Incomplete Self-Enforcing Labor Contracts

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Abstract

TBA

1. INTRODUCTION

It is a well-known fact that, when the elasticity of labor supply of the representative agent is set to match the estimates from micro-data, the standard real business cycle model cannot account for the empirical volatility and correlation of hours, wages and output. In particular, given that productivity shocks are transitory, the theory predicts that hours and wages will approximately trace out the static labor supply curve over the business cycle. In turn, given that the empirical individual labor supply curve is relatively rigid, the theory predicts that the standard deviation of both wages and output will be much greater than the standard deviation of hours. Empirically, in the post-war U.S. economy total hours have the same volatility as output while wages are nearly acyclical (Kydland, 1995). (*too hard*)

*(The existing literature has attacked the puzzle....)

Azariadis (1975) and Baily (1974) noticed that—when workers are risk-averse and cannot get insurance against shocks to their productivity in the formal markets due to a moral hazard problem—then firms find profitable to step-in and offer implicit insurance to their employees. In the implicit employment-insurance contract, the wage combines the compensation for the service of labor and an insurance premium or indemnity, depending on the realization of the state of nature. Therefore, the wage needs not to be equal the marginal product of labor at every date and state and typically it is less volatile than the marginal product.

While the theory of implicit contracts offers a natural explanation for the relative volatility of wages and output observed in the U.S. business cycle, it cannot account for the relative volatility of

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hours and output. To see why this is the case, notice that the wage and number of working-hours that the optimal employment-insurance contract specifies for every realization of the state of nature must be ex-post Pareto-efficient. Because of the first welfare theorem, also in a spot labor market the equilibrium wage and working-hours are Pareto-efficient for every realization of the state of nature. Therefore, as long as the income effect on leisure is negligible, total hours in the implicit contract economy are the same as in the spot labor market economy.

*(Moreover they cannot explain wage rigidity on the extensive margin)

In this paper, we elaborate on the idea that a firm and its employees are trading labor and insurance simultaneously and develop a theory of employment-insurance contracts that has implications not only for the distribution of resources between the firm and its employees, but also for the total amount of labor employed in equilibrium. More specifically, we consider a model economy populated by risk-neutral firms and risk-averse workers that have no access to the financial markets (inter-date and inter-state trade). The labor market is competitive—in the sense that firms post employment-insurance contracts and workers decide which productive location to visit—and frictional—in the sense that the set of jobs that a worker is qualified to fill at a certain firm becomes known only after a time-consuming application process. Finally, we take the view that the enforcement of contracts is limited in the sense that both the firm and the worker can unilaterally renege on the contract at any point in time.

In this model economy, when the firm goes to the labor market and posts an employment contract to attract applicants for its vacant positions, it will also attract some workers that are qualified for jobs that are not vacant yet. And when the firm can choose between two or more workers to fill-in the same position, it will pick the cheapest employee irrespective of any seniority considerations. This moral-hazard problem creates a tension between the goals of risk-sharing and hiring whenever the state of nature is such that the value of the contract for attracting the efficient number of applicants falls below the continuation value of the full-insurance contract for some of the senior employees. The optimal employment-insurance contract efficiently trades-off risk-sharing and hiring by prescribing not only what wage the firm should pay its employee at every date and state, but also what contract the firm should offer in the labor market.

As a first contribution to the analysis of optimal limited-commitment contracts in competitive search markets, this paper studies the simple case where workers are infinitely lived and firms are only active for two periods. Taking as given the value of the contracts offered by the firm to the workers hired in the first period, the optimal limited-commitment contract has always the same structure. If the second-period productivity of the firm turns out to be sufficiently high, the optimal contract does not distort either the risk-sharing or the hiring margins: senior employees receive the same wage as in the past and newly hired workers are paid the wage that attracts the efficient number of applicants. Moreover, because in this region of the productivity space the optimal hiring wage is greater than the wage that guarantees perfect consumption smoothing to senior employees, the firm does not strategically terminate any matches. If the realization of the firm's productivity falls in a non-empty interval to the left of the no-distortion region, the optimal contract prescribes a common wage for senior and junior employees. This firm-wide wage is smaller than the wage paid to senior employees in their first period of service and greater than the efficient-hiring wage. In this region, the optimal contract distorts both the risk-sharing and the hiring margin in order to avoid the moral hazard problem generated by imperfect commitment. To the left of the double-distortion region, there is an interval of productivity values conditional on which the optimal contract prescribes that the wage paid to senior employees is kept constant and the wage offered to new employees is distorted downwards. In this (possibly empty) region, senior employees face a positive probability of being replaced by cheaper untenured workers. Finally, if the second-period productivity is sufficiently low, the firm withdraws from the job market.

By distorting the hiring wage away from its ex-post optimal level, the contract under limited commitment has an impact on the level of employment within a firm and across the whole economy. From the perspective of an established firm, the level of net hiring is efficient when the productivity shock is positive, is inefficiently high in response to a small negative shock and is inefficiently low in response to a large negative shock. From the perspective of a potential entrant, an upward (downward) distortion in the wage posted by the established firms implies a lower (higher) probability of filling its positions at any given wage and, in turn, a lower (higher) return from entering the market creating new jobs. Even though we do not carry out the general equilibrium analysis, it is safe to argue that our theory implies that—in response to a small negative aggregate productivity shock—the creation of new vacancies is smaller and the increase in the unemployment rate is greater than in the full-commitment economy. On the other hand, when the economy is hit by a positive productivity shock, the creation of new vacancies and the fall in the unemployment rate is efficient. Overall, our theory of wage determination offers a mechanism that amplifies unemployment fluctuations and dampens wage movements and, in this sense, it successfully accomplishes the mission that the implicit contract literature had set itself to thirty years ago.

1.1. Related Literature

An earlier strand of literature modifies the basic implicit contracts model by assuming that the realization of the state of nature is privately observed by the firm. Under asymmetric information, the ex-ante optimal contract is subject to the truth-telling constraints and it need not be ex-post efficient in every state, potentially opening the door for involuntary unemployment and large fluctuation in working-hours in response to small productivity shocks. In practice, Chari (1983) and Green and Kahn (1983) prove that—as long as leisure is a normal good—the optimal employment-insurance contract under asymmetric information is such that there is involuntary employment (i.e. the marginal product of labor is smaller than the worker's marginal valuation of their leisure in every state of the world) and the volatility of hours is lower than under perfect information.

Our paper also relates and contributes to the literature on long-term risk-sharing contracts with limited commitment (Thomas and Worrall, 1988; Beaudry and DiNardo, 1991; Kocherlakota, 1996). These papers study the extent to which two parties with heterogeneous degrees of risk aversion or facing imperfectly correlated income shocks manage to share their risks given that at any point in time each (or one) of them can unilaterally decide to leave the relationship. In these models, the parties continuation values once they leave the relationship are determined by a stochastic process that is exogenous to the two trading partners. They find that as long as the parties' participation constraints are not binding, the allocation of consumption is the same under the limited-commitment as under the full-commitment. On the other hand, in our model, the continuation values outside the relationship are affected by the parties' actions—more specifically, by the firm's choice of what contract to offer in the open market—and therefore we can study how the optimal limited-commitment contract distorts the extent of risk-sharing and the parties' actions that affect the outside options. Perhaps unsurprisingly, we find that at the optimum both margins are distorted: workers and firms share too little risk and firms do not attract the efficient number of applicants.

Last but not least, our theory of employment-insurance contracts under limited commitment speaks to the labor-search literature. Shimer (2005) noticed that the standard random search model à la Mortensen and Pissarides cannot simultaneously account for the empirical behavior of unemployment, vacancies and wages over the business cycle. This quantitative observation has brought about a wealth of theoretical studies that modify the basic random search framework in order to add novel channels of transmissions from underlying productivity shocks to unemployment fluctuations (Hall, 2005; Kennan, 2004; Menzio, 2004; Hagedorn and Manovskii, 2005). The fundamental difference between all these studies and our theory is that we drop the assumption of random search and replace it with competitive search, which allows us to endogenize the arrival rate of applicants to the firm and to create a trade-off between risk-sharing and efficient hiring.

1.2. Structure of the Paper

Section 2 lays out the environment of the economy. Section 3 studies the optimal insurance-employment contract under full-commitment. Section 4 derives the characterization of the optimal contract under limited commitment. Section 5 compares the cyclical properties of the general equilibrium under the alternative assumptions of full and limited commitment. In Section 6 we discuss some of the key assumptions of the model and briefly conclude.

2. PHYSICAL ENVIRONMENT

The economy is populated by a continuum of workers with measure 1. Each worker's preferences over streams of consumption $\tilde{c} = \{c_t\}_{t=0}^{\infty}$ can be represented by an additively separable utility function

$$U(\tilde{c}) = \sum_{t=0}^{\infty} \beta^t \cdot u(c_t), \tag{1}$$

where β belongs to the interval $(0,1)$ and $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and strictly concave. If unemployed at the beginning of period t, the worker observes the entire distribution of labor contracts offered in the market and chooses which firm to visit. After reaching the firm, the worker observes which positions are currently filled and which are vacant and chooses which one to apply for. If q is the expected number of applicants for that position, the worker has a probability $\lambda(q)$ of being selected, where the function $\lambda : \mathbb{R}_+ \to (0,1)$ is differentiable, strictly decreasing and such that $\lim_{q\to 0} \lambda(q) = 1$ and $\lim_{q\to\infty} \lambda(q) = 0$. If the worker is selected and hired, the wage w_t and the probability of maintaining the match alive until the beginning of period $t+1$ are set according to the contract. If not hired, the worker receives $b > 0$ units of the consumption good as unemployment benefit and she continues searching in period $t+1$. Next, consider a worker who is employed at the beginning of period t. With probability $\sigma \in (0,1)$ the worker is forced to leave the job, collect the unemployment benefit b and search for employment in period $t+1$. In addition, the contract might endogenously specify that, in the current date and state, the match should be destroyed with positive probability. Also in this case, the worker collects the unemployment benefit and looks for a new job in period $t+1$. Finally, if the match survives, then the worker and the firm exchange labor services for consumption goods according to their contractual agreement. Following the earlier literature on insurance-employment contracts, we assume that the market for contingent claims is imperfect: at date t, workers can neither trade units of the consumption good in future dates/states nor their future labor services.

In period t, there is a large measure of idle firms that can enter the market and become active by investing $I > 0$ units of the consumption good. In addition, in period t, there is a measure f_{t-1} of established firms that have entered the maket in the past and are still active. Every active firm is composed by a large number of jobs N. Consider a firm entering the market at date t. First, the firm advertises an employment contract that rules over the relationship with workers hired during period t. The value of the contract determines the expected number of applicants and, in turn, the employment level in period t. More specifically, if the firm attracts an average of q applicants per job, each of its N position is filled with probability $\eta(q) = q \cdot \lambda(q)$, where the function $\eta : \mathbb{R}_+ \to (0,1)$ is assumed to be strictly increasing and such that $\lim_{q\to 0} \eta(q) = 0$ and $\lim_{q\to\infty} \eta(q) = 1$. Assuming that the law of large numbers applies, the firm employs $N_{1,t} = N \cdot \eta(q)$ workers and produces $p_{1,t} \cdot N_{1,t}$ units of the consumption good, where $p_{1,t} > 0$ is the firm-specific output-per-worker. Next, consider a firm that entered the market at date $t-1$. First, the firm advertises a contract that regulates the relationship with workers hired in period t. The employment level at the beginning of the period and the value of the contract offered to new hires jointly determine the expected number of applicants for filled and unfilled positions and, in turn, the firm's work force. If the firm employs $N_{2,t}$ workers, it produces $N_{2,t} \cdot p_{2,t}$ units of the consumption good. For the sake of simplicity, we assume that firms activated before date $t-1$ are no longer productive in period t. The objective of a firm is to maximize the expected sum of profits discounted at the interest rate $R = 1/\beta$.

The system is subject to firm-specific shocks, aggregate shocks and sunspots. If x_l is the realization of the aggregate shock in period t, then the productivity of newly created firms is a discrete random variable $\tilde{p}_{1,t}$ which takes the value p_k with probability $\pi_{1,l,k}$, for $k = 1, ...K$. Similalrly, if x_l is the realization of the aggregate shock in period t, then the productivity of newly created firms is a random variable $\tilde{p}_{2,t}$ which takes the value p_k with probability $\pi_{2,l,k}$, for $k = 1, ...K$. In turn, the aggregate shock $\tilde{x}_t$ is a random variable that takes the value x_l with probability $\pi_{m,l}$ if $x_{t-1} = x_m$. Finally, $\tilde{\theta}_t$ is a random variable without intrinsic economic content which is distributed as a uniform over the interval $[0, 1]$. The realization of firm-specific shocks, aggregate shocks and sunspots is publicly obervable after established firms post the employment contract and before the entry of new firms.

3. FIRST BEST CONTRACT

Consider a firm that has entered the market at date t and has to choose which contract to advertise. It is convenient to decompose the decision problem of the firm into two subsequent problems. First, the firm chooses the value of the contract W_1 and how many workers to hire $N_1 \in [0, N]$, subject to the constraint imposed by the labor supply curve. Secondly, taking as given the initial level of employment N_1, the firm selects the optimal contract ω among those that deliver the value W_1. In this section, we formalize and characterize the solution to the latter problem under the assumptions that all parties can perfectly commit to their contractual agreements and that contracts are complete.

3.1. The Optimal Contract

A contract ω between the firm and a worker hired at date t specifies: (i) the wage $w_{1,1}$ that the worker receives during her first period of employment; (ii) the probability $\delta_1 \in [0, 1]$ that the match is destroyed before the beginning of period $t+1$; (iii) the wage $w_{1,2}\left(\zeta_{t+1}\right) \in \mathbb{R}_+$ that the worker receives in period $t+1$, conditional on being employed and on the the realization $\zeta_{t+1} = \{x_{t+1}, p_{2,t+1}, \theta_{t+1}\}$ of the aggregate shock, the firm-specific shock and the sunspot; (iv) the severance benefit $s_{1,2}\left(\zeta_{t+1}\right) \in (-b, \infty)$ that the worker receives in period $t+1$, conditional on not being employed; (v) the probability $\delta_2\left(\zeta_{t+1}\right) \in [0, 1]$ that the worker is not employed in period $t + 1$, conditional on the firm not having found a replacement for her; (vi) the probability $\rho\left(\zeta_{t+1}\right) \in [0, 1]$ that the worker is not employed, conditional on the firm having found a replacement for her; (vii) the wage $w^e_{2,2}\left(\zeta_{t+1}\right) \in \mathbb{R}_+$ paid to an applicant hired as a replacement of the worker. A contract ω is feasible if its expected value to the worker is greater or equal than W_1.

Suppose that—given the realization $\zeta_{t+1} = \{x_{t+1}, p_{2,t+1}, \theta_{t+1}\}$ at date $t + 1$—the contract ω prescribes the wages $w_{1,2}$ and $w^e_{2,2}$, the severance payment $s_{1,2}$ and the job destruction probabilities

δ_2 and ρ. If in period $t+1$ and in state ζ_{t+1}, an unemployed worker applies to a filled position, she is hired with probability $\lambda(q^e) \cdot \rho$ and returns to the unemployment pool with probability $1-\lambda(q^e) \cdot \rho$. In the first case, she receives the wage $w^e_{2,2}$, consumes it and applies for another job in period $t+1$. In the second case, she collects the unemployment benefit b, consumes it and applies for another job in period $t+1$. Therefore, the applicant's expected utility is

$$W_{2,2}\left(w^e_{2,2}, \rho|\zeta_{t+1}\right) = \lambda(q^e)\,\rho \cdot \left[u\left(w^e_{2,2}\right) - u(b)\right] + u(b) + \beta E\left(Z_{t+2}\right), \tag{2}$$

where $E(Z_{t+2})$ is the expected value of searching for a job in period $t+2$. Conjecture that there exists a one-to-one mapping between the realization of the aggregate shock x and the contemporaneous value of searching Z. Under this conjecture, we can express the expected value of searching $E(Z_{t+2})$ as a function of x_{t+1} only. Next, notice that the optimality condition for the workers' application strategy requires that, if q^e is strictly positive, then $W_{2,2}(.)$ has to be equal to the value Z_{t+1} of searching for a job in period $t+1$. Using this necessary condition, we can express the average queue length q^e as $q\left(w^e_{2,2}, \rho\right)$, where the function $q(w, \rho)$ is defined as

$$q\left(w, \rho|\zeta_{t+1}\right) = \begin{cases} \lambda^{-1}\left\{\dfrac{u(z_{t+1}) - u(b)}{\rho\left[u(w) - u(b)\right]}\right\} & \text{if } \rho u(w) \geq u(z_{t+1}) - (1-\rho)\,u(b), \\ \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

and $u(z_{t+1})$ is the flow value of search $Z_{t+1} - \beta \cdot E(Z_{t+2})$.

In period $t+1$ and in state ζ_{t+1}, a senior employee is exogenously displaced with probability σ, replaced by a junior hire with probability $\eta(q^e) \cdot \rho$, endogenously laid-off with probability δ_2. When any of these three events occurs, the worker receives the severance payment $s_{1,2}$, collects the unemployment benefit b and applies for a new job in period $t+2$. On the other hand, if the senior worker remains employed, she consumes $w_{1,2}$ units of the consumption good and then looks for another job. Therefore, in period $t+1$ and in state ζ_{t+1}, a senior worker's expected utility is

$$\begin{aligned} W_{1,2}\left(\omega_2\left(\zeta_{t+1}\right)|\zeta_{t+1}\right) = \quad & \left[1 - \sigma + \eta(q^e)\,\rho + (1 - \eta(q^e))\,\delta_2\right] \cdot u(w_{1,2}) + \\ & \left[\sigma + \eta(q^e)\,\rho + (1 - \eta(q^e))\,\delta_2\right] \cdot u(b + s_{1,2}) + \beta E(Z_{t+2}), \end{aligned} \tag{4}$$

where $\omega_2\left(\zeta_{t+1}\right)$ is equal to the vector $\left\{w_{1,2}, s_{1,2}, \delta_2, \rho, w^e_{2,2}\right\}$.

Suppose that, in period $t+1$ and in state ζ_{t+1}, the firm offers the wage $w^u_{2,2}$ to each worker hired to man an unfilled position. If in period $t+1$ and in state ζ_{t+1}, an unemployed worker applies to an unfilled position, she is hired with probability $\lambda(q^e)$ and returns to the unemployment pool with probability $1-\lambda(q^e)$. Therefore, the average number of applicants q^u per unfilled position is equal to $q\left(w^u_{2,2}, 1\right)$. Denoting with v^e the number of filled positions $N_1 \cdot \delta_1 \cdot (1-\sigma)$ and with $v^u = N - v^e$ the number of filled positions, we can express the firm's profit in period $t+1$ and in state ζ as

$$P_2\left(\omega_2\left(\zeta_{t+1}\right), w_{2,2}^u\left(\zeta_{t+1}\right)|\zeta_{t+1}\right) =$$

$$v^e \cdot \left\{\eta\left(q^e\right)\left(p_{2,t+1} - \rho w_{2,2}^e - (1-\rho)\, w_{1,2}\right) + (1-\eta\left(q^e\right))(1-\delta_2)\left(p_{2,t+1} - w_{1,2}\right)\right\} + \tag{5}$$

$$v^u \cdot \eta\left(q^u\right)\left(p_{2,t+1} - w_{2,2}^u\right) - N_1\delta_1 \cdot \left[\sigma + \eta\left(q^e\right)\rho + (1-\eta\left(q^e\right))\delta_2\right] \cdot s_{1,2}.$$

Denote with $Q\left(\zeta_{t+1}|\zeta_t\right)$ the distribution of $\{x_{t+1}, p_{2,t+1}, \theta_{t+1}\}$ conditional on the realization $\zeta_t = \{x_t, p_{1,t}, \theta_t\}$. Then, the sum of period t profits $N_1 \cdot (p_1 - w_1)$ and the discounted expected value of (5) over $Q\left(\zeta_{t+1}|\zeta_t\right)$ represents the value of the contract ω to the firm. Similarly, the sum of the worker's utility $u(w_1)$ in the first period and the discounted expected value of (6) over $Q\left(\zeta_{t+1}|\zeta_t\right)$ represents the value of the contract ω to the worker. Therefore, a contract ω is optimal if and only if it solves the constrained optimization program

$$\max_{\omega} N_1\left(p_1 - w_1\right) + \beta \int \left\{\max_{w_{22}^u} P_2\left(\omega_2\left(\zeta_{t+1}\right), w_{2,2}^u\left(\zeta_{t+1}\right)|\zeta_t\right)\right\} dQ\left(\zeta_{t+1}|\zeta_t\right), \qquad \text{s.t.}$$

$$W_1 \leq u(w_1) + \beta\left(1-\delta_1\right)\int \left\{W_{1,2}\left(\omega_2\left(\zeta_{t+1}\right)|\zeta_{t+1}\right)\right\} dQ\left(\zeta_{t+1}|\zeta_t\right) + \beta\delta_1 E(Z_{t+1}). \tag{6}$$

3.2. Properties of an Optimal Contract

Under a complete full-commitment labor contract, the allocation of a worker's time can be chosen independently from her wage and her severance transfer. Therefore, an optimal complete full-commitment contract must prescribe an efficient employment rule—i.e. δ_1, δ_2 and ρ are strictly positive if and only if the match has no surplus—and a system of wages and transfers that perfectly insures the worker—i.e. $s_{1,2}$ and $w_{1,2}$ are such that the worker's marginal utility of consumption across dates and states is constant. Moreover, under a complete full-commitment labor contract, the consumption of a senior employee can be chosen independently from the wages promised to new hires. Therefore, any optimal contract must prescribe an efficient replacement policy—i.e. a wage $w_{2,2}^e$ such that the ex-ante surplus of the match between the firm and a senior employee is maximized—and an efficient hiring policy—i.e. a wage $w_{2,2}^u$ that maximizes the expected profit associated with an unfilled vacancy.

Given the assumption that a firm is only active for two periods, a contract ω prescribes an efficient employment policy if $\delta_2\left(\zeta_{t+1}\right)$ is set to 1 *(0)* for all $p_{2,t+1}$ smaller *(greater)* than b, if $\rho\left(\zeta_{t+1}\right)$ is set to 1 *(0)* when either *(neither)* $p_{2,t+1}$ or $w_{2,2}^e\left(\zeta_{t+1}\right)$ are smaller than b, if δ_1 is set to 1 if and only if $E(\max\{p_{2,t+1}, b\})$ is smaller than $E(z_{t+1})$. Given the specification of worker's preferences in (1), a contract ω perfectly insures the worker if, for all ζ_{t+1}, $w_{1,2}\left(\zeta_{t+1}\right)$ and $b + s_{1,2}\left(\zeta_{t+1}\right)$ are equal to w_1. Because the flow value of search z_{t+1} is always strictly greater than the unemployment benefit

b, replacing a senior employee with a new hire cannot increase the ex-ante surplus of the match. These results are summarized and formally proved in the following proposition.

Proposition 3.1: (First Best Contract) *Let* $\{\omega^*, w^{u*}_{2,2}(\zeta_{t+1})\}$ *be a solution to (6).*
(i) The contract ω^* *employs labor efficiently:* δ^*_1 *is equal to* 1 *(*0*) if* $E(\max\{p_{2,t+1}, b\})$ *is greater (smaller) than* $E(z_{t+1})$*,* $\delta^*_2(\zeta_{t+1})$ *and* $\rho(\zeta_{t+1})$ *are equal to* 1 $(= 0)$ *if* $p_{2,t+1}$*is smaller (greater) than* b*.*
(ii) The contract ω^* *perfectly insures workers:* $w^*_{1,2}(\zeta_{t+1})$ *and* $b + s_{1,2}(\zeta_{t+1})$ *are both equal to* w^*_1 *for all* ζ_{t+1}*.*
(iii) The contracts ω^* *and* $w^{u*}_{2,2}(\zeta_{t+1})$ *guarantee that hiring is efficient:* $w^{u*}_{2,2}(\zeta_{t+1})$ *maximizes* $\eta(q^e)(p_{2,t+1} - w^u_{2,2})$ *and* $w^e_{2,2}(\zeta_{t+1})$ *is smaller or equal than* z_l*.*

Proof. See the Technical Appendix. ||

A couple of remarks about Proposition 3.1 are worthwhile. First, notice that any optimal contract ω^* can be implemented by specifying a wage w^*, which the firm pays to the worker in any date/state when she is employed, a severance benefit s^*, which the firm pays to the worker in any date/state when she is laid-off, and by assigning to the firm the right to unilaterally terminate employment and complete discretion in choosing the hiring wages. Secondly, notice that part (i) implies that—in our environment as in the models analyzed by the early literature on implicit contracts—allowing for employment-insurance contracts has consequences for the distribution of the gains from trade between the firms and the workers, but it is irrelevant as far as the allocation of labor is concerned.

4. INCOMPLETE SELF-ENFORCING CONTRACT

In this section, we modify the contractual framework in two directions. First, we assume that the cost of legally enforcing participation to a labor contract is prohibitively high. Under this assumption, the contract posted by a firm has to be self-enforcing, i.e. at every stage of the employment relationship, the prescriptions of the contract have to be such that both the firm and the worker prefer to follow them than to leave the match. Secondly, we assume that courts cannot verify whether a worker has been hired to replace a senior employee or to fill an open position. Under this assumption, contracts are incomplete, i.e. the prescriptions of the contract between a firm and a worker cannot be made contingent upon the reason of employment. The discussion of this framework is deferred until Section 6.

4.1. The Optimal Contract

Consider a firm that has entered the market at date t and promised a contract ω with expected utility W_1 to all of the N_1 applicants hired during its first period of activity. The contract ω is self-enforcing if—at any stage of the employment relationship and after any history—both parties prefer to behave according to ω than to break their match. Therefore, the contract ω is self-enforcing if: (i) in every state where the worker is laid-off with positive probability, the severance payment $s_{1,2}$ is equal to zero; (ii) in every state ζ_{t+1} where the worker is employed with positive probability, the wage $w_{1,2}$ belongs to the interval $[b, p_{2,t+1}]$; (iii) in every state ζ_{t+1} where the worker is replaced with positive probability by a new applicant, the wage $w_{2,2}^e$ belongs to the interval $[b, p_{2,t+1}]$; (iv) conditional on the firm having found a qualified substitute for the worker, the replacement probability ρ is equal to 1 if $w_{1,2}$ is greater than $w_{2,2}^e$ and ρ is equal to 0 if $w_{1,2}$ is smaller than $w_{2,2}^e$; (v) if the contract specifies that the match is not destroyed at the end of period t, then both parties must be weakly better off entering period $t+1$ inside the employment relationship rather than outside of it. In addition, the incompleteness assumption implies that the wages $w_{2,2}^u$ and $w_{2,2}^e$ take on the same value $w_{2,2}$ in every state of the world ζ_{t+1}. Finally, a contract ω is said to be feasible if its expected value to the worker is greater or equal than W_1.

Taking as given the employment level N_1 and the promised value W_1, an incomplete contract ω is optimal if it maximizes the firm's expected profits

$$\max_{\omega} N_1 (p_1 - w_1) + \beta \int \left\{ P_2 \left(\omega_2 \left(\zeta_{t+1} \right), w_{2,2} \left(\zeta_{t+1} \right) | \zeta_t \right) \right\} dQ \left(\zeta_{t+1} | \zeta_t \right), \tag{7}$$

subject to the feasibility constraint

$$W_1 \leq u(w_1) + \beta (1 - \delta_1) \int \left\{ W_{1,2} \left(\omega_2 \left(\zeta_{t+1} \right) | \zeta_{t+1} \right) \right\} dQ \left(\zeta_{t+1} | \zeta_t \right) + \beta \delta_1 E(Z_{t+1}), \tag{8}$$

the period-$(t+1)$ self-enforcement constraints

$$\begin{gathered} s_{1,2} \left(\zeta_{t+1} \right) = 0, \; w_{1,2} \left(\zeta_{t+1} \right) \in [b, p_{2,t+1}] \text{ if } \delta_2 \left(\zeta_{t+1} \right) < 1, \\ w_{2,2} \left(\zeta_{t+1} \right) \in [b, p_{2,t+1}] \text{ if } p_{2,t+1} \geq z_{t+1}, \; w_{2,2} \left(\zeta_{t+1} \right) \leq [b, z_{t+1}] \text{ if } p_{2,t+1} < z_{t+1} \\ \rho \left(\zeta_{t+1} \right) = \begin{cases} 1 \text{ if } w_{1,2} \left(\zeta_{t+1} \right) > w_{2,2} \left(\zeta_{t+1} \right), \\ 0 \text{ if } w_{1,2} \left(\zeta_{t+1} \right) \leq w_{2,2} \left(\zeta_{t+1} \right), \end{cases} \end{gathered} \tag{9}$$

and the period-t self-enforcement constraints which require that if $\delta_1 < 1$ then

$$\begin{gathered} \int \left\{ W_{1,2} \left(\omega_2 \left(\zeta_{t+1} \right) | \zeta_{t+1} \right) \right\} dQ \left(\zeta_{t+1} | \zeta_t \right) \geq E(Z_{t+1} | \zeta_t), \\ \int \left\{ P_2 \left(\omega_2 \left(\zeta_{t+1} \right), w_{2,2} \left(\zeta_{t+1} \right) | \zeta_t \right) \right\} dQ \left(\zeta_{t+1} | \zeta_t \right) \geq \\ \int \left\{ \max_{w} N \cdot \eta \left(q \left(w, 1 | \zeta_{t+1} \right) \right) \cdot \left(p_{2,t+1} - w \right) \right\} dQ \left(\zeta_{t+1} | \zeta_t \right). \end{gathered} \tag{10}$$

Consider two contracts ω^1 and ω^2 that satisfy the constraints (8)–(10) and such that $w_{1,2}^1 \left(\zeta_{t+1} \right) > w_{2,2}^1 \left(\zeta_{t+1} \right)$ and $w_{1,2}^2 \left(\zeta_{t+1} \right) < w_{2,2}^2 \left(\zeta_{t+1} \right)$ in some state ζ_{t+1}. Then, it is immediate to see that there exists an $\alpha \in (0,1)$ such that the contract ω^{α} defined as $\alpha \cdot \omega^1 + (1 - \alpha) \cdot \omega^2$ is such that $w_{1,2}^{\alpha} \left(\zeta_{t+1} \right)$

is srictly greater than $w^{\alpha}_{2,2}\left(\zeta_{t+1}\right)$ and $\rho\left(\zeta_{t+1}\right)$ is strictly smaller than 1. The latter example implies that the set of feasible incomplete self-enforcing contracts is not convex and the optimum cannot be characterized using Kuhn-Tucker conditions.

In the following pages we implement a local/global solution strategy. First, we consider a contract ω^* and a state $\{x_{t+1}, p_{2,t+1}, \Theta\}$, with $\Theta \subset [0,1]$, that has positive but arbitrarily small probability mass. Taking as given the prescriptions of the contract in every other state, we solve for the optimal wages w_1, $w^{nr}_{1,2}\left(x_{t+1}, p_{2,t+1}, \Theta\right)$ and $w^{nr}_{2,2}\left(x_{t+1}, p_{2,t+1}, \Theta\right)$ subject to the constraint $w^{nr}_{1,2}(.) \leq w^{nr}_{2,2}(.)$. Then, we solve for the optimal wages w_1, $w^{r}_{1,2}\left(x_{t+1}, p_{2,t+1}, \Theta\right)$ and $w^{r}_{2,2}\left(x_{t+1}, p_{2,t+1}, \Theta\right)$ subject to the constraint $w^{r}_{1,2}(.) > w^{r}_{2,2}(.)$. Under mild regularity assumptions, these two optimization problems can be shown to have a unique solution that can be characterized with local conditions. Secondly, we compare the value of the two alternative contracts. If the contract ω^* is optimal and prescribes that ρ is equal to 0 in state $\{x_{t+1}, p_{2,t+1}, \Theta\}$, then the wages $\{w_{1,2}(.), w_{2,2}(.)\}$ have to be equal to $\left\{w^{nr}_{1,2}(.), w^{nr}_{2,2}(.)\right\}$ and ω^* has to be preferred to an alternative contract that prescribes $\rho\left(x_{t+1}, p_{2,t+1}, \Theta\right) = 1$. These necessary conditions are enough to characterize the optimal contract up to the period$-t$ wage w_1.

4.2. Necessary Conditions for Optimality

Let ω^* be a feasible incomplete self-enforcing contract that maximizes (7) and satisfies the constraints (10) with a strict inequality. Let ζ_{t+1} be a state $\{x_{t+1}, p_{2,t+1}, \Theta\}$ that has positive and arbitrarly small probability mass μ. Suppose that the contract ω^* prescribes $\delta^*_2 = 0$, $w^*_{1,2} \leq w^*_{2,2}$ and $\rho^* = 0$ in state ζ_{t+1}.

Consider the alternative contract $\hat{\omega}$ that prescribes $\hat{\delta}_2 = \delta^*_2$, $\hat{w}_{1,2} \leq \hat{w}_{2,2}$ and $\hat{\rho} = 0$ at date $t+1$ and in state ζ_{t+1}, that replicates the contract ω^* at date $t+1$ and in any state different from ζ_{t+1} and that sets the wage $\hat{w}_1$ to make the worker indifferent between $\hat{\omega}$ and ω^*. The contract $\hat{\omega}$ is feasible as long as the wages $\left(\hat{w}_{1,2}, \hat{w}_{2,2}\right)$ belong to the set $\Gamma^{nr}\left(x_{t+1}, p_{2,t+1}\right)$, where $\Gamma^{nr}(.)$ is defined as

$$\Gamma^{nr}(.) = \{w_{1,2} \in [b,p], w_{2,2} \in [b,p] \text{ if } p \geq z,\ w_{2,2} \in [b,z] \text{ if } p < z, w_{1,2} \leq w_{2,2}\}. \tag{11}$$

If the firm switches from ω^* to $\hat{\omega}$, there are three effects on expected profits: (i) at date $t+1$ and in state ζ_{t+1}, the value of each of the $N - N_1\delta_1(1-\sigma)$ open vacancies goes from $\eta\left(q\left(w^*_{2,2}, 1|.\right)\right) \cdot \left(p - w^*_{2,2}\right)$ to $\eta\left(q\left(\hat{w}_{2,2}, 1|.\right)\right) \cdot \left(p - \hat{w}_{2,2}\right)$; (ii) at date $t+1$ and in state ζ_{t+1}, the value of each of the $N_1\delta_1(1-\sigma)$ filled vacancies goes from $\left(p - w^*_{1,2}\right)$ to $\left(p - \hat{w}_{1,2}\right)$; (iii) at date t, the wage increases by $\hat{w}_1 - w^*_1$, which is approximately equal to $u'(w^*_1)^{-1} \cdot \beta \cdot \mu \cdot N_1\delta_1(1-\sigma) \cdot \left(u(\hat{w}_{1,2}) - u(w^*_{1,2})\right)$. Overall, by switching from ω^* to $\hat{\omega}$, the change in the firm's profits is approximately equal to the difference between $\beta \cdot \mu \cdot T^{nr}\left(w^*_1, \hat{w}_{1,2}, \hat{w}_{2,2}|x_{t+1}, p_{2,t+1}\right)$ and $\beta \cdot \mu \cdot T^{nr}\left(w^*_1, w^*_{1,2}, w^*_{2,2}|x_{t+1}, p_{2,t+1}\right)$, where $T^{nr}(.)$ is defined as

$$T^{nr}(.|.) = N_1\delta_1(1-\sigma)\cdot\left\{\frac{u(w_{1,2})-u(b)}{u'(w_1)}+(p-w_{1,2})\right\}+[N-N_1\delta_1(1-\sigma)]\cdot H(w_{2,2}|x_{t+1},p_{2,t+1}),$$

$$H(w|x,p)\equiv h(w|.)\cdot(p-w)\equiv\eta\left(q\left(w,1|\zeta\right)\right)\cdot(p-w). \tag{12}$$

The function $T^{nr}(.)$ can be interpreted in two ways. On the one hand, $T^{nr}(.)$ is the weighted sum of the firm's profits and worker's utility at date $t+1$ and in state ζ_{t+1}, when the wages offered to senior and junior employees are $\{w_{1,2}, w_{2,2}\}$ and $w_{1,2}\leq w_{2,2}$. On the other hand, $T^{nr}(.)$ is the sum of the insurance and hiring value of the contract at date $t+1$ and in state ζ_{t+1}, when $w_{1,2}\leq w_{2,2}$. The first term in (12) is the insurance value of the contract, which is maximized for $w_{1,2}$ equal to w_1. The second term in (12) is the hiring value of the contract, which is the product between the number of unfilled positions and the expected value $H(w|x,p)$ of each one of them.

Next, consider a contract $\tilde{\omega}$ that prescribes $\tilde{\delta}_2=\delta_2^*$, $\tilde{w}_{1,2}>\tilde{w}_{2,2}$ and $\tilde{\rho}=1$ at date $t+1$ and in state ζ_{t+1}, that replicates the contract ω^* at date $t+1$ and in any state different from ζ_{t+1} and that sets $\tilde{w}_1$ to make the worker indifferent between $\tilde{\omega}$ and ω^*. The alternative contract $\hat{\omega}$ is feasible as long as the wages $(\tilde{w}_{1,2},\tilde{w}_{2,2})$ belong to the set $\Gamma^r(x_{t+1},p_{2,t+1})$, where $\Gamma^r(.)$ is defined as

$$\Gamma^r(.)=\{w_{1,2}\in[b,p], w_{2,2}\in[b,p] \text{ if } p\geq z,\ w_{2,2}\in[b,z] \text{ if } p<z, w_{1,2}>w_{2,2}\}. \tag{13}$$

By switching from ω^* to $\tilde{\omega}$, the change in the fim's profits is approximately equal to the difference between $\beta\cdot\mu\cdot T^r(w_1^*,\tilde{w}_{1,2},\tilde{w}_{2,2}|.)$ and $\beta\cdot\mu\cdot T^{nr}\left(w_1^*,w_{1,2}^*,w_{2,2}^*|.\right)$, where the function $T^r(.)$ is defined as

$$T^r(.|.)=N_1\delta_1(1-\sigma)\cdot(1-h(w_{2,2}|.))\cdot\left\{\frac{u(w_{1,2})-u(b)}{u'(w_1)}+(p-w_{1,2})\right\}+N\cdot H(w_{2,2}|.). \tag{14}$$

The function $T^r(.)$ can be interpreted as the weighted sum of the firm's profits and worker's utility at date $t+1$ and in state ζ_{t+1}, when the wages offered to senior and junior employees are $\{w_{1,2},w_{2,2}\}$ and $w_{1,2}>w_{2,2}$. Alternatively, $T^r(.)$ can be interpreted as the sum of the insurance and hiring value of the contract, when the firm has an incentive to replace senior employees with junior hires.

Finally, because the contract ω^* is optimal and μ is arbitrary, it must be the case that $T^{nr}\left(w_1^*,w_{1,2}^*,w_{2,2}^*|.\right)$ is greater or equal than both $T^{nr}(w_1^*,\hat{w}_{1,2},\hat{w}_{2,2}|.)$ and $T^r(w_1^*,\tilde{w}_{1,2},\tilde{w}_{2,2}|.)$. By generalizing this argument and extending it to all local deviations, we obtain the following set of necessary conditions.

Lemma 4.1: (Necessary Conditions for Optimality) *Let* ω^* *be a solution to (7) such that* $\delta_1^*=0$ and *the constraints (10) are not binding.*
(i) If at date $t+1$ *and in state* $\{x_{t+1},p_{2,t+1},\theta\}$*, the contract prescribes that* $\delta_2^*=0$, $w_{1,2}^*\leq w_{2,2}^*$ *and* $\rho^*=0$*, then almost surely*

$$\begin{aligned} T^{nr}(w_1^*,w_{1,2}^*,w_{2,2}^*|.) &\geq \max_{(w_{1,2},w_{2,2})\in\Gamma^{nr}(.)} T^{nr}(w_1^*,w_{1,2},w_{2,2}|.) \text{ and} \\ T^{nr}(w_1^*,w_{1,2}^*,w_{2,2}^*|.) &\geq \max_{(w_{1,2},w_{2,2})\in\Gamma^{r}(.)} T^{r}(w_1^*,w_{1,2},w_{2,2}|.). \end{aligned} \tag{15}$$

(ii) If at date $t+1$ and in state $\{x_{t+1}, p_{2,t+1}, \theta\}$, the contract prescribes that $\delta_2^* = 0$, $w_{1,2}^* > w_{2,2}^*$ and $\rho^* = 1$, then $T^r(w_1^*, w_{1,2}^*, w_{2,2}^*|.)$ satisfies almost surely the inequalities (15.a) and (15.b).
(iii) If $p_{2,t+1}$ is greater (smaller) than b, then $\delta_2^*\left(\zeta_{t+1}\right)$ is equal to 0 (1).

Proof. See the Technical Appendix. ||

4.3. The Optimal Contract Without Replacement

Given a vector $\{w_1, x, p\}$ in $\mathbb{R}_+^3$, consider the maximization problem

$$\max_{w_{1,2}, w_{2,2}} T^{nr}(w_1, w_{1,2}, w_{2,2}|x, p), s.t. (w_{1,2}, w_{2,2}) \in \Gamma^{nr}(x, p). \tag{16}$$

Assume that the matching function $\eta(q)$ is such that the value $H(w_{2,2}|x, p)$ of a vacant position is a twice-differentiable and strictly concave function of the hiring wage $w_{2,2}$ over the interval $[z, p]$.

Let the vector $\{w_1, x, p\}$ be such that the $p > z$. Because the objective function $T^{nr}(.)$ is continuous and the feasible set $\Gamma^{nr}(.)$ is non-empty, continuous and compact-valued, the set of solutions to (16) is non-empty, upper hemi-continuous with respect to $\{w_1, x, p\}$ and compact-valued. Because the value of a vacant position $H(w_{2,2}|x, p)$ is equal to 0 for all $w_{2,2} \leq z$ and strictly greater than zero for $w_{2,2} \in (z, p)$, there are no solutions to (16) such that $w_{2,2}$ is smaller than z. Therefore, the strict concavity of the objective function $T^{nr}(.)$ over the rectangle $[b, p] \times [z, p]$ is sufficient to guarantee the uniqueness of the solution to (16). Finally, because the concave programme (16) satisfies Slater's condition, the Kuhn-Tucker conditions are necessary and sufficient for optimality.

Denote with λ the multiplier associated with the constraint $(w_{1,2} - w_{2,2}) \leq 0$. If at the optimum the no-replacement constraint is not binding, then the wage $w_{1,2}^{nr*}(p|.)$ offered to senior employees must maximize the insurance value of the contract, while the wage $w_{2,2}^{nr*}(p|.)$ must maximize the hiring value of the contract. More formally, if $\lambda^* = 0$, then $\left(w_{1,2}^{nr*}(p|.), w_{2,2}^{nr*}(p|.)\right)$ satisfies the Kuhn-Tucker conditions if and only if

$$\begin{aligned} w_{1,2}^{nr*}(p|.) &= w^I(p|.) \equiv \arg\max_{w \in [b,p]} \left\{ \tfrac{u(w) - u(b)}{u'(w_1)} + (p - w) \right\}, \\ w_{2,2}^{nr*}(p|.) &= w^H(p|.) \equiv \arg\max_{w \in [z,p]} H(w; p). \end{aligned} \tag{17}$$

On the other hand, if at the optimum the no-replacement constraint is binding, then senior and junior employees are paid the same wage, which efficiently trades-off the distortions on the insurance and on the hiring margins of the contract. More formally, if $\lambda^* > 0$, then $\left(w_{1,2}^{nr*}(p|.), w_{2,2}^{nr*}(p|.)\right)$ satisfies the Kuhn-Tucker conditions if and only if

$$\begin{aligned} & w_{1,2}^{nr*}(p|.) = w_{2,2}^{nr*}(p|.) = w_2^{nr*}(p|.) = \{w_2 \text{ if } \Phi^{nr}(w_2) = 0,\ z \text{ if } \Phi^{nr}(z) < 0,\ p \text{ if } \Phi^{nr}(p) > 0\}, \\ & \Phi^{nr}(w_2) = \tfrac{dT^{nr}(.)}{dw_{1,2}} + \tfrac{dT^{nr}(.)}{dw_{2,2}}|_{w_{1,2}=w_{2,2}=w_2} = \left(\tfrac{u'(w_2)}{u'(w_1)} - 1\right) + \tfrac{N - N_1(1-\sigma)}{N_1(1-\sigma)} \cdot H'(w_2|x, p). \end{aligned} \tag{18}$$

Finally, because the solution to (16) is unique, the multiplier λ^* is zero if and only if the optimal hiring wage $w^H(p|.)$ is greater or equal than the optimal insurance wage $w^I(p|.)$.

The optimal hiring wage $w^H(p|.)$ belongs to the open interval (z, p) for all $p > z$. Moreover, because the value $H(w|x,p)$ of an unfilled vacancy is a strictly concave function of w and the cross-derivative $H_{w,p}$ is strictly positive, the optimal hiring wage $w^H(p|.)$ is a continuous strictly increasing function of the firm's productivity p. Together, these two properties imply that the hiring wage $w^H(p|.)$ is strictly greater than the optimal insurance wage $w^I(p|.)$ if and only if the realization of firm's productivity p is greater than the cutoff k_1, which is implicitely and uniquely defined by

$$w^H(k_1|x) = w^I(k_1|x) = w_1 \text{ if } w_1 \geq z, \quad k_1 = z \text{ if } w_1 < z. \tag{19}$$

The optimal firm-wide wage $w_2^{nr*}(p|.)$ belongs to the open interval $\left(w^H(p|.), w^I(p|.)\right)$ for all realizations of firm's productivity $p \in (z, k_1)$. Conversely, $w_2^{nr*}(p|.)$ belongs to the interval $\left(w^I(p|.), w^H(p|.)\right)$ for $p > k_1$. Because the function $T^{nr}(.)$ is strictly concave in $(w_{1,2}, w_{2,2})$ and the sum of the cross-derivatives $T^{nr}_{w_{1,2},p}(.) + T^{nr}_{w_{2,2},p}(.)$ is strictly positive, the optimal firm-wide wage $w_2^{nr*}(p|.)$ is a continuous and non-decreasing function of p. More specifically, the wage $w_2^{*nr}(p|.)$ is equal to the marginal product of labor whenever p is smaller or equal than the cutoff k_2, where k_2 is implicitely defined as

$$\left(\frac{u'(k_2)}{u'(w_1)} - 1\right) + \tfrac{N - N_1(1-\sigma)}{N_1(1-\sigma)} \cdot H'(k_2; x, k_2) = 0. \tag{20}$$

For p greater than k_2, the firm-wide wage $w_2^{nr*}(p|.)$ is strictly increasing in p.

In light of the previous analysis, we can conclude the following. If at date $t+1$ and in state $\{x, p, \theta\}$, the firm's productivity p lies in the interval $[z, k_2]$ and ρ^* is equal to zero, then both senior employees and new applicants are paid according to their marginal product, i.e. $w_{1,2}^{nr*}(p|.) = w_{2,2}^{nr*}(p|.) = p$. If p lies in the interval $[k_2, k_1]$, then senior employees and new applicants are offered an identical wage such that there is under-provision of consumption insurance—in the sense that $w_{1,2}^{nr*}(p|.)$ is smaller than $w^I(p|.)$—and over-recruitment—in the sense that $w_{2,2}^{nr*}(p|.)$ is strictly greater than $w^H(p|.)$. Finally, if the firm's productivity p is greater than k_1, then senior employees receive a wage that is strictly smaller than the wage offered to new recruits. Specifically, the continuation wage $w_{1,2}^{nr*}(p|.)$ optimizes the provision of consumption insurance to senior employees and the wage $w_{2,2}^{nr*}(p|.)$ maximizes the value of each unfilled vacancy to the firm. The wages $\left(w_{1,2}^{nr*}(p|.), w_{2,2}^{nr*}(p|.)\right)$ are represented in Figure 1.

4.4. The Optimal Contract With Replacement

Consider the problem of maximizing the weighted sum of worker's utility and firm's profits in perod $t+1$ and in state $\{x, p\}$, given that the firm replaces senior employees with junior hires independently from the ordering of $w_{1,2}$ and $w_{2,2}$. Formally, given a vector $\{w_1, x, p\}$ in $\mathbb{R}^3_+$ with $p > z(x)$, consider the constrained optimization problem

$$\max_{w_{1,2},w_{2,2}} T^r(w_1, w_{1,2}, w_{2,2}|x,p), s.t. (w_{1,2}, w_{2,2}) \in [b,p] \times [z,p] \quad = \tag{21}$$

$$\max_{w_{2,2}\in[z,p]} \left\{ N \cdot H(w_{2,2}|.) + N_1 (1-\sigma) (1 - h(w_{2,2}|.)) \cdot \max_{w_{1,2}\in[b,p]} \left[\tfrac{u(w_{1,2})-u(b)}{u'(w_1)} + (p - w_{1,2}) \right] \right\}.$$

Assume that the fundamentals of the model are such that the maximand in (21) is a strictly concave function of $w_{2,2}$ over the interval $[z,p]$.

First, consider the second-stage optimization problem in (21). For every tuple $\{w_1, x, p, w_{2,2}\}$, the objective function is differentiable and strictly concave with respect to $w_{1,2}$ and the feasible set is non-empty and convex. Therefore, the solution $w^{r*}_{1,2}(p|.)$ to the second-stage problem is unique. For every $w_{2,2}$ in the interval $[z,p]$, $1-h(w_{2,2}|.)$ is strictly positive. Therefore, the solution $w^{r*}_{2,2}(p|.)$ to the second-stage problem is independent from $w_{2,2}$. Next, consider the first-stage optimization problem. For every triple $\{w_1, x, p\}$, the objective function is differentiable and strictly concave with respect to $w_{2,2}$ and the feasible set is non-empty and convex. Therefore, the solution $w^{r*}_{2,2}(p|.)$ to the first-stage problem is unique. The necessary and sufficient conditions for the optimality of $\left(w^{r*}_{1,2}(p|.), w^{r*}_{2,2}(p|.)\right)$ are

$$w^{r*}_{1,2}(p|.) = w^I(p|.),$$

$$w^{r*}_{2,2}(p|.) = \{w_{2,2} \text{ if } \Phi^r(w_{2,2}) = 0,\ z \text{ if } \Phi^r(z) < 0,\ p \text{ if } \Phi^r(p) > 0\}, \tag{22}$$

$$\Phi^r(w_{2,2}) = \tfrac{dT^r(.)}{dw_{2,2}} = \tfrac{N}{N_1(1-\sigma)} \cdot H'(w_{2,2}|.) - \left[\tfrac{u(w^I_{1,2}(p|.))-u(b)}{u'(w_1)} + \left(p - w^I(p|.)\right) \right] \cdot h'(w_{2,2}).$$

The system of equations in (22) has a clear economic interpretation. The wage offered to senior employees $w^{r*}_{1,2}(p|.)$ maximizes the insurance value of the contract, while taking as given the employment risk created by the firm's replacement policy. The wage offered to junior employees $w^{r*}_{2,2}(p|.)$ optimally trades-off the cost of the distortions on the insurance and the hiring margins created by the firm's replacement policy. Specifically, the wage $w^{r*}_{2,2}(p|.)$ is strictly smaller than the optimal hiring wage $w^H(p|.)$ and their distance is decreasing with the probability of exogenous separation σ.

Because the function $T^r(.)$ is strictly concave in $w_{2,2}$ and the cross-derivative $T^r_{w_{2,2},p}(.)$ is strictly positive, the wage $w^{r*}_{2,2}(p|.)$ offered to junor employees is a continuous and non-decreasing function of firm's productivity p. More specifically, the firm withdraws from the labor market by offering the wage $w^{r*}_{2,2}(p|.) = z$ whenever p is smaller or equal than k_3, where the cutoff k_3 is the unique solution to

$$\frac{N}{N_1(1-\sigma)} \cdot H'(z|.) - \left[\tfrac{u(w^{r*}_{1,2}(k_3|.))-u(b)}{u'(w^*_1)} + \left(p - w^{r*}_{1,2}(k_3|.)\right) \right] \cdot h'(z|.) = 0. \tag{23}$$

For all p greater than k_3, the wage $w^{r*}_{2,2}(p|.)$ is strictly increasing. The wages $\left(w^{r*}_{1,2}(p|.), w^{r*}_{2,2}(p|.)\right)$ are represented in Figure 2.

Finally, we need to relate the solution of the maximization problem (21) to the maximizers of $T^r(.)$ over the set $\Gamma^r(.)$. If for some $\{x, p\}$, the solution to (21) is such that the hiring wage is smaller than the continuation wage paid to senior employees, then $\left(w^{r*}_{1,2}(p|.), w^{r*}_{2,2}(p|.)\right)$ is also a maximizer of $T^r(.)$ over $\Gamma^r(.)$. On the contrary, if the solution to (21) is such that the hiring wage is greater or equal than the continuation wage paid to senior employees, then $\left(w^{r*}_{1,2}(p|.), w^{r*}_{2,2}(p|.)\right)$ does not belong in $\Gamma^r(.)$. Nevertheless, notice that the weighted sum of parties' utilities with replacement of senior employees is strictly smaller than without replacement, i.e. $T^r(.) < T^{nr}(.)$ for all $p > z$. Therefore, if $w^{r*}_{1,2}(p|.) \leq w^{r*}_{2,2}(p|.)$, then the optimal contract ω^* prescribes $\rho^* = 0$ and the wages $w^{nr*}_{1,2}(p|.)$ and $w^{nr*}_{2,2}(p|.)$.

4.5. Charcaterization Results

If contracts were complete, it would be optimal to offer $w^H(p|.)$ to those junior workers hired to fill open positions, to offer $w^I(p|.)$ to senior employees and to minimize their risk of unemployment by offering an unattractive salary to junior employees hired as substitutes. In general, this ex-post efficient allocation is not feasible when contracts are incomplete. When the hiring wage is constrained to be independent from the nature of hiring, the optimal contract either minimizes the employment risk of senior employees by restricting $w_{2,2}$ to be greater than $w_{1,2}$ or introduces employment risk and allows for an unconstrained hiring wage. In the previous pages, we have derived the properties of the contract with and without employment risk. In this subsection, we finally identify which alternative is best as a function of firm's productivity.

When the realization of firm's productivity is sufficiently high, the wage required to maximize the value of the v^u open positions is greater than the wage required to optimize the provision of insurance to senior employees. In this case, the ex-post efficient allocation can be implemented with incomplete contracts. Specifically, the incomplete self-enforcing contract can prescribe to offer $w^H(p|.)$ to junior employees and $w^I(p|.)$ to senior employees. Because $w^H(p|.)$ is greater than $w^I(p|.)$, none of the applicants tries to get a position currently held by a senior and the incomplete contract does not introduce any employment risk. As discussed in the previous sub-sections, the critical level of productivity at which the optimal hiring wage curve crosses the optimal insurance wage is k_1. These remarks lead to the following proposition.

Proposition 4.2: *Let ω^* be an optimal incomplete limited-commitment contract such that the constraints (10) are not binding. For all $\zeta_{t+1} = \{x_{t+1}, p_{2,t+1}, \theta_{t+1}\}$ such that $p_{2,t+1}$ is greater than $k_1(.)$, the contract ω^* prescribes that: (i) $\rho^*\left(\zeta_{t+1}\right)$ is equal to 0, (ii) $w^*_{1,2}\left(\zeta_{t+1}\right)$* is equal to $w^I(p_{2,t+1}|.)$; *(iii) $w^*_{2,2}(\zeta_{t+1})$ is equal to $w^H(p_{2,t+1}|.)$.*

Proof. See the Technical Appendix. $||$

When the productivity of the firm falls short of k_1, the ex-post efficient allocation is not feasible under incomplete contracts. In fact, if the firm was to offer the optimal hiring wage $w^H(p|.)$ to junior employees and promise the optimal insurance wage $w^I(p|.)$ to senior employees, there would be some unemployed workers applying to positions currently held by seniors. And the firm would lay-off and replace as many tenured workers as possible because $w^I(p|.)$ is greater than $w^H(p|.)$. As discussed in Section 4.3, this moral hazard problem can be eliminated by distorting the hiring and the insurance wages away from the optimum and pay a tenure-independent firm-wide wage w_2^{nr*}. On the one hand, the firm-wide wage is strictly greater than the optimal hiring wage $w^H(p|.)$ and therefore an inefficiently high number of applicants is attracted to the firm's vacancies. On the other hand, the firm-wide wage is strictly smaller than the optimal insurance wage $w^I(p|.)$ and therefore imposes some extra consumption risk on the workers. Alternatively, the firm can eliminate the consumption risk by offering $w^I(p|.)$ to senior employees and reduce the employment risk by distorting the hiring wage downwards. If the firm's productivity is below the cutoff k_1 but arbitrarily close to it, the cost of distorting the hiring and insurance wages by setting a firm-wide wage w_2^{nr*} becomes arbitrarily small. On the contrary, the cost of distorting the hiring wage downwards and imposing some employment risk on senior employees does not vanish as the firm's productivity approaches the cutoff k_1. If the wage $w_{2,2}^{r*}$ converges towards z, the employment risk vanishes but nobody applies to the firm's open positions. If the wage $w_{2,2}^{r*}$ converges towards $w^H(p|.)$, the efficient number of applicants is attracted towards the firm's openings but senior employees are laid-off too often. These remarks lead to the following proposition.

Proposition 4.3: *Let ω^* be an optimal incomplete limited-commitment contract such that the constraints (10) are not binding. There exists an $\epsilon(.) > 0$ such that for all $\zeta_{t+1} = \{x_{t+1}, p_{2,t+1}, \theta_{t+1}\}$ with $p_{2,t+1}$ in the interval $(k_1(.) - \epsilon(.), k_1(.))$, the contract ω^* prescribes that: (i) $\rho^*\left(\zeta_{t+1}\right)$ is equal to 0, (ii) $w_{1,2}^*\left(\zeta_{t+1}\right)$ is equal to $w_2^{nr*}(p_{2,t+1}|.)$; (iii) $w_{2,2}^*(\zeta_{t+1})$ is equal to $w_2^{nr*}(p_{2,t+1}|.)$; (iv) $w_2^{nr*}(p_{2,t+1}|.)$ is strictly greater than $w^H(p_{2,t+1}|.)$ and strictly smaller than $w^I(p_{2,t+1}|.)$.*

Proof. See the Technical Appendix. ||

Minimizing the risk of unemployment by setting a firm-wide wage is not always the optimal way to cope with the moral hazard created by contractual incompleteness. To illustrate this point consider a realization of firm's productivity in the interval between the cutoffs k_2 and k_3. On the one hand, if the firm offers a common wage for senior and junior employees, it is optimal to set w_2^{nr*} equal to the productivity of labor p. On the other hand, if the firm offers different wages for workers with different tenure, it is optimal to offer an attractive and profitable wage to junior employees, i.e. $w_{2,2}^{r*}(p|.) \in (z, p)$, and to provide perfect consumption insurance to senior employees, i.e. $w_{2,2}^{r*}(p|.) = w^I(p|.)$. Notice that the payoffs generated by setting a firm-wide wage can be replicated with an appropriate selection of tenure-specific wages, i.e. $w_{2,2}^{r*}(p|.) = z$ and $w_{1,2}^{r*}(p|.) = p$. By revealed preferences, the weighted sum of worker's and firm's payoffs must be greater under the second alternative.

Proposition 4.4: *Let ω^* be an optimal incomplete limited-commitment contract such that the constraints (10) are not binding. For all $\zeta_{t+1} = \{x_{t+1}, p_{2,t+1}, \theta_{t+1}\}$ such that $p_{2,t+1}$ is greater than $k_2(.)$ and smaller than $k_3(.)$, the contract ω^* prescribes that: (i) $\rho^*\left(\zeta_{t+1}\right)$ is equal to 1, (ii)*

$w^*_{1,2}\left(\zeta_{t+1}\right)$ *is equal to* $w^I(p_{2,t+1}|.)$*; (iii)* $w^*_{2,2}(\zeta_{t+1})$ *is equal to* $w^{r*}_{2,2}(p_{2,t+1}|.)$*, where* $w^{r*}_{2,2}(p_{2,t+1}|.)$ *belongs to* $(z_{t+1}, p_{2,t+1})$*.*

Proof. See the Technical Appendix. ||

Under what conditions on the fundamentals of the economy is the cutoff k_2 smaller than k_3? Suppose that the probability of filling an open position $h(w|.)$ can be represented as the product between an efficiency parameter A and a concave function $\hat{h}(w|.)$. The optimal hiring wage $w^H(p|.)$ does not depend on the efficiency of the matching process because A affects equally the marginal cost and the marginal benefit of increasing $w_{2,2}$. Similarly, the cutoffs $k_1(.)$ and $k_3(.)$ do not depend on the efficiency parameter A. On the other hand, the cutoff $k_2(.)$ is decreasing in the efficiency of the matching process. Indeed, for A sufficiently small $k_2(.)$ is equal to the first period wage w^*_1, while $k_3(.)$ remains strictly smaller than w^*_1. When search frictions are sufficiently large, the optimal incomplete contract prescribes inefficient separations in some states of the world.

When the productivity of the firm falls between the flow value of search z and $\min\{k_2(.), k_3(.)\}$, the optimal incomplete contract is indeterminate. On the one hand, if the firm offers a common wage to all its employees, it is optimal to set w^{nr*}_2 equal to the productivity of labor p. On the other hand, if the firm offers different wages for workers with different tenure, it is optimal to offer an unattractive wage to junior employees, i.e. $w^{r*}_{2,2}(p|.) \leq z$, and to provide perfect consumption insurance to senior employees, i.e. $w^{r*}_{2,2}(p|.) = w^I(p|.)$ which in turn is equal to the productivity of labor p. From the perspective of the firm and its senior employees, the two alternatives are identical.

Proposition 4.5: *Let* ω^* *be an optimal incomplete limited-commitment contract such that the constraints (10) are not binding. For all* $\zeta_{t+1} = \{x_{t+1}, p_{2,t+1}, \theta_{t+1}\}$ *such that* $p_{2,t+1}$ *is greater than* z *and smaller than* $\min\{k_2(.), k_3(.)\}$*, the contract* ω^* *prescribes one of the following: (i)* $\rho^*\left(\zeta_{t+1}\right)$ *is equal to* 0 *and* $w^*_{1,2}\left(\zeta_{t+1}\right)$*,* $w^*_{2,2}(\zeta_{t+1})$ *are both equal to* $p_{2,t+1}$*; (ii)* $\rho^*\left(\zeta_{t+1}\right)$ *is equal to* 1*,* $w^*_{1,2}\left(\zeta_{t+1}\right)$ *is equal to* $w^I(p_{2,t+1}|.)$ *and* $w^*_{2,2}(\zeta_{t+1})$ *is smaller ro equal to* z*.*

Proof. See the Technical Appendix. ||

For realization of productivity p_k in the interval $[x_2, x_1 - \epsilon]$, the optimal contract cannot be completely characterized unless we are willing to impose more structure on the fundamentals of the economy. Indeed, it is possible to construct examples such that θ^*_k is equal to 1 for all realizations of p over the interval $[x_2, x_1 - \epsilon]$, examples such that θ^*_k is equal to 0, and examples such that the interval can be divided into two or more subsets where θ^*_k takes on different values. Figure 3 illustrates the continuation and the hiring wages, i.e. $\theta^*_k \cdot w^{nr*}_{i,2}(p_k) + (1 - \theta^*_k) \cdot w^{r*}_{i,2}(p_k)$ for $i = 1, 2$, in the case where x_3 is smaller than x_2 and θ^*_k is equal to 0 over the entire interval $[x_2, x_1 - \epsilon]$.

5. GENERAL EQUILIBRIUM ANALYSIS

In this section, we derive the general equilibrium effects of the limited-commitment and incompleteness assumptions. For the purposes of such general equilibrium analysis, it is convenient to introduce three modifications to the framework described in Section 2. First, we assume that workers can search during the same period in which they are laid-off. Secondly, we rescale the payoffs of a firm and a worker who trade during the second period of the firm's life. Specifically, the worker's payoff is $(1-\beta\cdot(1-\sigma))^{-1}\cdot(u(w_{1,2})+\beta\sigma Z)$ and the firm's payoff is $(1-\beta\cdot(1-\sigma))^{-1}\cdot(p-w_{1,2})$. Finally, we assume that established firms are subject to aggregate productivity shocks, while the productivity of newly created firms is acyclical. The first two modifications are introduced to make the expected discounted value of a match time-independent as if the firm's horizon was infinite. The third modification allows us to carry out the general equilibrium analysis without relying on numerical simulations.

5.1. Characterization of the General Equilibrium

Suppose that the equilibrium value of search Z is constant across all states of the economy in the ergodic set of the dynamical system. Given this conjecture, consider the economic system at some arbitrary date t. Every newly created firm advertises the incomplete limited-commitment contract $\omega_{1,t}$. The contract $\omega_{1,t}$ is the solution $\omega^*(N_{1,t}, W_{1,t}; Z)$ of the constrained optimization problem (LCC), where the promised value $W_{1,t}$ maximizes the firm's profits and $N_{1,t}$ is determined by the worker's application strategy

$$\begin{aligned} W_{1,t} = \quad & \arg\max\nolimits_{W_1 \geq Z} P_1\left(\omega^*\left(N\cdot\eta\left(q\left(W_1,1;Z\right)\right),W_1\right);Z\right), \\ N_{1,t} = \quad & N\cdot\eta\left(q\left(W_{1,t},1;Z\right)\right)\equiv N\cdot\eta\left(q_{1,t}\right). \end{aligned}$$

Every established firm offers to its new employees a contract $\omega_{2,t}$ worth $W_{2,t}(p_{2,t},\theta_t)$. The promised value $W_{2,t}(p_{2,t},\theta_t)$ is equal to $(1-\beta\cdot(1-\sigma))^{-1}\cdot(u(w_{2,2}(p_{2,t},\theta_t))+\beta\sigma Z)$ and is determined by the contract $\omega_{1,t-1}$ offered by the firm at date $t-1$. Similarly, the continuation value of the contract offered by an established firm to its senior employees is $(1-\beta\cdot(1-\sigma))^{-1}\cdot(u(w_{1,2}(p_{2,t},\theta_t))+\beta\sigma Z)$. Assuming that the realization of the state of nature $(p_{2,t},\theta_t)$ is such that $w_{1,2}(p_{2,t},\theta_t)\leq w_{2,2}(p_{2,t},\theta_t)$, then the number of applicants $q_{2,t}$ for each vacant position is $q(W_{2,t}(p_{2,t},\theta_t),1;Z)$.

The measure of vacancies at established firms is given by $f_{t-1}\cdot(N-N_{1,t-1}(1-\sigma))$, where f_{t-1} is the number of firms created at date $t-1$. The measure of vacancies at new firms is given by $f_t\cdot N$, where f_t is indirectly determined by the free entry condition

$$\begin{aligned} \left(P_1\left(N_{1,t},W_{1,t};Z\right)-I\right)\cdot f_t &= 0, \\ P_1\left(N_{1,t},W_{1,t};Z\right)-I &\leq 0. \end{aligned}$$

Therefore, the measure of applicants at established and new firms is given by the sum of $f_{t-1}\cdot(N-N_{1,t-1}(1-\sigma))\cdot q_{2,t}$ and $f_t\cdot N\cdot q_{1,t}$. On the other hand, at date t the measure of workers that search for a job is given by the unemployment rate u_t. Therefore, the clearing condition for the labor market requires that

$$u_t = f_{t-1} \cdot (N - N_{1,t-1}(1-\sigma)) \cdot q_{2,t} + f_t \cdot N \cdot q_{1,t}.$$

The flow of workers ϕ_t^{ue} from unemployment to employment is given by the sum of matches created at old and new firms. The flow of workers ϕ_t^{eu} from employment to unemployment is given by the measure of matches destroyed. The change $u_{t+1} - u_t$ in the unemployment rate is given by the difference between these two flows. Formally, we have that

$$\phi_t^{ue} = f_{t-1} \cdot (N - N_{1,t-1}(1-\sigma)) \cdot \eta(q_{2,t}) + f_t \cdot N \cdot \eta(q_{1,t}),$$

$$\phi_t^{eu} = f_{t-1} \cdot N_{1,t-1}(1-\sigma) + f_{t-1} \cdot (N - N_{1,t-1}(1-\sigma)) \cdot \eta(q_{2,t}) + \sigma \cdot f_t \cdot N_{1,t},$$

$$u_{t+1} - u_t = \phi_t^{eu} - \phi_t^{ue}.$$

5.2. Dynamical System

Given the conjecture that Z is constant across states, equation (18) implies that the value $W_{1,t}$ of the contract offered by newly created firms and the number of applicants per vacancy $q_{1,t}$ that it draws are both constant—i.e. $W_{1,t} = W_1$, $q_{1,t} = q_1$. The latter remark implies that the optimal incomplete contract $\omega_{1,t}$ offered by new firms is constant across states of the world—i.e. $\omega_{1,t} = \omega_1$ and $W_{2,t+1}(p_2,\theta) = W_2(p_2,\theta)$. Therefore, at date t, there are $N - N \cdot q_1(1-\sigma)$ vacancies available at old firms and each of them receives $q(W_2(p_{2,t},\theta_t), 1; Z)$ applicants.

Using the previous observations and the market clearing condition (20), the number of entering firms f_t can be expressed as a function of the rate of unemployment u_t, the number of established firms f_{t-1} and the realization of the state of nature $(p_{2,t},\theta_t)$. More specifically, we have

$$f_t(u_t, f_{t-1}, p_{2,t}, \theta_t) = \frac{u_t - f_{t-1} \cdot (N - N_1 \cdot (1-\sigma)) \cdot q(W_2(p_{2,t},\theta_t), 1; Z)}{N \cdot q_1}.$$

If (22) is substituted into equation (21), then the next-period unemployment rate u_{t+1} can be expressed as a function of the state variables $\{u_t, f_{t-1}, p_{2,t}, \theta_t\}$, i.e.

$$u_{t+1}(u_t, f_{t-1}, p_{2,t}, \theta_t) = u_t + N_1(1-\sigma)[f_{t-1} - f_t(u_t, f_{t-1}, p_{2,t}, \theta_t)].$$

Because $p_{2,t}$ and θ_t are i.i.d. over time, equations (22) and (23) are sufficient to describe the dynamics of the economy from any starting point $\{u_0, f_{-1}, p_{2,0}, \theta_0\}$.

According to the equilibrium condition (19), whenever a positive measure of firms enters the market the ex-post profits $P_1(N_{1,t}, W_{1,t}; Z)$ are equal to the investment cost I. But, if the value of search is constant across states, then $(N_{1,t}, W_{1,t})$ are determined by the firm's optimality condition (18) and can be expressed as functions of Z only.Therefore, in every state such that a positive measure of firms enters the market, the value of search Z is pinned down by the free entry condition (19). We can conclude that, if the fundamentals of the economy are such that there is entry at every state in the ergodic set of the system, then the conjecture that the value of search Z is constant is vindicated.

5.3. The GE Effect of Incompleteness

Let the stochastic process of productivity be as follows. In every period t, the productivity $p_{1,t}$ of labor at newly established firms is p^*. In every period t, the productivity $p_{2,t}$ at old firms takes the value p^* with probability $1-2\pi$ and the value $p^* - \alpha$ $(p^* + \alpha)$ with probability π. Moreover, suppose that aggregate shocks are zero probability events and they have small magnitude.

The optimal incomplete contract under limited-commitment prescribes that the wage paid to senior employees is almost surely kept constant—i.e. $w^*_{1,2}(p^*, \theta) = w^*_1$ for all $\theta \in [0, 1]$—and that new hires are almost surely offered the wage that maximizes the value of unfilled vacancies—i.e. $w^*_{2,2}(p^*, \theta) = w^*(p^*)$ for all θ. Because the productivity of labor is almost surely constant over time, the continuation wage offered to senior employees is almost surely equal to the hiring wage—i.e. $w^*_{1,2}(p^*, \theta) = w^*_{2,2}(p^*, \theta)$ for all θ and $x_1 = p^*$. The contract also specifies the wages conditional on a positive or a negative shock to productivity, even though these are zero probability events. In case the economy is hit by the positive productivity shock, Proposition 4.3 implies that senior employees are offered the full-insurance wage w^*_1 and the firm advertises the optimal hiring wage $w^*(p^* + \alpha)$. If the economy is hit by the negative productivity shock, Proposition 4.4 implies that senior and junior employees are paid the common wage $s^{nr*}(p^* - \alpha)$ which is strictly greater than the optimal hiring wage $w^*(p^* - \alpha)$ and strictly smaller than the full-insurance wage w^*_1.

Alternatively, consider the assumptions of complete contracts and limited commitment. If contracts are complete, the firm can specify a different hiring wage depending on whether the junior employee is replacing a senior worker or not. Therefore, by promising to the applicants a sufficiently high wage in case they replace a senior employee, the firm can commit to a long-term employment relationship. In the environment described in this section, the optimal complete contract under limited commitment prescribes that senior employees are paid the wage w^*_1 for every realization of the state of nature and that new applicants are offered the wage $w^*(p_2)$ if they apply and fill a vacant position and the wage w^*_1 if they apply and fill a position held by a senior.

Suppose that the economy is hit by the negative productivity shock at date t. Maintaining the limited-commitment assumption, established firms offer a strictly greater wage and attract strictly more applicants per vacancy if contracts are incomplete. In turn, equation (22) implies that a smaller number of firms enters the market and the total measure of vacancies is smaller if contracts are incomplete. Moreover, equations (21) and (22) imply that fewer matches are created, more unemployed workers remain unemployed during period t and the the unemployment rate at date $t+1$ is higher under the assumption of incomplete contracts. On the other hand, if the economy is hit by the positive productivity shock, the complete and incomplete contracts are identical and so are the unemployment dynamics.

6. DISCUSSION

1. a game theoretic interpretation of the contract.

We can interpret the contract as a sub-game perfect equilibrium. A worker only observes his own history of employment and the values posted by the firm in the market. When the firm or the worker deviate, grim strategies are triggered. The contract studied in the paper need not be explicit: it can be implicit. How can applicants be assigned to vacancies as we assumed in the model, if they do not know where the unfilled jobs are? Imagine a case where the wage of applicants is lower, then they need no information. They just apply at random. Otherwise, they can ask the firm what jobs to try out for and the firm will direct them to the vacant jobs.

2. The smart reader can realize that now it is not without loss of generality that the contract only depends on the individual worker-firm history. One can imagine various schemes to buy commitment. We rule those out by assumption. The issue will be discussed later. In general, you could think that transfers can be conditioned on firm-aggregate outcomes in order to achieve some commitment. This is a restrictive assumption, because you can imagine the first best contract can be implemented with limited commitment by making the wage of non-replaced workers increasing in the turnover rate. Such an arrangement seems unappealing because easily subject to renegotiation. Once the allocation of applicants to jobs is realized, then the firm and a subset of the workers can improve upon the original contract by allowing the firm to replace the workers for which there is a qualified replacement.

3. savings: not a big deal because of σ. The firm cannot get the value 1 to 1. Also there might be differences in the return to savings between firm and worker. In general why should you be at the corner?

4. why do we need search frictions? yes, we need search frictions to have unemployment;

5. there are no welfare effects: the limited-commitment contract is constrained Pareto-efficient.

A. TECHNICAL APPENDIX

Proof of Proposition 3.1: (i) Suppose that there is a realization of the aggregate shock x_l, of the firm-specific shock $p_k > b$ and a subset Θ_1 of [0,1] with positive measure such that the contract ω^* prescribes $\delta_2^* > 0$. Consider an alternative contract $\hat{\omega}$ that makes the same prescriptions as ω^* with the exception that, when $(x_{t+1}, p_{2,t+1}, \theta_{t+1})$ belongs to (x_l, p_k, Θ_1), then $\hat{\delta}$ is equal to 0 and $\hat{w}_{1,2}$ is equal to $\delta_2^* \cdot \left(b + s_{1,2}^*\right) + (1 - \delta_2^*) \cdot w_{1,2}^*$. Since the utility function is concave, the contract $\hat{\omega}$ gives the worker at least the same expected utility as ω^* and, therefore, is feasible. Moreover, the firm's profit under $\hat{\omega}$ exceeds the profit under ω^* by

$$\pi_{m,l}\pi_{k,l}N_1\delta_1(1-\sigma)\int_{\Theta_1}\left\{\left(1-\eta(q^e\left(\zeta_{t+1}\right))\right)\delta_2\left(\zeta_{t+1}\right)(p_k-b)\right\}d\theta > 0. \tag{24}$$

The contract ω^* is not optimal: a contradiction. In a similar way, we can prove the remainder of part (i).

(ii) Suppose that there is a realization of the aggregate shock x_l, of the firm-specific shock p_k and a subset Θ_1 of [0,1] with positive measure μ such that the contract ω^* prescribes $w_{1,2}^* \neq b + s_{1,2}^*$. Consider an alternative contract $\hat{\omega}$ such that $\hat{\omega}_2\left(\zeta_{t+1}\right)$ is equal to $\omega_2^*\left(\zeta_{t+1}\right)$ if $\zeta_{t+1} \notin (x_l, p_k, \Theta)$ and $\hat{\omega}_2\left(\zeta_{t+1}\right)$ is equal to $\left\{\hat{w}_{1,2}, \hat{s}_2, \delta_2^*, \rho^*, w_{2,2}^e\right\}$ if $\zeta_{t+1} \in (x_l, p_k, \Theta_1)$. Specifically, let $\hat{w}_{1,2}$ be the solution to the equation

$$u\left(\hat{w}_{1,2}\right) = \mu^{-1}\int_{\Theta_1}\left\{\left[1-\sigma+\eta\left(q^e\right)\rho+\left(1-\eta\left(q^e\right)\right)\delta_2^*\right]\cdot\left[u\left(w_{1,2}^*\right)-u\left(b+s_{1,2}^*\right)\right]+u\left(b+s_{1,2}^*\right)\right\}d\theta \tag{25}$$

and $\hat{s}_2$ is equal to $\hat{w}_{1,2} - b$. By construction, the contract $\hat{\omega}$ gives the same expected utility to the worker as ω^* and, therefore, is feasible. Moreover, the concavity of the utility function $u(c)$ implies that $\hat{w}_{1,2}$ is strictly smaller than the average of $w_{1,2}$ and $b + s_{1,2}$. Therefore, firm's profit under $\hat{\omega}$ is strictly greater than under ω^*. The contract ω^* is not optimal: a contradiction. In a similar way, we can prove the remainder of part (ii).

(iii) The result follows immediately from the firm's objective function. ||

Proof of Lemma 4.1:. (i)–(ii) Suppose that there is a state $\{x_{t+1}, p_{2,t+1}, \Theta_1\}$ where the optimal contract ω^* violates inequality (13.a). Denote with $\left\{w_{1,2}^{nr}, w_{2,2}^{nr}\right\}$ the couple of wages that maximizes $T^{nr}(w_1^*, w_{1,2}, w_{2,2}|.)$, subject to $b \leq w_{1,2} \leq w_{2,2} \leq p_{2,t+1}$. Consider a sequence $\{\Theta_n\}_{n=1}^{\infty}$ of subsets of Θ_1 such that the probability mass of the state $\{x_{t+1}, p_{2,t+1}, \Theta_n\}$ converges to zero and the Δ_n converges to a strictly postive value, where Δ_n is defined as

$$\begin{aligned}
&\Delta_n = \mu\left(\Theta_n\right)^{-1}\beta\textstyle\int_{\Theta_n} T^{nr}\left(w_1^*, w_{1,2}^{nr}, w_{2,2}^{nr}|.\right) - \\
&\mu\left(\Theta_n\right)^{-1}\beta\textstyle\int_{\Theta_n}\left\{\mathbf{1}(w_{1,2}^*\left(\Theta_n\right) \leq w_{2,2}^*\left(\Theta_n\right)) \cdot T^{nr}\left(w_1^*, w_{1,2}^*\left(\Theta_n\right), w_{2,2}^*\left(\Theta_n\right)|.\right)\right\}d\theta - \\
&\mu\left(\Theta_n\right)^{-1}\beta\textstyle\int_{\Theta_n}\left\{\mathbf{1}(w_{1,2}^*\left(\Theta_n\right) > w_{2,2}^*\left(\Theta_n\right)) \cdot T^{r}\left(w_1^*, w_{1,2}^*\left(\Theta_n\right), w_{2,2}^*\left(\Theta_n\right)|.\right)\right\}d\theta.
\end{aligned} \tag{26}$$

For every n, consider the contract $\hat{\omega}_n$ that prescribes $\hat{\delta}_2 = \delta_2^*$, $w_{1,2}^{nr} \leq w_{2,2}^{nr}$ and $\hat{\rho} = 0$ at date $t+1$ and in state $\{x_{t+1}, p_{2,t+1}, \Theta_n\}$, that replicates the contract ω^* at date $t+1$ and in any state different from $\{x_{t+1}, p_{2,t+1}, \Theta_n\}$ and that specifies a wage $\hat{w}_1$ such that the worker is indifferent between $\hat{\omega}$ and ω^*. By assumption, the contract ω^* is such that the constraints (10) are not binding. Therefore, for $\mu\left(\Theta_1\right)$ sufficiently small, the contract $\hat{\omega}$ satisfies the constraints (10) as well. Moreover, the contract $\hat{\omega}$ satisfies (8) and (9) by contruction. The contract $\hat{\omega}$ is feasible. By switching from ω^* to $\hat{\omega}_n$, the firm's profits change by $\mu\left(\Theta_n\right) \cdot \Delta_n$ plus a higher order function of $(w_1^* - \hat{w}_n)$. Because $|w_1^* - \hat{w}_n|$

is bounded above by $\mu(\Theta_n) \cdot (u(p_{2,t+1}) - u(b))$, there exists an N such that for all $n \geq N$, the firm strictly prefers $\hat{\omega}_n$ to ω^*. A contradiction. Part (ii) of Lemma 4.1 is proved in the same fashion.

(iii) Suppose that there is a realization of the aggregate shock x_l, of the firm-specific shock $p_k > b$ and a subset Θ_1 of [0,1] with positive measure μ such that the contract ω^* prescribes $\delta_2^* > 0$. Consider an alternative contract $\hat{\omega}$ that replicates ω^* with the exception that $\hat{\delta}$ is set equal to 0 and $\hat{w}_{1,2}$ is set equal to $\delta_2^* \cdot b + (1 - \delta_2^*) \cdot w_{1,2}^*$ when $\zeta_{t+1} \in \{x_l, p_k, \Theta_1\}$. By construction, the wage $\hat{w}_{1,2} \in [b, p_k]$. Since the utility function is concave, the contract $\hat{\omega}$ gives the worker at least the same expected utility as ω^* in every state of the world and it satisfies the constraints (8) and (10.a). Moreover, in state $\{x_l, p_k, \Theta_1\}$ the firm's profit under $\hat{\omega}$ exceeds the profit under ω^* by

$$\mu^{-1} N_1 (1 - \sigma) \int_{\Theta_1} \left\{ \left(1 - \eta(q^e\left(\zeta_{t+1}\right))\right) \delta_2\left(\zeta_{t+1}\right)(p_k - b) \right\} d\theta > 0. \tag{27}$$

The previous inequality implies that the contract $\hat{\omega}$ satisfies the constraint (10.b). Therefore, the contract $\hat{\omega}$ is feasible and afford the firm with strictly more profits than ω^*: a contradiction. In a similar way, we can prove the remainder of part (iii). ||

Proof of Proposition 4.2: For all $p \geq x_1$, the no-replacement outcome of the lottery $\left(w_{1,2}^{nr*}(p), w_{2,2}^{nr*}(p)\right)$ is given by (12). By definition, the couple $\left(w_{1,2}^{nr*}(p), w_{2,2}^{nr*}(p)\right)$ maximizes the objective function $T^{nr}(w_1^*, w_{1,2}, w_{2,2}; p)$ over the triangle $\left\{(w_{1,2}, w_{2,2}) \in [b,p]^2 : w_{2,2} \geq w_{1,2}\right\}$. Given the concavity of the objective function $T^{nr}(.)$ with respect to $(w_{1,2}, w_{2,2})$, (12) maximizes the objective function $T^{nr}(.)$ over the square $[b,p]^2$. Moreover, because the function $T^r(.)$ is strictly smaller than $T^{nr}(.)$ for every $(w_{1,2}, w_{2,2}) \in [b,p]^2$, it follows that $T^{nr}(w_1^*, w_{1,2}^{nr*}(p), w_{2,2}^{nr*}(p); p)$ is strictly greater than the maximum of $T^r(.)$ over the triangle $\{(w_{1,2}, w_{2,2}) \in [b,p]^2 : w_{2,2} < w_{1,2}\}$. Therefore, (12) satisfies condition (10) in Lemma 4.1 and the result follows. ||

Proof of Proposition 4.3: Without loss of generality we can assume that w_1^* is greater than b or otherwise the interval (b, x_1) is empty. For all $p \in (x_2, x_1)$, the no-replacement outcome of the lottery $\left(w_{1,2}^{nr*}(p), w_{2,2}^{nr*}(p)\right)$ is given by $(s^{*nr}(p), s^{*nr}(p))$. For all $p \in (x_3, x_1)$, the replacement outcome $\left(w_{1,2}^{r*}(p), w_{2,2}^{r*}(p)\right)$ is given by $(w_1^*, s^{*r}(p))$. Therefore, for all p in the interval $(\max\{x_2, x_3\}, x_1)$, the difference between the weighted sum of utilities generated by the two allocations is

$$\begin{aligned} & T^{nr}(w_1^*, w_{1,2}^{nr*}(p), w_{2,2}^{nr*}(p); p) - T^r(w_1^*, w_{1,2}^{r*}(p), w_{2,2}^{r*}(p); p) = \\ & N_1(1-\sigma) \cdot \left[\tfrac{u(s^{*nr}(p)) - u(w_1^*)}{u'(w_1^*)} + (w_1^* - s^{*nr}(p))\right] + \\ & N_1(1-\sigma) h\left(s^{*r}(p)\right) \cdot \left[\tfrac{u(w_1^*) - u(b)}{u'(w_1^*)} + (s^{*nr}(p) - w_1^*)\right] + \\ & N\left[H(s^{*nr}(p); p) - H(s^{*r}(p); p)\right]. \end{aligned} \tag{28}$$

For $p \to x_1$, the wage $s^{*nr}(p)$ converges to $w_1^* = w^*(x_1)$. If $s^{*r}(p)$ converges to $w^*(x_1)$, then the limit of (28) is strictly positive because the first and third term vanish and the second term converges to a strictly positive value. If $s^{*r}(p)$ converges to b, then the limit of (28) is strictly positive because the first and second terms vanish and the third term converges to a strictly positive value. If $s^{*r}(p)$ converges to any value in the interval $(b, w^*(x_1))$, then the limit of (28) is strictly positive because both the limits of the second and the third term are strictly positive. By continuity, there exists a left neighborhood of x_1 where (28) is strictly positive. ||

Proof of Proposition 4.4: Using the definition (**??**), it is immediate to verify that the cutoff x_2 belongs to the open interval (b, w_1). Using the definition (17), it is immediate to verify that the cutoff x_3 belongs to the interval $[b, w_1]$. Therefore, for all $p \in (x_3, x_2)$, the no-replacement outcome of the lottery $\left(w_{1,2}^{nr*}(p), w_{2,2}^{nr*}(p)\right)$ is given by (p, p) and the replacement outcome $\left(w_{1,2}^{r*}(p), w_{2,2}^{r*}(p)\right)$ is given by $(p, s^{*r}(p))$. Because $(p, s^{*r}(p))$ is the unique solution to the maximization problem (15), then $T^r(w_1^*, p, s^{*r}(p); p)$ is strictly greater than $T^r(w_1^*, p, b; p)$. In turn, because $h(b; p) = 0$, $T^r(w_1^*, p, b; p)$ is equal to $T^{nr}(w_1^*, p, p; p)$. ||

Proof of Proposition 4.5: The result follows from the argument used in the proof of Proposition 4.4. ||

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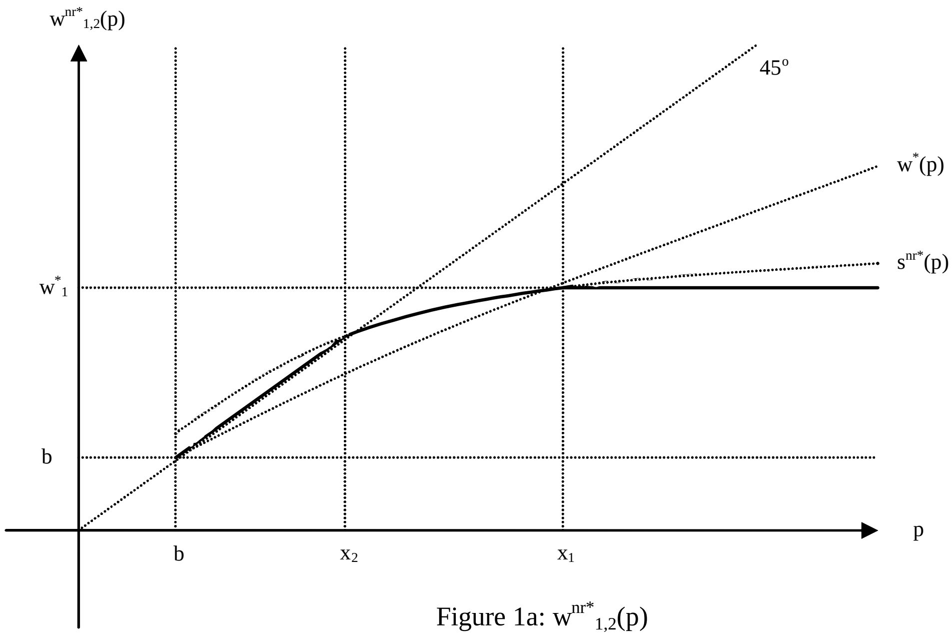

Figure 1a: $w^{nr*}_{1,2}(p)$

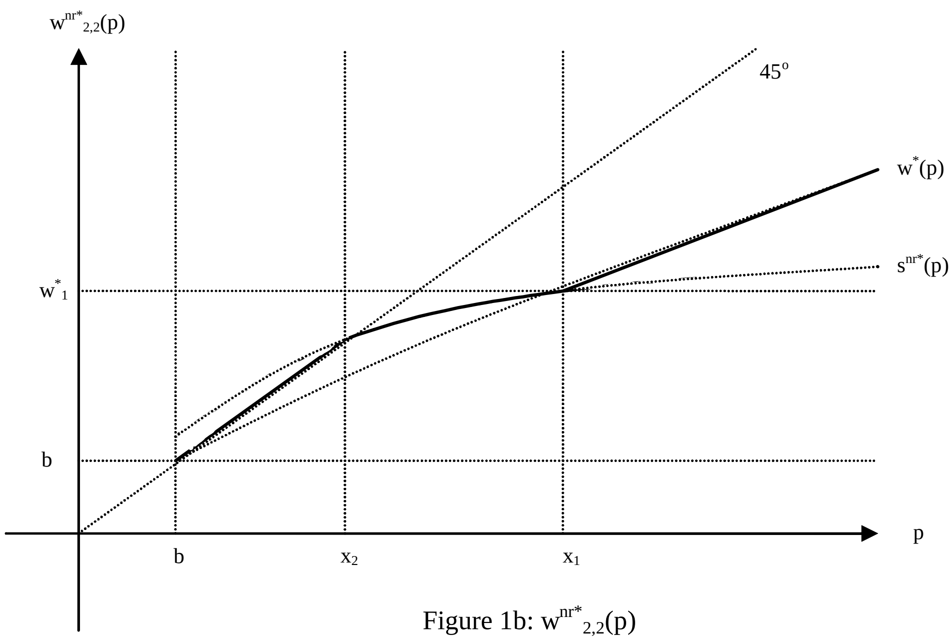

Figure 1b: $w^{nr*}_{2,2}(p)$

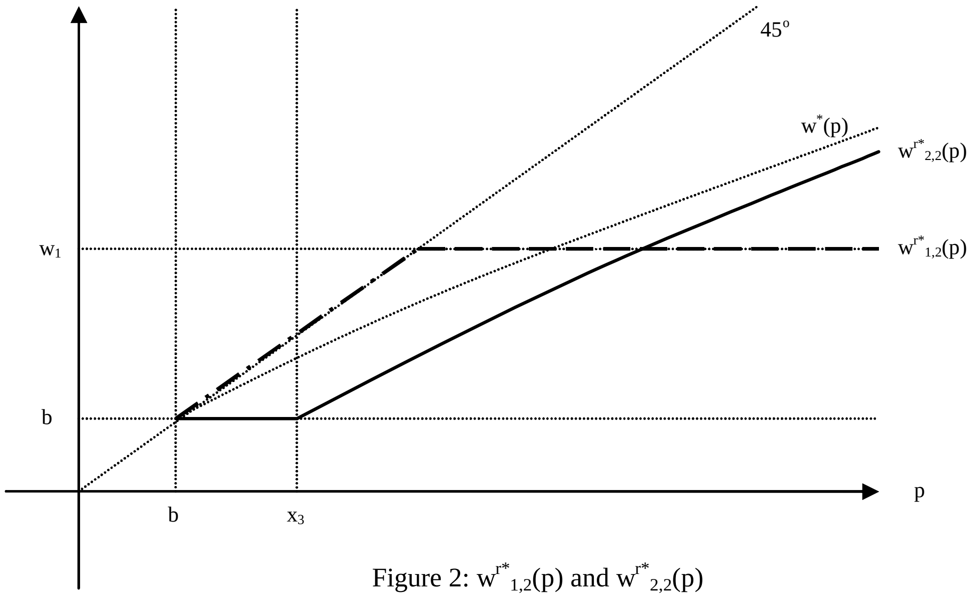

Figure 2: $w^{r*}_{1,2}(p)$ and $w^{r*}_{2,2}(p)$

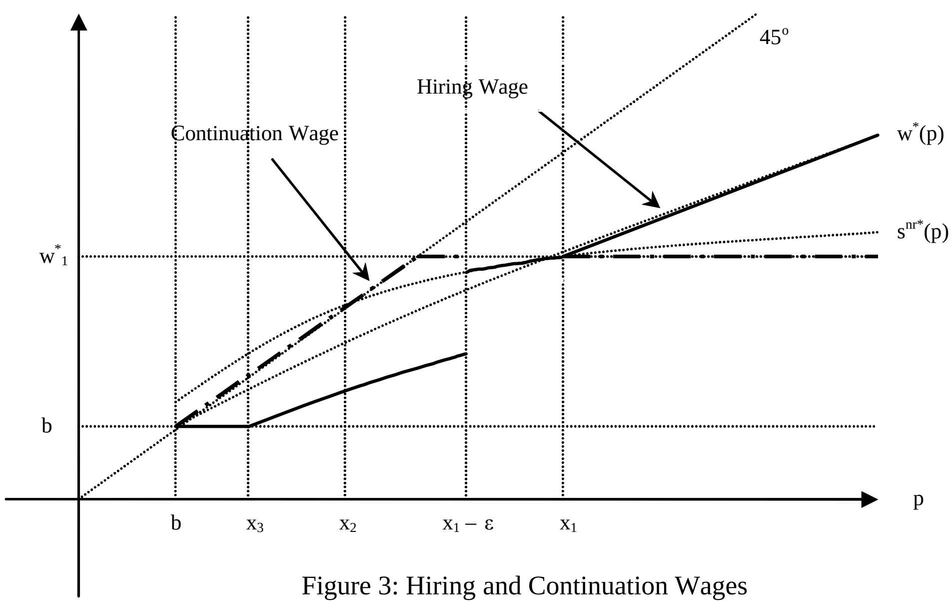

Figure 3: Hiring and Continuation Wages