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The dynamics of city formation: finance and governance*

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Abstract:
This paper examines city formation in a country whose urban population is growing steadily over time, with new cities required to accommodate this growth. In contrast to most of the literature there is immobility of housing and urban infrastructure, and investment in these assets is taken on the basis of forward-looking behavior. In the presence of these fixed assets cities form sequentially, without the population swings in existing cities that arise in current models, but with swings in house rents. Equilibrium city size, absent government, may be larger or smaller than is efficient, depending on how urban externalities vary with population. Efficient formation of cities with internalization of externalities involves local government intervention and borrowing to finance development. The paper explores the institutions required for successful local government intervention.

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I: Introduction

Understanding city formation and the financing requirements of cities is critical to effective policy formulation in developing countries that face rapid urbanization. The rapid growth of urban populations in developing countries is well known, but what is less well known is the growth in the number of cities. Between 1960 and 2000 the number of metro areas over 100,000 in the developing world grew by 185%, i.e. almost tripled (Henderson and Wang, 2005). Moreover the UN's projected two billion person increase in the world urban population over the next 45 years ensures this growth in city numbers will continue. How do we start to think about whether the proliferation of cities in developing countries is following an efficient growth path, and how policies may assist or constrain achievement of better outcomes?

We start with two fundamental premises which define the research agenda. The first is that city formation requires investment in non-malleable, immobile capital, in the form of public infrastructure, housing, and business capital. Owner-occupied housing capital is immobile and long lived, depreciating at a gross rate well under 1% a year and a net rate after maintenance of almost zero. Urbanization also involves heavy investment in roads, water mains, sewers and the like that are immobile and depreciate slowly. The second premise is that, in developing countries, a key local public finance need is for cities to tax and to borrow, and/or to use central government funds to finance infrastructure investments and subsidize the development of industrial parks so as to attract businesses (World Bank, 2000).

Why does immobility of capital matter to the analysis of city formation? Consider a context in which the urban population of a country is growing steadily with ongoing rural-urban migration, as resources shift out of agriculture into urban industrial production. In most of the literature city formation is modelled assuming perfect mobility of all resources (Henderson, 1974 and Anas 1992). Under this assumption, the analysis of the city formation process has three unsatisfactory aspects, at odds with the data. The first is that initial urbanization is characterized by huge swings in population of cities. In a country with just one type of city, urbanization proceeds by the first city growing until at some point a second city forms, with the timing depending on the details of the city formation mechanism and institutions. Regardless of that timing, when a second city forms the first city loses half its population who migrate instantly to that second city. Then the first city resumes growth and the second city grows in parallel until a third city forms, at which point both existing cities lose 1/3 of their population who migrate to this new third city. And so on. Of course, when the number of cities gets very large, the population swings of established cities may diminish. However, that still leaves the second unsatisfactory aspect to this process: when new cities form in this context, they jump instantly to some large size, rather than grow over time to a steady state size.

The third unsatisfactory aspect concerns outcomes under likely institutional environments. With perfect mobility of resources, an efficient city formation process requires the intervention of "large agents" such as city governments or large scale land developers who, through subsidies and zoning, co-ordinate the en masse movement of population from old cities into a new city. Many developing countries lack the local institutions required to co-ordinate en mass movements of population. With no large agents and with perfect mobility of resources, there is massive coordination failure in these models. New cities only form when existing cities become so enormously oversized that living conditions deteriorate to the point where any individual would be better off defecting from existing cities to form a "city" of size one. In these models national urban population growth brings about this dismal Malthusian outcome of enormously oversized cities, a not particularly plausible result.

This paper will start at the opposite extreme to the existing literature and assume that there are substantial sunk capital costs of housing and associated urban infrastructure. Forward-looking agents, in our case housing builders, anticipate income streams that will be earned in new and in old cities and make investment decisions accordingly. These assumptions ensure that there are neither swings nor jumps in urban population, and forward-looking behavior also resolves the coordination problem.

Our setting is an economy with a single final good and with a continuous flow of migrants moving out of agriculture into cities. Each new city starts off small and grows through rural to urban migration until the next new city becomes the target of migrants. In the base case we have pure sequential growth, where first one city grows to its final stationary size, then a second city starts from zero population and grows continuously to its stationary size, and then a third and so on. Extending the base case to include heterogeneity across cities and technological progress, we show that, while cities still grow in sequence from scratch to some level, they also experience later repeated episodes of further growth, in parallel with other cities. We present evidence in section 5.2 that urban growth in countries follows this broadly sequential pattern.

While having immobility of capital in our model gives patterns of city formation that are more consistent with "reality", the imposition of this key assumption has further interesting economic implications. Immobility affects how housing prices within cities vary over time, as growth proceeds. When a new city forms the prices of fixed assets in old cities may adjust in order to maintain occupancy in both new and old cities. This analysis of how asset prices vary within existing cities as a new city grows is a fundamental insight of the paper, as is the key role of competitive housing builders in the city formation process.

Outcomes depend on the institutional regime under which urbanization occurs. We start the paper by analyzing the benchmark case of socially efficient city, and show that efficient city size is larger than in models in which resources are perfectly mobile. We then look at equilibrium city formation under a regime in which there are no 'large agents' or city governments – all individuals are price-takers in all markets. The simple coordination failures that arise in perfect mobility models of city formation without city governments do not occur here. Agents are forward looking, correctly anticipating population flows and housing market conditions in new and old cities. In this context investment in durable capital provides a commitment device for competitive builders to solve coordination failure problems. However, these agents can't internalize city externalities, so equilibrium city size may be larger or smaller than socially optimal depending, in an intuitive manner, on the way in which externalities vary with city size. We then turn to city formation with large agents: private developers or public city governments. Such cities borrow in order to offer subsidies to attract migrants during the period in which urban scale economies are not fully developed. As we will see, timely formation requires development of institutions governing land markets, leases, and taxation.

The equilibria under the two market regimes, the first without and the second with city governments, generate different income streams and housing price paths. As such, they yield predictions about how housing price paths would differ across economies operating under the two different regimes. For the regime without city governments, rental prices are constant in growing cities. However rental and asset prices vary in stationary cities, starting high when another new city first starts to grow, then declining, and then rising again towards the end of the growth interval of the new city. That in principle is a testable hypothesis, where one could compare price paths in more stationary cities in developing countries with those in the current fastest growing city. In the second case, with fully functional city governments, housing prices are invariant across cities, whether growing or stationary. As such, one role of city governments with full tax powers is to use their tax, tax exemption and subsidy powers to smooth out realized incomes, and as a result also smooth housing price cycles. In summary, for developing countries with rapid new city formation, the paper will predict housing price cycles in stagnant cities as new cities grow under one set of institutional arrangements, but not under the other.

This idea of looking at housing price cycles is supported empirically by Glaeser and Gyourko (2005), who examine the USA in the modern era in which the country is fully urbanized. They look at the impact of city specific economic shocks. They find positive shocks are associated with strong population growth of receiving cities, but have fairly modest effects on housing prices in those cities. Cities that experience negative shocks have very sharp price declines but rather modest or even zero population effects. While our situation is different – the driving force is urbanization in developing countries not shocks in an urbanized world – the implication is the same. In the face of growth of another city, stationary cities may retain population through price changes, while growing cities have stable prices.

In terms of other relevant theoretical literature, there are growth models with city formation (e.g. Black and Henderson 1999 and Rossi-Hansberg and Wright 2004), but they assume that cities form with perfect mobility of all resources and without a local public sector that borrows to finance development. Incorporating immobility and financing considerations requires a different approach. The effect of having durable, immobile capital on single city growth has been tackled in Brueckner (2000). However, the only papers that examine new city formation as the population grows with durable capital are a thesis chapter of Fujita published in 1978 and Cuberes (2004). Fujita examines planning, but not market solutions. Cuberes in a paper written simultaneously and independently of ours has an empirical focus, with a motivating model that has only two cities in total that ever form in the economy. Cuberes doesn't analyze the role of institutions in driving different types of equilibria, housing price cycles, and the general topics in this paper. This paper develops a model of city formation under immobility of capital, building on Venables (2005) who illustrates that population immobility will affect the city formation process.

There is a complementary paper on city formation with durable capital by Helsley and Strange (1994) in which large land developers form cities simultaneously in a static context, using durable capital as a strategic commitment device. We have a dynamic context and for much of the paper there are no large land developers; but the Helsley and Strange paper reinforces the idea that durable capital functions as a commitment device to overcome coordination failure.

2. The model

In order to isolate the key elements in the urbanization process, we make four simplifying assumptions. First we assume a small open economy where agents can borrow and lend at a fixed interest rate δ in world capital markets. We do not embed the process in a closed economy model with capital accumulation and an endogenous interest rate. Second, we assume that the urban sector grows in population by a constant amount, ν, each instant, as if there were a steady stream of population out of agriculture and into the urban sector. Constancy of this rate is not critical to the concepts developed in the paper. For example, having the migration rate to cities respond to rural-urban income differences would affect the rate of population flow into cities and affect our precise calculations of the rate of cities' population growth. But it would not affect the process of how new cities form or the analysis of policies and institutions. Third, in the base case, we abstract from ongoing technological change which would tend to increase equilibrium and efficient city sizes over time. Finally to derive the key results, we assume that all cities form under identical circumstances – technology, amenities, and industrial composition. While these four assumptions define our base case, in section 5.1 we demonstrate the implications of relaxing them, and show how are main insights are robust to these changes.

We start with a description of a city in the economy, setting out both the urban agglomeration benefits and the urban diseconomies associated with city population growth. Cities form on a "flat featureless plain" with an unexhausted supply of identical city sites, and land is available for urban development at zero opportunity cost. There are $n(t)$ workers in a particular city at date t and we define a worker's income, $y(t)$, as

$$y(t) \;=\; x(n(t)) \;-\; cn(t)^{\gamma-1} \;+\; s(t). \tag{1}$$

As we will see, the first term is the worker's output, the second is land rent plus commuting costs, and the final term is any subsidy (or, if negative, a tax) that the worker receives. This real income expression contains all the components of earnings, subsidies, and expenditures that vary directly with city size. This sum is then available to be spent on final consumption and on housing. We discuss each of the components of (1) in turn, as well as housing.

Production: Firms in a city produce a single homogenous good with internal constant returns to scale but subject to citywide scale externalities. With constant returns we simply assume that each worker is also a firm. The city work force is n and per worker output is $x(n)$, with $x'(n) > 0$ and bounded away from infinity at $n = 0$, and $x''(n) < 0$. This represents urban scale economies where per worker output rises at a decreasing rate with city population, as workers benefit from interaction with each other.[1] Output per worker may continue to rise indefinitely with n, or may pass a turning point as congestion sets in.

Commuting and land rent: The second term on the right-hand side of equation (1) is land rent plus commuting costs in a city of size $n(t)$. It generalizes the standard approach in the urban systems literature (Duranton and Puga, 2004). All production in a city takes place in the city's central business district (CBD), to which all workers commute from residential lots of fixed size. Free mobility of workers requires all workers in the city to have the same disposable income after rent and commuting costs are paid. Thus, there is a land rent gradient such that, at all points within the city, land rent plus commuting costs per person equal the commuting costs of the edge worker whose rent is zero. Edge commuting costs take the form $cn(t)^{\gamma-1}$ (the term in (1)) which is derived, along with expressions for rent and commuting costs, in Appendix 1. The parameter c measures the level of commuting costs and γ combines relevant information on the shape and commuting technology of the city. If commuting costs per unit distance are constant then, in a linear city $\gamma = 2$, and in a circular or pie shaped city $\gamma = 3/2$. Our generalization encompasses these cases, and also allows commuting costs to be an iso-elastic function of distance, as shown in Appendix 1. We require merely that $\gamma > 1$, so average commuting costs, as well as average land rent, rise with city population. Integrating over the commuting costs paid by people at each distance from the centre and over their rents gives the functions $TC(n)$ and $TR(n)$ reported in Table 1.

Housing: A plot of city land can be occupied by a worker only after a capital expenditure of H has been incurred. This represents the construction of a house, although it could also include other aspects of infrastructure such as roads and water supply. The housing construction, sale, and rental markets are all assumed to be perfectly competitive, and the spot market rent of a house at time t is denoted $h(t)$, this paid in addition to the rent on land. Throughout the paper we assume that the two rent components are separable; housing rent, $h(t)$, is paid separately from land rent which is determined by the city land rent gradient. We also assume that the two sources of income can be taxed separately. Thus, house builders may rent land from land owners with an infinite lease and pay land rents according to the perfectly foreseen city land rent gradient. Alternatively builders could initially buy the land from the land owners, capitalizing the land rents. And a model with owner-occupancy where residents buy land and housing would yield equivalent results. Land owners are people outside the urban sector, although the same results on city formation hold if they are nationwide Arrow-Debreu share holders in the land of all cities.[2]

Subsidies and taxes: The final term in equation (1) denotes a per worker subsidy at rate $s(t)$ (tax if negative) to workers in the city at date t. We will investigate use of this under different city governance structures. We note that since workers are also firms, the subsidy could be viewed as going to firms, a common element of city finance.

Table 1 summarizes key relationships in a city with population n. The left-hand block of the table reports the basic relationships between commuting costs and land rent, derived in Appendix 1. The right-hand block defines relationships which we will use repeatedly through the paper. Total surplus, $TS(n)$, is the output minus commuting costs of a city of size n; notice that this is defined without including housing costs. Average surplus $AS(n)$ and marginal surplus $MS(n)$ follow from this in the obvious way. $LS(n)$ is the surplus per worker net of average land rent paid to landowners, $LS(n) \equiv AS(n) - AR(n)$; it follows that $y(t) = LS(n(t)) + s(t)$ (from equation 1). Finally, $EX(n) = MS(n) - LS(n)$ is the production externality associated with adding a worker-firm to the city: it is the increase in output of all other workers in the city when a further worker is employed.

Table 1: Commuting costs, land rents, and surplus.

edge commuting cost = land rent+commuting cost	$cn^{\gamma-1}$	Total surplus: $TS(n)$	$nx(n) - n^{\gamma}(c/\gamma)$
Total commuting costs: $TC(n)$	$n^{\gamma}(c/\gamma)$	Average surplus: $AS(n)$	$x(n) - n^{\gamma-1}(c/\gamma)$
Total land rent: $TR(n)$	$n^{\gamma}c(\gamma-1)/\gamma$	Marginal surplus: $MS(n)$	$x(n) + nx'(n) - n^{\gamma-1}c$
Average land rent: $AR(n)$	$n^{\gamma-1}c(\gamma-1)/\gamma$	Labour surplus $LS(n) \equiv AS(n) - AR(n)$	$x(n) - n^{\gamma-1}c$
		Externality: $EX(n) = MS(n) - LS(n)$	$nx'(n)$

The shapes of these functions are critical, and we state assumptions that are sufficient for the propositions that follow.

A1: $LS(n)$ is strictly concave in n with unique interior maximum at n_L, $x'(n_L) = n_L^{\gamma-2}c(\gamma-1)$, and such that as $n \to \infty$, $LS(n) < LS(0)$.

It follows that $AS(n)$ is strictly concave (since $\gamma > 1$), but we also assume:

A2: $AS(n)$ has interior maximum at n_A, $x'(n_A) = n_A^{\gamma-2}c(\gamma-1)/\gamma$.

A1 and A2, together with $\gamma > 1$ and our assumptions on $x(n)$ imply that: **(I)** $n_A > n_L$. **(ii)** $MS(0) = AS(0)$. **(iii)** $MS(n)$ intersects $AS(n)$ from above at n_A. $MS(n)$ is initially increasing and is decreasing for all $n > n_A$, since after n_A, $MS'(n) = 2AS'(n) + nAS''(n) < 0$. However, it is also convenient to assume explicitly that the total surplus curve, *TS(n)*, has the textbook S-shape as in Fujita (1978), or that:

A3: Starting from $n = 0$, $MS(n)$ is strictly increasing in n until it peaks, after which it is strictly decreasing.

These relationships are illustrated in Figure 1. The average surplus curve is strictly concave with maximum at point n_A. Marginal surplus and average surplus start at the same point, then marginal lies above average until they intersect at n_A. Surplus net of land rents, $LS(n)$, lies below $AS(n)$, with maximum at $n_L < n_A$.

Our analysis also requires an assumption that the magnitude of housing construction costs, H, be large enough to ensure that housing is never left empty; this prevents jumps in city size. The issue arises in different contexts and here we state a condition sufficient to apply in all cases:

A4: $\delta H \geq AS(n_A) - AS(0)$.

The interest charge on housing per worker is therefore at least as large as the difference between the maximum level of surplus per worker and its level in a new city with zero initial population. This implies housing costs are high relative to net agglomeration benefits of cities, and the aptness of the assumption could be debated empirically. However we note that assumption A4 is an all-purpose sufficient condition; in each of the situations we examine lower relative magnitudes of housing costs are necessary.

With these ingredients in place we look next at the social welfare maximum to establish a benchmark and basic concepts, then at the competitive equilibrium without city government (sections 4, 5 and 6), before turning to analysis of city government (section 7).

3. Socially optimal city formation

At date 0 new urban population, arriving at ν per unit time, starts to flow into one or more new cities. How should this population be allocated across cities, and how large should cities consequently be? After we had solved this problem, we discovered that Fujita (1978) had already solved a more complex version of the same problem in a regional planning context, albeit without the market and institutional analyses we focus on. So our analysis of the social planner problem is geared to providing intuition for this benchmark case.

City formation can be sequenced in a number of ways. One possibility is that cities grow sequentially, so at any point in time there is only one growing city which, t periods into its growth, has population $n(t) = \nu \cdot t$. A second possibility is that population goes temporarily into existing cities and then, at some date, a new city forms with a jump to some discrete size. This has the advantage of delivering returns to scale instantaneously, as the new city can jump to efficient size n_A giving the maximum value of *AS*; old cities also have population approximately equal to n_A. But the cost of jumps is that, after residents have left to form the new city, some housing is left empty in old cities. This cost depends on the magnitude of the sunk housing costs, *H*, and in Appendix 2 we prove that assumption A4 is sufficient to ensure that jumps of any type, large or small, are inefficient. A third possibility is that cities develop in parallel, with several cities growing simultaneously. Such an outcome is inefficient, as gains from increasing returns are slowed, and this too is demonstrated in Appendix 2. We therefore focus on the first case in which population enters each city in turn, without jumps, until each city's growth period is complete.

The new city is born at a date $t = 0$, grows at rate ν for $t \in [0, T\,]$ with population $n(t) = \nu t$, and is then stationary at final size νT. Population then flows into another new city for *T* periods, and so on. The problem is to choose *T* to maximize the present value of the total surplus earned in all future cities, net of house construction costs. In general we are going to write expressions in a form that combines production, commuting, and rent into the functions *TS*, *AS*, and others given in Table 1. For this first problem we also write out the elements of the problem in full, but then immediately combine terms into *TS*, *AS*, and *MS* expressions.

The present value of the total surplus in all future cities from time 0 can be expressed as

$$\begin{aligned}\Omega &= \frac{1}{1-e^{-\delta T}}\left[\int_0^T (\nu t x(\nu t) - (\nu t)^{\gamma} c/\gamma) e^{-\delta t} dt + \frac{(\nu T x(\nu T) - (\nu T)^{\gamma} c/\gamma) e^{-\delta T}}{\delta}\right] - \int_0^{\infty} \nu H e^{-\delta t} dt \\ &= \frac{1}{1-e^{-\delta T}}\left[\int_0^T TS(\nu t) e^{-\delta t} dt + \frac{TS(\nu T) e^{-\delta T}}{\delta}\right] - \int_0^{\infty} \nu H e^{-\delta t} dt.\end{aligned} \tag{2}$$

The term in square brackets is the present value of the output minus commuting costs (total surplus, see Table 1) in a city founded at date zero. While a city is growing the surplus at any instant is $TS(\nu t)$. At date *T* the city stops growing and has a surplus at every instant from then on of $TS(\nu T)$. At that point another city is founded and the process repeats itself. The term multiplying the square brackets is the sum of the geometric series $1 + e^{-\delta T} + e^{-\delta 2T}$ ...; it therefore sums the present value of all such future cities. The final term is total housing costs. These do not depend on T because, as long as all houses remain occupied at all dates, the total number of houses built is simply equal to the total inflow of new workers, regardless of where they live.

Optimization with respect to T gives first order condition,

$$\frac{[1-e^{-\delta T}]^2}{\delta e^{-\delta T}} \frac{d\Omega}{dT} = -\left[\int_0^T TS(\nu t)e^{-\delta t}dt + \frac{TS(\nu T)e^{-\delta T}}{\delta}\right] + MS(\nu T)\frac{[1-e^{-\delta T}]\nu}{\delta^2} = 0. \qquad (3)$$

Integrating by parts, this first order condition can be expressed as

$$\int_0^T [MS(\nu t) - MS(\nu T)]\, e^{-\delta t}dt = 0 \qquad (4)$$

(see Appendix 3, equation A9).

We denote the welfare maximizing population as n_{opt} and the corresponding solution of the first order condition as T_{opt} ($\equiv n_{opt}/\nu$). First order condition (4) is readily interpreted. It says that city size must be chosen so that the present value of the marginal surplus from adding a worker to a new city, $MS(\nu t)$, equals the present value over the same time frame of the marginal surplus from adding the worker to an existing city, $MS(\nu T)$. We can now state our first proposition:

Proposition 1. There exists a unique optimal city size. This city size is larger than that which maximizes surplus per worker, i.e. $T_{opt} > T_A$ ($\equiv n_A/\nu$).

Proof: See Appendix 3. ■

The intuition underlying the result that $T_{opt} > T_A$ comes from the fact that cities in this economy do not jump to their optimal size, but instead grow to it. Since a new city undergoes a period where average surplus is low, it is optimal to expand the growing city beyond size n_A before switching to a new city. Thus, in an efficient solution, average surplus follows a rising path as the city grows and then falls somewhat before it is optimal to start the development of a new city. Furthermore, it is possible that $x'(\nu T_{opt}) < 0$; i.e. it could be efficient to expand to a size at which negative externalities dominate positive ones. The magnitude of the gap between T_{opt} and T_A is smaller the lower is the interest rate, as can be seen from the second term in equation (4') in Appendix 3. As the discount rate goes to zero, the optimal city size approaches the size where $MS(\nu T)$ and $AS(\nu T)$ are equal and hence intersect at the maximum of AS, so $T_{opt} \rightarrow T_A$.

4. Competitive equilibrium without city governments

Given this benchmark, we now turn to equilibria with different forms of governance. We look first at the equilibrium in which there are no large agents – neither governments nor large property developers. We seek to find the equilibrium steady state city size, i.e. the length of time T for which a new city grows before it becomes stationary and growth commences in the next new city.[3] In the steady state all cities will have the same value of T, but in setting out the analysis and looking to extensions later in the paper, we develop notation for individual cities. The first is city 1 and it attracts population for T_1 periods, the second city 2 for T_2 periods, and so on. In the base case, technology is identical in all cities, although subsidy rates may differ, so the subsidy function for the ith city is $s_i(t)$. In our base equilibrium in which there is no government, $s_i(t) = 0$, but we carry the terms in the analysis for future reference.

There are three types of economic agents. (I) Landowners, who are completely passive. They are price takers, simply receiving rent according to the city land rent gradient, as discussed in Section 2. (ii) Workers, who are perfectly mobile between cities and must occupy a house in the city in which they work. This mobility implies that their real income net of housing rent is the same in all cities, so at any date $LS(\nu t) + s_i(t) - h_i(t)$ is the same for all occupied cities, i. (iii) Perfectly competitive 'builders' who provide housing. From Section 2, housing is available on a spot rental market, and house construction incurs sunk cost H. The private decision to build is based on a comparison of H with the future rents that a house will earn. Given this, builders' forward-looking decisions of whether to build in new versus old cities, based upon anticipated future rents, is central to the analysis of the competitive equilibrium city size.

The equilibrium condition for supply of housing in a growing city is that the construction cost equals the present value of rents earned. Thus, in city 1, at any date $\tau \in [0, T_1]$ at which construction is taking place

$$H = \int_{\tau}^{\infty} h_1(t) e^{-\delta(t-\tau)} dt = \int_{\tau}^{T_1} h_1(t) e^{-\delta(t-\tau)} dt + \hat{H}_1 e^{-\delta(T_1-\tau)}. \tag{5}$$

The first equality simply defines cost, H, to equal the present value of instantaneous rents, $h_1(t)$, in city 1 forever. The second equality, for this base case where each new city will grow from scratch to its final stationary size, breaks that present value into two components. The first is the present value of rents earned while the city is growing up to T_1. In the second, $\hat{H}_1$ is the present value (discounted to date T_1) of rents earned from date T_1 onwards. Construction takes place at all dates in the interval $[0, T_1]$, implying two things. First, for $t \in [0, T_1]$, $h_1(t) = \delta H$, which comes from differentiating equation (5) with respect to τ. Essentially the zero profit condition on construction, (5), means that housing rent in a growing city must be constant, equal to the interest charge on the capital cost. Second, $\hat{H}_1 = H$, necessary for construction to break even at the last date at which it occurs, T_1.

Although house rent is constant at $h_1 = \delta H$ in a growing city, rent will vary with time in each stationary city to give a path of rent that clears the housing market in that city. Consider the rents earned on housing in city 1 in the period in which city 2 is growing $t \in [T_1, T_1 + T_2]$. Workers are fully mobile between cities, and rents will adjust to clear the housing market, i.e. to hold mobile workers indifferent between living in stationary city 1 or in growing city 2. Thus, city 1 housing rent during the period in which city 2 is growing, denoted $h_{12}(t)$, must equate real incomes net of housing rent across cities which, using equation (1), means that they satisfy

$$LS(\nu(t-T_1)) + s_2(t) - \delta H = LS(\nu T_1) + s_1(T_1) - h_{12}(t), \qquad t \in [T_1, T_1+T_2]. \tag{6}$$

As terms on the left-hand side of this vary through the growth cycle of city 2, $t \in [T_1, T_1 + T_2]$, so rent in city 1 must adjust to hold workers indifferent, so that city 1 housing stock continues to be occupied. Of course, the condition holds only for $h_{12}(t) \geq 0$; if the income gap between cities is too great then rents in stationary cities go to zero and housing in these cities is left empty as workers migrate to the growing city. Our assumption A4 is sufficient to secure $h_{12}(t) \geq 0$ (see below).

Equation (6) defines stationary city rent $h_{12}(t)$ for the period $t \in [T_1, T_1 + T_2]$ in which city 2 is growing. During time interval $t \in [T_1 + T_2, T_1 + T_2 + T_3]$ city 3 is growing and housing rents in both the stationary cities, cities 1 and 2, are set by the path of returns in city 3, so $h_{13}(t) = h_{23}(t)$, determined by an equation analogous to (6), and so on. Extending this analysis through infinitely many time periods, the present value (discounted to date T_1) of these rents, $\hat{H}_1$, is given by

$$\hat{H}_1 = \sum_{i=2}^{\infty} \int_{\Gamma_{i-1}}^{\Gamma_i} h_{1i}(t) e^{-\delta(t-T_1)} dt. \tag{7}$$

where Γ_i is the date at which city i stops growing, $\Gamma_i \equiv \sum_{j=1}^{i} T_j$. The key equilibrium condition is that the date T_1 at which city 1 becomes stationary, is the value of T_1 at which this present value equals construction costs, $\hat{H}_1 = H$. This date is a function of all future T_i, $i > 1$, and these dates are in turn determined by equations analogous to (7) and $\hat{H}_i = H$.

To solve, we invoke a symmetric steady state (with function $s(\cdot)$ the same in all cities), where symmetry follows from the sequential nature of the process: each new city forms under exactly the same circumstances as the previous one. We rewrite equation (6) to give house rents in all old cities in a symmetric steady state. Thus, if all old cities had growth period T then house rent in each such city at date t in the growth cycle of a city born at date 0 is given by $\hat{h}(t)$, defined by

$$\hat{h}(t) = LS(\nu T) + s(T) + \delta H - [LS(\nu t) + s(t)], \qquad t \in [0, T]. \tag{8}$$

These growth cycles repeat indefinitely, so summing their present value over all future cycles gives (discounting to date 0, the date at which the last old city stopped growing):[4]

$$\hat{H} = \frac{1}{1-e^{-\delta T}} \int_0^T \hat{h}(t) e^{-\delta t} dt = \frac{1}{1-e^{-\delta T}} \int_0^T [LS(\nu T) + s(T) + \delta H - [LS(\nu t) + s(t)]] e^{-\delta t} dt. \tag{9}$$

Setting $\hat{H} = H$ and integrating, the H terms cancel out so that housing market equilibrium requires

$$\int_0^T [LS(\nu t) + s(t) - LS(\nu T) - s(T)] e^{-\delta t} dt = 0. \tag{10}$$

The value of T solving equation (10) is the last date, T, at which it is profitable to build a house in a growing city. Prior to T, real income in the growing city is large enough to make building still profitable; a moment after T, real income in that city would have fallen sufficiently to make building in a new city relatively more attractive, so builders switch to the next new city.

Equation (10) defines the equilibrium, focussing on the last date at which it is profitable to build in a city. However, for future reference it is helpful to also write down an inequality condition which ensures that, for all $t \in [0, T]$, builders do not want to switch construction to an alternative existing city, a condition we will invoke in other parts of the paper. A necessary condition for builders not to switch is that, at each date τ in which building is occurring in a new city,

$$\int_{\tau}^{T}[\delta H - \hat{h}(t)]e^{-\delta(t-\tau)}dt = \int_{\tau}^{T}[LS(\nu t) + s(t) - LS(\nu T) - s(T)]e^{-\delta(t-\tau)}dt \geq 0, \quad \tau \in [0, T]. \quad (11)$$

We label this inequality the 'no-switch' condition. It says that builders cannot earn more rent from building in an old city, with rent path $\hat{h}(t)$, than from building in the one that is currently growing where rents are δH. Notice that in this case the differential housing rent expression is defined just to run up to T; beyond T, in the equilibrium, new and old cities would both give the same present value rent $\hat{H}$. The expression holds with equality at $\tau = 0$ and $\tau = T$, the dates at which builders switch cities, as in (10).

We now summarise results for the competitive equilibrium without city government $s(t) = s(T) = 0$. We label the value of T solving equation (10) T_{eq}, with corresponding population size $n_{eq} = \nu T_{eq}$. This gives the following proposition:

Proposition 2: Without city government there exists a unique steady-state equilibrium city size n_{eq}. Workers' real income increases and then decreases during the growth of a city, with this variation in real income being transmitted to all existing cities via variation in housing rent.

Proof: The left-hand side of equation (10) takes value zero at $T = 0$. Its gradient is given by $\partial(\int_{0}^{T}[LS(\nu t) - LS(\nu T)]e^{-\delta t}dt)/\partial T = -LS'(\nu T)[1 - e^{-\delta T}]\nu/\delta$ and is therefore decreasing until T_L and strictly increasing thereafter (by strict concavity of the function LS, assumption A1). The value of the integral is therefore strictly increasing through zero at $T = T_{eq}$, ensuring existence of a unique solution. Housing rents satisfy equation (8), so that income net of housing rent in both new and old cities is $LS(\nu t)$ - δH. The no-switch condition (11) is satisfied with inequality for all $\tau \in (0, T)$ since LS is initially strictly increasing and then decreasing, with equality 0 and T. Finally we note existence of the equilibrium requires $h(t) > 0$ for all t. This condition will be satisfied if $\delta H > LS(\nu T_L) - LS(\nu T_{eq})$. Given the right-hand side is less than $AS(\nu T_A)$ - $AS(0)$ in assumption A4, (the peak value of LS is less than that for AS), this is a weaker condition on H than is A4.■

The time paths of income and rent are illustrated in Figure 2. The top line gives the output minus land rent and commuting costs of a worker in a city founded at date 0, $LS(\nu t)$. During the life of the city this rises to a peak at T_L, and then starts to decline until date T_{eq} is reached, after which it is stationary. The worker also pays housing rent which, during the growth of the city is simply δH. The worker's real income net of housing costs is the difference between these, given by the middle line $LS(\nu t)$ - δH, which varies over the life of the city.

In the time interval $[T_{eq}, 2T_{eq}]$ another city is growing and offering its inhabitants the income schedule $LS(\nu t)$ - δH. Workers in the stationary city are mobile, and remain in the stationary city only if rents follow the path $\hat{h}(t)$ (equation (8)). Thus, there are housing rent cycles in old cities as the housing market adjusts to conditions in the current growing city. As illustrated in Figure 2 house rents in old cities jump up when a new city is born as this city is initially unattractive; they are then U-shaped, reaching δH at the point where the new city is the same size as old ones. The process repeats indefinitely with periodicity T_{eq}, so stationary cities have a rent cycle in response to the possibility of migration to the growing city.

Viewing Figure 2, one might ask why, once a new city starts, builders do not continue to build in old cities in which rents are higher. Once building starts in a new city (at dates T_{eq}, $2T_{eq}$, etc), the no-switch condition is satisfied along the equilibrium path of house rents so it is profitable to continue building there. Initial builders in the new city know that they will be followed by further builders in that city. The key is that housing investment is irreversible; any further housing built in old cities cannot be moved to a new city when rents in old cities start to fall. One can also use Figure 2 to gain further insight into the equilibrium by considering alternative T to T_{eq} as candidate equilibrium values.[5]

There are two other points about this equilibrium we will refer back to later in the paper. First, without subsidies, we can rearrange (10) to read

$$\int_0^T LS(\nu t)e^{-\delta t}dt \; + \; \frac{e^{-\delta T}}{\delta} LS(\nu T) \; = \; \frac{1}{\delta} LS(\nu T). \tag{12}$$

This equation arises in Venables (2005), where once migrants have chosen a city, thereafter they are perfectly immobile. The left-hand side is the present value of income for the first person in a new city; and the right-hand side is the present value of the alternative, entering an old city. These are equalized at the switch point of migration into a new city, where migrants are indifferent between migrating permanently to a new versus an old city. Housing versus population immobility yield identical outcomes.

Second, in the proof of Proposition 2, we showed that the no-switch condition holds with strict inequality for all $\tau \in (0, T)$. Inspection of (11) shows that this implies $\int_\tau^T [LS(\nu t) - LS(\nu T)]e^{-\delta(t-\tau)}dt > 0$. This condition states that entrants to a city during its growth path, $\tau \in (0, T)$, receive a higher present value of income than if they chose an old city, or started a new one. Entrants at these intermediate dates get a "surplus", by avoiding the low incomes of a start-up city.[6] This surplus plays a key role in the analysis of city government behavior later, where surpluses are, in essence, taxed away.

Finally, we note some of the comparative static properties of the equilibrium. It is possible to show that a faster rate of population inflow, ν, reduces T_{eq}, although it has an ambiguous effect on city size νT_{eq}. The discount rate, δ has an unambiguous effect, with a higher discount rate giving larger city size. This can be seen by totally differentiating (10) to give

$$\frac{\partial\left(\int_0^T [LS(\nu t) \; - \; LS(\nu T)]e^{-\delta t}dt\right)}{\partial T} dT \; = \; \left(\int_0^T t[LS(\nu t) \; - \; LS(\nu T)]e^{-\delta t}dt\right) d\delta. \tag{13}$$

The partial derivative on the left-hand side is positive in the neighborhood of T_{eq}, as noted in the proof of proposition 2. To show that the right-hand side is positive, we observe that, compared to (10) with $s(t) = s(T) = 0$, the term in square brackets which switches negative to positive is now weighted by t. Since (10) holds, with weighting the right-hand side of (13) must be positive, so $dT/d\delta > 0$. The result that an increase in the discount rate leads to larger cities is intuitive, since a higher discount rate puts more weight on the low income levels that are initially earned in a new city, discouraging city formation. This suggests that in a model with capital market imperfections, where private agents discount the future more heavily than is socially optimal, equilibrium cities will tend to be larger relative to the optimum.

5. Extensions

We have assumed so far that all cities have identical and stationary underlying technologies. As a consequence they all have identical terminal size and growth is strictly sequential. In this section we do two things. First, we illustrate how outcomes are altered by relaxing key assumptions to allow for heterogeneity of cities, technical change within cities, and a time when national urban population growth ceases. Heterogeneity plays the key role of generating an urban size distribution, as in the literature on Zipf's Law. Relaxation of our assumptions also alters growth patterns, to better mimic patterns in the data. Patterns retain the essential aspect of sequencing: cities form in sequence and when a new city forms it has an interval of solo growth where it absorbs all migrants. However now after cities have completed their initial solo growth spurt, they experience later episodes of continued growth. Second, we relate our findings on urban growth to the evidence, drawing in particular on the work of Cuberes (2004).

5.1 Heterogeneous cities and technical change

To allow technology to vary across cities and time, we replace the function $LS(n)$ by $LS_i(n_i(t), t)$, which gives the income of a worker, after land rent and commuting costs but before housing costs, in city i of size $n_i(t)$ at date t from the start of the national urban growth process. Spatial variations in this function might arise because of differences in production amenities across urban sites such as natural harbors or inland waterways that affect shipping costs and received prices. Or local natural geography may affect land availability and commuting costs or production technologies. Variations through time might be due to changes in production or commuting technology which facilitate increases in city size, where the latter is emphasized in the literature on city development (Mills, 1972).

To solve examples for these more complex cases we use a simple algorithm that, at any instant, allocates perfectly mobile migrants to the city offering the highest present value of rents to builders, so the resulting equilibrium is consistent with builder profit maximizing behaviour. Below for a case with just heterogeneity, we also specify algebraic equations which may be used to solve directly for the lengths of all growth episodes. With free migration between any pair of cities, *i* and *j*, as in equation (6), free mobility at each instant requires $LS_i(n_i(t), t) - h_i(t) = LS_j(n_j(t), t) - h_j(t)$, where for simplicity we drop the *s*(t) terms that are zero in this analysis. Builders build in city *i* at time τ only if the present value of rents in city *i* is equal to or greater than in any other city, a generalized application of housing no-switch analysis. From the free mobility condition this requires

$$\int_\tau^\infty [h_i(t) - h_j(t)]e^{-\delta t} = \left[\int_\tau^\infty LS_i(n_i(t), t)e^{-\delta(t-\tau)}dt - \int_\tau^\infty LS_j(n_j(t), t)e^{-\delta(t-\tau)}dt\right] \geq 0 \quad \forall\ j \neq i. \quad (14)$$

Note on the RHS of the equality in (14) the integral expressions also define the present value of incomes in a city from τ on, forever, where for example

$$V_i(\tau) \equiv \int_\tau^\infty LS_i(n_i(t), t)e^{-\delta(t-\tau)}dt \quad (15)$$

The algorithm allocates migrants to the city, or cities, offering the highest rents or equivalently highest value of $V_i(\tau)$. Any cities offering a lower present value of rents, or a lower $V_i(\tau)$, than the current maximum are either stationary or not yet founded. The equilibrium from this algorithm is therefore a set of city population levels $n_i(t)$ such that

$$\begin{aligned} \dot{n}_i(\tau) &\geq 0 \quad if \quad V_i(\tau) = \max_j[V_j(\tau)] \quad \forall\ j \\ \dot{n}_i(\tau) &= 0 \quad if \quad V_i(\tau) < \max_j[V_j(\tau)] \\ \textstyle\sum_i \dot{n}_i(\tau) &= \nu \end{aligned} \quad (16)$$

where ν is the exogenous change in overall urban population at any instant. If two cities, *i* and *j*, both offer the maximum and are growing at the same time, from earlier analysis of equation (5), rents during growth equal δH. Thus while city *i* and *j* grow at the same time

$$LS_i(n_i(t), t) = LS_j(n_j(t), t). \qquad (17a)$$

We illustrate the equilibrium in this general setting by constructing an example that differs from the symmetric case in three main ways. First, while total urban population growth continues at rate ν, it only does so for a fixed length of time, $\bar{T}$, and then ceases. Second, cities are heterogeneous in the level of the functions $LS_j(n_j(t), t)$. Third, there is continuous technical progress at a constant rate during $t \in [0, \bar{T}]$; this progress is specified to shift *LS* functions up and out, so as to continuously increase optimal city sizes.

Figures 3 a, b, c illustrate outcomes for one example. The functional forms used in examples are given in Appendix 4. Figure 3a gives the population of each city as a function of time, where in equilibrium cities are occupied in order, starting with the city with the highest natural amenity level. We see that, with our functional forms, just five cities develop; and the system becoming stationary at $\bar{T}$. Two points stand out. The first, as anticipated, is that once the system is stationary there is a city size distribution; heterogeneity produces different long-run city sizes. Second, each city first experiences an initial period of solo growth, a key aspect of sequential city formation; but then it alternates stationary periods with periods of continued growth. For example, consider the first city to develop (city 1). Its first stationary period arises during the initial growth period of city 2. However, before the city 3 commences growth, city 1 has a second growth period, concurrent with growth in city 2, and so on. This pattern of repeated growth is induced both by heterogeneity and by technical progress. Heterogeneity means that, before initiating a later (and less efficient) city, each existing city expands further. Technical progress as we have modelled it here increases agglomeration benefits, and existing cities adjust to this by having repeated episodes of growth.

Figure 3b gives the values $V_i(\tau)$ along the equilibrium path. Each city is growing when it offers the maximum value of $V_i(\tau)$, and we see that, for example, in the interval when cities 1, 2, and 3 are all growing, they are offering new migrants the same present values of incomes and builders thus the same present value of rents. The equilibrium path of house rents where mobile workers are indifferent between all cities at all dates is illustrated in Figure 3c for city 1 and, as in Figure 2, we see rent cycles. Thus, at the date builders switch to city 2, where house rents are δH from equation (5), current incomes in city 2 are small compared to city 1. New migrants only choose city 2 because rents spike in city 1, given that builders have ceased construction there.

Pure Heterogeneity.

For a situation with pure heterogeneity, the pattern of initial solo growth of a new city with later episodes of resumed growth can be proved directly and the model readily solved algebraically. Assume city sites are ranked so $LS_1(n) > LS_2(n) > LS_3(n) > LS_4(n) \ldots \forall\, n$, where builders always choose the highest ranked unoccupied site to start a new city. The solution has the characteristic that first city 1 grows solo; then city 2 grows solo; then city 1 and 2 grow together for a time interval before they stop; then city 3 forms and grows solo; and so on. Before each new city has its period of solo growth, all existing cities have an episode of simultaneous growth of the same length for each city.[7] In Appendix 4, we show that an equilibrium where a new city forms without an episode of resumed growth of all cities cannot be an equilibrium.

We start by examining the equations defining relationships for cities 1 and 2. If T_2 is the length of time city 2 grows solo at which point cities 1 and 2 start growing together, then, from equation (17)

$$LS_1(\nu T_1) \;=\; LS_2(\nu T_2). \qquad \text{(17b)}$$

Note, given $LS_1(n) > LS_2(n)$, that equation (17b) implies $T_1 > T_2$. When a city is growing builders cover costs so the present value of future rents equals H. However, since this is true for builders in city 1 at dates T_1 and $T_1 + T_2$, it must be the case also that the present value of rents over the interval $[T_1, T_1 + T_2]$ covers the present value of interest charges δH over the same interval. This requires (from worker mobility between city 1 and 2 in this interval) that

$$\int_{T_1}^{T_1+T_2} [\delta H \;-\; h_{12}(t)]\, e^{-\delta(t-T_1)} dt \;=\; \int_{T_1}^{T_1+T_2} [LS_2(\nu t) \;-\; LS_1(\nu T_1)]\, e^{-\delta(t-T_1)} dt \;=\; 0 \qquad \text{(18a)}$$

where $h_{12}(t)$ is the rent in city 1 in this interval. Of course (18a) corresponds to the no-switch condition in (11). Given (17b), equation (18a) can be written as

$$\int_{T_1}^{T_1+T_2} [LS_2(\nu t) \;-\; LS_2(\nu T_2)]\, e^{-\delta(t-T_1)} dt \;=\; 0 \qquad \text{(18b)}$$

Equations (17b) and (18b) solve for T_1 and T_2 directly.

At $T_1 + T_2$ there are two possibilities. Either a new, third city forms immediately which is ruled out in the Appendix. Or, as is the case, cities 1 and 2 grow for a time interval T_{12} before city 3 forms. When city 3 forms now equation (18b) applies directly, replacing 2 subscripts by 3. At the end of T_3, when all three cities start to grow simultaneously, it must also be the case that

$$LS_3(\nu T_3) \; = \; LS_1[\nu(T_1 + \int_{T_1+T_2}^{T_1+T_2+T_{12}} \alpha_{12}^1(t)dt)] \; = \; LS_2[\nu(T_2 + \int_{T_1+T_2}^{T_1+T_2+T_{12}} (1 - \alpha_{12}^1(t)dt)] . \qquad (17c)$$

where $\alpha^1(t)$ is the share of city 1 in the ν flow at any instant during the T_{12} episode of simultaneous growth of cities 1 and 2. Equations (18b) and (17c) solve for $T_{12},\ T_3,\ \int_{T_1+T_2}^{T_1+T_2+T_{12}} \alpha^1(t)\, dt$.

This process continues forward as we move to successively lower order cities. Each forms after an episode of parallel growth of existing cities. The model is solved by repeated application of equations (18) and (17) (where (17) expands to add on more cities with equal LS_i`s as more cities form).[8] The process ends with an episode of parallel growth if population growth terminates.

These cases illustrate how our approach can be generalised and how the main insights are robust. Other extensions are possible, but not pursued here. For example, cities might be subject to occasional, unanticipated shocks such as improvements in urban transport technology [9]. Cities might also be subject to repeated city specific shocks, as in the work of Cordoba (2004), Rossi-Hansberg and Wright (2004), and Duranton (2004), who seek to generate models of Zipf's law.[10]

5.2 Evidence

The data support the idea of sequential city formation, with new cities growing from scratch without population losses for existing cities. In terms of population losses, for example, from 1900-1990 when the USA moved from being 40% urban to 75% urban, Black and Henderson (2003) show that almost no metro areas and certainly no medium or larger ones experienced significant (over 5%) population losses between decades. In a worldwide data set for 1960-2000 covering all metro areas over 100,000, Henderson and Wang (2005) identify 25 countries that start with just one metro area in 1960 and have more metro areas form during 1960-2000. Of these 25 initial metro areas, 22 experience no population losses in subsequent decades. Of the 3 that do, none lose population at the decade when new metro areas enter the picture; and each has a special circumstance (Phnom Penh in the 1970's and Latvia and Estonia where all cities lose population from 1990-2000).

In an exhaustive study, Cuberes (2004) shows more generally that the data support the key features of sequential growth. Cuberes covers city populations in 52 countries drawn from various sources, using primarily metro area level data. The start date for each country depends on data availability, most lying in the range 1880 to 1930, with the earliest being 1790 (USA) and the latest 1953 (Uruguay). Cuberes first presents strong evidence that cities grow in sequence. He ranks the 5 largest cities at the start date for each country from 1 (largest) to 5 (smallest) and then plots which ranked city has the highest growth rate in each decade. Sequential growth should have city 1 growing fastest in the earliest decade(s), then city 2 in the next decade(s), then city 3 and so on. The data are noisy, but for most countries a regression line against time and rank of the fastest growing city (at each decade) has the hypothesized positive slope. In particular for today's developed countries, 12 out of 16 have a positive slope, and for today's medium and low income countries (where data start in the more modern era, post-1925), 14 of 17 countries have positive slopes (Cuberes, Table 5).

Correspondingly, Cuberes also shows that individual cities tend to have early periods of rapid growth (from their date of entry as a city), followed by slow growth and/or stagnation. Taking the starting top 5 cities in each country he shows there is an-inverted U-relationship between the share of the 5 cities in total national or total national urban population and calculates the date at which these share peaks – early for today's high income countries and later for remaining countries. And the inverted-U relationship between urban primacy (share of the largest city) and time is already well-established in the literature (e.g, Junius, 1999).

6. Equilibrium versus Optimum

With this discussion of the equilibrium in place we now move on to draw out some of its properties. The first issue is the efficiency of equilibrium. Are equilibrium city sizes too large or too small, and exactly what policy responses might be appropriate? We address this question for our base case in which cities are symmetric and there is no technical progress. In this case analysis can be easily based on comparison of the social optimum with the equilibrium.

Proposition 3: The competitive equilibrium without city government gives larger cities than optimum, $T_{eq} > T_{opt}$, if

$$\int_0^{T_{opt}} [EX(\nu t) - EX(\nu T_{opt})] e^{-\delta t} dt > 0. \tag{19}$$

and conversely.

Proof: Subtracting equation (10) from equation (4), equation (19) may be rewritten as

$$\int_0^{T_{opt}} [EX(\nu t) - EX(\nu T_{opt})] e^{-\delta t} dt = \int_{T_{opt}}^{T_{eq}} [LS(\nu t) - LS(\nu T_{eq})] e^{-\delta t} dt + \int_0^{T_{opt}} [LS(\nu T_{opt}) - LS(\nu T_{eq})] e^{-\delta t} dt$$

From Proposition 2, the integral terms on the right-hand side are positive iff $T_{eq} > T_{opt}$. Thus $T_{eq} > T_{opt}$ iff the term on the left-hand side is positive. ■

The interpretation of (19) is direct. Cities are too large [small] if the *present value* of externalities created by a marginal migrant in a new city is greater [less] than the present value of externalities created by that migrant in a stationary city, over the new city's growth interval. The condition depends on how externalities vary with city size. For example, if the value of the externality declines monotonically with city size, the present value of externalities in a new city exceeds that in an old city. Noting that builders ignore these externalities in choosing when to start building in a new city, new cities start up too late and stationary sizes are too large, because the ignored benefits of diverting migrants to a new city exceed the ignored benefits of adding people to an old city. Conversely if the externality is increasing in city size, as with the commonly used case in which $x(n)$ is isoelastic, then new cities form too early and old cities are too small, given that the relatively high externalities in an old city compared to those in a new city are not internalized.

The fact that this equilibrium without city governments can result in smaller city sizes than the social optimum contrasts with traditional perfect mobility analyses where a new city only forms when the real income of a worker in a growing city falls to the level of $LS(0)$ (i.e. $LS(0) = LS(\nu t)$), where it pays people to leave the city, regardless of whether others follow. The co-ordination problem in static models where timely new city formation requires en mass co-ordinated movement of workers well before the date at which $LS(0) = LS(\nu t)$, is solved here because our agents, in particular builders, commit to new city development through initial fixed H investments, and are sequentially rational. Then the comparison of equilibrium with optimum size just turns on the present value of marginal externalities in new versus old cities, as one would expect from applied welfare economics in a dynamic context.

For a national government to implement an optimum in principle is straightforward. Suppose that a national government announces a subsidy schedule in which subsidies are a function of city size. Builders thinking of starting construction in a new city know migrants to the city are guaranteed this schedule as the city grows, and then when it is stationary. The subsidies are financed out of lump sum national taxes which could be on the entire population, on all urban residents, or on land rents.

Proposition 4. If the national government enacts a Pigouvian subsidy schedule for residents of all cities, $s(t) = EX(\nu t)$, then the competitive equilibrium without city governments will be socially optimal.

Proof: In equation (10) if $s(t) = EX(\nu t)$, then equation (4) for an optimum will be satisfied, given that $LS(\nu t) + EX(\nu t) = MS(\nu t)$. ■

The proposition is intuitive, since the only distortion present in the competitive equilibrium is workers' failure to internalize the externalities they create for other workers. This solution, like the competitive one without city governments, has fluctuating housing rents according to equation (8) where now

$$\hat{h}(t) = LS(\nu T_{opt}) + s(T_{opt}) + \delta H - [LS(\nu t) + s(t)] = MS(\nu T_{opt}) - MS(\nu t) + \delta H$$

In fact, the swings in housing prices and hence also real income will be greater than without national government intervention, since MS has larger swings than LS.[11]

Note it is not essential that the subsidy path employed by the national government follow the Pigouvian one. By comparing (4) and (10) it is only necessary that $s(t)$ be constructed to satisfy

$$\int_0^T [s(t) - s(T)] e^{-\delta t} dt = \int_0^T [EX(\nu t) - EX(\nu T)] e^{-\delta t} dt.$$

Thus, the present value of subsidies in a new city compared to an old city must be set equal to the difference between the present value of externalities in a growing and a stationary city. As an example of an alternative subsidy path, the national government could set $s(T) = 0$. Then the present value of subsidies offered over the growth of a new city should equal the difference between the present value of the externality in that city, and the present value of the externality in the old city, which can be positive or negative according to whether competitive equilibrium cities, absent policy, are too large or too small as in Proposition 3.

7 Competitive equilibrium with private government

We now assume that national government is inactive, and turn to the case where each city has its own government which has the abilities to tax land rents, borrow in capital markets, and subsidize worker-firms. We look first at private local governments, or the large developer case, before turning to public governments. While the analysis will confirm the usual result in static models that local governments can implement efficient solutions, in a dynamic context the impact of local governance has distinct features. Local governance affects the income distribution between early and later entrants to a city, dramatically changes housing market outcomes, involves debt accumulation by local governments, and requires institutions that support such financing.

Following Henderson (1974) we assume that, at any instant, there is an unexhausted supply of potential large developers who each own all the land that will ultimately be used in their individual city and who collect all land rents in their city. However, they face competition from existing and other potential new cities and are induced to offer migrants subsidies to enter their city. These subsidies are guaranteed for all time. We continue to assume that housing is constructed by perfectly competitive builders and rented on a spot rental market and to separate out housing rents from land rents. Land rents paid to the developer at each instant equal the rent from the urban land rent gradient, while rents on housing cannot be taxed by the developer.

To find the equilibrium we proceed in three steps. First, we consider the behavior of a single developer establishing a new city; the developer's city can attract population only if it offers migrants a sufficiently high income that they enter this city rather than an old one, *and* migrants pay housing rents sufficient to induce building in the new city. Second, subject to this constraint, the developer announces a subsidy schedule and size of the city to maximize the present value of profits, defined as land rents net of subsidies. The size chosen must be consistent with building and migration decisions so, for example, builders would not choose to continue building beyond the announced size. Finally, we move from the decision of a single developer to the full equilibrium with free entry of developers. Competition between potential new developers bids the present value of profits down to zero. This zero profit condition ensures that no more than one developer actually enters at any date, validating our focus on a single developer at step one.

We have already developed the apparatus for the first of these steps. Suppose that all old cities have population size νT and building in the new city starts at date 0. The developer in choosing a subsidy schedule $s(t)$ is constrained by no-switch conditions: the subsidy schedule must produce an income path in the developer's city and a corresponding housing rent path in old cities such that, at all dates $\tau \in [0, T]$, builders do not want to resume building in old cities. That is, the present value forever of rents at each instant in the new city, $h(t)$, must be at least as great as those in old cities $\hat{h}(t)$, where rents are defined from free migration (equation (6))

$$\begin{aligned}\int_\tau^\infty [h(t) - \hat{h}(t)]e^{-\delta(t-\tau)}dt &= \int_\tau^\infty [LS(\nu t) + s(t) - \bar{y}]e^{-\delta(t-\tau)}dt \\ &= \int_\tau^T [LS(\nu t) + s(t) - \bar{y}]e^{-\delta(t-\tau)}dt + [LS(\nu T) + s(T) - \bar{y}]\frac{e^{-\delta(T-\tau)}}{\delta} \geq 0.\end{aligned} \tag{20}$$

The first equality defines rents based on income flows in the new city, $LS(\nu t) + s(t)$, versus those in old cities, $\bar{y}$, which are viewed as fixed by this new city developer. The second equality breaks the income stream in the new city into the part where the city is growing and the part beyond T where it is stationary and income is $LS(\nu T) + s(T)$. Noting that while the city is growing $h(t) = \delta H$, and noting that in a symmetric equilibrium $\bar{y} = LS(\nu T) + s(T)$, in equilibrium equation (20) reduces to the no-switch condition from (11) where

$$\int_{\tau}^{T}[\delta H - \hat{h}(t)]e^{-\delta(t-\tau)}dt = \int_{\tau}^{T}[LS(\nu t) + s(t) - \bar{y}]e^{-\delta(t-\tau)}dt \ \geq \ 0 \qquad (11a)$$

The objective of the developer is to maximize rent net of subsidy payments, subject to the constraint above. The instruments are the subsidy schedule $s(t)$, $t \in [0, T]$ together with terminal date T at which the city stops growing and after which the subsidy $s(T)$ is constant. Thus, we solve the program,

$$\max_{s(t),\, T} \quad R \equiv \int_{0}^{T}\left[TR(\nu t) - \nu t s(t)\right]e^{-\delta t}dt + \left[TR(\nu T) - \nu T s(T)\right]\frac{e^{-\delta T}}{\delta}$$

$$s.t. \quad \int_{\tau}^{T}[LS(\nu t) + s(t) - \bar{y}]e^{-\delta(t-\tau)}dt \ + \ \left[LS(\nu T) + s(T) - \bar{y}\right]\frac{e^{-\delta(T-\tau)}}{\delta} \ \geq \ 0, \quad \tau \in [0, T]. \qquad (21)$$

We show in Appendix 5, by setting up the Lagrangean, that the constraints hold with equality for all $\tau \in [0, T]$. It follows that, differentiating with respect to τ across the constraints, $s(t) + LS(\nu t) = \bar{y}$. While the level of the subsidy is yet to be determined (depending on $\bar{y}$), its shape is set to deliver a flat income path. Intuition about the constraint can be understood by thinking about the equilibrium of Section 4. In that case the no-switch condition (11) holds with equality at the dates on which the city commenced and ceased growing, and holds with inequality at intermediate dates. As we noted, that implies at these intermediate dates workers in the growing city have higher present values of incomes than if they were in an old city or were the initial entrants to a new city. In the present case, optimization by the developer extracts this surplus, so that incomes of all entrants are the same and equal to those in old cities. This also implies that house rents in old cities are constant and equal to those in new cities at δH. A fundamental impact of having large agents, developers or city government, is to smooth out income streams and housing price cycles for growing and stationary cities respectively. This is potentially an empirically testable hypothesis.

To solve the optimization problem we therefore use the constraint, $s(t) + LS(\nu t) = \bar{y}$, in the objective, together with the fact that $TR(\nu t) + \nu t LS(\nu t) = TS(\nu T)$ from Table 1, to give,

$$\max_{T} \quad R = \int_{0}^{T}\left[TS(\nu t) - \nu t\bar{y}\right]e^{-\delta t}dt + \left[TS(\nu T) - \nu T\bar{y}\right]\frac{e^{-\delta T}}{\delta}. \qquad (21')$$

Choice of T gives first order condition

$$MS(\nu T) = \bar{y}. \tag{22}$$

This condition gives the optimal value of T for a single developer, depending on $\bar{y}$.

The third and final step of the analysis comes from the assumption that there is free entry of large developers. Their profits, R, must therefore be zero. Consequently $\bar{y}$ and subsidy levels must be bid up to the point at which this condition holds. Equilibrium is therefore characterized by substituting (20) into (19') and setting the consequent level of R equal to zero to give,

$$R\delta/\nu = \int_0^T \left[MS(\nu t) - MS(\nu T)\right] e^{-\delta t} dt = 0. \tag{23}$$

where the expression is derived from integrating by parts (see equations A9 and A10 in Appendix 3). The value of T solving equation (23) characterizes city size in the large developer case. This gives the following proposition:

Proposition 5. A unique steady state equilibrium with competitive private city governments supports the social optimum. Workers' real income is constant through time in all cities at level $\bar{y} = MS(\nu T_{opt})$. This income consists of wages net of land rent and commuting costs, $LS(\nu t)$, plus subsidy payments $s(t) = MS(\nu T_{opt}) - LS(\nu t)$ from a guaranteed schedule.

Proof: By a comparison of (23) with (4) we know that developers set $T = T_{opt}$, which we showed earlier has a unique value. Paying $s(t) = \bar{y} - LS(\nu t)$ supports the constant real income path, $\bar{y} = MS(\nu T_{opt})$. There remain two issues. First, we wrote the optimization problem with workers flowing into the city at rate ν, giving population νt. Could a developer profitably engineer a jump in population? The best possible jump is to instantaneously create a city of size n_A and maximal real income, $AS(n_A)$. However, this is not profitable. Creating this new city would reduce house rents according to equation (8), inducing residents to stay in the old city; to induce inter-city migration the developer would have to offer migrants enough income to drive rents in old cities to zero. But doing so is not profitable; assumption A4 is sufficient to ensure that $MS(\nu T_{opt}) > AS(n_A) - \delta H$, where $AS(n_A) - \delta H$ is the maximum income net of housing rent demanded by builders which a new city jumping to n_A can pay migrants. Second there is the issue of why the subsidy path needs to be guaranteed. Consider dates $t > T_{opt}$. At such dates all housing construction in the city is sunk, so any reduction in $s(t)$ would be exactly matched by a reduction in house rents. The developer can therefore expropriate whoever owns the housing stock. In order for housing construction to take place, the developer has to commit to not do this, so the full time path of subsidy payments to workers must be guaranteed. ■

7.1. Financing city development

Critical to being able to offer the constant real income $\bar{y} = MS(\nu T_{opt})$ at each instant is the ability of the developer to borrow and accumulate debt, so as to smooth income streams. The path of debt incurred by the developer is implicit in the equilibrium outlined above, but here we draw it out explicitly. City debt at date $\tau \in [0, T]$ is given by the value of cumulated subsidy payments less land rents collected, or

$$D(\tau) = \int_0^\tau \big(\nu t s(t) - TR(\nu t)\big) e^{\delta(\tau - t)} dt \tag{24}$$

$$= e^{\delta\tau} \int_0^\tau \big(\nu t[\bar{y} - AS(\nu t)]\big) e^{-\delta t} dt \tag{24'}$$

where integration and the discount factor cumulate past expenditures and the interest on them. Equation (24') comes from noting that $s(t) = \bar{y} - LS(\nu t) = \bar{y} - AS(\nu t) + AR(\nu t)$. The debt path is described by the following corollary.

Corollary 1. Total city debt rises monotonically with city growth up to the last instant of development where the increase is zero. Per person debt is declining towards the end of a city’s growth path. Post-growth, land rents collected exactly equal subsidies paid plus interest payments on the debt.

Proof: See Appendix 5. ■

The underlying paths describing city finance are illustrated in Figure 4 which shows an example of a subsidy schedule, per worker interest charges on the debt, and the level of subsidy minus land rent. Note that subsidies per worker decline and then rise again, mimicking the inverse of the *LS* path so as to maintain constant income. Debt accumulates according to the gap between rents collected versus subsidies paid plus accumulated interest on the debt. Using this information there is a second corollary to Proposition 5.

Corollary 2. Given a balanced budget constraint for the developer from time T on, which requires that rents equal subsidies plus interest on the public debt, the developer by setting $T = T_{opt}$ maximizes the real income payable to residents once the city is stationary.

Proof. See Appendix 5. ■

We argued above that developer profit maximization constrained by the no-switch condition (11) meant that the developer extracted the surplus earnings of later entrants compared to initial ones, so a constant income is paid to all entrants. The corollary confirms what then must result from competition: profits are bid away so that the developer pays the highest constant income possible subject to a zero profit constraint and the requirement that debt be paid off. A complimentary perspective on the developer is that competition to form the current new city requires the developer to pay the highest income possible to the initial residents (so they do not go to other potential new cities), subject to the no-switch constraint that building does not later resume in old cities. That constraint requires that later residents are paid no less than initial ones; i.e. everyone is paid a constant income. If later residents were paid less, then rents would rise in old cities and induce builders to construct houses there because the no switch condition is violated. Equivalently, the developers' solution mimics the outcome of a situation where the city's objective is simply to maximise per worker income once the city is stationary, given the debt repayment constraint. It is the debt repayment constraint that yields a city size and borrowing interval defined by T_{opt}.

Of course, it is essential to this argument that the city developers can neither renounce its debt, nor expropriate house-owners. Clearly, there is an incentive for a stationary city – one which has finished borrowing and in which house construction is complete – to renounce debt. The only collateral for this debt are the city assets, in our model just the housing stock. However builders will not provide housing if it is likely to be seized by debt holders, and lenders will not lend unless there is assurance of repayment. Furthermore, once house construction is complete there is also an incentive for developers to expropriate house-owners by cutting subsidy payments. The final incidence of this falls on housing rent, and builders will not provide housing if they are vulnerable to such expropriation.

These are fundamental problems in city finance, especially in developing countries, where localities have been able to renounce debt, with national governments sometimes then taking on the debt. Home ownership may mitigate the problems, as might constitutional requirements imposed by higher level governments. We assume that these default possibilities and moral hazard issues do not arise.

7.2. Other aspects of the developer equilibrium

First we have the Henry George theorem amended to a dynamic context.

Corollary 3: In the large developer equilibrium, (I) the present value of land rents collected in a city equal the present value of subsidies paid out, so land rents cover all public expenditures. (II) The present value of externalities, $\int_0^T EX(\nu t)e^{-\delta t}dt$, created by the marginal entrant from a city's initial occupation onward equals the present value of subsidies, $\int_0^T s(t)e^{-\delta t}dt$, paid to that entrant.

Proof: See Appendix 5. ■

Note that, in contrast to the static version of the Henry George theorem, there is no equality between subsidies paid at any instant and externalities at that instant. Nor is there any equality between the present value of total subsidies and total externalities. In fact, it is possible to show that total externalities, $\int_0^T \nu t EX(\nu t)e^{-\delta t}dt$, exceed total subsidies, $\int_0^T \nu t s(t)e^{-\delta t}dt$, given $\int_0^T \nu t[MS(\nu t) - MS(\nu T)]e^{-\delta t}dt > 0$ (see the second section of Appendix 2). What matters is the present value of the externalities created by the marginal entrant.

We have several further comments on the developer equilibrium. First, since developers earn zero profits, the order in which they develop is arbitrary. Second, we could construct an identical equilibrium in which the developer owns the housing and subsidy payments are not guaranteed. We need to adjust our equations to subtract the present value of housing costs from the developer's profits and reduce $s(T)$ by δH, so that in the zero profit condition all housing terms net out to zero. Once the developer owns housing, in order to retain residents and cover all costs including the debt payments (which now increase by housing costs), the developer would choose to offer the $s(t)$ schedule, net of housing costs, that we constructed above.

The third comment concerns the starting point for the steady-state, the first city. As long as potential developers are always out there, the first city must follow the specified income and $s(t)$ paths in order to survive competition from a city that would replace it otherwise. But what about a historical, pre-existing (pre-developer) city founded before a country starts the urbanization process? Once new cities form, competition among developers forces the efficient solution with constant real income, $\bar{y} = MS(\nu T_{opt})$. Corollary 2 tells us that it is the best a developer can offer new migrants. In the historical city, whatever its size, housing and possibly land rents (depending on its institutions) will adjust so residents also now receive $\bar{y} = MS(\nu T_{opt})$. And population adjustments may occur also. If for example, the historical city is sufficiently oversized, keeping all residents might require negative housing rents. In that case the city would lose population (and some builders at the city fringe will go out of business), until non-negative rents are restored.

The final comment concerns the use of the large developer paradigm to model city formation. Do the results apply to cities that are controlled by public governments? Suppose local public governments set policies at each instant to reflect the choices of voters, under a perfect information electoral process in which only current city residents vote. Thus at each instant the city government chooses policy to maximize the real income of current residents, and these payments are constrained by the city's debt obligations. City residents vote recognising the implications of their decisions for city debt, for the future growth of the city, and for the incentives of forward-looking house-builders to undertake construction. We assume city governments can fully tax away land rents (as set by the land rent gradient for each city), and are unable to renounce debt or expropriate house-owners. In a companion paper we explore what happens if cities can't borrow, can't tax, or face debt limits.

Assuming city governments can fully tax land rents and borrow without legislated debt limits, voters choose $s(t)$ for the city at each instant, where the relevant voting population grows as the city grows. Voters also choose, at some date, to stop growth, this setting the value of T after which $s(T)$ is constant. A dynamic voting game is a difficult problem and our treatment and findings are limited to showing that, under our assumptions, local public governments can duplicate the outcome of the large developer case.

Proposition 6. A steady state solution with city governments exists and supports the social optimum. Workers' real income is at a constant maximal level in all cities, $\bar{y} = MS(\nu T_{opt})$.

Proof: The income path $\bar{y} = MS(\nu T_{opt})$, as constructed in proposition 1, is feasible. Two sorts of deviation from this path are possible. First, voters may choose to halt growth at some date other than T_{opt}. However, from corollary 2, given cumulated debt, T_{opt} is the date that maximises the income flow at all future dates. A referendum on the date at which to stop growth therefore chooses $T = T_{opt}$. The second possible deviation is that at some date prior to T_{opt} residents choose a subsidy rate that pays themselves an income level greater than $\bar{y} = MS(\nu T_{opt})$. Such a payment increases debt and debt service obligations so what the city can pay in the future is reduced, or $y(T) < \bar{y}$. Any change in the stopping date, $T \neq T_{opt}$, further reduces the city's capacity to both pay $\bar{y}$ and meet debt service (by Corollary 2). This means that once the city is stationary, in order to stop out-migration to other stationary cities (equation (9)) builders would have to offer lower rents and thus would be unable to collect enough rents to cover housing costs. Knowing this, forward-looking builders will cease building (at a non-optimal time for residents) if residents deviate to pay themselves a subsidy so income at any instant exceeds $\bar{y}$. ■

As in the discussion of equilibrium without city governments, sequentially rational builders play a critical role. If citizens try to borrow excessively then it is builders who foresee that the city is not sustainable and will not supply housing. The equivalence of the private and public city outcomes is analogous to that obtained in the static literature.

8. Concluding comments.

In this paper we have developed a dynamic model to analyze the problem of city formation and city size in an economy in which total urban population is increasing. We think that this environment is relevant for many developing countries experiencing rapid urbanization. The dynamic context has a number of advantages. It yields sequential formation of cities, where new cities grow from scratch to a stationary size, rather than instantly jump to that size, as is more consistent with the worldwide data on city formation and growth. It enables the competitive equilibrium to be analyzed free of simple coordination failures. It allows us to see how prices of fixed housing assets vary between growing and stationary cities. And it enables us examine a role for city borrowing and debt accumulation.

We find that socially optimal city size is larger than in a static model; cities should grow beyond the point at which surplus per worker is maximized. The competitive equilibrium with no large agents may support cities that are larger or smaller than socially efficient, depending on how externalities vary with city size. In the competitive equilibrium housing prices in stationary cities cycle with the growth of a new city, mirroring the evolution of per worker income in the growing city, potentially an empirically testable finding. We looked at extensions involving heterogeneity of cities and technical change to show our basic results are robust to heterogeneity which generates a size distribution of cities and to technical change which generates on-going increases in agglomeration benefits. Then we show that large developers or public city government can internalize these externalities and support the social optimum. These institutions also smooth the time path of housing prices in stationary cities and of after-tax income paths in new cities. But the institutional requirements for such equilibria are strong. A companion paper explores what happens if cities face taxation or borrowing constraints.

Appendix 1: Commuting costs and rent gradients

Population at distance z from the CBD is kz^{θ} and commuting costs from this distance are cz^{η}, where $\theta = 0$ or 1, in respectively a linear or circular city and $\eta \geq 1$. Total population in a city of radius $\bar{z}$ is:

$$n = \int_0^{\bar{z}} kz^{\theta} dz = \bar{z}^{1+\theta} k/(1+\theta), \text{ so } \bar{z} = \left[n(1+\theta)/k\right]^{1/(1+\theta)}. \tag{A1}$$

Edge commuting costs are:

$$c\bar{z}^{\eta} = c\left[n(1+\theta)/k\right]^{\eta/(1+\theta)}. \tag{A2}$$

Total commuting costs are

$$TC = \int_0^{\bar{z}} kcz^{\eta} z^{\theta} dz = \frac{ck\bar{z}^{1+\theta+\eta}}{1+\theta+\eta} = \frac{ck}{1+\theta+\eta}\left[\frac{n(1+\theta)}{k}\right]^{\frac{1+\theta+\eta}{1+\theta}}. \tag{A3}$$

We define the parameter $\gamma \equiv (1+\theta+\eta)/(1+\theta) > 1$ and choose units such that $k = 1+\theta$. Edge commuting costs and total commuting costs in Table 1 follow directly. Total land rent is population, n, times edge commuting cost times minus total commuting costs.

Appendix 2: The social optimum

In this appendix we develop three properties that define the form we give to equation (2).

1) Inefficiency of jump solution

If workers can be moved between cities at zero cost then a new city size can jump to some size. Suppose that instead of growing continuously through interval $[0, T]$ a new city jumps at date ζ to population $\nu\zeta$. For $t \in [0, \zeta]$ new migrants accumulate in old cities. The cost of this jump is that new migrants have to be accommodated in existing cities until they jump to a new city where housing is then built for them. The present value of the extra housing costs incurred over $[0, T]$ is:

$$C = \delta H\nu\zeta\int_0^T e^{-\delta t}dt + H\nu\zeta e^{-\delta\zeta} - H\nu\int_0^{\zeta} e^{-\delta t}dt.. \tag{A4}$$

The first term is the cost of holding $\nu\zeta$ houses in old cities. The remaining terms give the cost saving in the new city from the fact that $\nu\zeta$ units of housing are constructed at date ζ (second term) rather than being constructed continuously through $t \in [0, \zeta]$. Using integral (A10) below, this expression integrates to:

$$C = \delta H\left[\nu\zeta\int_{\zeta}^{T} e^{-\delta t}dt + \int_{0}^{\zeta}(\nu\zeta - \nu t)e^{-\delta t}dt\right] \tag{A5}$$

which can be interpreted directly as the present value of the rental income foregone on having empty houses. The term outside the square brackets is the rental income per house. Inside, the first term is the number of houses that are empty for $t \in [\zeta, T]$, and the second the number that are empty at date $t \in [0, \zeta]$, each discounted back to date zero.

The benefit of jumping is the value of putting new workers arriving during $t \in [0, \zeta]$ in an established city with average surplus $AS(\nu T)$ rather than in a growing city with surplus $AS(\nu t)$,

$$B = \int_{0}^{\zeta}\nu t AS(\nu T)e^{-\delta t}dt - \int_{0}^{\zeta}\nu t AS(\nu t)e^{-\delta t}dt. \tag{A6}$$

Hence, the net benefit is

$$\begin{aligned} C - B &= \delta H\left[\nu\zeta\int_{\zeta}^{T} e^{-\delta t}dt + \int_{0}^{\zeta}(\nu\zeta - \nu t)e^{-\delta t}dt\right] - \int_{0}^{\zeta}\nu t[AS(\nu T) - AS(\nu t)]e^{-\delta t}dt \\ &= \delta H\left[\nu\zeta\int_{\zeta}^{T} e^{-\delta t}dt + \int_{0}^{\zeta}(\nu\zeta - 2\nu t)e^{-\delta t}dt\right] + \int_{0}^{\zeta}\nu t\left[\delta H - [AS(\nu T) - AS(\nu t)]\right]e^{-\delta t}dt. \end{aligned} \tag{A7}$$

Setting (A7) equal to zero defines the value of H above which it is not profitable to jump. In the second line, the first square bracketed expression is positive. [Note the second integral in that first square brackets is positive. In that integral; the term in parentheses under that integral is declining in *t;* the integral is zero without discounting; and thus it is positive with discounting.]. A sufficient condition for $C > B$, then is that the second square bracketed term is also positive. Assumption A4 ensures this.

2) Inefficiency of simultaneous development of multiple new cities

Redefine the problem in equation (2) to have β cities form at time 0 and each grow at a rate ν/β for a length of time T. Optimising gives the same first order condition for T as in the text (although different optimal values, as functions depend on $\nu t/\beta$). The first order condition for β is

$$[1-e^{-\delta T}]\frac{\beta}{\nu}\frac{d\Omega}{d\beta} = \delta^{-1}\left[\int_0^T (MS(\frac{\nu}{\beta}t)-MS(\frac{\nu}{\beta}T))e^{-\delta t}dt\right] - \left[\int_0^T (MS(\frac{\nu}{\beta}t)-MS(\frac{\nu}{\beta}T))te^{-\delta t}dt\right] \qquad \text{(A8)}$$

On the RHS, given eq (4), the first term in square brackets is zero (implied by the first order condition defining the optimal T). Following the analysis of equation (4) and assumption A3, given the first term is zero, then the second term in square brackets must be positive (negative items in the expression get low t weights and positive ones high t weights). Thus the whole condition is negative, indicating that increases in β reduce welfare. β is bounded below by one, the case we solve for in the text. Having multiple cities growing at different rates (i.e. a non-symmetric outcome) doesn't change the principle. Whatever the growth rates, we still want to minimize the number of cities.

3) Inefficiency of stopping and restarting growth

Why not halt a city's growth before an optimal T, start a new city and then later resume growth of the first city? Consider any sequence of this type where at any instant only one city is growing. Pick a date at which growth cycles of all existing cities are complete. At that date, the undiscounted total surplus is the same regardless of the sequencing. However starting with the first city, given discounting, prematurely stopping growth advances [delays] the date of low [high] *MS* values, lowering the present value of total surplus.

Appendix 3: Derivations, Section 3

1) Derivation of equation (4)

Integration by parts gives the expression:

$$\left[\int_0^T TS(\nu t)e^{-\delta t}dt + \frac{TS(\nu T)e^{-\delta T}}{\delta}\right] = \left(\frac{\nu}{\delta}\right)\int_0^T MS(\nu t)e^{-\delta t}dt. \qquad \text{(A9)}$$

Using this in equation (3) gives equation (4).

2) Proof of Proposition 1

In the static model welfare is maximized at the peak of the *AS* schedule, T_A ($\equiv n_A/\nu$) where $AS(\nu T_A) = MS(\nu T_A)$. Writing $TS(\nu t) = \nu t AS(\nu t)$ and using

$$\left[\frac{1-e^{-\delta T}}{\delta^2}\right] = \int_0^T te^{-\delta t}dt + \left[\frac{Te^{-\delta T}}{\delta}\right], \qquad \text{(A10)}$$

to get an expression for $\left[1-e^{-\delta T}\right]/\delta^2$, first order condition (4) can be written as

$$\int_0^T [MS(\nu T) - AS(\nu t)]\nu t e^{-\delta t}dt + [MS(\nu T) - AS(\nu T)]\nu T e^{-\delta T}/\delta = 0. \tag{4'}$$

To prove that $T_{opt} > T_A$, suppose not. If $T_{opt} < T_A$, then from assumption A2 and its implications, both terms in square brackets in (4') are greater than zero (see Figure 1). At $T = T_A$ the first term is strictly positive and the second zero. At $T > T_A$ the second term is negative and strictly decreasing. The first term is strictly decreasing and eventually becomes negative given that $MS(\nu T_A)$ declines continuously for $T > T_A$. Thus equation (4') has a unique solution at $T_{opt} > T_A$. Note that at the solution to the first order condition, the second derivative of the objective is $d^2\Omega/dT^2 = MS'(\nu T)\nu^2/(\delta(e^{\delta T}-1))$. This is negative for $T > T_A$, ensuring a maximum. ■

Appendix 4:

1) The example in Figure 3

The figure was constructed with both heterogeneity and technical change using:

$$LS_i(n_i, t) = 5a_i + \left(n_i e^{0.0002t} - 0.4(n_i/a_i)^2\right)/2 - cn_i^{\gamma - 1}$$

with $c = 0.2$, $\gamma = 2$, and efficiency level a_i following sequence {1.0, 0.975, 0.95, 0.925, 0.9, 0.875, 0.85} $\delta = 0.008$ and $\nu = 0.012$ for $t \in [0, 1000]$ and $\nu = 0$ thereafter.

2) Heterogeneity and resumed growth

Here we provide the elements of a proof under heterogeneity that there is no positioning of relative values of $LS_2(\nu T_2),\ LS_1(\nu T_1),\ LS_3(\nu T_3)$, etc. that can constitute an equilibrium, absent intervals of parallel growth. The key is to note that competition among builders requires that, in equilibrium once cities cease their interval of solo growth, the present value of future rents must equal the opportunity cost of housing (noting opportunity costs are covered while a city is growing), or $H_1=H_2=...=H$ from equation (7). The proof has two parts.

a) We can't have $LS_1(\nu T_1) > LS_2(\nu T_2),\ LS_2(\nu T_2) > LS_3(\nu T_3)$, and so on. If, say the first inequality holds it would have been better to resume building in city 1 rather than continuing in city 2 during city 2's solo growth interval, once city 2 has grown for longer than the t where $LS_1(\nu T_1) = LS_2(\nu t)$.

That is, the based on the definitions of $H_{i.}$'s (see below and in Section 4), H_1 would exceed H_2. The same arguments apply in comparing city 2 and 3 and so on.

b) What about $LS_1(\nu T_1) \le LS_2(\nu T_2),\ LS_2(\nu T_2) \le LS_3(\nu T_3)$, etc. ? To show these can't hold, we define the equations that need to hold for $H_1 = H$, $H_2 = H$, etc. While we write the equations as though there are never episodes of resumed growth, resumed growth simply truncates the expressions, since resumed growth in equilibrium requires the present values of future rents equal opportunity costs in all cities. From $H_1=H$, we know

$$\int_0^{T_2} [LS_1(\nu T_1) - LS_2(\nu t)]e^{-\delta t}dt + e^{-\delta T_2}\int_0^{T_3} [LS_1(\nu T_1) - LS_3(\nu t)]e^{-\delta t}dt$$
$$+ e^{-\delta(T_2+T_3)}\int_0^{T_4} [LS_1(\nu T_1) - LS_4(\nu t)]e^{-\delta t}dt + \ldots = 0 \qquad \text{(A11)}$$

Similarly from $H_2=H$ we know

$$\int_0^{T_3} [LS_2(\nu T_2) - LS_3(\nu t)]e^{-\delta t}dt + e^{-\delta T_3}\int_0^{T_4} [LS_2(\nu T_2) - LS_4(\nu t)]e^{-\delta t}dt + \ldots = 0 \qquad \text{(A12)}$$

If (A12) holds with $LS_1(\nu T_1) \le LS_2(\nu T_2)$, then the collective of the second, third and all remaining parts of (A11) are non-positive. Then, for (A11) to hold the first part must be nonnegative. But that is impossible. Compared to (A12), the first term in (A11) has a smaller positive part ($(LS_1(\nu T_1))$ and a larger negative part ($LS_2(\nu t)$). The fact that $T_2>T_3$ only increases the negativity since it adds in terms where $LS_2(\nu t) > LS_2(\nu T_2) \ge LS_1(\nu T_1)$. The same arguments apply in moving forward to the 4th, 5th and so on cities. Therefore an equilibrium with heterogeneity can't have $LS_3(\nu T_3),\ LS_2(\nu T_2),\ LS_1(\nu T_1)$ as permanently stationary values. Each solo growth episode is followed by simultaneous growth of all cities.

Appendix 5: Derivations, Section 7

1) Optimization problem (19)

The Lagrangean corresponding to (21) is:

$$L \equiv \int_0^T [TR(\nu t) - \nu ts(t)]e^{-\delta t}dt + [TR(\nu T) - \nu Ts(T)]\frac{e^{-\delta T}}{\delta}$$

$$+ \int_0^T \lambda(\tau)\left[\int_\tau^T [LS(\nu t) + s(t) - \bar{y}]e^{-\delta(t-\tau)}dt + [LS(\nu T) + s(T) - \bar{y}]\frac{e^{-\delta(T-\tau)}}{\delta}\right]d\tau \tag{A13}$$

where the function is written with the constraint as an integral over τ from 0 to *T* with multipliers λ(τ). The first order condition for *s(t)* at date *z* is

$$\nu z e^{-\delta z} = \int_0^z \lambda(\tau)e^{-\delta(z-\tau)}d\tau \tag{A14}$$

from which $\nu e^{-\delta z} = \lambda(z)$. This is strictly positive at all dates, so the constraint binds. Note (A13) can also be maximized to solve for the optimal *T*, where the solution has
$s(T) = EX(\nu T),\ \bar{y} = LS(\nu T) + EX(\nu T) \equiv MS(\nu T)$, as in the text.

2) Proof of Corollary 1

For the debt expressions, from equation (24),

$$D(\tau) = e^{\delta\tau}\int_0^\tau (\nu t\bar{y} - TS(\nu t)])e^{-\delta t}dt. \tag{A15}$$

Using (A9) and (A10) this can be integrated to give

$$\delta D(\tau) = \nu\bar{y}\left[\frac{e^{\delta\tau} - 1}{\delta} - \tau\right] + TS(\nu\tau) - \nu e^{\delta\tau}\int_0^\tau MS(\nu t)e^{-\delta t}dt. \tag{A16}$$

At date $\tau = T_{opt}$ this expression reduces to

$$\delta D(T_{opt}) = TS(\nu T_{opt}) - MS(\nu T_{opt})\nu T_{opt}. \tag{A17}$$

(derived using equation (4) and noting $\bar{y} = MS(\nu T_{opt})$). This says that debt service is equal to total surplus minus real income payment to workers, equal in turn to rents minus subsidies, so
$\delta D(T) + \nu Ts(T) = TR(\nu T)$.

Differentiating (A16) with respect to time, τ,

$$D'(\tau)\frac{e^{-\delta\tau}}{\nu} = \bar{y}\left[\frac{1-e^{-\delta\tau}}{\delta}\right] - \int_0^\tau MS(\nu t)e^{-\delta t}dt = \int_0^\tau[\bar{y} - MS(\nu t)]e^{-\delta t}dt. \qquad \text{(A18)}$$

Given that $\bar{y} = MS(\nu T)$, $D'(\tau) \to 0$ as $\tau \to T_{opt}$, from eq.(4). In the last term in (A18), the term in the integral is positive for small τ, and then eventually declines monotonically, given assumption A3 (see Figure 1). Thus the integral starts positive, increases, and then decreases monotonically until it is zero at T_{opt}, implying that total debt is always increasing up to T_{opt}. Since $D'(\tau) \to 0$ as $\tau \to T_{opt}$, it must be the case that, with strictly positive population growth until T_{opt}, debt *per worker* peaks at some point and then declines.■

3) Proof of Corollary 2

Steady state income is $y = AS(\nu T) - \delta D(T)/\nu T$. Using (A16) and rearranging gives

$$y = \frac{e^{\delta T}}{T}\left[\int_0^T MS(\nu t)e^{-\delta t}dt - \delta\bar{y}\left(\frac{1-e^{-\delta T}}{\delta^2} - \frac{Te^{-\delta T}}{\delta}\right)\right]. \qquad \text{(A19)}$$

Optimizing with respect to T and then substituting in $\bar{y}=MS(\nu T_{opt})$ we get

$$\frac{e^{\delta T}}{T}[\delta T-1]\int_0^T[MS(\nu t)-MS(\nu T_{opt})]e^{-\delta t}dt + [MS(\nu T)-MS(\nu T_{opt})] = 0$$

This equation is satisfied at $T=T_{opt}$, as is the second order condition.■

4) Proof of Corollary 3

The first result follows trivially from imposing the zero profit equilibrium condition on the objective function in eq. (21'). For the second, given $\bar{y} = MS(\nu T) = LS(\nu T) + s(T)$ and given the definition of the externality in Table 1, from T onwards the subsidy equals the Pigouvian tax on externalities or $s(T) = EX(\nu T)$. Thus for (II) the remaining requirement is to show that the present value of externalities from 0 to T of a marginal entrant,

$\int_0^T EX(\nu t)e^{-\delta t}dt = \int_0^T[MS(\nu t)-LS(\nu t)]e^{-\delta t}dt$,

equals that entrant's present value of subsidies $\int_0^T s(t)e^{-\delta t}dt = \int_0^T[MS(\nu T)-LS(\nu t)]e^{-\delta t}dt$.

Comparing the two, the LS terms cancel out and the two remaining terms are equal from (23).■

Endnotes:

1. These urban scale economies can be given a variety of micro foundations; see Duranton and Puga (2004).

2. In this case we would add an income term to (1), the worker's share in national urban land rents, which is perceived as fixed by any worker.

3. New development does not occur simultaneously in more than one new city for 'stability' reasons. Having migrants go to more than one new city is not robust to population perturbations because urban agglomeration effects cause real incomes to rise with city scale

4. If we continue with the notation in (5) - (7), substituting (6) into (7), the equilibrium condition for housing construction at date T_1 is

$$H = \hat{H}_1 = \sum_{i=2}^{\infty} \int_{\Gamma_{i-1}}^{\Gamma_i} \left([LS(\nu T_1) + s_1(T_1)] + \delta H - [LS(\nu(t - \Gamma_{i-1}) + s_i(t)] \right) e^{-\delta(t-T_1)} dt.$$

Imposing symmetry, so all T's and $s(\cdot)$ schedules are the same, evaluating gives equation (9). To see this, in evaluating the expression pull $e^{\delta T_1}$ outside the summation sign and recognize that the summation infinitely repeats a fixed expression so that expression is multiplied by $e^{-\delta T} + e^{-\delta 2T}$ +....which equals $1/(1- e^{-\delta T}) - 1$.

5. Suppose that the first city stops growing just before T_{eq}. Then its $LS(\nu T_{eq})$ would be somewhat greater, which shifts up its $\hat{h}(t)$ curve at all future dates, given the path $LS(\nu t) - \delta H$ of new growing cities. This means that future house rents in this city would be somewhat higher, making it profitable to continue building, rather than stopping and switching to a new one. And if we looked at a potential equilibrium where all cities operated with a lower T_{eq}, not only are the $\hat{h}(t)$ curves shifted up, their later parts where rents are less than opportunity costs are cut off, furthering the incentive to continue building in cities until T_{eq} is reached. Similarly, if a builder supplies housing beyond T_{eq}, that lowers $LS(\nu T_{eq})$ and shifts down the $\hat{h}(t)$ path the builder will receive once the city is stationary, lowering rents so that their present value will no longer cover housing cost.

6. We note also that T_{eq} gives a size which maximizes the present values at date of their entry of the incomes of all entrants. For entrants at date τ, the present value of income net of housing costs is

$$\int_{\tau}^{T} LS(\nu t) e^{-\delta(t-\tau)} dt + \frac{e^{-\delta(T-\tau)}}{1 - e^{-\delta T}} \int_0^T LS(\nu z) e^{-\delta z} dz - H$$

The first term is the present value to entrants at time τ of their income during the remaining growth time of the city. In the second term, the integral expression gives the present value of income net of housing costs for any resident of the city in steady state during the growth cycle of each successive new city. This cycle repeats indefinitely but only starts after a time length (T -

τ) (hence the term $e^{-\delta(T-\tau)}/(1-e^{-\delta T})$ before the second integral). Maximizing this expression with respect to T gives eq. (10), for any τ. The intuition is that changing T only changes final income and the future income net of rent cycles after the city stops growing; these changes apply to everyone regardless of date of entry.

7. With technical progress there is an algebraic solution but it is much more complicated. The problem is that technical progress causes incomes in cities with stationary populations to continue to grow, possibly at different rates (given technical change is interacted with size). These cities therefore may restart growth at different times (higher quality sooner), and may also stop parallel growth at different times. In this more complicated case, to solve the model using housing no-switch conditions, requires the application of these conditions between all pairs of existing cities.

8. Multiple cities growing in parallel each absorb a fraction of the ν flow. The fraction each city absorbs over the interval of parallel growth is solved by the expansion of (17c) to more cities.

9. Suppose in our symmetric world where urban sites are identical, at the time of unanticipated technological change, there are *m* cities of size n_L and a new growing city. Depending on whether the new growing city was at relatively advanced stage in its growth process and on whether, when the technological change shifts *LS* up and out, at their n_L size, existing cities are on the downward verus upward portion of the new LS curve, we have three possible patterns: (i) the new city may continue solo growth, followed by simultaneous growth of all cities before a new city forms; (ii) old cities may grow in parallel for an interval before the pattern in case (i) sets in; or (iii) old cities may grow in sequence until either the case (i) or (ii) pattern sets in. Cases (i) and (ii) cover the situations where after technical change existing cities are on the downward sloping portions *LS* curves. Case (i) occurs if the new growing city is big enough and case (ii) if it is not. What is the difference between the two cases? Define technical change to occur at time T_1 where there are *m* stationary cities and define $T=n_L/\nu$. We apply the no-switch condition from (11) (with s(t) set to 0) to determine whether, at T_1, building first continues in the new city (versus resumes the bigger existing cities). Where *LS** denotes the new *LS* curve, building continues in the new city and the first case holds if

$$\int_0^{(m+1)T-T_1} LS^*(\nu t)e^{-\delta t}dt \ \geq \ LS^*(\nu T)(1-e^{(m+1)T-T_1})/\delta\,.$$

The equation says the benefits of growing the new city until it catches up to old cities at time $(m+1)T$ exceed those of putting a building in any old cities in that time interval. If the LHS is less than the right at time T_1 then old cities grow until the two sides are equal and then the sequence of case (i) kicks in. Case (iii) occurs if, after technical change, existing cities are on the *upward* sloping part of their new *LS ** curves, rather than downward sloping. Before we apply case (i) or (ii), there is an interval where all old cities each grow in sequence to exploit the rising part of the *LS** curve and the length each old city grows in sequence is defined by an application of equations (5) - (10). The length of this interval T_b is defined by

$$\int_0^{T_b} LS(\nu(T+t))\ e^{-\delta t} dt \ = \ LS(\nu(T+T_b))\ (1-e^{-\delta T_b})/\delta \,.$$

10. To generate Zipf's law these papers have to impose an arbitrary lower bound on how far unlucky cities (those hit with a succession of bad draws) can decline. Here it might seem that durable capital would give that lower bound naturally; but that is not the case. If conditions are bad enough not even rent declines can hold people in a very unlucky city; it would be deserted.

11. These large swings raise the possibility that the condition for house rents to be non-negative could be violated, in the sense that Assumption A4 does not ensure that $\delta H > MS(\nu T_{opt}) - MS(\nu t)$ for all t.

References:

Abdel-Rahman, H, and A. Anas (2004), "Theories of Systems of Cities"," in J.V. Henderson and J-F Thisse (eds), *Handbook of Urban and Regional Economics, Volume 4, Cities and Geography*, North Holland, in press.

Anas, A (1992), "On the Birth and Growth of Cities," *Regional Science and Urban Economics*, 22, 243-258

Black D. and J.V. Henderson (1999), "A Theory of Urban Growth," *Journal of Political Economy*, 107, 252-284.

Black D. and J.V. Henderson (2003), "Urban Evolution in the USA", *Journal of Economic Geography*, 3, 343-372

Brueckner J. (2000) "Urban Growth Models with Durable Housing" in *Economics of Cities* (J-M Huriot and J-F Thisse (eds), Cambridge University Press.

Cordoba J-C (2004), "On the Size Distribution of Cities", http://www.ruf.rice.edu/~jcordoba/cordoba

Cuberes D. (2004), "The Rise and Decline of Cities" http://home.uchicago.edu/~dcuberes/David%20Cuberes_files/page0002.htm

Duranton G, and Puga D (2004) "Micro-Foundations of Urban Agglomeration Economies" in J.V. Henderson and J-F Thisse (eds), *Handbook of Urban and Regional Economics, Volume 4,* *Cities and Geography*, North Holland.

Duranton G. (2004), “Urban Evolutions: the Still, the Slow and the Fast”, http://158.143.49.27/~duranton/research.html#CitySize

Fujita M (1978), *Spatial Development Planning*, Chapter 6 and Appendix F. Amsterdam: North Holland.

Glaeser E.L. and J. Gyourko (2005) “Urban Decline and Durable Housing,” *Journal of Political Economy*, forthcoming

Helsley R. and W. Strange(1994), “City Formation with Commitment,” *Regional Science and Urban Economics*, 24, 373-390.

Henderson, J.V. (1974), “The Sizes and Types of Cities”, *American Economic Review*, 61, 640-56.

Henderson J.V. and H-G Wang (2005) “Urbanization and City Growth” http://www.econ.brown.edu/faculty/henderson/papers.html

Junius K. (1999) “Primacy and Economic Development: Bell Shaped or Parallel Growth of Cities”, *Journal of Economic Development*, 24(1), 1-22

Mills E.S. (1972), *Studies in the Structure of the Urban Economy*, Johns Hopkins University Press.

Rossi-Hansberg E. and M.L.J. Wright (2004) “Urban Structure and Growth” http://www.stanford.edu/~erossi/papers.html.

Venables, A.J. (2005), "Spatial Disparities in Developing Countries: Cities, Regions, and International Trade", *Journal of Economic Geography*, forthcoming

World Bank (2000), *Entering the 21st Century: World Development Report 1999/2000*, Oxford University Press.

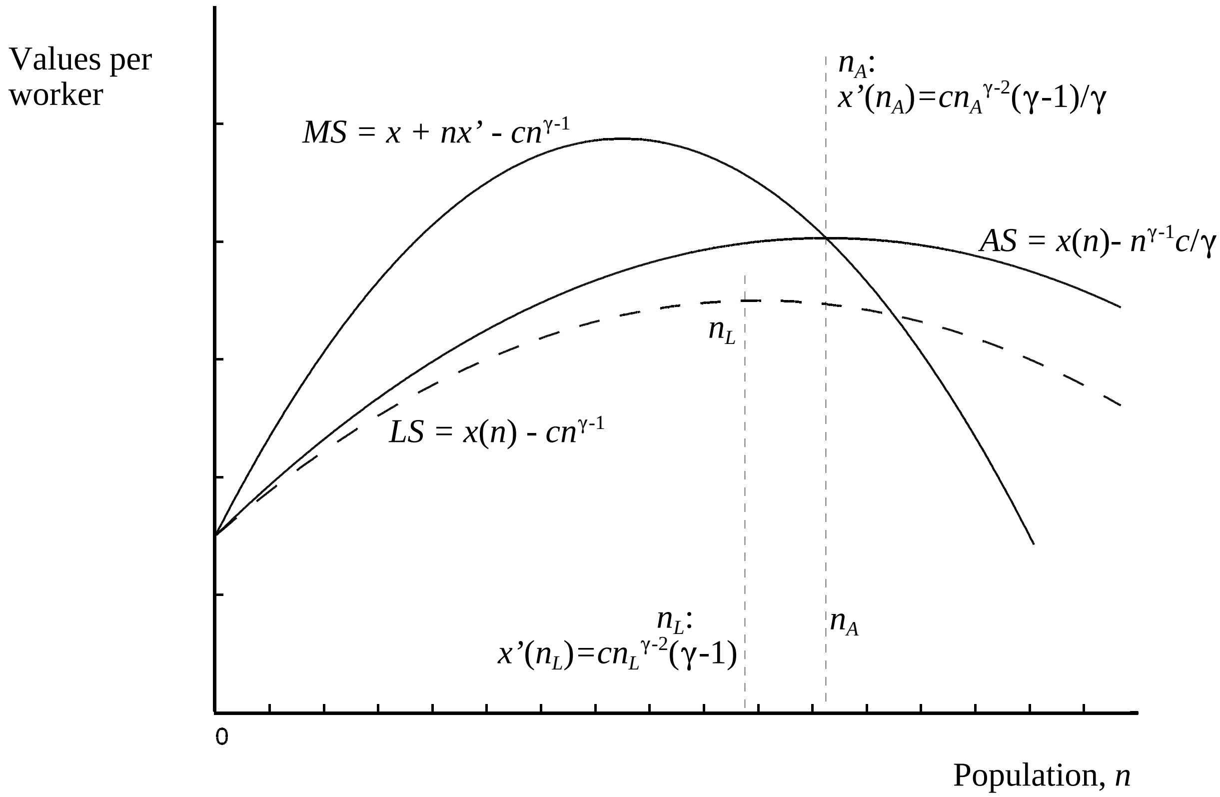

Figure 1: Surplus per worker

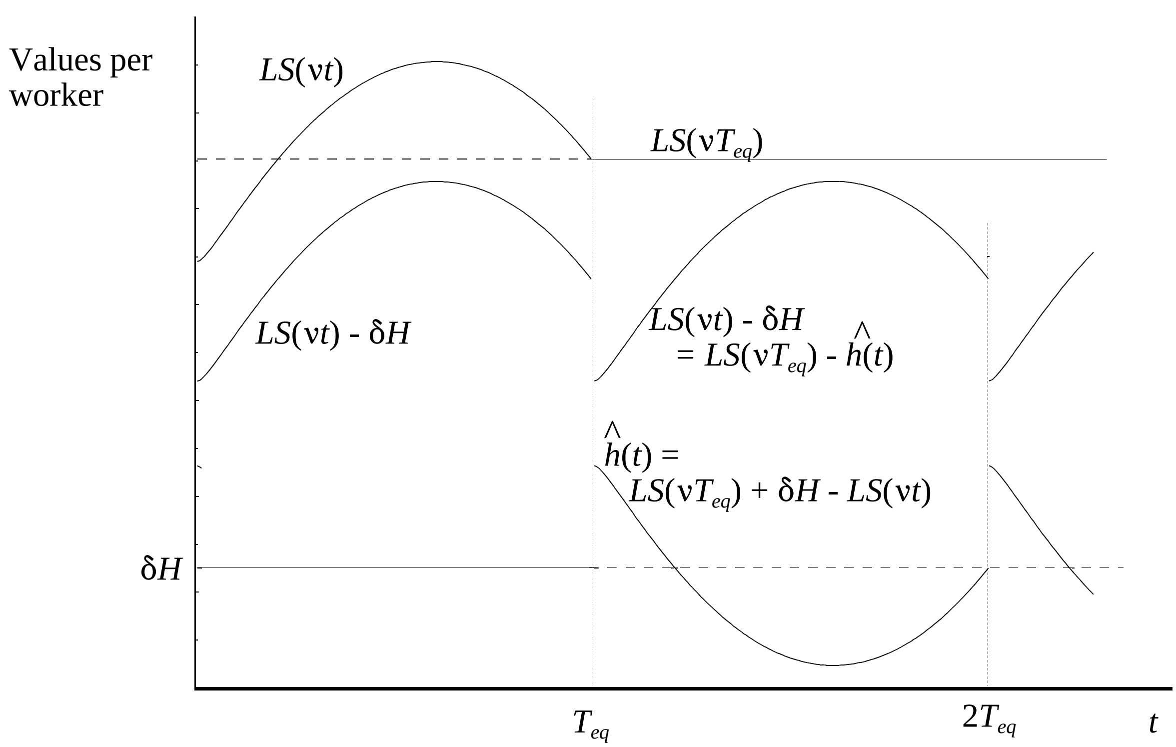

Figure 2: Income and housing rent in the competitive equilibrium:

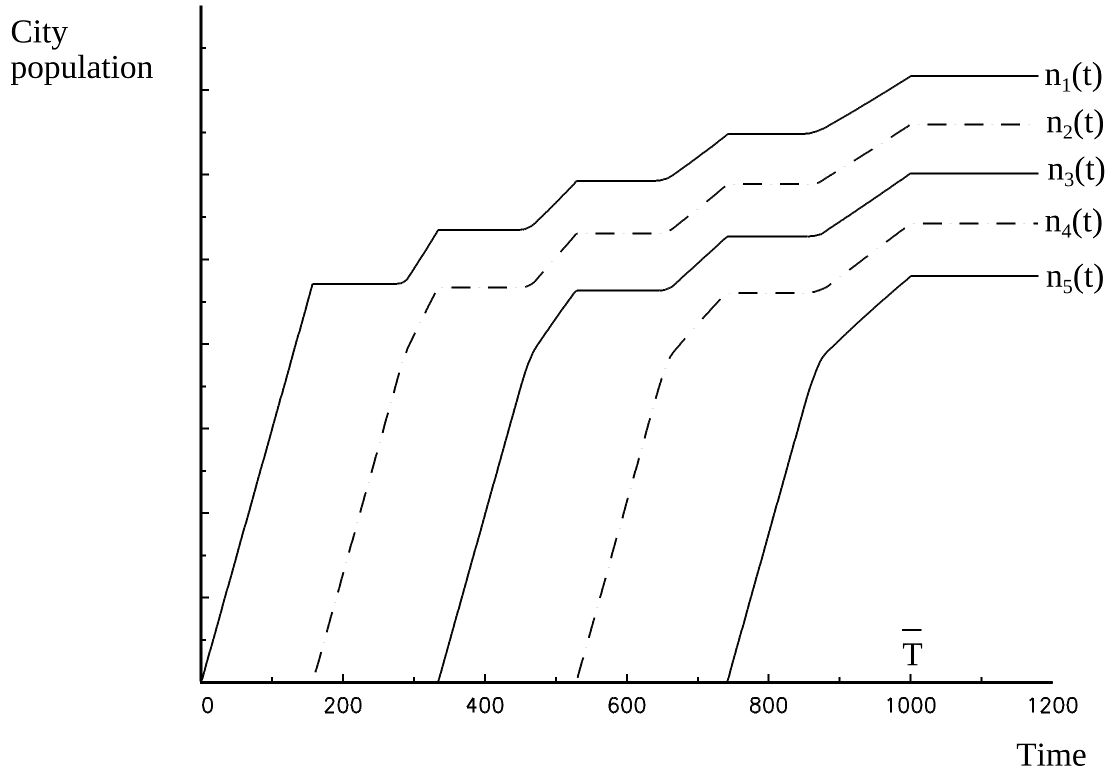

Figure 3a: City size

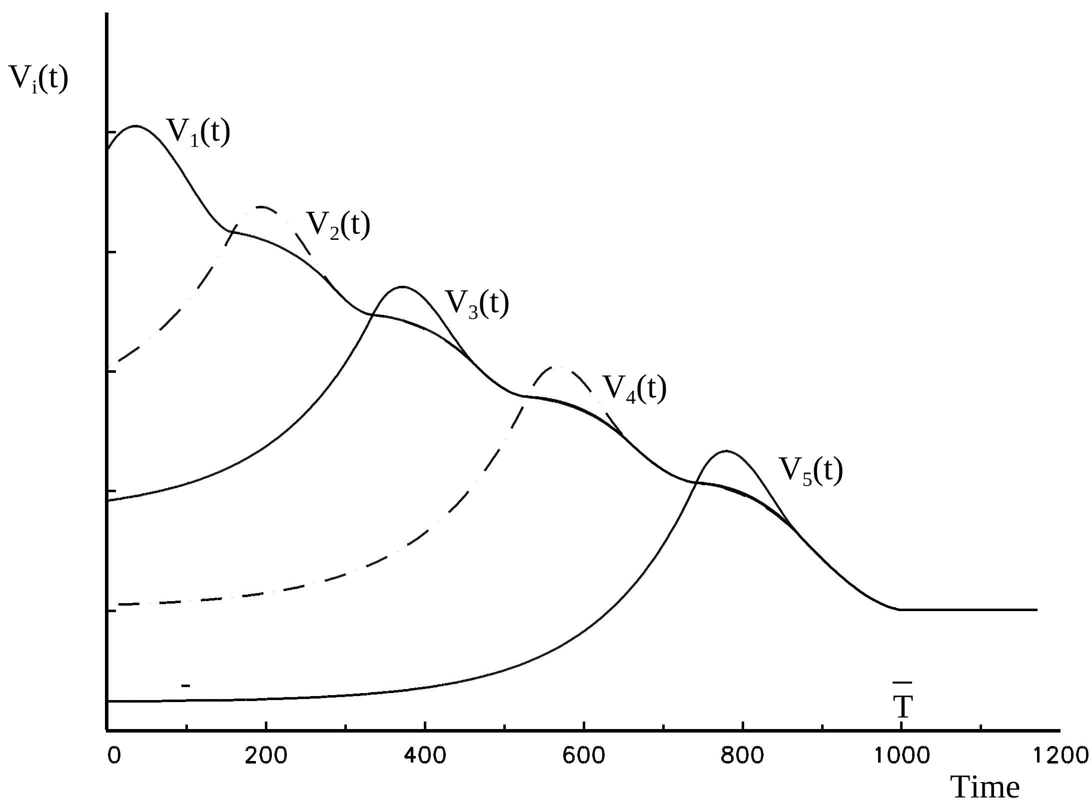

Figure 3b: Present values

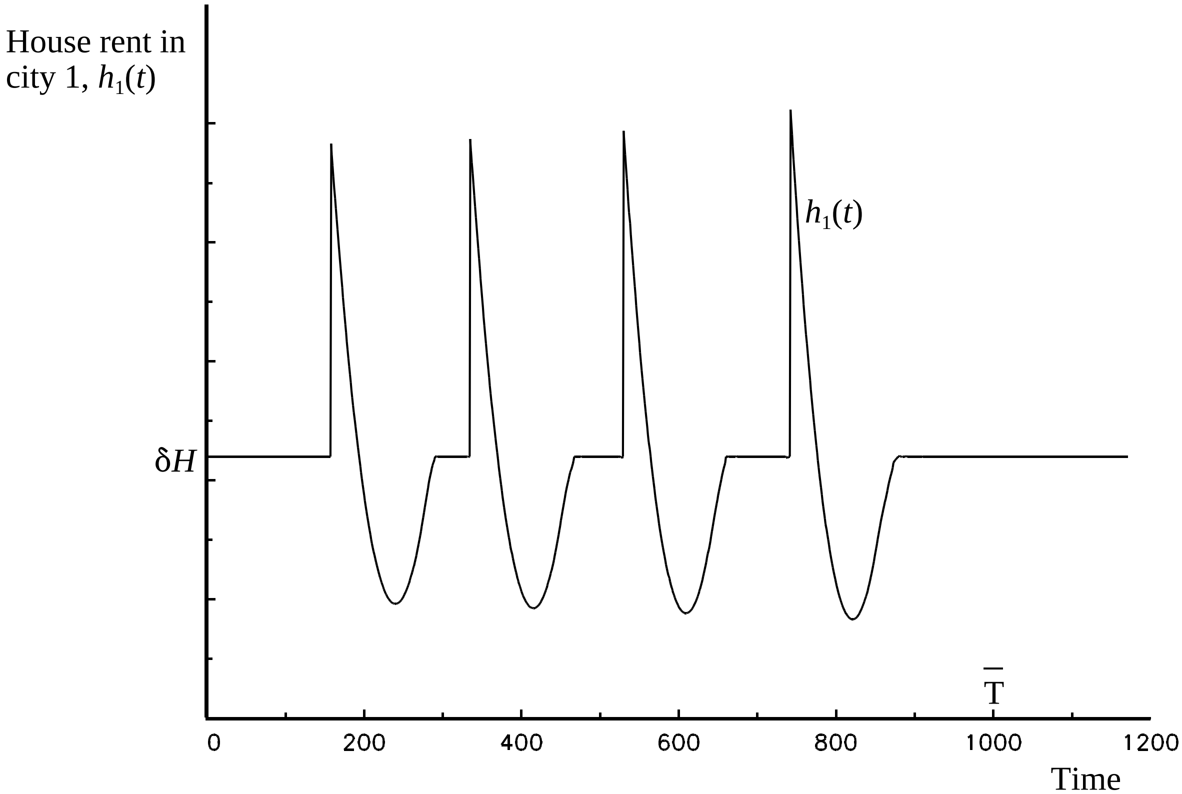

Figure 3c: City 1 house rents

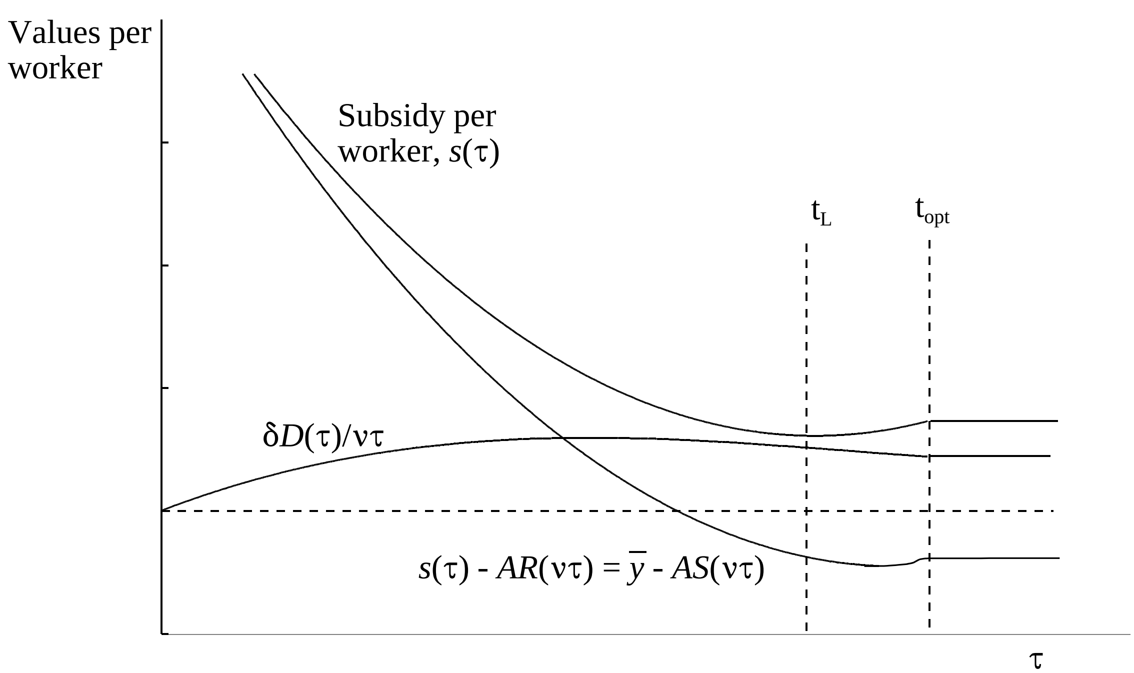

Figure 4: Large developer’s subsidy and debt service (per worker)