Multivariate probit regression using simulated maximum likelihood

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Overview

- Introduction and motivation
- The model and the method of Simulated Maximum Likelihood (SML)
- The **`mvprobit`** program (**`ml`**, method **`lf`**)
- Illustrations – **`mvprobit`** in action
- Further remarks about program use

Introduction and motivation

- Evaluation of probit model likelihood functions requires calculation of Normal probability distribution functions.
- Algorithms exist for accurately calculating accurate univariate and bivariate Normal pdfs, but not for trivariate or higher dimensional Normal distributions (at least not in Stata). Instead, ...
- Recent literature on calculating multivariate Normal pdfs using simulation-based methods
- Here: multivariate probit model estimated using simulated ML ('GHK' simulator): **`mvprobit`**
  - cf. **`triprobit`** at SSC-IDEAS

The model

M equation multivariate probit model:

$y_{im}^{*} = \beta_m' X_{im} + \varepsilon_{im}$, $m = 1, \ldots, M$

$y_{im} = 1$ if $y_{im}^{*} > 0$ and 0 otherwise

ε_{im}, $m = 1, \ldots, M$, are error terms distributed as multivariate normal, each with a mean of zero, and variance-covariance matrix V, where V has values of 1 on the leading diagonal and correlations $\rho_{jk} = \rho_{kj}$ as off-diagonal elements.

- Structure like a SUR model but depvars are binary (and need not have same set of X in every equation)
- M different choices at a point in time OR choices on one item at M points in time (panel model with free correlations)

Estimation principles ($M = 3$)

Log-likelihood function for a sample of N independent observations:

$$L = \Sigma_i \, w_i \log\Phi_3(\mu_i; \Omega)$$

w_i is an optional weight

$\Phi_3(\mu_i; \Omega)$ is standard trivariate normal cdf, where

$\mu_i = (K_{i1}\beta_1'X_{i1} \, , K_{i2}\beta_2'X_{i2} \, , K_{i3}\beta_3'X_{i3})$

with $K_{ik} = 2y_{ik} - 1$, for each j, $k = 1,\ldots,3$.

Ω has elements Ω_{jk} , where $\Omega_{jj} = 1$ for $j = 1, \ldots, 3$; $\Omega_{21} = \Omega_{12} = K_{i1}K_{i2}\rho_{21}$, $\Omega_{31} = \Omega_{13} = K_{i3}K_{i1}\rho_{31}$, $\Omega_{32} = \Omega_{23} = K_{i3}K_{i2}\rho_{32}$.

- Log-likelihood function depends on the trivariate standard normal distribution function $\Phi_3(.)$!
- Evaluate using Geweke-Hajivassiliou-Keane (GHK) smooth recursive conditioning simulator

The GHK simulator

- Exploits the fact that a multivariate normal distribution function can be expressed as the product of sequentially conditioned univariate normal distribution functions – which can be easily and accurately evaluated.
- Trivariate case: 8 joint probabilities corresponding to the eight possible combinations of successes (y_{im} = 1) and failures (y_{im} = 0). Focus on Pr(every outcome is a success):

$$\Pr(y_1 = 1, y_2 = 1, y_3 = 1) = \Pr(\varepsilon_1 \le \beta_1' X_{1,}\ \varepsilon_2 \le \beta_2' X_{2,}\ \varepsilon_3 \le \beta_3' X_3)$$

$$= \Pr(\varepsilon_3 \le \beta_3' X_3 |\ \varepsilon_2 < \beta_2' X_{2,}\ \varepsilon_1 < \beta_1' X_1)$$

$$\times\ \Pr(\varepsilon_2 < \beta_2' X_2 \mid \varepsilon_1 < \beta_1' X_1) \times \Pr(\varepsilon_1 < \beta_1' X_1)$$

- Expression involves conditioning upon unobservable variables (that are correlated with each other).
- However if a good approximation for these conditional distributions can be found, then the likelihood function only requires evaluation of univariate integrals.
- How may the approximations be derived?

The GHK simulator (ctd.)

Cholesky decomposition of the covariance matrix for the errors:

$$E(\varepsilon\varepsilon') \equiv V = Cee'C$$

where C is the lower triangular Cholesky matrix corresponding to V and $e \sim \Phi_3(0, I_3)$, i.e. three uncorrelated standard normal variates.

Hence:

$$\varepsilon_1 = C_{11}e_1$$
$$\varepsilon_2 = C_{11}e_1 + C_{22}e_2$$
$$\varepsilon_3 = C_{31}e_1 + C_{32}e_2 + C_{33}e_3$$

and C_{jk} is the jkth element of matrix C.

$\Rightarrow$ rewrite Pr(three successes) as

$$\Pr(\varepsilon_1 \leq \beta_1'X_{1,}\ \varepsilon_2 \leq \beta_2'X_{2,}\ \varepsilon_3 \leq \beta_3'X_3)$$
$$= \Pr[e_3 \leq (\beta_3'X_3 - C_{32}e_2 - C_{31}e_1)/C_{33} \mid e_2 < (\beta_2'X_2 - C_{21}e_1)/C_{22}, e_1 \leq \beta_1'X_1/C_{11}]$$
$$\times \Pr[e_2 \leq (\beta_2'X_2 - C_{21}e_1)/C_{22} \mid e_1 \leq \beta_1'X_1/C_{11}] \times \Pr[e_1 \leq \beta_1'X_1/C_{11}].$$

The standard normal variates, e, that now appear in the decomposition are uncorrelated with each other (by construction).

The first two conditional probabilities can be further rewritten as unconditional probabilities defined in terms of truncated standard normal variates: ….

The GHK simulator (ctd.)

Rewriting in terms of truncated standard normal variates:

$$\Pr(\varepsilon_1 \le \beta_1'X_1, \varepsilon_2 \le \beta_2'X_2, \varepsilon_3 \le \beta_3'X_3)$$
$$= \Pr[\varepsilon_3 \le (\beta_3'X_3 - C_{32}e_2^* - C_{31}e_1^*)/C_{33}] \times \Pr[\varepsilon_2 \le (\beta_2'X_2 - C_{21}e_1^*)/C_{22}]$$
$$\times \Pr[\varepsilon_1 \le \beta_1'X_1/C_{11}]$$
$$= Q_3 \times Q_2 \times Q_1, \text{ say,}$$

where e_1^* and e_2^* are truncated univariate standard normal variates with upper truncation points at $\beta_1'X_1/C_{11}$ and $(\beta_2'X_2 - C_{21}e_1^*)/C_{22}$ respectively.

Computation of Q_1 is straightforward and, if one had some specific values for e_1^* and e_2^*, then one could also compute Q_2 and Q_3, and hence the overall multivariate probability.

Similar arguments for all the other probabilities of success/failure (and for $M > 3$).

The GHK simulator (ctd.)

GHK simulator:

(i) Derive values for e_1^*, e_2^* via random draws from upper truncated standard normal distributions with truncation points as above (use random number generator, plus inversion formula);

(ii) Recursively compute a multivariate probability value from the Qs.

(iii) Replicate R times, and then

(iv) Calculate the simulated probability as the arithmetic mean of the values of the simulated probabilities from each replication.

SML estimator: ML with the multivariate normal probabilities calculated at each iteration using the GHK simulator $\Rightarrow$ numerically intensive!

- SML estimator consistent as $N \to \infty$ and $R \to \infty$. (Asymptotic, like ML!)
- Simulation bias $\to 0$ when R raised with N. [$(R/\sqrt{N}) \to \infty$ sufficient.]

The **mvprobit** program

mvprobit *equation1 equation2 ... equationM* [**weight**] [**if** *exp*] [**in** *range*] [, **draws**(#) **seed**(#) **beta0** **atrho0**(*matrix_name*) **robust** **cluster**(*varname*) **constraints**(*numlist*) **level**(#) **maximize_options**]

where each equation is specified as

([eqname:] depvar [=] [varlist] [, noconstant])

by may be used; all types of weights allowed; has standard features of estimation commands including access to estimated results; no limit on *M* (in principle)

Options

draws(#**):** number of random draws in simulation (*R*). Default = 5.

seed(#**):** initial value of random-number seed used in simulation process. (Default = 123456789).

beta0: estimates of the marginal probit regressions are reported.

atrho0(*matrix_name***):** starting values for the off-diagonal elements of the correlation matrix *V* that differ from the default starting values (all zero).

Remaining options: same as the corresponding ones for **biprobit**.

Prediction using **mvppred**

mvppred newvarname_prefix [if *exp***] [in** *range***] [,**
statistic]

where **statistic** is one of

xb the linear prediction for each equation; the default.

stdp the standard error of the linear predictions for each equation.

pmarg the marginal success probability for each equation.

pall the joint probabilities: (i) Pr(y_{im} = 1, for all $m = 1,\ldots,M$), and (ii) Pr(y_{im} = 0, for all $m = 1,\ldots,M$).

Only one statistic may be chosen at a time.

For statistics **xb**, **stdp**, and **pmarg**, results are stored in the variables `newvarname_prefix`*i*, for equations $i = 1, \ldots, M$.

For the **pall** statistics, results are stored in the variables
`newvarname_prefix`*1s* for predicted probability (i) and
`newvarname_prefix`*0s* for predicted probability (ii).

[Options for prediction restricted to the 'all successes' and 'all failures' cases = the only two cases that could be programmed without M being fixed. (Number of joint probabilities is 2^M.)]

Illustrations

(1) (i) syntax, options etc., and (ii) accuracy of SML relative to ML (**`mvprobit`** vs. **`biprobit`**)

– 'School' data (Pyndyck & Rubenfeld; Stata manuals), $N = 95$, $M = 2$

(2) Simulated data, $N = 5000$, $M = 4$.

'School' data:

`private:` whether children attend a private school,

`vote:` whether the household head had voted for an increase in the property tax,

`years:` # years family has been at the present residence

`logptax:` log(property tax); `loginc:`log(income).

`biprobit` estimates:

```
. use http://www.stata-press.com/data/r7/school.dta, clear

. biprobit (private=years logptax loginc) (vote=years logptax loginc), nolog

Seemingly unrelated bivariate probit              Number of obs   =         95
                                                  Wald chi2(6)    =       9.59
Log likelihood = -89.254028                       Prob > chi2     =     0.1431

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
private      |
       years |  -.0118884   .0256778    -0.46   0.643    -.0622159    .0384391
     logptax |  -.1066962   .6669782    -0.16   0.873    -1.413949    1.200557
      loginc |   .3762037   .5306484     0.71   0.478     -.663848    1.416255
       _cons |  -4.184694   4.837817    -0.86   0.387    -13.66664    5.297253
-------------+----------------------------------------------------------------
vote         |
       years |  -.0168561   .0147834    -1.14   0.254    -.0458309    .0121188
     logptax |  -1.288707   .5752266    -2.24   0.025    -2.416131   -.1612839
      loginc |    .998286   .4403565     2.27   0.023     .1352031    1.861369
       _cons |  -.5360573   4.068509    -0.13   0.895    -8.510188    7.438073
-------------+----------------------------------------------------------------
     /athrho |  -.2764525   .2412099    -1.15   0.252    -.7492153    .1963102
-------------+----------------------------------------------------------------
         rho |  -.2696186   .2236753                     -.6346806    .1938267
------------------------------------------------------------------------------
Likelihood ratio test of rho=0:     chi2(1) =  1.38444    Prob > chi2 = 0.2393
```

```
. predict p11,p11
. predict p00,p00
. predict xbb1, xb1
. predict xbb2, xb2
. predict stdpb1, stdp1
. predict stdpb2, stdp2
. predict pmargb1, pmarg1
. predict pmargb2, pmarg2
```

Predictions of joint & marginal probabilities, linear predictions, etc.

mvprobit estimates (*R* = 100) are close to ML estimates:

```
. mvprobit (private = years logptax loginc) (vote=years logptax loginc), nolog
> dr(100)

Multivariate probit (SML, # draws = 100)          Number of obs   =         95
                                                  Wald chi2(6)    =       9.64
Log likelihood = -89.220805                       Prob > chi2     =     0.1405

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
private      |
       years |  -.0118233   .0256205    -0.46   0.644    -.0620386    .0383921
     logptax |  -.1033056   .6672673    -0.15   0.877    -1.411126    1.204514
      loginc |   .3695001   .5303571     0.70   0.486    -.6699806    1.408981
       _cons |  -4.140149   4.837923    -0.86   0.392     -13.6223    5.342006
-------------+----------------------------------------------------------------
vote         |
       years |  -.0172153   .0148029    -1.16   0.245    -.0462285     .011798
     logptax |  -1.280732   .5725493    -2.24   0.025    -2.402908   -.1585563
      loginc |   .9956743    .437904     2.27   0.023     .1373982     1.85395
       _cons |  -.5627991   4.055359    -0.14   0.890    -8.511157    7.385558
-------------+----------------------------------------------------------------
    /atrho21 |  -.2811165   .2396506    -1.17   0.241    -.7508231      .18859
-------------+----------------------------------------------------------------
       rho21 |  -.2739382   .2216667    -1.24   0.217    -.6356397    .1863855
------------------------------------------------------------------------------
Likelihood ratio test of  rho21 = 0:
             chi2(1) =  1.45088   Prob > chi2 = 0.2284
```

```
. mvppred pall, pall
(Pr(all zeros), Pr(all ones) will be stored in variables pall0s, pall1s)
. mvppred xbm, xb
(xb will be stored in variables xbmi, i = 1,...,#eqs)
. mvppred stdpm, stdp
(stdp will be stored in variables stdpmi, i = 1,...,#eqs)
. mvppred pmargm, pmarg
(pmarg will be stored in variables pmargmi, i = 1,...,#eqs)
```

Predictions

SML predictions very similar to ML counterparts

```
. su pall1s p11 pall0s p00 xbm1 xbb1 xbm2 xbb2 /*
>         */ stdpm1 stdpb1 stdpm2 stdpb2 pmargm1 pmargb1 pmargm2 pmargb2

    Variable |     Obs        Mean    Std. Dev.       Min        Max
-------------+-----------------------------------------------------
      pall1s |      95    .0513848    .0293697   .0006823   .1675037
         p11 |      95    .0514965    .0295284   .0006783   .1691212
      pall0s |      95    .3252403    .1496049   .0397381   .8772917
         p00 |      95    .3241522    .1485598    .040815    .882799
        xbm1 |      95   -1.273431    .2017621  -1.930628  -.8744448
        xbb1 |      95   -1.275218    .2041617  -1.937996  -.8695227
        xbm2 |      95    .3308476    .4381363  -1.365069   1.519954
        xbb2 |      95    .3313479    .4383708   -1.37215    1.52224
      stdpm1 |      95    .3404043    .1805338    .185824   1.046698
      stdpb1 |      95    .3405953    .1807651   .1859334   1.049172
      stdpm2 |      95    .2541217    .1181758    .141525   .7976922
      stdpb2 |      95    .2546185    .1186652   .1415696   .8023056
     pmargm1 |      95    .1057509    .0322633   .0267645    .190938
     pmargb1 |      95    .1055308     .032576   .0263119   .1922807
     pmargm2 |      95    .6216642    .1502225   .0861157   .9357387
     pmargb2 |      95    .6218135    .1500823   .0850083   .9360256
```

Less similarity between SML and ML with smaller *R*, we found.
More on choice of *R* below.

N = 95 is 'small', so potential finite sample biases anyway.
Raise *N* and *M* in simulated data example …

Simulated data: $M = 4$, $N = 5000$

```
. set seed 12309
. set obs 5000
obs was 0, now 5000

. matrix R = (1, .25, .5, .75 \ .25, 1, .75, .5 \ .5, .75, 1, .75 \ .75, .5, .7
> 5, 1)

. drawnorm u1 u2 u3 u4, corr(R)

. corr u*
(obs=5000)

             |       u1       u2       u3       u4
-------------+------------------------------------
          u1 |   1.0000
          u2 |   0.2587   1.0000
          u3 |   0.5077   0.7483   1.0000
          u4 |   0.7523   0.5093   0.7589   1.0000

. ge x1 = uniform()-.5
. ge x2 = uniform() + 1/3
. ge x3 = 2*uniform() + .5
. ge x4 = .5*uniform() - 1/3

. * Equations
. ge y1s = .5 + 4*x1 + u1
. ge y2s = 3 + .5*x1 - 3*x2 + u2
. ge y3s = 1 - 2*x1 + .4*x2 -.75*x3 + u3
. ge y4s = -6 + 1*x1 - .3*x2 + 3*x3 - .4*x4 + u4
.
. ge y1 = y1s>0
. ge y2 = y2s>0
. ge y3 = y3s>0
. ge y4 = y4s>0
```

Correlation structure (V)

Equations

`mvprobit` estimates (R = 75)

Estimates quite close to those in the ‘true’ model

NB warning message at iteration 1 not a problem as model later converged

LR test of MVP against M independent univariate probits (V identity matrix)

```
. mvprobit (y1=x1) (y2=x1 x2) (y3 = x1 x2 x3)(y4=x1 x2 x3 x4), dr(75)

Iteration 0:   log likelihood = -8681.8526
Warning: cannot do Cholesky factorization of rho matrix
Iteration 1:   log likelihood = -7922.4199
Iteration 2:   log likelihood = -7749.5212
Iteration 3:   log likelihood = -7746.5769
Iteration 4:   log likelihood = -7746.5734
Iteration 5:   log likelihood = -7746.5734

Multivariate probit (SML, # draws = 75)           Number of obs   =       5000
                                                  Wald chi2(10)   =    5561.10
Log likelihood = -7746.5734                       Prob > chi2     =     0.0000

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
y1           |
          x1 |   3.991634   .0962586    41.47   0.000      3.80297    4.180297
       _cons |   .5078066   .0233257    21.77   0.000      .462089    .5535241
-------------+----------------------------------------------------------------
y2           |
          x1 |   .5413448   .0704179     7.69   0.000     .4033283    .6793614
          x2 |  -2.848419   .0781794   -36.43   0.000    -3.001648   -2.695191
       _cons |   2.867846   .0734467    39.05   0.000     2.723894    3.011799
-------------+----------------------------------------------------------------
y3           |
          x1 |  -2.011164   .0708565   -28.38   0.000     -2.15004   -1.872288
          x2 |   .5341271   .0642228     8.32   0.000     .4082527    .6600015
          x3 |  -.7438451   .0311735   -23.86   0.000    -.8049441   -.6827462
       _cons |   .9133876   .0725944    12.58   0.000     .7711051     1.05567
-------------+----------------------------------------------------------------
y4           |
          x1 |   1.125271   .0891551    12.62   0.000     .9505303    1.300012
          x2 |  -.3030878   .0826195    -3.67   0.000    -.4650191   -.1411565
          x3 |   2.898115   .0779738    37.17   0.000     2.745289    3.050941
          x4 |  -.4352364   .1598866    -2.72   0.006    -.7486084   -.1218644
       _cons |  -5.751499   .1685475   -34.12   0.000    -6.081846   -5.421152
-------------+----------------------------------------------------------------
    /atrho21 |   .2621885   .0317031     8.27   0.000     .2000516    .3243253
-------------+----------------------------------------------------------------
    /atrho31 |   .5646645   .0345949    16.32   0.000     .4968597    .6324692
-------------+----------------------------------------------------------------
    /atrho41 |   .9396437   .0571792    16.43   0.000     .8275746    1.051713
-------------+----------------------------------------------------------------
    /atrho32 |   .9737444   .0406156    23.97   0.000     .8941392     1.05335
-------------+----------------------------------------------------------------
    /atrho42 |   .5670195   .0424459    13.36   0.000     .4838271    .6502119
-------------+----------------------------------------------------------------
    /atrho43 |   1.007126   .0537097    18.75   0.000     .9018571    1.112395
-------------+----------------------------------------------------------------
       rho21 |   .2563413   .0296198     8.65   0.000     .1974249    .3134127
-------------+----------------------------------------------------------------
       rho31 |   .5114301   .0255462    20.02   0.000     .4596439    .5597501
-------------+----------------------------------------------------------------
       rho41 |   .7350585   .0262846    27.97   0.000     .6791715    .7824714
-------------+----------------------------------------------------------------
       rho32 |   .7503451   .0177483    42.28   0.000     .7134322    .7831052
-------------+----------------------------------------------------------------
       rho42 |    .513167   .0312682    16.41   0.000     .4493034    .5718126
-------------+----------------------------------------------------------------
       rho43 |   .7645708   .0223127    34.27   0.000     .7172009    .8049075
------------------------------------------------------------------------------
Likelihood ratio test of  rho21 = rho31 = rho41 = rho32 = rho42 = rho43 = 0:
             chi2(6) =  1870.56   Prob > chi2 = 0.0000
```

Using `mvprobit`: further remarks

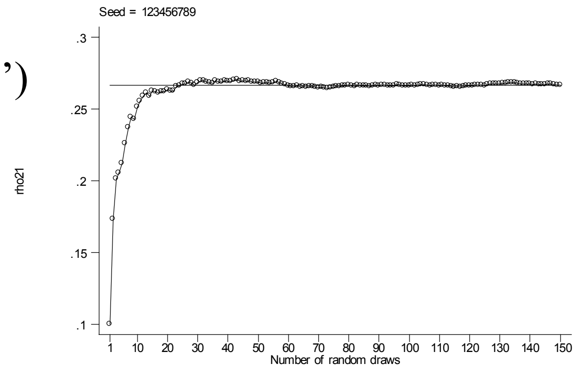
- Choosing number of random draws: higher R increases accuracy but also run-time;
- Choosing seed (different random numbers give different simulated probabilities)

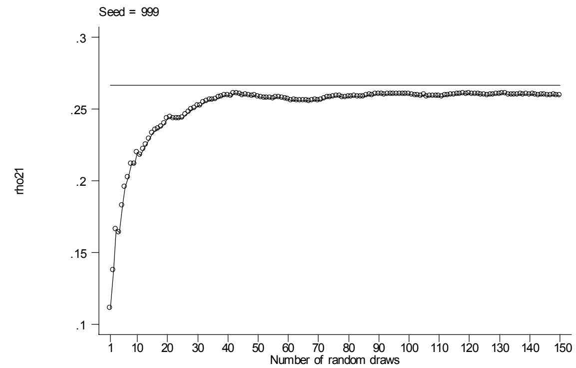
Re-estimated the earlier models for each value of $R = 1, \ldots, 150$, and several alternative seed values

SML estimate of ρ_{21}: R = 1, …, 150; seed = 123456789, 999

(a) two-equation model, generated data (N = 5000)

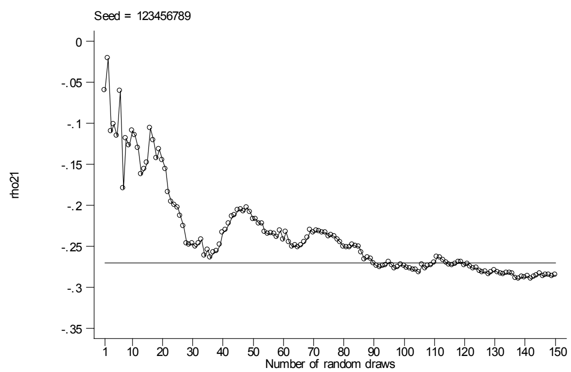
ρ_{21}('true')

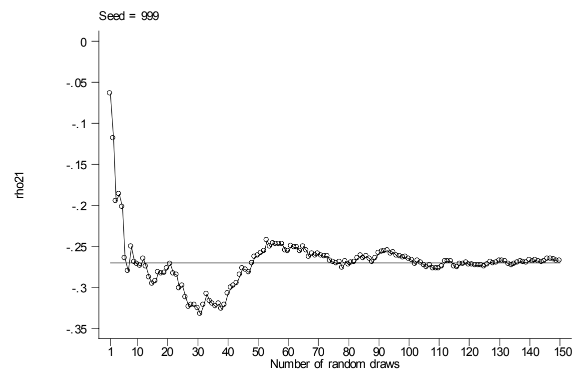




(a) two-equation model, 'School' data (N = 95)

ρ_{21}(ML)





With 'large' N, SML estimator OK regardless of seed when $R > \sqrt{N}$

Using **`mvprobit`**: further remarks

Run-time:

- Four-equation model took c. 2.25 hours using Stata 7/SE on Pentium P4/1.4Ghz; c. 5.3 hours using Stata 7/IC on Sun Solaris)
- increases linearly in R (for given N, M)
- increasing number of explanatory variables, *ceteris paribus,* has no big effect
- **`atrho`** option use reduced run-time relatively little (since got good estimates of V within a few iterations anyway)
- Runtime increases substantially as M increases (several days/weeks if N = 1000s and $M > 6$ or 7!)

Capacity:

- $R*M$ temporary variables created: **`set memory`**
- For 'large' models (many covariates, many M): **`set matsize`**