the lowest quality seller, so that trade must occur eventually, if not in a given period. Also related are Taylor (1999), Hendel and Lizzeri (1999), and Hendel, Lizzeri and Siniscalchi (2005).

In the bargaining literature, the closest paper is Evans (1989), which shows that, with correlated values, the bargaining may result in an impasse when the buyer is too impatient relative to the seller. Our assumption of short-run buyers is less general, since it is implies that the buyer is actually myopic.[2] However, Evans considers the case of binary values. Moreover, there is no gain from trade in case of a low value. In our set-up, Proposition 1 holds instead quite generally, and trade is always strictly efficient. In his appendix, Vincent (1989) provides another example of equilibrium in which bargaining breaks down. As in Evans, there are only two possible values in his object. While there are gains of trade for both values in his case, it is not known whether his example admits other equilibria, potentially exhausting all gains of trade eventually. Other related contributions include Cramton (1984), Gul and Sonnenschein (1988) and Vincent (1990). Other related contributions include Cramton (1984), Gul and Sonnenschein (1988) and Vincent (1990).

Nöldeke and van Damme (1990) and Swinkels (1999) develop an analogous distinction in Spence's signalling model. Both consider a discrete-time version of the model, in which education is acquired continuously and a sequence of short-run offers submit offers that the worker can either accept or reject. Nöldeke and van Damme considers the case of public offers, while Swinkels focuses mainly on the case of private offers. Nöldeke and van Damme shows that there is a unique equilibrium outcome that satisfies the never a weak best response requirement, and that

[1] More precisely, equality obtains whenever there is a positive measure of goods' qualities traded, since there is a continuum of sellers in their model.

[2] There is no difficulty in generalizing Proposition 1 to the case of an impatient, but not myopic buyer, but we feel that there is no much gain from such generality.

the equilibrium outcome converges to the Riley outcome as the time interval between consecutive periods shrinks. With private offers, Swinkels proves that the sequential equilibrium outcome is unique, and shows that, in contrast to the public case, it involves pooling in the limit. While the set-up is rather different, the logic driving these results is similar to ours, at least with public offers. Indeed, in both papers, when offers are observable, firms (buyers) are deterred from submitting mutually beneficial offers because rejecting such an offer sends a strong signal to future firms (buyers) that is so attractive that only very low types would prefer to accept the offer immediately.

2 The model

We consider a dynamic game between a single seller, with one unit for sale, and a countable infinity of potential buyers, or buyers for short. Time is discrete, and indexed by $n = 1, \ldots, \infty$. At each time n, one buyer makes an offer for the unit. Each buyer makes an offer only at one time, and we refer to buyer n as the buyer who makes an offer in period n, provided the seller has accepted no previous offer. After observing the offer, the buyer either accepts or turns down the offer. If the offer is accepted, the game ends. If the offer is turned down, a period elapses and it is the next buyer's turn to submit an offer.

The (reservation) value of the unit is seller's private information. The reservation value to the seller is $c(x)$, where the random variable x is determined by nature and uniformly distributed over the interval $[\underline{x}, 1]$, $\underline{x} \in [0, 1)$. We interpret x as an index, such as the quality of the good. The valuation of the unit to buyers is common to all of them, and is denoted $v(x)$. Buyers do not observe the realization of x, but its distribution is common knowledge. We assume that c and v are strictly positive, strictly increasing and differentiable, with bounded derivatives.[3] In fact, we shall need the slightly stronger assumption that v is strongly increasing over $[\underline{x}, 1]$, that is,

$$\inf_{x \neq x'} \frac{v(x) - v(x')}{x - x'} > 0.$$

Observe that the assumption that x is uniformly distributed is with little loss of generality, since few restrictions are imposed on the functions v and c.

We assume that gains from trade are always positive with $\inf_x v(x) - c(x) > \nu$ for some $\nu > 0$. In examples and extensions, we shall often restrict attention to the case in which $v(x) = x$ and $c(x) = \alpha x$, with $\underline{x} > 0$, i.e. the reservation value to the seller is a fraction $\alpha \in (0, 1)$ of the valuation x to the buyers. The seller is impatient, with discount factor $\delta < 1$. We are particularly interested in the case in which δ is sufficiently large, and lower bounds on δ play an important role in our analysis. In each period in which the seller owns the unit, he derives a per-period

[3] In fact, the main results (Propositions 1-4) hold provided only that v be strongly increasing and c be non-decreasing and Lipschitz continuous.

gross surplus of $(1-\delta)\,c\,(x)$. Therefore, the seller can always guarantee a gross surplus of $c\,(x)$ by never selling the unit.

Buyer n submits an offer p_n that can take any real value. An outcome of the game is a triple (x, n, p_n), with the interpretation that the realized index is x, and that the seller accepts buyer n's offer of p_n (implying that he rejected all previous offers). The case $n = \infty$ corresponds to the outcome in which the seller rejects all offers (set p_∞ equal to zero). The seller's von Neumann-Morgenstern utility functions over outcomes is his net surplus:

$$\sum_{i=1}^{n-1} (1-\delta)\,\delta^{i-1} c\,(x) + \delta^{n-1} p_n - c\,(x) = \delta^{n-1}\,(p_n - c\,(x))\,,$$

when $n < \infty$, and zero otherwise. An alternative formulation that is equivalent to the one above is that the seller derives no per-period gross surplus from owning the unit, but incurs a production cost of $c\,(x)$ at the time he accepts the buyer's offer. It is immediate that this interpretation yields the same utility function.

Buyer n's utility function over outcomes, given outcome (x, n, p_n), $n < \infty$, is equal to:

$$v\,(x) - p_n,$$

and the utility of all other buyers is set to zero. If $n = \infty$, all buyers' utility is zero. All players are risk-neutral. We define the players' expected utility over lotteries of outcomes, or payoff for short, in the standard fashion. We allow for mixed strategies on the part of all players.

We consider both the case in which offers are public, or observable, and the case in which previous offers are private, or not observable. It is worth pointing out that the results for the case in which offers are public would also hold for any information structure (about previous offers) in which each buyer $n > 1$ observes the offer made by buyer $n-1$.

A history $h^{n-1} \in H^{n-1}$ in case no agreement has been reached at time n is a sequence $(p_1, \ldots, p_{n-1})$ of offers that were submitted by the buyers and rejected by the seller (set H_0 equal to $\{\varnothing\}$). A strategy for the seller is a sequence of jointly measurable functions σ_S^n : $[\underline{x}, 1] \times H^{n-1} \times \mathbb{R} \to [0, 1]$, mapping the realized valuation v, the history h^{n-1}, and buyer n's offer p_n into a probability of acceptance. In the public case, a strategy for buyer n is a function $\sigma_B^n : H^{n-1} \to \mathcal{P}\,(\mathbb{R})$, mapping the history h^{n-1} into a probability distribution over offers, where the set $\mathbb{R}$ of real numbers is endowed with the Borel structure. In the private case, a strategy for buyer n is an element $\sigma_B^n \in \mathcal{P}\,(\mathbb{R})$. A sequential equilibrium is defined in the standard fashion. Given some sequential equilibrium, a buyer's offer is *serious* if it is accepted by the seller with positive probability. An offer is *losing* if it is not serious. Clearly, the specification of losing offers in a sequential equilibrium is to a large extent arbitrary. Therefore, statements about uniqueness are understood to be made up to the specification of the losing offers. Finally, an offer is a *winning* offer if it is accepted with probability one.

Observe that, whether offers are public or private, the seller's optimal strategy must be of the cut-off type. That is, if $\sigma_S^n\,(x, h^{n-1}, p_n) > 0$ for some v, then $\sigma_S^n\,(x', h^{n-1}, p_n) = 1$ for all $x' > x$.

The proof of this skimming property can be found in Fudenberg and Tirole (Chapter 10, Lemma 10.1), for instance. If $\sigma_S^n(x', h^{n-1}, p_n) = 0$ for all $x' < x$, and $\sigma_S^n(x', h^{n-1}, p_n) = 1$ for all $x' > x$, the valuation x is called the *marginal valuation* (at history (h^{n-1}, p_n) given the strategy profile). Since the specification of the action of the seller with marginal valuation does not affect payoffs, we also identify equilibria which only differ in this regard. For definiteness, in all formal statements, we shall follow the convention that the seller with marginal valuation rejects the offer so that the set of valuations still assigned positive probability following a rejection is a closed set, and for conciseness, we shall omit to specify that some statements only hold 'with probability one'. For instance, we shall say that the seller accepts the offer when he does so with probability one. Standard arguments also establish that buyers never submit any offer that is strictly larger than $c(1) = \bar{c}$, the highest possible reservation value to the seller.

The model considered by Akerlof (1970) is not quite the static version of this game, as the market mechanism adopted there is Walrasian. Much closer is the second variant analyzed by Wilson (1980), although he considers a continuum of buyers. We briefly sketch here the static equivalent of the dynamic game described above if there is only one potential buyer, who submits a take-it-or-leave-it offer. The game then ends whether the offer is accepted or rejected, with payoffs specified as before (with $n = 1$). Clearly, the seller accepts any offer p provided $p > c(x)$. Therefore, the buyers offers $c(x^*)$, where x^* maximizes

$$\int_{\underline{x}}^{x} (v(t) - c(x))\, dt,$$

over $x \in [\underline{x}, 1]$. More generally, given $t \in [\underline{x}, 1)$, let $x^*(t)$ denote the marginal valuation given the optimal offer when the distribution is uniform over $[t, 1]$. Observe that $x^*(t) > t$ for all $t \in [\underline{x}, 1)$.

3 Observable offers

3.1 Main result

In this section, we maintain the assumption that offers are public. Observe that the equation

$$\int_{x}^{1} (v(t) - \bar{c})\, dt = 0$$

admits either no or exactly one solution x in $[\underline{x}, 1)$ since its integrand is strictly positive (negative) above (below) the unique root of $v(t) = \bar{c}$. Indeed, if such a solution x_1 exists, it is an element of $[\underline{x}, 1)$. Obviously, this solution x_1 exists if and only if:

$$\int_{\underline{x}}^{1} (v(t) - \bar{c})\, dt < 0,$$

that is, if it is unprofitable for the first buyer to submit an offer that is accepted with probability one by the seller. More generally, given $1 =: x_0 > x_1 > \cdots > x_k$, define x_{k+1} as the unique solution in $[\underline{x}, x_k)$, if any, of the equation

$$\int_{x}^{x_k} (v(t) - c(x_k))\, dt = 0.$$

Clearly, this process must eventually stop. The resulting finite sequence $\{x_k\}_{k=0}^{K}$, $x_k \subset [\underline{x}, 1]$, all k, is easy to compute for special functions v and c. For instance, if $c(x) = \alpha v(x) = \alpha x$, $x_k = \beta^k$, where $\beta := 2\alpha - 1$. The sequence $\{x_k\}$ plays an important role in Proposition 1.

Proposition 1: Assume that $x_K > \underline{x}$. There exists a unique equilibrium outcome provided $\delta > \bar{\delta}$, for some $\bar{\delta} < 1$. On the equilibrium path, the first buyer submits the offer $c(x_K)$, which the seller accepts if and only if $x < x_K$. If the offer is rejected, all buyers $n > 1$ submit a losing offer.

Proof: The proof is by induction, and is broken into two steps.

We first claim that there exists some value $s_1 \in [\underline{x}, 1)$ such that, in any equilibrium, for any n, and any history h^{n-1} such that the marginal valuation is x, buyer n offers $\bar{c}$ if and only $x > s_1$(which the seller accepts with probability one). Given any history for which $x > s_1$, suppose that buyer n's equilibrium offer is accepted by the seller precisely if $t < s$, and let $p(s)$ denote this offer. Obviously, $p(s) \geq c(s)$. Therefore, for any s, buyer n's payoff $V_n(h^{n-1})$ must satisfy:

$$V_n\left(h^{n-1}\right) = (1-x)^{-1} \int_{x}^{s} \{v(t) - p(s)\}\, dt \leq (1-x)^{-1} \int_{x}^{s} \{v(t) - c(s)\}\, dt,$$

Observe that the integral is differentiable, with derivative $v(s) - c(s) - c'(s)(s-x) > \nu - (1-x)\sup_t c'(t)$, which is strictly positive provided x is close enough to one. Therefore, for x close enough to one, this upper bound is strictly increasing in s, so that it achieves a strict maximum at $s = 1$. Since the payoff is bounded above by this upper bound, and it is equal to this upper bound for $s = 1$, it follows that $s = 1$ is then also a strict maximum for the payoff. Thus, there exists a s_1 as claimed.

Next, we claim that $s_1 = x_1$. Suppose not. If $s_1 < x_1$, the contradiction is immediate, because offering $\bar{c}$ results in a net loss. So suppose $s_1 > x_1$, and consider an equilibrium in which for some n and some history h^{n-1}, the marginal valuation is $x := s_1 - \varepsilon > x_1$, $\varepsilon > 0$, and buyer n's offer is accepted by the seller precisely if $t < s$, for $s < 1$. As before, let $p(s)$ denote the corresponding offer.

Observe first that it cannot be that $s \in (s_1, 1)$. Indeed, since buyer $n+1$ then offers $\bar{c}$ whenever the offer is rejected, it must otherwise be the case that $p(s)$ solves $p(s) - c(s) = \delta(\bar{c} - c(s))$, so that buyer n's payoff is:

$$V_n\left(h^{n-1}\right) = (1-x)^{-1} \int_x^s \{v(t) - \delta\bar{c} - (1-\delta)c(s)\}\, dt,$$

which we argue is strictly convex in s on the interval $(s_1, 1)$. Indeed, the integral is differentiable in s, with derivative:

$$(v(s) - \delta\bar{c} - (1-\delta)c(s)) - (1-\delta)c'(s)(s-x).$$

As $\delta \to 1$, the numerator tends to $v(s) - \bar{c}$, which is strongly increasing. Therefore, there exists $\bar{\delta} < 1$ such that the numerator is strictly increasing in s provided $\delta > \bar{\delta}$.[4] It follows that $V_n\left(h^{n-1}\right)$ is strictly convex in s over $(s_1, 1)$, and cannot achieve a maximum there. Thus, $s \leq s_1$. This implies, however, that:

$$V_n\left(h^{n-1}\right) \leq \int_{s_1-\varepsilon}^{s_1} (v(t) - c(\underline{x}))\, dt \leq (v(1) - c(\underline{x}))\,\varepsilon,$$

which tends to 0 as $\varepsilon \to 0$. At the same time, since $s_1 - \varepsilon > x_1$, the payoff $V_n\left(h^{n-1}\right)$ is bounded away from 0, since buyer n can always offer $\bar{c}$. The contradiction follows by taking ε sufficiently small.

To conclude, we have shown that in any equilibrium, for any n and h^{n-1} such that the marginal valuation is strictly above x_1, buyer n offers $\bar{c}$, which is accepted with probability one by the seller.

Suppose that we have shown that, in any equilibrium, for any n and any h^{n-1} such that the marginal valuation is $x \in (x_k, x_{k-1}]$, buyer n offers $c(x_{k-1})$, and, if the offer is rejected, all future buyers submit losing offers. Suppose that $\underline{x} < x_k$, and consider now any equilibrium, any n and any h^{n-1} such that the marginal valuation is $x \in (x_{k+1}, x_k]$, $k < K$, or $w \in (\underline{x}, x_k]$ if $k = K$.

Suppose first that $x < x_k$. We claim that, if buyer n's offer is accepted precisely if $t < s$, then it must be that $s \leq x_k$. Indeed, the payoff that results from a choice of s in $(s_j, s_{j-1}]$, $j \leq k$, is strictly convex in s over each such subinterval (the corresponding price solving $p(s) - c(s) = \delta(c(x_{j-1}) - c(s)))$ by the same argument as above, and it is strictly negative for each possible choice $s = x_{j-1}$, $j \leq k$, by definition of x_j, given that $s < x_k \leq x_j$.

Suppose now that $s = x_k$, and assume that buyer $n-1$'s equilibrium offer p_{n-1} was a serious offer accepted if and only if $t < x_k$. We claim that buyer n and all subsequent buyers must submit a losing bid. Suppose not. Then it must be that $p_n > c(x_k)$. Without loss of generality, consider the history h^{n-1} for which p_n is maximized. More precisely, let $\bar{p}_n = \sup_{n,h^{n-1}} p_n$, and

[4] The conclusion would not necessarily hold if v were only assumed to be strictly increasing. This is the one place where the stronger assumption is needed.

pick (n, h^{n-1}) such that $|p_n - \bar{p}_n| < \varepsilon$ for $\varepsilon = (1-\delta)(\bar{p}_n - c(x_k))/2$. Observe that if buyer $n-1$ deviates and submits a serious offer accepted if and only if $t < s$, where $s < x_k$, the price $p(s)$ he must offer cannot exceed $c(s) + \delta(c(x_k) + \delta(\bar{p}_n - c(x_k)) - c(s))$, since the next serious offer itself cannot exceed $c(x_k) + \delta(\bar{p}_n - c(x_k))$ by the previous claim. Now, $p_{n-1} \geq \delta p_n + (1-\delta) c(x_k)$. Therefore, $\forall s < x_k$,

$$\begin{aligned}
p_{n-1} - p(s) &\geq \delta\bar{p}_n - \delta\varepsilon + (1-\delta) c(x_k) - (1-\delta) c(s) - \delta^2 \bar{p}_n - \delta(1-\delta) c(x_k) \\
&\geq \delta(1-\delta)(\bar{p}_n - c(x_k)) - \delta\varepsilon \\
&\geq \delta(1-\delta)(\bar{p}_n - c(x_k))/2,
\end{aligned}$$

implying that, for s close enough to x_k, such an offer yields a profitable deviation.

This establishes that, if $x < x_k$, then all future equilibrium offers must be rejected by the seller if $t > x_k$. Furthermore, buyer n can secure a payoff arbitrarily close to $\int_x^{x_k} (v(t) - c(x_k))\, dt$ by offering a price arbitrarily close to $c(x_k)$. The same argument as in the case $k = 0$ establishes that, if $x > s_{k+1}$, for some $s_{k+1} < x_k$ close enough to x_k, the payoff of any offer that is rejected by the seller if and only if $t < x$, for $x < x_k$, is strictly less than this supremum. From this it follows that an optimal offer, and therefore an equilibrium, only exists if indeed buyer n offers αv_k and all future buyers submit losing offers. The proof that s_{k+1} is in fact x_{k+1} is virtually identical to the case $k = 0$ and is omitted. ■

For completeness, let us briefly comment on the knife-edge case in which $\underline{x} = x_K$. Then as long as the marginal valuation is $\underline{x}$, any randomization over the offers $\{c(x_K), c(x_{K-1})\}$ is optimal, the payoff of either offer being zero. Because $\underline{x} = x_K$, equilibrium considerations do not uniquely 'pin down' the mixture, as is done in the proof above for the case $\underline{x} < x_K$ in which the marginal valuation is x_k, $k \leq K$, after an equilibrium offer that is serious. Indeed, the only reason why the equilibrium (as opposed to the equilibrium outcome) for the case $\underline{x} < x_K$ is not unique is that nothing pins down the behavior when the marginal valuation is x_k, $k \leq K$, *following an out-of-equilibrium offer*. Beyond this indeterminacy, the case $\underline{x} = x_K$ is identical to the case $\underline{x} < x_K$; in particular, along the equilibrium path, the seller will reject all offers provided $t \geq x_{K-1}$.

The comparison to the static case is immediate: if $\underline{x}$ is sufficiently close to x_K, then the probability of agreement is arbitrarily small, and the outcome is more inefficient than in the static case. On the other hand, if $\underline{x}$ is sufficiently small relative to x_K, then the probability of agreement is larger than in the static case, as it must be that $x^*(\underline{x}) < x_K$, since the payoff from offering $c(x_K)$ can be chosen arbitrarily close to zero.

3.2 Patient Single buyer

Proposition 1 assumes that each buyer makes only one offer. However, its proof goes through with a single, long-lived buyer provided the buyer's discount factor is small enough, fixing the seller's discount factor. On the other hand, the result is no longer valid if the long-lived buyer and seller share the same discount factor $\delta < 1$. In that case, we know from Vincent (1989) that there exists a (genericallly) unique Perfect Bayesian Equilibrium, and that, in this equilibrium, bargaining ends after a finite sequence of serious offers, one of which is accepted. Furthermore, Vincent exhibits an example with binary values in which delay does not vanish as the time interval between successive offers tends to zero.[5]

For sake of illustration, we describe here the equilibrium in the case in which $c(x) = \alpha v(x) = \alpha x$. Define the sequences:

$$x_0 = 0,\ x_{n+1} = \frac{\alpha}{1-\alpha} + \frac{\delta x_n^2}{x_{n-1}};\ z_0 = 1,\ z_n = \prod_{k=1}^{n} \frac{x_k - 1}{x_k}.$$

We show in appendix that there exists a unique equilibrium of the game with a long-lived buyer with discount factor δ. With probability one, agreement is reached in finite time. If the marginal valuation is $s \in [z_{n+1}, z_n)$, the buyer offers

$$p = (1-\delta)(1-\alpha)\frac{x_n^2}{x_n - 1}s + \delta^n \alpha,$$

which the seller accepts if and only if:

$$t < \frac{x_n}{x_n - 1}s.$$

The expected payoff of the buyer is then:

$$\frac{1}{2}\left((1-\delta)(1-\alpha)\frac{x_n^2}{x_n - 1} - 1\right)s^2 + \delta^n \alpha s - \frac{2\alpha - 1}{2}\delta^n.$$

We solve here for the case in which F is the uniform distribution. All proofs are gathered in Appendix.

In fact, the maximal number of offers in equilibrium, N, converges as $\delta \to 1$, so that, as the time interval between successive offers tends to zero, agreement is immediate, contrasting with the binary example studied by Vincent (1989). [To see this, observe that, for all n, the value of x_n tends to a well-defined limit strictly larger than one, and therefore $z_{n+1} - z_n$ tends to a strictly positive limit; given $\underline{x}$, it then follows that, for δ large enough, the duration of bargaining is independent of δ.]

3.3 Finite Horizon

Proposition 1 implies that all buyers but the first one submit losing offers. Yet when the game has finite horizon, this conclusion is blatantly false when $\underline{x} < x_1$. In particular, if the seller

[5] In fact, Vincent (1989) proves this result more generally for the case in which the buyer is at least as patient as the seller.

rejects all previous offers with positive probability, the last buyer must submit a serious offer. Indeed, his problem reduces then to the static case, for some specific $\underline{x}$. Whether agreement is reached with probability one before the last buyer, or the last buyer submits a serious bid, the qualitative conclusions of the finite horizon game seem to cast some doubt on the pertinence of Proposition 1. Therefore, the analysis of the game of the finite-horizon is not only an important extension that includes the static benchmark as a special case, but also a robustness test: as the length of the horizon increase, do the equilibria of the game with finite-horizon converge to the infinite-horizon equilibrium? For simplicity, we state our results here only for the case in which $v(x) = x$ and $c(x) = \alpha x$, with $\underline{x} > 0$, only their extension to the case of general functions is immediate.

Proposition 2: The equilibrium strategies of the finite horizon game converge pointwise to the equilibrium strategies in the infinite-horion game, as the length goes to infinity. In particular, the Perfect Bayesian equilibrium is essentially unique, and is in pure strategy. In that equilibrium, the strategy of buyer i is associated with thresholds $0 < s_i^0 < s_i^1 < \cdots < s_i^{k_i} = 1$. The strategy σ_i has the following form:

- if $\underline{v}_i \leq s_i^0$, buyer i offers a price $b_i\underline{v}_i$, and attracts all types up to $c_i\underline{v}_i$ for some $c_i \geq 1$ and $b_i \geq \alpha$. Thus, $\sigma_i(\underline{v}_i) = c_i\underline{v}_i$.
- if $\underline{v}_i \in (s_i^k, s_i^0)$, $\sigma_i(\underline{v}_i) = s_{i-1}^{l_k}$ for some l_k: buyer i offers a price which does not depend on the specific value of $\underline{v}_i$ in that interval, and attracts all types up to $s_{i-1}^{l_k}$.

4 Unobservable offers

In this section, we maintain the assumption that all offers are unobservable, or private. The main result is the following.

Proposition 3: There exists $\bar{\delta} < 1$, for all $\delta > \bar{\delta}$, in all sequential equilibria of the infinite horizon game with private offers, trade occurs with probability one eventually.

Proof: Given $x \in [\underline{x}, 1]$ and some equilibrium, let $F_n(x)$ denote the (unconditional) probability that the seller is of type $t \leq x$ and has rejected all offers submitted by buyers $i = 1, \ldots, n-1$. Suppose for the sake of contradiction that trade does not occur with probability one eventually, i.e. $\lim_{n\to\infty} F_n(x) \neq 0$ for some $x < 1$. Let $F = \lim_{n\to\infty} F_n$, and define $x_\infty := \inf\{x : F(x) > 0\}$. By assumption, $x_\infty < 1$. Let $x^* = \inf\{x : c(x) = v(x_\infty) - \varepsilon/2\}$ ($x^* := 1$ if $c(1) < v(x_\infty) - \varepsilon/2$), and let $x' := (x^* + x_\infty)/2$. Since c' is bounded, it follows that there exists some $v > 0$ such that $x' > (1+v)x_\infty$. Observe that, given the definition of F, there exists $N \in \mathbb{N}$ such that, for all $n > N$,

$$F_n((1+v/2)x_\infty) - F_n(x_\infty) > 2v(1)(F_n(x_\infty) - F_n(\underline{x}_n))/\varepsilon,$$

where $\underline{x}_n := \inf\{x : F_n(x) > 0\}$. Pick any $n > N$. Consider an offer equal to $c((1+v)x_\infty)$. Either such an offer is accepted, in which case player n's (unconditional) equilibrium payoff is at least $\left(F_n\left((1+v/2)\,x_\infty\right)-F_n\left(x_\infty\right)\right)\varepsilon/2$. In this case, player n's lowest offer must strictly exceed x_∞, a contradition given the definition of x_∞. Or such an offer is rejected, which means that the discounted (total) probability that at least as high an offer is submitted in the future must be at least $\left(c\left((1+v)\,x_\infty\right)-c\left((1+v/2)\,x_\infty\right)\right)/\left(\bar{c}-c\left((1+v/2)\,x_\infty\right)\right)$. Since this must be true for all such n, it follows that $F_n\left((1+v/2)\,x_\infty\right)\to 0$, contradicting the definition of x_∞.■

Thus, offers that are accepted by seller's types arbitrarily close to one are eventually submitted. Further, the next lemma shows that offers that are accepted by *all* types must be submitted with positive probability. To do so, we introduce the following notation. As before, given some equilibrium, $F_n(x)$ denote the (unconditional) probability that the unit is of index $t\le x$ and that the seller has rejected all offers submitted by buyers $i=1,\dots,n-1$. Set $\underline{x}_n := \inf\left\{x : F_n(x)>0\right\}$. Also, let S_n denote the set of marginal types corresponding to the set of offers in the support of buyer n's strategy. That is, $x\in S_n$ if and only if there exists an offer p_n submitted by buyer n with positive probability that the seller accepts if and only if his type is less than or equal to x.

Lemma 1: There exists $\bar{\delta}<1$, for all $\delta>\bar{\delta}$, in all sequential equilibria of the infinite horizon game with private offers, some buyer submits a winning offer with positive probability: there exists $N\in\mathbb{N}$ such that $\max S_N=1$.

Proof: From Proposition 3, we know that $\lim_n F_n(1)=0$. Fix some N such that $F_N(1)<\varepsilon/\bar{c}'$, where $\bar{c}'>0$ is an upper bound on the derivative of c over $[\underline{x},1]$. Let:

$$\tilde{V}_n(x) := \int_{\underline{x}}^{x}\left\{v(t)-c(x)\right\}dF_n(t)\,.$$

Observe that $\tilde{V}_n(x)$ is an upper bound to the (unconditional) payoff buyer n obtains when submitting an offer for which x is the marginal type, with equality if and only if either it is a losing offer ($F_n(x)=0$), or a winning offer ($x=1$). For sake of contradiction, suppose that the lemma's statement is false. Therefore, the interval $[T_N,1]$, where $T_N := \max_{n<N}\max S_n$, has nonempty interior, and the function $F_N(x)$ is differentiable on this interval, with derivative equal to one. Hence, on that interval, $\tilde{V}_N$ is differentiable and its derivative equals:

$$\tilde{V}_N'(x)=v(x)-c(x)-F_N(x)\,c'(x)\,,$$

which is strictly positive. Therefore $\tilde{V}_N$ is strictly increasing on $(T_N,1)$, so that buyer n's payoff cannot be maximized on this open interval. Hence, any such buyer can only submit either a winning offer -which is ruled out by assumption- or offers for which the marginal type is less than T_N, implying that such all marginal types are bounded away from one, contradicting Proposition 3.■

We will see later that the conclusion of the lemma can be further strengthened. Indeed, either some buyer submits a winning offer with probability one, or infinitely many buyers each submit a winning offer with probability bounded away from zero.

Let $N := \min\{n \in \mathbb{N} : V_n(1) \geq 0\}$. By the previous lemma, such an integer exists. Observe that buyer N need not actually submit a winning offer with positive probability.

Lemma 2: All buyers $n < N$ submit a serious offer with positive probability.

Proof: Suppose that player $n < N$ does not submit a serious offer w.p.p. (i.e., with positive probability). This implies, in particular, that his payoff is zero. Observe that some player $n' > n$ must submit a serious offer w.p.p., by Proposition 2. Let n' be the first such buyer, and let x' be the marginal type indifferent between accepting and rejecting the highest offer among the offers buyer n' may submit. Observe that the price that buyer n needs to submit so that s' is indifferent between accepting and rejecting this offer is strictly smaller than the price n' needs to submit. Observe also that the expected value of the unit, conditional on an offer being accepted by all seller's types less than x' is the same for buyer n and n'. Therefore, buyer n gets a strictly positive payoff from submitting such an offer, a contradiction. ■

The next lemma is repeatedly used in the sequel.

Lemma 3: Suppose that $\min S_{n+1} > \underline{x}_{n+1}$. Then there exists $\bar{\delta} < 1$, for all $\delta > \bar{\delta}$, $S_n \cap (\underline{x}_n, \min S_{n+1}) = \emptyset$.

Proof: For $x > \underline{x}_n$, let $p_n(x)$ denote the offer that buyer n must submit for type x to be the marginal type. For $x < \min S_{n+1}$, the hypothesis of the lemma implies that $p_n(x) = (1-\delta)c(x) + C$ for some constant C. The (unconditional) payoff of buyer n from submitting an offer for which the marginal type is x is given by:

$$\int_{\underline{x}}^{x} (v(t) - p(x))\, dF_n(t).$$

Clearly, if $v(x) = p(x)$ for some $x > \underline{x}_n$, offering $p(x)$ results in a strict loss, so in order to find the maximizers of this payoff, we can restrict attention to x such that $v(x) > p(x)$. Since c' is bounded and v is strongly increasing, this implies that we can restrict attention to x above some threshold $\bar{x}$, provided δ is sufficently close to one. Since F_n is convex (given the cream-skimming property), it is piecewise continuously differentiable on $(\underline{x}_n, \min S_{n+1})$, with derivative, on each subinterval, given by:

$$\frac{dF_n(x)}{dt}(v(x) - p(x)) - (1-\delta)\, c'(x)\, F_n(x),$$

which is strictly increasing for all δ sufficiently large, since v is strongly increasing, while F_n is convex. Therefore, the payoff is strictly convex on each of those subintervals, a result that holds more generally over $(\bar{x}, \min S_{n+1})$ since at all points this interval, $D_-F_n(x) < D_+F_n(x)$. Therefore, this payoff is first strictly negative and then strictly convex in x, so that it admits no maximum in this interval. ■

An immediate consequence of Lemma 3 is that no buyer before N, with the possible exception of the first buyer, uses a pure strategy.

Lemma 4: There exists $\bar{\delta} < 1$, for all $\delta > \bar{\delta}$, no buyer $n = 2, \ldots, N-1$ uses a pure strategy.

Proof: Suppose that buyer $n = 2, \ldots, N-1$ uses a pure strategy. Choose δ close enough to one so that Lemma 2 and 3 are valid. By Lemma 2, buyer n's unique offer must be serious. Let $x < 1$ denote the marginal type indifferent between accepting and rejecting this offer. By Lemma 3, buyer $n-1$ cannot submit any offer for which the marginal type would be in $(\underline{x}_n, x)$. This implies that the expected value conditional on the marginal type being equal to x is the same both for buyer n and buyer $n+1$. Since buyer n's offer is serious, buyer $n-1$ must thus submit a losing offer with positive probability, and therefore have a zero equilibrium payoff. However, the offer he must submit such that the marginal type equals x is strictly less than the unique offer submitted by buyer n (because of discounting), a contradiction. ■

Given some equilibrium, let $N' := \inf \{n \in \mathbb{N} : \min_n S_n = 1\}$, $N' := \infty$ if $\min_n S_n < 1$ for all n.

Lemma 5: There exists $\bar{\delta} < 1$, for all $\delta > \bar{\delta}$, $S_n \subset \{\underline{x}_N, 1\}$, for all $n = N, \ldots, N'$. Moreover, if $N > 1$, $N' > N$.

Proof: Consider the first statement. For the sake of contradiction, suppose that buyer $n \in \{N, \ldots, N'\}$ submits w.p.p. an offer for which the marginal type is $x \in (\underline{x}_N, 1)$. Then buyer $n+1$ cannot submit a winning offer with probability one, given Lemma 3, and thus both buyer $n+1$ and buyer $n+2$'s (unconditional) payoffs are strictly positive. This implies that $\min S_{n+2} > \underline{x}_{n+2} \geq \min S_{n+1} > \underline{x}_{n+1}$, which contradicts Lemma 3. Consider the second statement. Suppose that $N' = N$. Then by Lemma 3, $S_{N-1} \subset \left\{\underline{x}_{N-1}, 1\right\}$, so that the expected value of the unit, conditional on submitting a winning offer $\bar{c}$, is the same for buyer $N-1$ and buyer N, violating the definition of N. ■

Therefore, for discount factors close enough to one, we can break down any equilibrium into two phases.

- in the first phase (buyers 1 through $N-1$), all buyers but the first one use a (nondegenerate) mixed strategy, and all submit a serious offer with positive probability.

- in the second phase (buyers N through N'), buyers use a (possibly degenerate) randomization over $\{\underline{x}_N, 1\}$.

This raises several questions.

First, when is $N > 1$? If $\underline{x} < x_1$, where x_1 is as defined in the observable case, $N > 1$, since submitting a winning offer results in a strict loss for the first buyer. More generally, if $\underline{x} < x_k$, then $N > k$. On the other hand, the same argument as in the proof of Lemma 5 (first statement) yields that $N = 1$ if $\underline{x} \geq x_1$, and in fact, buyer 1 submits a winning offer with probability one if $\underline{x} > x_1$.

Second, does an equilibrium necessarily exist? The answer is positive, and follows from Glicksberg's fixed point theorem. As the set-up is not quite static, we record the result here and provide a proof in appendix.

Proposition 4: A (mixed-strategy) equilibrium exists.

Third, is an equilibrium necessarily unique? We shall show that the answer is negative, by means of example.

Third, is S_n finite, for $n < N'$? The analysis from the two-period case (see next section) suggests otherwise, although it may still be true that is S_n is finite for *some* equilibrium.

Fourth, is N' finite? We claim that there always exist an equilibrium in which N' is finite, but also that there always exists an equilibrium in which N' is infinite.

Lemma 6: There exists $\bar{\delta} < 1$, for all $\delta > \bar{\delta}$: an equilibrium exists in which N' is finite; further, if $\underline{x} \leq x_1$, an equilibrium exists in which N' is infinite; if $\underline{x} > x_1$, $N' = 1$ in every equilibrium.

Proof: Consider any equilibrium for δ close enough to one, so that Lemma 1-5 hold. Let μ_n denote the probability that buyer n submits a losing offer, along that equilibrium. Pick $\rho \geq 1$ and N' such that:

$$\sum_{n=N}^{N'} \left(\prod_{j=N}^{n-1} \rho \mu_j \right) \delta^{n-N} (1 - \rho \mu_n) + \left(\prod_{j=N}^{N'-1} \rho \mu_j \right) \delta^{N'-N} = \sum_{n=N}^{\infty} \left(\prod_{j=N}^{n-1} \mu_j \right) \delta^{n-N} (1 - \mu_n),$$

and $\rho \mu_n \leq 1$, for $n < N'$. That is, from the point of view of any buyer $n < N$, the discounted probability that a winning offer is submitted is the same in the equilibrium than for the strategy profile that only differs from the equilibrium in the probabilities with which buyers $n \geq N$ submit winning offers, namely $1 - \rho \mu_n$ rather than $1 - \mu_n$ if $n < N'$, and 1 rather than $1 - \mu_n$ if $n \geq N'$. Therefore, the payoff function of any such buyer has not changed, so that his original strategy remains optimal. From the point of view of any buyer $n \geq N$, the discounted probability that a winning offer is submitted (after them) has weakly increased, so that the payoff from any offer they submit has weakly decreased. This payoff remains the same, however, for the winning offer $\bar{c}$, so that their original strategy remains optimal as well. The second statement follows a similar construction, while the third was explained in the text. ■

Observe that the different equilibria exhibited in the proof of Lemma 6 are payoff-equivalent, from the point of the view of the seller. Nevertheless, the examples of multiple equilibria given in the next subsection show that this need not be the case for all equilibria. However, it seems important to study payoffs and expected delay.

Lemma 7: There exists $\bar{\delta} < 1$, for all $\delta > \bar{\delta}$, the payoff of all buyers $n \geq N$ is zero. Moreover, if $\underline{x} < x_1$, for all n, and the unconditional payoff of any buyer is $O\left((1-\delta)^2\right)$, while the conditional payoff of any buyer is $O(1-\delta)$. Further, for each $M \in \mathbb{N}$, there exists $\bar{\delta} < 1$, for all $\delta > \bar{\delta}$ and all n, if buyer n's payoff is positive, then the payoff of all buyers $n' = n-M, \ldots, n-1, n+1, \ldots, n+M$ is zero.

Proof: To be completed. ■

While it seems natural to ask whether all buyers but possibly the first one make zero profit in some or all equilibria, we have been unable to answer this question.

Finally, Lemma 8 addresses the issue of expected delay.

Lemma 8:

4.1 Conjectures and Numerical Evaluations

We have been unable to explicitly solve for the equilibria of the game, except in special cases discussed below. As shown above, any equilibrium can be partitioned into two "phases". Provided that the first player's payoff from submitting an offer accepted with probability one is negative, there must be a first buyer (say, player $N > 1$) who must be precisely indifferent between submitting a losing offer or submitting an offer accepted with probability one. From this point on, all buyers randomize over those kinds of offers, and as long as the latter offer is not submitted with probability one by some buyer, the conditional beliefs of the seller do not change any more. The first phase (up to player $N - 1$) is more complicated. In particular, all buyers but the first must use a mixed strategy, with strictly positive probability assigned to at least two offers. While it is possible to rule out many configurations with simple considerations, a wide range of possibilities remain; in fact, simple examples of multiple equilibria can be constructed.

The simplest conjecture consistent with our partial characterization is that each buyer randomizes over two offers, the lower of the two being a losing offer. It is then easy to show that, in fact, the other offer is serious, and is accepted by all types up to x_i, where x_i is strictly increasing in i for $i < N$. (This randomization must be strict up to buyer $N - 1$).

From numerical simulations, it appears that such equilibria exists for all α and δ, but *only if* $\underline{x}$ is sufficiently high. See figure 1. For lower values of $\underline{x}$, this does not work, because buyer 2 strictly gains from submitting an offer accepted with small but positive probability. This problem can be remedied by assuming instead that buyer 2's lower offer is serious as well, so that only the low offer of buyers $n \geq 3$ is a losing offer (in this revised conjecture, buyer 2's payoff is still zero). Such equilibria exist, and indeed, they can be constructed for lower values of $\underline{x}$ than is consistent with the first conjecture. However, it is again necessary that $\underline{x}$ be sufficiently high, for otherwise the same problem arises with buyer 4.

It seems therefore natural to amend the conjecture further, by considering the case in which, for a subsequence of the buyers in the first phase, the low offer is serious, while it is losing for the others. (It is easy to see that no two consecutive buyers can belong to that subsequence). Unfortunately, the resulting systems of equations is untractable, even numerically.

To be completed.

References

[1] Akerlof, G., 1970. The Market for 'Lemons': Qualitative Uncertainty and the Market Mechanism, *Quarterly Journal of Economics*, **84**, 488-500.

[2] Cramton, P., 1984. Bargaining with Incomplete Information: An Infinite-Horizon Model with Continuous Uncertainty, *Review of Economic Studies*, **51**, 573-591.

[3] Evans, R., 1989. Sequential Bargaining with Correlated Values, *Review of Economic Studies*, **56**, 499-510.

[4] Fudenberg, D. and J. Tirole, 1991. *Game Theory*. Cambridge, MA: MIT Press.

[5] Gul, F. and H. Sonnenschein, 1991. On Delay in Bargaining with One-Sided Uncertainty, *Econometrica*, **56**, 601-611.

[6] Hendel, I. and A. Lizzeri, 1999. Adverse Selection in Durable Goods Markets, *American Economic Review*, **89**, 1097-1115.

[7] Hendel, I., Lizzeri, A. and M. Siniscalchi, 2005. Efficient Sorting in a Dynamic Adverse-Selection Model, *Review of Economic Studies*, **72**, 467-497.

[8] Janssen, M.C.W., and S. Roy, 2002. Dynamic Trading in a Durable Good Market with Asymmetric Information, *International Economic Review*, **43**, 1, 257-282.

[9] Nöldeke, G. and E. van Damme, 1990. Signalling in a Dynamic Labour Market, *Review of Economic Studies*, **57**, 1-23.

[10] Swinkels, J., 1999. Educational Signalling with Preemptive Offers, *Review of Economic Studies*, **66**, 949-970.

[11] Taylor, C. 1999. Time-on-the-Market as a Signal for Quality, *Review of Economic Studies*, **66**, 555-578.

[12] Vincent, D., 1989. Bargaining with Common Values, *Journal of Economic Theory*, **48**, 47-62.

[13] Vincent, D., 1990. Dynamic Auctions, *Review of Economic Studies*, **57**, 49-61.

[14] Wilson, C., 1980. The Nature of Markets with Adverse Selection, *Bell Journal of Economics*, **11**, 108-130 .