A Dynamic Model of Labor Supply, Consumption/Saving, and Annuity Decisions Under Uncertainty[†]

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Abstract

This paper presents a dynamic model of labor/leisure, consumption/saving and annuity decisions over the life cycle. Such a dynamic model provides a framework for considering important policy experiments related to the reforms in Social Security. We address the role of labor supply in a life cyle utility maximization model, extending the classical optimal lifetime consumption problem under uncertainty first formalized in Phelps (1962) and later in Hakansson (1970). We introduce the labor decision in the finite horizon consumption/saving problem and solve numerically the stochastic dynamic programming utility maximization problem of the individual. Analytical solutions are infeasible when the individual is maximizing utility over consumption and leisure, given non-linear marginal utility. We illustrate how such a model captures changes in labor supply over the life cycle and show that simulated consumption and wealth accumulation paths are consistent with empirical evidence. We also present a model of endogenously determined annuities for the consumption/saving and labor/leisure framework with capital uncertainty in the presence of bequest motives and Social Security. This provides new insights into the "annuity puzzle" and the effects of Social Security on labor supply and savings behavior.

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1 Introduction

This paper presents a dynamic model of the joint labor/leisure, consumption/saving and annuity decisions over the life cycle. We introduce several models of the life cycle decision making of the individual, in increasing level of complexity and closeness to reality, in order to provide a framework of policy analysis for considering important policy experiments related to the reforms of Social Security. We introduce in this consumption/saving and labor/leisure framework the possibility of endogenously choose annuities under capital uncertainty and in the presence of Social Security and bequest motives. The model provides new insights regarding the "annuity puzzle" and the effects of social insurance on labor supply and wealth accumulation.

We begin the analysis with a simple model, which ignores, for almost all purposes, the individual's labor supply decision. In this model, consumption and saving over the life of the individuals are analyzed in detail. Modigliani and Brumberg (1954, 1980), Friedman (1957), Beckmann (1959), Phelps (1962), and Ando and Modigliani (1963) represent seminal contributions to the analysis of this classic problem in economics.

Phelps presents an infinite horizon model of the consumption/saving decision under investment uncertainty (providing the framework for the model that we solve first), and he derives closed-form solutions for several models with varying assumptions regarding the utility function. Hakansson (1970) provides a refinement and extension of Phelps' work, allowing for a choice among risky investment opportunities and the possibility to borrow and lend.[1]

We first present a finite horizon version of the simplest model and report closed-form solutions for the consumption decision rule. We then solve this model numerically with two objectives in mind: first, to validate the techniques that will be used exclusively in the more realistic model which introduces labor supply, annuities, and Social Security; and second, to determine whether an accurate characterization of the finite horizon problem is as difficult to obtain as it is for the infinite horizon case. Rust (1999a) discusses the complications involved in attempting to replicate Phelps' (1962) solutions using numerical dynamic programming.[2] The unboundedness of the utility functions used complicates the numerical approach, and even when using the most sophisticated techniques under the assumption of logarithmic utility, the problem remains quite challenging.

[1] Levhari and Srinivasan (1969) re-examine Phelp's model and include a dynamic portfolio choice. Merton (1969) generalizes Phelps's model to the continuous time case and also allows for a portfolio selection decision. Samuelson (1969) analyzes the lifetime portfolio selection problem in discrete time. Fama (1970), assuming "perfect markets," shows that a two-period model provides most of the insights of the multi-period model of consumption decisions.

[2] See also Benítez-Silva et al. (2000a) for a discussion of this issue and a comparison of numerical methods for solving consumption, inventory and product introduction problems.

The numerical approach for the finite horizon case is fairly well behaved. Even in the absence of the bequest motive, the numerical solution approximates the closed form solution quite well, using either the logarithmic utility function or the CRRA utility function. We show both analytically and numerically that the finite horizon solution of the consumption/saving problem with bequests converges to the infinite horizon model (without bequests). We also show simulated solution paths for consumption and wealth accumulation.

Modified versions of this benchmark model of the consumption/saving decision have been used extensively in the literature with different objectives. Hubbard and Judd (1987) provide a partial and general equilibrium discussion of the importance of social insurance in a model with uncertainty and borrowing constraints. Thurow (1969) invokes credit market restrictions to reconcile the prediction of the life cycle model with the empirical evidence. Zeldes (1989a) and Deaton (1991) study the role of liquidity constraints using extensions of this model in a finite and infinite horizon framework, respectively. Beckmann (1959) presents a dynamic programming model that introduces income uncertainty (but with no labor decision), Sandmo (1970) explores the role of income and capital uncertainty in a two period consumption/saving model, and Miller (1974) presents the infinite horizon version of such a model concentrating on income uncertainty, finding that agents would always consume less when income is stochastic. Nagatani (1972) also introduces income uncertainty to justify the close relationship between consumption and income in the data, and Zeldes (1989b) solves a similar model using numerical techniques, since closed-form solutions are unavailable when using a constant relative risk aversion utility function.

Skinner (1988) explores the importance of precautionary savings in a model with risky income, approximating the optimal consumption path via Taylor expansions. Carroll (1997) presents a theory of *buffer-stock saving* where individuals maintain contingency funds to hedge against income uncertainty. Some empirical evidence presented by Carroll (1994) and Carroll and Samwick (1997) seems to support certain implications of this theory. Hubbard et al. (1994, 1995) analyze and solve with numerical techniques, a multi-period model of the consumption decision with uncertain lifetimes, and stochastic wages and medical expenses. They emphasize the importance of precautionary savings and the role of social insurance. Attanasio and Weber (1995), Attanasio and Browning (1995), and Attanasio et al. (1997) highlight the importance of considering the effects of changes in demographics and labor supply behavior in a life cycle model if we are to match the empirical evidence. However, they still model labor supply as exogenous. More recently Gourinchas and Parker (1999) have estimated the consumption/saving model using simulation techniques, and Cagetti (1999) has focused on wealth accumulation. Dynan et al. (1999) explore saving behavior across income groups, Banks et al. (1998) analyze income and expenditure patterns around the time of retirement, Palumbo (1999) highlights the importance of taking into account uncertain medical expenses to explain the slow rates of dissaving among the elderly, and Cifuentes (1999) uses the consumption/saving model to discuss the effects of Pension reform.[3]

None of these models considers explicitly the labor supply decision of the individual, and thus, our work can be considered an attempt to complement and extend those models by considering labor decisions as indeed endogenous to the life cycle consumption/saving problem. Although this is not a completely novel consideration, our models attempt to incorporate realism by considering several sources of uncertainty, introducing annuities and Social Security, and providing a general framework which allows for policy experimentation.[4]

More recently, an increasing number of papers have incorporated the labor decision in general equilibrium models of the economy in their analysis of the effects of Social Security reform. Huggett and Ventura (1997), Bütler (1998), İmrohoroğlu et al. (1994, 1999a, 1999b), and De Nardi et al. (1999) are just a few examples. However, since they do not focus on individual decision-making, and because the general equilibrium approach requires a number of strong assumptions to make the problem solvable, there are many aspects of the life cycle model still to be addressed.

At the heart of our work is the allowance of agents to make their labor/leisure decision along with their consumption/saving decision in a utility maximizing framework in finite horizon. Individuals can work full-time, part-time, or not at all at any point during their life, and they can consume continuously subject to a budget constraint (they can not borrow against future income). They can also accumulate wealth over their life at an uncertain rate of return which we model as draws from a log-normal distribution. Following our piecemeal approach to solving these models, we first introduce wages as deterministic; that is, agents know their exact profile of wages from day one. This effectively maintains the value function as dependent only on wealth, making the model a fairly simple extension of the consumption/saving model. We consider an isoelastic and Cobb-Douglas utility function on consumption and leisure, and given the unavailability of closed-form solutions when the marginal utility is non-linear, we solve the problem numerically by backward induction using dynamic programming techniques. We will assume throughout most of the analysis that the constant relative risk aversion parameter is larger than one, effectively implying that con-

[3] Browning and Lusardi (1996) present an comprehensive survey of the consumption/saving literature and focus on saving behavior. See also Deaton (1992) for an illuminating presentation of consumption models.

[4] Heckman (1974), Heckman and MaCurdy (1980), MaCurdy (1981, 1983), Bodie and Samuelson (1989), Bodie, Merton and Samuelson (1992), Low (1998, 1999), Flodén (1998), and French (2000) tackle this issue in theoretical and/or empirical contexts.

sumption and leisure are substitutes.[5] We also have to parameterize the within-period valuation of consumption versus leisure, a parameter that has an important effect on the labor supply decisions, as will become clear from our discussion in the following sections. We show that this model already captures paths of consumption, labor, and wealth accumulation, consistent with the literature and empirical regularities.

We next introduce labor income uncertainty, allowing for the wages to be stochastic. We start by characterizing the wage realizations as independent and identically distributed draws from a log-normal distribution, with a mean at each point in time that matches both the deterministic profile considered previously and a standard deviation consistent with research on the variability of income. This new source of uncertainty increases the computational burden of solving the model by a single order of magnitude, since now the value function also depends on the uncertain draws of wages. The numerical techniques used can still handle the problem, but computing time increases as the "curse of dimensionality" makes its appearance. We then allow for serial correlation in the wages following the empirical evidence on the topic. We solve models with different serial correlation factors and compare the results to those of the previous models. We then simulate the solutions with certain starting values of the state variables and average out the simulations to compute a path for consumption, labor, and wealth accumulation over the life cycle.

We next tackle the problem of introducing annuities and a Social Security system to this framework.[6] The strategy is to first introduce in the consumption/saving model with bequests the possibility of partial or total annuitization by individuals and then extend the model and introduce endogenous labor and Social Security. Agents endogenously decide to annuitize at any point of their lives part or all of their wealth; that is, they can purchase a sure income stream for the remainder of their lives at a price which takes into account the rate of return on the income purchased.[7] The cost of the annuity cannot exceed current wealth in the period that they annuitize, and the decision to annuitize is unique and non-reversible. This last assumption effectively means that they can only annuitize once in their life.[8] We do not, however, force them to do so at a given age or stage of their lives.[9]

To solve this model we have to take into account that agents are choosing the optimal time to annuitize and the size of the annuity along with their consumption/saving decision, forcing us to

[5] See the discussion in Heckman (1974) and Low (1998).

[6] Rust (1999b) presents a survey of models that try to incorporate uncertainty and insurance mechanisms in models of social insurance.

[7] The literature refers to this type of annuity as *single premium immediate life annuity.*

[8] This is a fairly realistic assumption, as emphasized in TIAA-CREF (1999).

[9] This model complements and extends the framework introduced in Friedman and Warshawsky (1990).

carry the annuity value as another state variable of the problem. This is an important exercise because we introduce this kind of insurance in the simplest possible stochastic model of lifetime decision making and show that agents do react to the availability of this insurance. We also analyze the effects of higher interest differentials between the uncertain rates of returns on wealth and the return of the annuities, as well as the effect of having a higher bequest motive.

We then extend this model to consider the labor/leisure decision as endogenous and introduce Social Security. The full model provides important insights into the classic and important question of whether social schemes affect the behavior of individuals, and this model of endogenous annuities provides some insights into the "annuity puzzle," the question as to why the annuity market is so narrow. Our results suggest that the low rates of annuitization can be the product of optimal decision making by individuals in a life cycle model which endogenizes the labor/leisure decision and accounts for Social Security.[10]

In the next section we solve analytically and numerically the finite horizon version of the consumption/saving benchmark model and simulate its implied consumption and wealth accumulation paths. Section 3 introduces the endogenous labor/leisure model, presents its numerical solution, and provides a discussion of its results. In section 4, we extend the life cycle models of consumption/saving and labor/leisure decisions to allow for endogenous annuitization and the presence of Social Security, and we present the insights from the extended model. Section 5 summarizes the main results and discusses extensions currently being considered and implemented.

2 The Consumption/Saving Model

In this section we solve a finite horizon version of the consumption/saving problem analyzed in Phelps (1962).[11] Agents choose consumption according to the following utility maximizing framework:

$$\max_{0 \leq c_s \leq w} E_t \left[\sum_{s=t}^{T} \beta^{s-t} u(c_s) \right] , \tag{1}$$

where β is the discount factor, which includes the mortality probabilities, c represents consumption, and w is wealth at the beginning of the period. Savings accumulate at an uncertain interest rate of

[10] This important result is consistent with the conclusions of Bodie and Samuelson (1989), and Bodie, Merton and Samuelson (1992). They all emphasize the role of labor supply flexibility in making risky investments more attractive. We find that the counterpart of that is that life annuities are a less atractive investment once the more complete model is considered.

[11] Phelps solved the infinite horizon problem analytically assuming no labor income and using different forms of the utility function.

return $\tilde{r}$ such that $w_{t+1} = \tilde{r}(w_t - c_t)$. Utility depends only on consumption.[12] We can express and solve this problem using Dynamic Programming and Bellman's principle of optimality. We solve it by backward induction starting in the last period of life, in which the individual solves

$$V_T(w) = \max_{0 \leq c \leq w} log(c) + K\ log(w - c)\ , \tag{2}$$

assuming a logarithmic utility function where $K \in (0, 1)$ is the bequest factor.[13] By deriving the first order condition with respect to consumption we find that

$$c_T = \frac{w}{1+K}\ , \tag{3}$$

and from this we can write the analytical expression for the last period value function:

$$V_T(w) = log(\frac{w}{1+K}) + K\ log(\frac{wK}{1+K})\ . \tag{4}$$

We can then iterate by backward induction and write the next to last period value function as:

$$V_{T-1}(w) = \max_{0 \leq c \leq w} log(c) + \beta\ E\ V_T(w - c)\ , \tag{5}$$

where the second term in the right hand side can be written as

$$E\ V_T(w - c) = \int_0^{\overline{r}} V_T(\tilde{r}(w - c)) f(\tilde{r})\ d\tilde{r}\ , \tag{6}$$

where $\tilde{r}$ is the stochastic return on capital accumulation. Then we can write

$$V_{T-1}(w) = \max_{0 \leq c \leq w} log(c) + \beta\ E\ log(\tilde{r}\ (\frac{w-c}{1+K})) + \beta\ K\ E\ log(\tilde{r}\ (\frac{(w-c)K}{1+K}))\ . \tag{7}$$

Here the logarithmic utility simplifies the problem. Again taking first order conditions with respect to consumption, we obtain an expression for the consumption rule in the next to last period of life:

$$c_{T-1} = \frac{w}{1+\beta+\beta K}\ . \tag{8}$$

[12] This is the standard characterization of the utility function. In a slightly different setup, Alessie and Lusardi (1997b) introduce habit formation, by considering a utility function that depends additionally on past consumption. See also Deaton (1992) for a discussion of such a model.

[13] Agents in this model care only about the absolute size of their bequests, leading to its been called the "egoistic" model of bequests. A bequest factor of one would correspond to valuing bequest in the utility function as much as current consumption. The importance of bequest motives is still an open issue in the literature. Here we take the position of acknowledging that bequests do exist and explore the implications of changing the importance of the bequest motive in the utility function. Hurd (1987, 1989), Bernheim (1991), Modigliani (1988), Wilhem (1996) and Laitner and Juster (1996) are some of the main references on the debate over the significance of bequests and altruism in the life cycle model. Kotlikoff and Summers (1981) stress the importance of intergenerational transfers in aggregate capital accumulation.

We then have an expression for V_{T-1} in the following form:

$$V_{T-1}(w) = log(\frac{w}{1+\beta+\beta K}) + \beta\, log(\frac{w\beta}{1+\beta+\beta K}) + \beta\, K\, log(\frac{w\beta K}{1+\beta+\beta K}) + \Upsilon\ , \tag{9}$$

where Υ gathers all the terms that do not depend on w. From here we can write V_{T-2} and again derive first order conditions, resulting in

$$c_{T-2} = \frac{w}{1+\beta+\beta^2+\beta^2 K}\ . \tag{10}$$

Through backward induction, we continue iterating to find c_{T-k}

$$c_{T-k} = \frac{w}{1+\beta+\beta^2+\beta^3+\ldots+\beta^k+\beta^k K}\ . \tag{11}$$

for any $k < T$. From these decision rules, we can observe that as T grows large, the finite horizon solution with bequests converges to the infinite horizon solution, since the influence of the bequest parameter becomes less important as the time horizon increases. In the infinite horizon case with logarithmic utility and no non-labor income, the simplified decision rule is $c = (1-\beta)w$, as shown in Phelps (1962). The derivation of the decision rules in the case of the CRRA utility function is similar though somewhat more involved and is presented in the Appendix. Benítez-Silva et al. (2000a) present, among others, the CARA utility case under certainty.

Our ability to derive an analytical solution for this model allows us to evaluate the effectiveness of our numerical methods, which are all that we have available in more complicated models. The exercise of solving the model numerically is also interesting on its own given that the infinite horizon version of this model has been shown to be quite difficult to replicate using numerical methods, even with the logarithmic utility function, as discussed in Rust (1999a) and Benítez-Silva et al. (2000a).

The numerical procedure is by nature very similar to the analytical approach, involving backward recursion starting in the last period of life. We discretize wealth and compute the optimal value of consumption for all those wealth levels using bisection. Bisection is an iterative algorithm with all the components of a nonlinear equation solver. It makes a guess, computes the iterative value, checks if the value is an acceptable solution, and if not, iterates again. The stopping rule depends on the desired precision given that the solution is bracketed by the nature of the algorithm and that the round-off errors will probably not allow us to increase the precision beyond a certain limit. In each iteration of the numerical solution, except for the final one where all uncertainty has been eliminated, we have to compute the expectation in equation (6), which is potentially the most computationally demanding step. For this we use Gaussian Legendre quadrature. We also compute the derivative of this expectation using numerical differentiation, also requiring quadrature as part of its routine. Here the analytical derivatives are simple to compute, but this is not always the case for more complicated models. We therefore wish to evaluate the accuracy of the numerical strategy.

Gaussian quadrature approximates the integral through sums using rules to choose points and weights based on the properties of orthogonal polynomials corresponding to the density function of the variable over which we are integrating, in this case the draws of the interest rates following a log-normal distribution. The points and weights are selected in such a way that finite-order polynomials can be integrated exactly using quadrature formulae. The weights used have the natural interpretation of probabilities associated with intervals around the quadrature points.[14] At this point we are considering a one dimensional problem, for which quadrature methods have been shown to be very accurate compared with other techniques of computing expectations (integrals) such as Monte Carlo integration and weighted sums.[15]

This all amounts to manipulating (6) through a change of variables such that we can write it as an integral in the $(0,1)$ interval and then approximate it by a series of sums depending on the quadrature weights and quadrature abscissae which we compute recursively, following readily available routines (e.g. Press et al. 1992).[16]

An additional numerical technique that we use to solve the model completely is function approximation by interpolation. Since savings in a given period are accumulated at a stochastic interest rate, next period's wealth will not necessarily fall in one of the grid points for which we have the value of the function already calculated. Ideally we would solve the next period's problem for any wealth level, but this is computationally infeasible. Therefore, we use linear interpolation to find the corresponding value of the function given the values in the nearest grid points.[17]

The bisection algorithm that uses the quadrature and interpolation procedures eventually converges to a maximum of the lifetime consumption problem for a given value of wealth in a given period (or reaches the pre-decided tolerance level). This procedure is repeated until the solution of the first-period problem is obtained.

Once we have solved the model, we have a decision rule for every level of wealth in our initial

[14] For a detailed characterization of quadrature methods we refer the reader to Tauchen and Hussey (1991), Judd (1998), and Burnside (1999).

[15] For an analysis of how different techniques perform in applied problems see Rust (1997).

[16] We can write $\int_r V(r)fr\ dr$ after a change of variables as $\int_0^1 V(F^{-1})du$, which can then be approximated by $\sum_{i=1}^N w_i V(F^{-1}(u_i))$, where w_i are the quadrature weights and u_i are the quadrature abscissae.

[17] More sophisticated interpolation procedures can be used such as splines or Chebyshev interpolation. They are not considered here, but we plan to conduct a sensitivity analysis of the procedures used at each step of the numerical computations.

grid. Here case we have chosen a grid space of 500 points; to gain accuracy more of these points are concentrated at low wealth levels where the function is changing rapidly. Figures 1 and 2 show the decision rule of the consumption/saving problem for wealth ranging from 0 to 100 units. For expositional purposes we have solved a 10-period model.

Figure 1 plots several decision rules given logarithmic utility. It first plots the numerical solutions for different time periods, denoted C_1, C_2, and so on. It also plots the solution of the infinite horizon problem borrowing from Phelps (1962), denoted by $CINF$ in the figure. We have chosen a discount factor of 0.95 and a bequest parameter of 0.6. Figure 2 plots the decision rule when we consider a CRRA, with risk aversion parameter equal to 1.5, $\beta = 0.95$, and bequest parameter equal to 0.6, we also plot the analytical solution of the infinite horizon problem, borrowing from Levhari and Srinivasan (1969). For both types of utility function we observe that the consumption rules increase with wealth and time and that in very few periods we are fairly close to the solution of the infinite problems.

Figure 3 and 4 are concerned with comparing the numerical solutions with the true analytical solutions derived above and in the Appendix. We plot in both figures the percentage difference between the two solutions in terms of the value of the true solution, for a sample of time periods. The numerical technique performs quite well. For about half of the range of values, the numerical solution is very accurate with deviations below 1%, for both types of utility functions. After that, errors are a bit larger, especially for early time periods. For the first period and for high levels of wealth the error reaches 12% to 13%, depending on the utility function. These differences are the result of the extrapolation methodology for accounting for wealth levels outside the grid of points we are solving over. We extrapolate linearly, what in some cases can lead to a better than average return for the individual, this leads our agents to underconsume in order to profit from this advantage.

In Figures 5 and 6 we simulate this model using the numerical solution for the CRRA utility function. We report the results of 5,000 simulations of an 11-period model with 500 grid points for wealth in the 0 to 200,000 range. We plot consumption and wealth paths with an initial wealth level of 10,000.[18] We also consider several values for the parameters of interest. In the first specification, γ is taken to be 1.5 (the parameter of relative risk aversion), and it is increased to 2.5 in the second specification (hg lines in the plots). We then increase the bequest parameter to 0.6, leaving $\gamma = 1.5$ (bq lines in the plots), and finally, we decrease the relative risk aversion parameter to 0.7 (lg lines

[18] This is approximately the net worth reported by Poterba (1998), using the Survey of Consumer Finances, for individuals at the beginning of their working lives.

in the figures).

We observe that people consume less at the beginning of their lives, with increased consumption in the final periods of life, given uncertain interest rates represented by draws from a truncated log-normal distribution. Consumption does, however, decrease if the risk aversion parameter is less than 1. Focusing on the pattern of wealth accumulation, we observe that individuals deaccumulate their wealth gradually. We also see that increasing the relative risk aversion parameter has the effect of making consumption less smooth (with higher wealth accumulation), while decreasing the parameter from the benchmark value of 1.5 leads to more smoothing (with lower wealth accumulation). We can also observe the expected effect of the bequest parameter: those with a higher concern for their offspring, represented by a higher valuation of bequests in the utility function, consume uniformly less over the life cycle than do those with a lower bequest parameter. This former population also accumulates more and for a longer period. These results regarding the effect of the bequest motives are consistent with, and in fact extend, the theoretical model of Hurd (1987) to the case of agents with various levels of bequest.

This model is meant to serve as a benchmark for the models discussed next and for the introduction of annuities and Social Security in Section 4.

3 Introducing the Labor/Leisure Decision

We next tackle the issue of extending the model of Section 2 to allow for an endogenous labor supply decision. Utility is now a function of consumption and leisure, and agents will optimally choose both in every period of their lives. They effectively solve

$$\max_{c_s, l_s} E_t \left[\sum_{s=t}^{T} \beta^{s-t} u(c_s, l_s) \right] , \tag{12}$$

again in finite horizon. The within-period utility function is assumed to be Isoelastic and Cobb-Douglas between consumption and leisure in time t:

$$u(c_t, l_t) = \frac{(c_t^{\eta} l_t^{1-\eta})^{1-\gamma}}{1-\gamma} , \tag{13}$$

where γ is the coefficient of *relative risk aversion* and η is the valuation of consumption versus leisure.[19] Consumption and leisure are substitutes or complements depending on the value of γ as discussed in Heckman (1974) and Low (1998), with the cutoff approximately equal to 1.[20] In most

[19] See Browning and Meghir (1991) for evidence on non-separability of consumption and leisure within periods.

[20] Heckman presents a model of perfect foresight and shows that by introducing the labor supply decision it is possible to reconcile the empirical evidence on consumption paths with the life cycle framework, without resorting

of our analysis we will assume values of γ larger than 1, implicitly assuming substitutability between consumption and leisure. We will assume that the agent has only three choices with respect to the labor decision: part-time, full-time, or out of the labor force.[21] It is also important to emphasize that for computational convenience we have chosen a lower bound on leisure equal to 20% of the available time during a given period.[22]

3.1 Deterministic Wages

First, we will assume that wages follow a deterministic path which peaks around age 50 and then smoothly decreases. Given that we allow for consumption and leisure to influence each other using a CRRA utility function, and considering that we are concerned with corner solutions for the labor decision, the model can only be solved numerically. To do so we employ the techniques presented in Section 2.

We use Dynamic Programming to characterize this problem and again solve by backward induction. The individual in the last period now solves

$$V_T(w) = \max_{(0 \le c \le w+\omega(1-l),l)} U(c,l) + K\, U(w + \omega(1-l) - c)] \,, \tag{14}$$

where ω represent wages and leisure (labor) is chosen among the three possible states. Once we obtain the optimal decision rules using the bisection algorithm, we then solve recursively. We can write the value function in the next to last period as

$$V_{T-1}(w) = \max_{(0 \le c \le w+\omega(1-l),l)} U(c,l) + \beta\, E\, V_T(w + \omega(1-l) - c) \,. \tag{15}$$

The value function still remains unidimensional since there is no uncertainty about the wages. We solve this model again by bisection, computing the expectations by quadrature and interpolating the values of the next period's value functions.

Once we have solved the model, we simulate it given starting wealth values. The capital uncertainty is characterized by draws from a truncated log-normal distribution. Figures 7-9 present plots of the paths of consumption, labor supply, and wealth accumulation resulting from this 30-period model, which we map into an age profile for expositional purposes. We set initial wealth

to credit market restrictions or uncertainty. Low's (1998, 1999) work is fairly close in nature to our analysis: he abstracts from capital uncertainty but allows borrowing. French's (2000) model is also close to our work, although it focuses on the retirement decision.

[21] We solve in this case a 30-period model to reduce the computational burden of the solution process, but we plan to work with a 65 period model in the near future.

[22] Different values of this parameter have essentially no effect on the solutions presented below.

equal to 10,000 units and consider varying levels of the relative risk aversion parameter, bequest motive, and the valuation of consumption versus leisure in the utility function.

These results have several interesting features. First, as can be seen from Figure 7, consumption tracks income for a significant amount of time before age 40, at which point the consumption path begins to flatten, finally decreasing by the end of the life cycle. We can also see that those who value leisure more ($\eta = 0.5$ *versus* $\eta = 0.7$, *eta* in the figure) receive lower wages because they work mostly part-time, although they are able to maintain an average consumption level higher than their part-time wage level starting at about age 40, since some individuals choose to work full-time. The pattern of labor supply is equally interesting. Agents with a high valuation of consumption seem to work full time most of their lives, except at the beginning when their wages are low and they have initial wealth to smooth consumption. Later in life, our model is able to pick up the decrease in labor supply due to lower wages. It is important to emphasize that those with higher bequest motives (*bq* in the figures, bequest parameter equal to 0.6 versus 0.1 for the other curves) work more on average than those with lower bequest parameters. In Figure 9 we show the wealth accumulation over the life cycle implied by the model. The pattern here is fairly close to the estimated, simulated, and reported results of several papers (e.g., Hubbard et al. 1994, Attanasio and Weber 1995, Attanasio et al. 1997, Alessie and Lusardi 1997a, Alessie et al. 1997, Poterba 1998, and Cagetti 1999) reflecting empirical data quite closely. We see little accumulation early in life, and then after age 40 agents begin to accumulate higher levels of wealth which only decreases near the end of life. We can also see from the graph that those with higher bequest motives start deaccumulating their wealth later in life and than those with higher valuation for leisure start to accumulate earlier in life. Finally, those that are more risk averse start accumulating later in life and end up accumulating less resources than the rest of individuals. This model is broadly consistent with some features of the data that show very low savings rates among young individuals, with an increase only later in life.[23]

3.2 Stochastic and Serially Correlated Wages

We next make the model more realistic by introducing income uncertainty, while maintaining the endogeneity of the labor/leisure decision.[24] We start by introducing stochastic *i.i.d.* wages from a

[23] We have also simulated a model with initial wealth equal to 50,000 units. In this case the model predicts very similar behavior, except at the beginning of life when wealthy individuals delay their entrance into the labor force and consume out of their initial endowment.

[24] We do not allow here for nonzero correlation between income shocks and asset returns. For a discussion of this possibility at the micro level see Davis and Willen (2000).

log-normal distribution with a changing mean that follows the deterministic profile used above.

This feature complicates the model because the value functions now depend on the uncertain wage realizations. We write the problem solved by the agents in the last period of life as

$$V_T(w,\omega) = \max_{(0\leq c\leq w+\omega(1-l),l)} U(c,l) + K\, U(w+\omega(1-l)-c)]\ , \tag{16}$$

where labor is again chosen among the three possible states. Once we obtain the decision rules numerically we can write the value function in the next to last period:

$$V_{T-1}(w,\omega) = \max_{(0\leq c\leq w+\omega(1-l),l)} U(c,l) + \beta\ E\ V_T(w+\omega(1-l)-c,\omega)\ . \tag{17}$$

The functions for the earlier periods are again obtained recursively. The expectation $E\ V_t(\omega(1-l)+w-c,\omega)$ appearing in the value functions for the different periods can be written as follows:

$$\int_0^{\overline{r}}\int_0^{\overline{\omega}} V(\tilde{r}(w+\tilde{\omega}(1-l)-c),\tilde{\omega})\ f(\tilde{\omega})d\tilde{\omega}\ f(\tilde{r})d\tilde{r}\ . \tag{18}$$

The interpolation of the values of the next period value function has to be carried out in two dimensions, a slightly more cumbersome and slower procedure. The double integrals are again solved by Gaussian Legendre quadrature, but we use iterated integration since we are assuming independence of wages and interest rates.[25]

Figures 10-12 show the consumption, labor, and wealth accumulation paths for this model. The main difference from the case of deterministic wages is that individuals start to save and accumulate later in life, and work on average a bit more later in life, ultimately accumulating a higher level of wealth before they enter the deaccumulation phase.[26]

Finally, we introduce serially correlated wages, such that

$$ln\ \omega_t\ =\ (1-\rho)\ \alpha_t + \rho\ ln\ \omega_{t-1}\ +\ \epsilon_t\ , \tag{19}$$

where α_t is a quadratic trend that mimics the one presented in the case of deterministic wages. The ϵ_t are *i.i.d.* draws from a normal distribution with mean 0 and variance σ_t^2. If ρ is 0, this reduces to the case of *i.i.d.* wages. The solution method does not change significantly from the last model, and only the careful manipulation of the serially correlated component has to be considered.

[25] Given that the value function depends on wealth and wages, we needed to discretize both variables in order to approximate the integrals, using 50 points for wealth and 50 points for wages. We found that using fewer points significantly affected the accuracy of the calculations, leading to possible erroneous conclusions.

[26] Lusardi (1998) presents empirical results pertaining to the role of income variance in a consumption/saving model. She finds that income variation seems to affect precautionary savings, but the final effect on wealth accumulation is not too large. Our results indicate that individuals could be using their labor supply to hedge the income uncertainty.

Figures 13-15 show the paths of the relevant variables. Our results do not show striking differences with the previous graphs. Consumption profiles again track income paths very closely up to age 45, when wealth accumulation starts in meaningful amounts. Higher serial correlation leads to accumulation and deaccumulation slightly later in life, since individuals seem to take advantage of the effects of serial correlation once their peak earnings years have been reached. The labor supply profile is also quite similar to those shown before, with individuals facing higher serial correlation in their wages working a bit longer than the rest. We plot the paths for different values of the serial correlation parameter. With high correlation, we plot the case of individuals starting with wealth of 10,000 units and initial wages of 30,000 units, the initial wage for those with low serial correlation is 20,000 units.

From the solution and simulation of these models we can conclude that a life cycle model with endogenized labor supply behaves quite consistently with the empirical data on wealth accumulation and consumption profiles and that wealth accumulation seems to start only in mid-life. Additionally, such a model captures the gradual exiting from the labor force by older individuals who face lower wages and who have a lower serial correlation of wages once they reach a certain age. This model seems well-suited for analyzing important policy issues regarding the effects on savings and labor supply of reforms in social insurance programs.

4 Endogenously chosen Annuities

In this section we extend the models presented in Section 2 and Section 3.1, the consumption/saving model and the extended model of endogenous labor with deterministic wages, by allowing individuals to purchase an annuity with a fraction or all of their wealth at any point in their lives. We also introduce a stylized Social Security system in the endogenous labor/leisure model. We endogenize the annuitization decision by providing the agents with the possibility of exchanging a certain number of dollars today for a stream of income over the rest of their lives. The annuity has a given rate of return we assume to be fixed. The cost of the annuity, calculated as the net present value of the promised stream of income, cannot exceed the total wealth of the agent at the time of the purchase of the annuity. This is a *single premium immediate life annuity.* The decision to annuitize is unique and non-reversible. These last two assumptions mean that individuals can only annuitize once in their lives. We do not, however, place any restriction on the timing of this annuity.[27]

[27] We do not consider at this point the role of taxes in the decision to annuitize, see Gentry and Milano (1998) for a discussion of the effects of taking taxes into account.

The first model presented here is similar to that of Friedman and Warshawsky (1990), although they focus on older individuals and on the issue of annuity pricing in order to explain the almost non-existence of a market for these instruments. Another difference is that they force individuals to invest a proportion of their wealth in an actuarially fair social annuity, without considering investment uncertainty. Brugiavini (1993) focuses on the role of longevity uncertainty in the purchase of annuities in a two/three period model. She also considers a model that allows for income uncertainty and the different behavior of employees and entrepreneurs. Mitchell et al. (1999) use the term structure of interest rate rather than a fixed interest rate, to calculate the expected present discounted value of the annuities in a model of uncertain lifetime. They find that retirees should value annuities even if they are not actuarially fair. Brown (1999a), extending the model of Yaari (1965), focuses on the role of annuities when individuals face an uncertain lifetime, using data on older Americans to construct a measure of the consumer's valuation of additional annuitization.[28] However, his model abstracts from capital uncertainty and does not endogenize the annuity decision in the general sense that we do. Brown (1999b) uses data on older individuals to test and ultimately reject the "Annuity Offset Model," the hypothesis that old individuals purchase term insurance to offset the excessive annuitization imposed by the government social programs. Kotlikoff and Spivak (1981) also use a Yaari-type model to emphasize the important role of the family as an incomplete annuities market, with the annuity decision made at exogenous points in time. Eichenbaum and Peled (1987) use a two period model to underline the over accumulation of private capital in a model of competitive annuities with adverse selection.[29]

The most important differences between our analysis below and that of previous research is the consideration of the labor/leisure decision and the introduction of a fairly realistic social security system, changes which yield striking effects on the results.

The agents are again choosing consumption in order to maximize utility over their lifetime but now have the choice of converting part of their wealth to an annuity. This annuity has a rate of return which can be lower than, greater than, or equal to the mean of the market rate, and it provides a stream of income until the time of death, which can be considered uncertain given that the mortality probabilities are embedded in the discount rate. The annuity premium $A(a)$, where

[28] Davies (1981) extends Yaari's model of uncertain lifetime and uses it to explain the low levels of deaccumulation by the elderly. Sheshinski (1999) also extending Yaari's theoretical model, accounts for retirement and consumption decisions.

[29] Walliser (1997, 1998) discusses the role of annuities in a social insurance framework, and Boskin et al. (1998) provide an overview of the role of annuities in the economy.

a is the annuity received every period and λ is the rate at which we discount it, is equal to

$$A = a[\frac{1-\lambda^{k+1}}{1-\lambda}] \ , \tag{20}$$

assuming that agents receive the first payment in the same period in which they annuitize. We again solve this model by backward induction using numerical Dynamic Programming techniques.

4.1 Endogenous Annuities in the Consumption/Saving Model

We first analyze the introduction of endogenous annuities in the consumption/saving model without labor. The decision in the last period of life is very similar to that of the consumption/saving problem, but now the value function depends not only on wealth but also on the value of the annuity, which enters the budget constraint:

$$V_T(w,a) = \max_{0 \leq c \leq w+a} U(c) + K\ U(w-c) \ . \tag{21}$$

In this last period we do not allow for the annuity decision to occur, since annuitizing would return exactly what they put into the annuity, assuming no transaction costs. But even if agents do not actually decide to annuitize, it is possible that they have annuitized earlier in their lives; thus, we must solve for the value function under as many combinations of wealth and annuity values as possible.[30] In the simulation part of the model, agents reaching the last period of life without having annuitized will not annuitize in the last period. Recall that agents still face capital uncertainty.

We can then write the next to last period value function as follows:

$$V_{T-1}(w,a) = \max_{0 \leq c \leq w - A(a) + a} U(c) + \beta\ E\ V_T(w,a) \ , \tag{22}$$

where $A(a) \leq w$. Here things are a bit more complex. Agents who have already annuitized will receive a stream a and subsequently find the optimal consumption rule. Agents who have not already annuitized are able to decide what portion of their wealth will be put into the annuity.

In order to solve this model we conduct a maximization in stages. First, for a given value of the annuity we compute the optimal consumption rule via bisection, and again use quadrature and interpolation to calculate the expectations (the integrals in the model). This is embedded in another bisection algorithm for calculating the optimal fraction of wealth to annuitize and the implied annuity to be received in the periods ahead, possibly 0. This can be written as

[30] As in the previous section, we discretize the two variables that enter the value function in order to approximate the integrals and again choose 50 grid points for each variable.

$$\max_{a} \max_{0 \le c \le w - A(a)} [U(c) + \beta\, E\, V(a, \tilde{r}(w - A(a) - c))] \ , \tag{23}$$

and again $A(a) \le w$. The first order condition for an optimum in the inner maximization is

$$U'(c_a) - \beta\, E\, [r\ V'(a, \tilde{r}(w - A(a) - c_a))] = 0 \ . \tag{24}$$

We solve this by the same methods explained above. The outer maximization solution method is very similar, but now the first order condition results from differentiating

$$U(c(a, w)) + \beta\, E\, [V\ (a, \tilde{r}(w - A(a) - c(a, w)))] \tag{25}$$

with respect to a. This results in the following first order condition:

$$U'(c(a, w))\frac{\partial c}{\partial a} + \beta\, E\, [\ \frac{\partial V}{\partial a} - \tilde{r}\frac{\partial V}{\partial \tilde{w}}\, [A'(a) + \frac{\partial c}{\partial a}]\,] = 0 \ , \tag{26}$$

which by the envelope condition reduces to this intuitively plausible $f.o.c.$:

$$E\, [\frac{\partial V_{t+1}(a, \tilde{w})}{\partial a}] = E\, [\tilde{r}\ \frac{\partial V_{t+1}(a, \tilde{w})}{\partial \tilde{w}}\ A'(a)] \ , \tag{27}$$

where $\tilde{w}$ is wealth next period. The left hand side of this expression can be understood as the marginal value of an additional unit of annuity and the right hand side as its marginal cost. The agent will try to set these equal when calculating the optimal annuity in every period.

Bisection searches over the values of the annuity (which imply optimal levels of consumption calculated by the inner bisection algorithm), using quadrature to calculate the expectations and again interpolating the values of the next period value function. The interpolation has to be performed in two dimensions, which complicates and slows down the procedure slightly.

Once we have solved this model, we simulate it and construct consumption, wealth accumulation, and annuity paths over the life cycle. The results are quite striking. In Figure 16 we replicate the model of Section 2 for a starting wealth value of 10,000 units in a 30-period model, which we then map into a lifetime age profile. Consumption is again increasing over the lifetime due to the investment uncertainty as well as the mortality uncertainty embedded in the discount rate. Figure 17 shows the consumption path resulting from averaging 500 simulations for individuals with a starting wealth value of 10,000 units. We can see the smoothness of the path compared with that of Figure 17, for the same starting value of wealth and the same parameter values. In fact, consumption is practically flat around 600 units. The contrast is sharper for increased values of the parameter of relative risk aversion.

In Figure 17 we also show the average annuity value received (a in the graph), which changes slightly at the beginning but remains mostly flat over the course of the lifetime. We report four different specifications: the first has a relative risk aversion parameter of 1.5, the second (hg in the plot) has $\gamma = 2.5$, the third (lr in the plot) is simulated with a 1% average interest rate of the annuity (2 percentage points lower than the benchmark), and the fourth (bq in the plot) considers a higher bequest motive, 0.6 versus 0.1 in the benchmark case.[31]

We observe that when the rate of return is lower, the annuitization and lifetime consumption are also lower, consistent with Friedman and Warshawsky (1990). We find that increasing the bequest motive results in a slightly smaller level of average annuities received at the beginning of life but has little effect thereafter, although consumption is less smooth. The implications for wealth accumulation are that individuals accumulate higher balances late in life when they have higher valuations of bequests. A higher relative risk aversion does not have a very significant effect, but from the figures we can conclude that consumption is less smooth and wealth accumulation is higher in the last third of the lifetime, something consistent with the idea that higher risk aversion should lead to less smooth paths for lifetime consumption and higher paths of accumulation.

Figure 19 reports wealth accumulation and the evolution of the value annuitized (A in the graph) at each stage of the life cycle. There is a clear difference between this wealth path and that of Figure 18, which replicates the model of Section 2, implying that once individuals are able to annuitize, they run down their wealth fairly smoothly over their life, with more risk averse individuals again benefiting the most. The annuitization happens very early in this endowment consumption/saving model with investment uncertainty, and for average agents, amounts to a third of their wealth in the initial periods. Depending on the realizations of interest rates (again, draws from a truncated log-normal distribution with mean higher than the rate of return of the annuity), agents sometimes annuitize later in life and in a lower proportion, a seemingly reasonable result. These results are also consistent with Mitchell et al. (1999) suggesting that our model extends their simplified stochastic life cycle model to a full dynamic characterization of the annuitization decision in the presence of bequest motives and capital uncertainty, allowing for annuitization to happen at any point in the life cycle and with any fraction of the individual's wealth.

In Figure 20 we present the simulations of the annuity model with the annuity made actuarially fair, in the sense that its rate of return is equal to the average return of the stochastic market

[31] Here we consider non-actuarially fair annuities. These can result from the fact that the insurance company selling the annuities faces transaction costs and adverse selection.

return.[32] The proportion of wealth annuitized in this case is not larger than 50%, and the annuitization does not always take place in the first period of life. We can also see that the patterns of wealth accumulation and consumption are somewhat different from those of the model with less than actuarially fair annuities, because people accumulate higher balances at the end of their life which allows them to consume more at the end of the life cycle.

These results have several interesting implications. First, in a simple model of consumption and saving decisions with income uncertainty, the possibility of annuitizing wealth is used by individuals to smooth their consumption stream almost entirely. If we interpret this mechanism as a pseudo-social insurance system, there is no doubt as to the importance of the effects that such a scheme has on the microeconomic behavior of agents. However, we want to explore the reasons for the lack of availability of such annuities in the current capital markets. Some researchers emphasize the issue of pricing, and some point to adverse selection; yet others blame it on the high capital returns to equities. Our average results seem to provide some insights into the "annuity puzzle," the question of why the annuity market is so narrow. If it is optimal for an average individual to annuitize between 30% and 50% of their wealth, as our model suggests, and Social Security accounts for approximately that proportion of their wealth (Friedman and Warshawsky 1990), it is very likely that the low demand for annuities that we observe is the result of optimal decision making by individuals. For some individuals Social Security would provide less than the optimal level of annuitization, causing them to buy additional annuity notes in the market. For others, S.S. would lead to over-annuitization and they might react by buying life insurance to offset the imposed annuity purchases through the social insurance system.

Interestingly, the most novel results and insights come from the extension of this model to endogenize the labor decision and introduce Social Security, which we consider next.

4.2 Endogenous Annuities in the Extended Framework

Our conjecture regarding the effects of extending the classical life cycle model with annuities to endogenize labor supply, in the same fashion as in Section 3, is twofold. First, such a model could help shed new light on long standing questions such as the effect of Social Security on the micro behavior of agents. Second, it is likely to provide further insights into the "annuity puzzle." The conjecture regarding annuities is that once we introduce labor supply we should see the annuity decision delayed in the life cycle, given that individuals use their labor as an insurance instrument

[32] The concept of "actuarially fair" falls slightly short to define the annuities we are allowing our agents to purchase, because they are also a riskless asset, as opposed to the risky alternative capital investment.

when they are young. The results confirm some of these conjectures, and go even further.

We once again proceed by numerical dynamic programming to solve a model of endogenous consumption/saving, labor/leisure, and endogenous annuities, employing backward induction. We can write the individual's problem in the last period of life as

$$V_T(w,a) = \max_{(0\leq c\leq w+\omega(1-l)\tau+a+ss,l)} U(c,l) + K\ U(w+\omega(1-l)\tau - c + a + ss)]\ , \tag{28}$$

where τ represents the Social Security tax we discuss below, and ss the Social Security benefits to which the individual is entitled. We then obtain the optimal decision rules using the sequential bisection algorithms discussed in the previous subsection and solve recursively. We can write the value function in the next to last period as

$$V_{T-1}(w,a) = \max_{(0\leq c\leq w+\omega(1-l)\tau-A(a)+a+ss,l)} U(c,l) + \beta\ E\ V_T(w,a)\ . \tag{29}$$

The computational burden of this model is similar to the model without labor supply since we assume a deterministic path of wages. We solve this model again by bisection, computing the expectations by quadrature and interpolating the values of the next period's value functions.

An additional extension, already considered in the formulation of (28) and (29), is to introduce Social Security, and we do so in a stylized manner. Assuming a deterministic path of wages, and further assuming that we are analyzing the behavior of an individual born in 1937, that will be 65 as of the year 2002, we can compute the benefits that such a person will receive had he worked at least 35 years since the age of 21.[33] We follow the formulae provided in SSA (2000) and assume that individuals can only start receiving benefits at age 65. We do, however, allow for work after that age and control for the earnings test provision to calculate the corresponding benefits.[34] We also tax wages at the current individual tax rate (6.2%), free from the Disability Insurance withholding (given that we do not model Disability Insurance in our framework), and add the taxes paid by the employer (6.2%), since those payments can be considered as discounted from a theoretical *before all taxes* salary.[35]

[33] This is a fairly simplified characterization of the current Social Security system, which in our model could result in an artificial trend towards early retirement, given that agents could potentially foresee the gains from not working or working less as age 65 approaches, because their benefits will remain the same regardless of their working history. Our results seem not to suffer from this problem, as we will see below. Thus, although stylized, our characterization of the current Social Security system is a good first approximation to evaluate behaviorally meaningful responses.

[34] See Myers (1993) for a comprehensive review of Social Security rules, and Friedberg (2000) for a discussion of the effects of the earnings test on labor supply. The Social Security Administration website is an excellent source of information not only for recipients, and future recipients, but also for researchers: `www.ssa.gov`.

[35] The Internal Rate of Return (IRR) of the Social Security system that we introduce is 1.6522%, a fairly realistic number given the assumptions made to compute the benefits. See the discussion in Geanakoplos et al. (1999).

The results from this model are presented in Figures 21-24. The figures present the paths of consumption, labor supply, annuities, and wealth chosen optimally by individuals over their life cycle. We can compare these results both with the ones presented in the previous subsection and with the ones in Section 3.1. The most important effects of introducing labor supply are twofold: first, the annuity decision is delayed from the initial periods of the life cycle (around the early 20's in the previous model) to around mid-life; second, and more importantly, the average individual now annuitizes a very small proportion of his or her wealth, becoming a marginal insurance instrument for these agents. This effect is even stronger once we introduce Social Security, with annuitization becoming even more marginal compared with the overall resources of individuals at any given age. This last result sheds new light on the "annuity puzzle," leading us to conclude that in a more complete dynamic framework it is less of a puzzle why annuities are less attractive as an insurance instrument than has been believed.

A very important issue to highlight at this point, which is also valid for the model presented in the previous subsection, is that this partial and residual annuitization by the average individual is consistent with the theory that says that an individual would in principle annuitize all its wealth if the annuity were actuarially fair. The behavior of a single individual in our model in front of the possibility of purchasing an actuarially fair annuity, given a state of the world and the expectations over the future states of the world, is more heterogenous than the average results (product of hundreds of simulations) show. Some individuals never annuitize, while others annuitize very late in life, but basically all of those who annuitize, no matter at what age, put 100% of their current wealth in that annuity, as a portfolio selection approach to this problem would tend to predict.

In some respects the extended model presented in this section is close to that of Bodie and Samuelson (1989), and Bodie, Merton and Samuelson (1992), in that it puts together the life cycle consumption/saving model with the portfolio decision, allowing for flexible labor supply. Their research concentrates in the continuous time case, and the effect of making labor supply flexible in the investment mix by individuals, and not on the consumption/saving and labor/leisure choices over the life cycle. Furthermore, they do not consider annuitization in their model. However, we consider that the insights from our work complement their results quite nicely. They find that allowing for labor supply flexibility in the traditional life cycle consumption/saving model leads to more investment in the risky asset, assuming negative correlation between investment returns and the labor income innovations. We find that even in the absence of this correlation, introducing labor supply in the model, allow us to explain why individuals would not choose annuities as their preferred investment product.

The richness of the model allows us to go even further, and we can show two very important effects of Social Security on behavior. First, we can see the labor supply response of individuals to the reception of Social Security benefits: they decrease their participation on average to part-time work when they reach age 65, but in some individual cases, complete retirement is chosen.[36] Second, the effect on savings and individual capital accumulation is striking, reducing them considerably. These results complement and extend the classic discussions of Feldstein (1974), and Kotlikoff (1979) regarding the effects of Social Security on saving behavior.[37] In both cases these results give important insights into classic questions regarding the role of Social Security in shaping individual behavior.[38]

Finally, we can compare the welfare of individuals with and without Social Security in the extended consumption/saving model with endogenous labor supply and endogenous annuities. We compute how much extra wealth we would have to give to individuals at the beginning of life to make them as well off in a world with Social Security as they would be in a world without public social insurance. Remember that these individuals have access to risky assets and fairly priced annuities. The results show that depending on the initial level of wealth and annuities the (negative) compensating variation can be substantial, suggesting that a young individual would be better off in a world without Social Security. For example, if initial wealth is around 10,000 units and initial annuity receipts are relatively small, individuals would have to see their wealth doubled to reach the same level of utility. If initial wealth is around 30,000 units then the compensation would have to be of a third of that amount, and about a fourth if initial resources are 50,000 units.

5 Conclusions

This paper has presented several models of life cycle consumption/savings and labor/leisure decision making under uncertainty. We first present a benchmark finite horizon consumption/saving problem and solve it analytically and then use numerical dynamic programming techniques to validate the methodology used throughout the paper. We find that the decision rule of the finite horizon model with bequests converges to the infinite horizon solution. We also find that numerical methods approximate the finite horizon version of Phelps (1962) model quite well, and better than they

[36] As discussed by Rust and Phelan (1997), to show the effects of Social Security on labor supply is a surprisingly difficult task. Task that their work and our model achieve successfully. French (2000) also highlights the effects of social insurance on the labor supply of the current and future old.

[37] Page (1998) provides a survey of the empirical literature which tackles this issue.

[38] İmrohoroğlu et al. (1999b) using a different model find results qualitatively similar to those reported here regarding the decline of labor supply and wealth accumulation once Social Security is introduced.

do the original infinite horizon problem. We then present a model that endogenizes labor supply, allowing first for deterministic wages, and then introducing income uncertainty. We conclude that the model is consistent with consumption and wealth accumulation profiles in the data and that precautionary savings can even increase when we consider that labor supply (another source of accumulating precautionary balances) is endogenous, a result consistent with Low (1998). The model also shows the reduction of labor force participation at the end of the life cycle.

The paper then introduces the possibility of endogenously choosing annuities in a consumption/saving framework with capital uncertainty and bequest, later extended to endogenize the labor/leisure decision. Agents can choose to annuitize part or all their wealth at any point of their lives, but they can do this only once. This model can be understood as a privatized system with no mandatory contributions, but with a one-time opportunity to annuitize. We then include a more traditional Social Security system. The solution is consistent with some early results in the literature and in a sense generalizes those models. We find that in the simple consumption/saving model agents do choose to annuitize part of their wealth and that they do so early in life, allowing them to smooth consumption considerably compared with the behavior observed in the benchmark model. We also analyze the effects of higher interest differentials between the uncertain rates of returns on wealth and the return on the annuities, as well as the effect of having a higher bequest motive. Once we take into account the labor decision, annuities are bought later in life and on average represent a small percentage of average wealth holdings, an effect which becomes even more clear when we introduce Social Security. We also show that labor supply and wealth accumulation react to the incentives set forth by Social Security, and that a young individual would have to be compensated with a substantial increase in wealth to be as well off in a world with S.S. as in a world without it. We claim that this complete model of endogenous consumption, labor, and annuity decisions provides important insights into the "annuity puzzle" since the lack of demand for annuities can be the result of optimal behavior once labor supply and Social Security are accounted for.

There are several possible extensions of the model(s) presented here. Benítez-Silva et al. (2000b) extend this model and that of Rust and Phelan (1997) to account for disability insurance and Medicare, and model more closely retirement incentives. We are also planning to allow for added uncertainty through health shocks which can be correlated with wages, as well as mortality uncertainty based on life tables, instead of embedding it in the discount factor. Another extension would explicitly consider borrowing, as in the consumption/saving literature. The model could eventually also allow for private pensions. Our model can also be used to estimate underlying parameter values following the simulation techniques in Gourinchas and Parker (1999), and French (2000) given data on the variables of interest.

Finally, another currently considered extension of this model attempts to integrate the job search decision into the life cycle dynamic maximization framework introduced here (See Benítez-Silva 2000). Both young and older workers search for new jobs while out of work and on the job in non-trivial proportions. This activity should be taken into account in a life cycle model given the importance of the outcomes for the future path of earnings, wealth accumulation, and lifetime utility. Such a unifying framework would extend the life cycle utility maximization model and reconcile these two bodies of literature, which although theoretically intertwined (See Seater 1977), have evolved in different directions.

Figure 1: Consumption Decision Rule. Log Utility

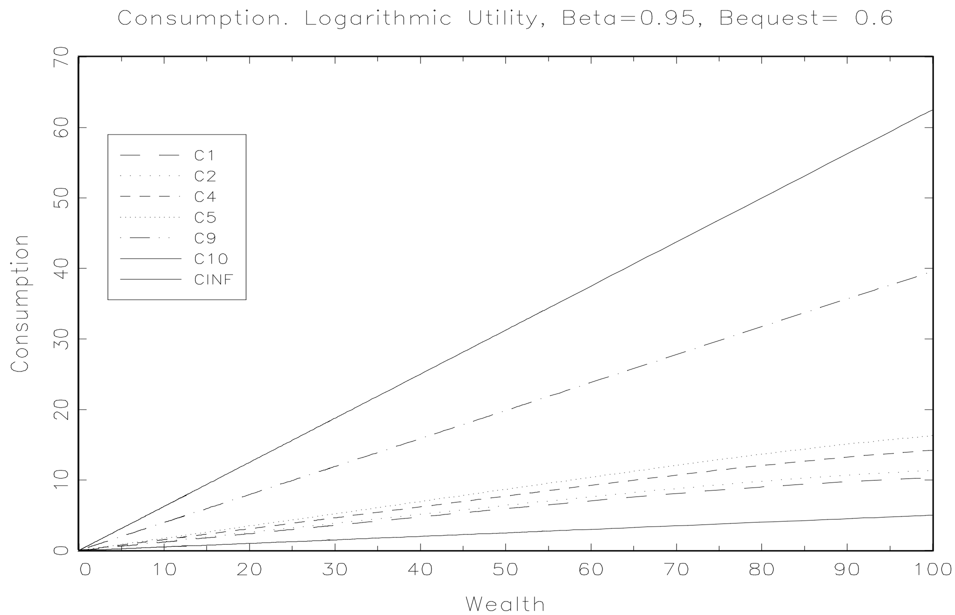


Figure 2: Consumption Decision Rule. CRRA Utility

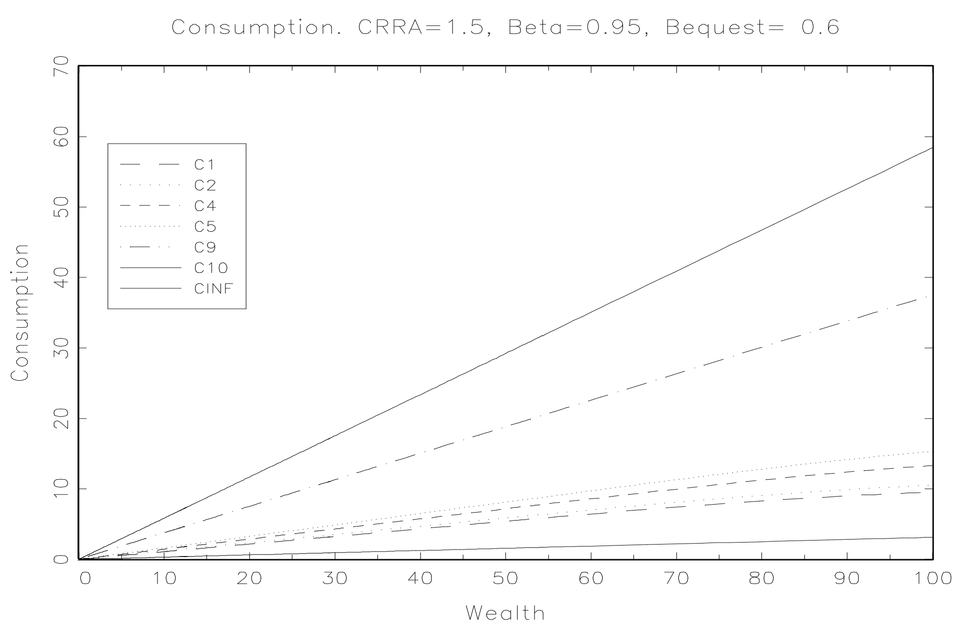


Figure 3: Computed vs. True Decision Rule. Log Utility

Figure 4: Computed vs. True Decision Rule. CRRA Utility

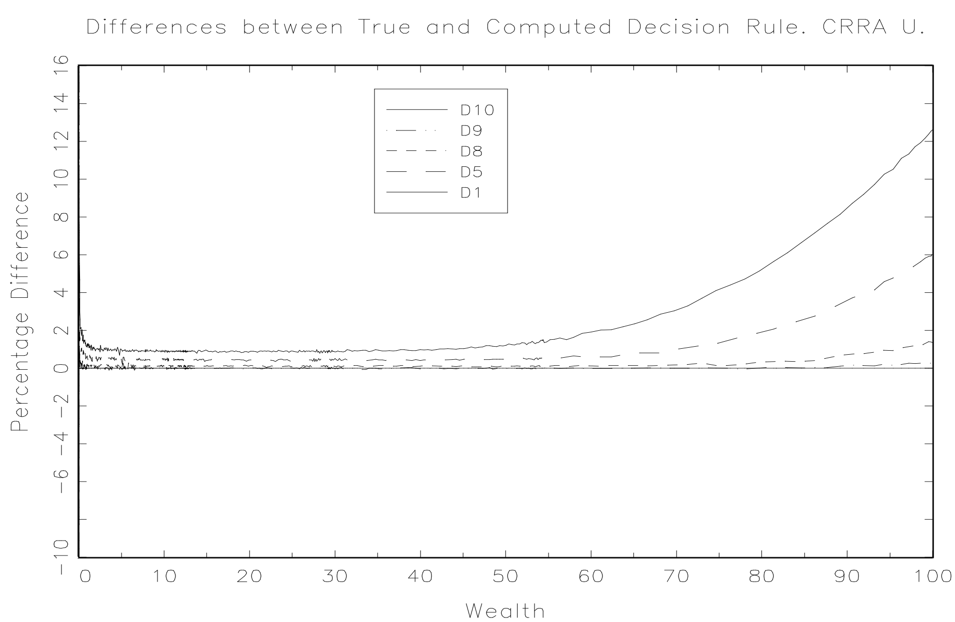


Figure 5: Simulated Consumption. CRRA utility

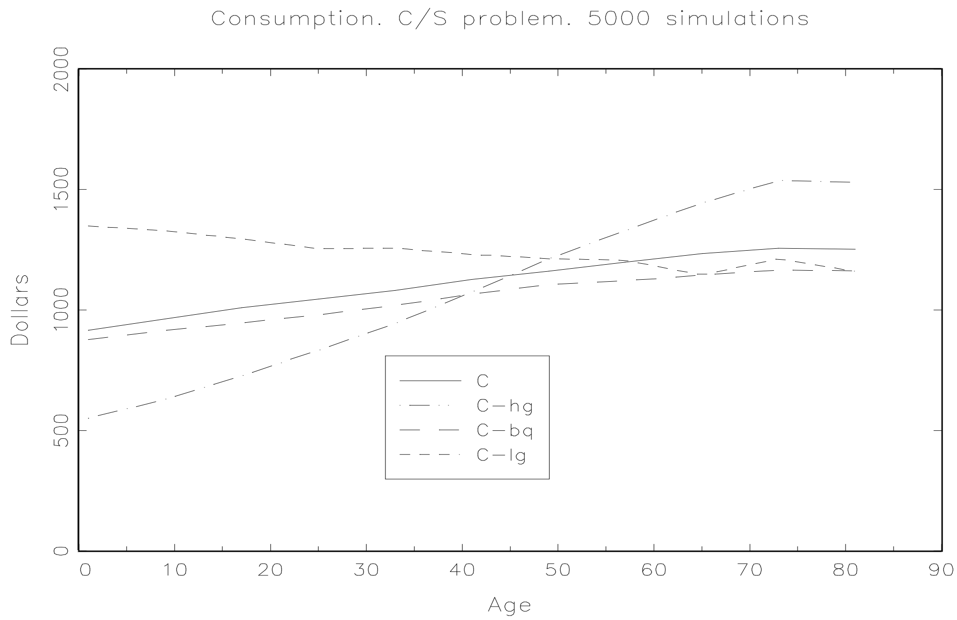


Figure 6: Simulated Wealth Accumulation. CRRA utility

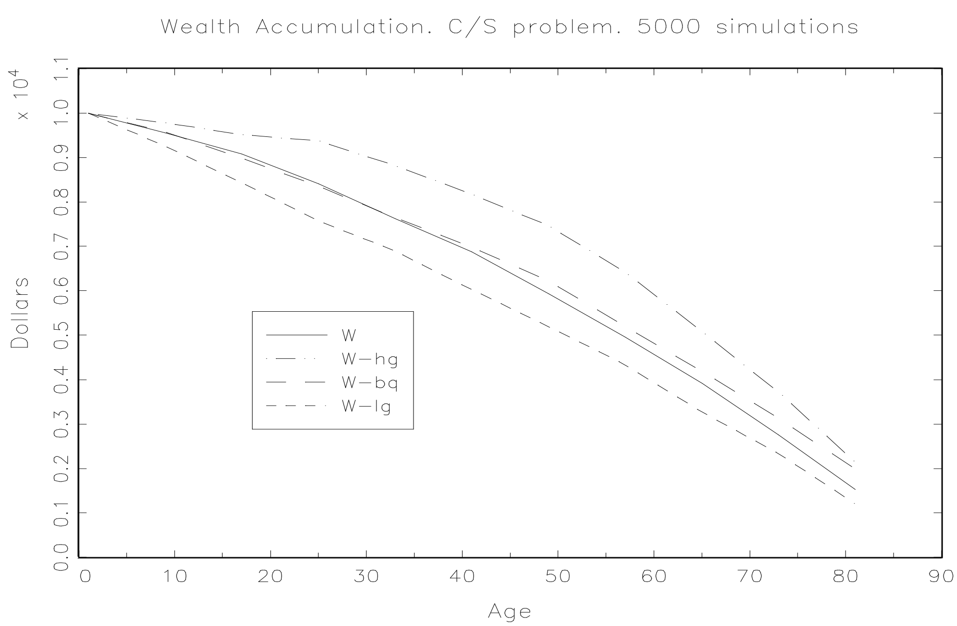


Figure 7: Simulated Consumption. Deterministic Wages

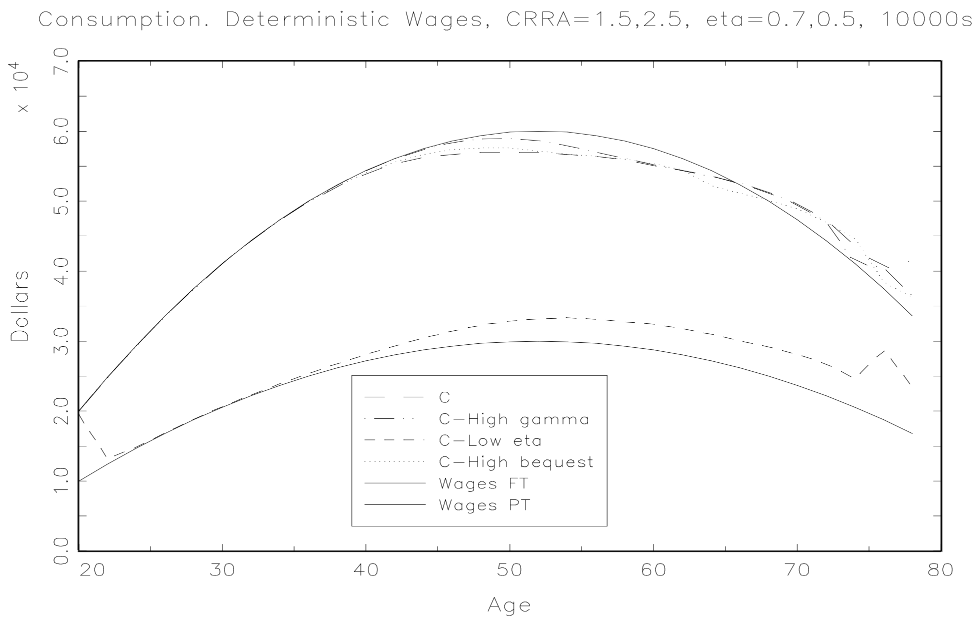


Figure 8: Simulated Labor Supply. Deterministic Wages

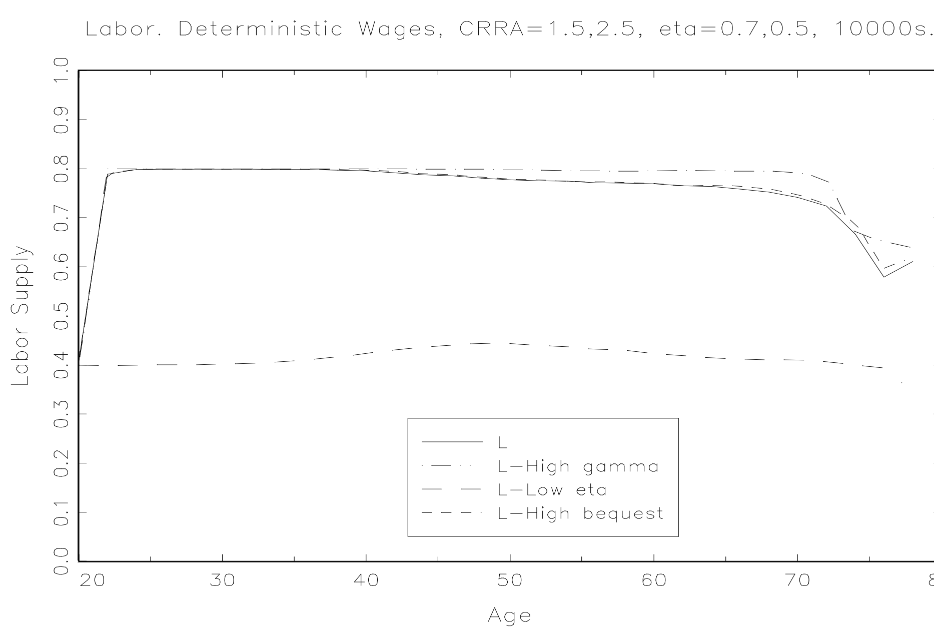


Figure 9: Simulated Wealth. Deterministic Wages

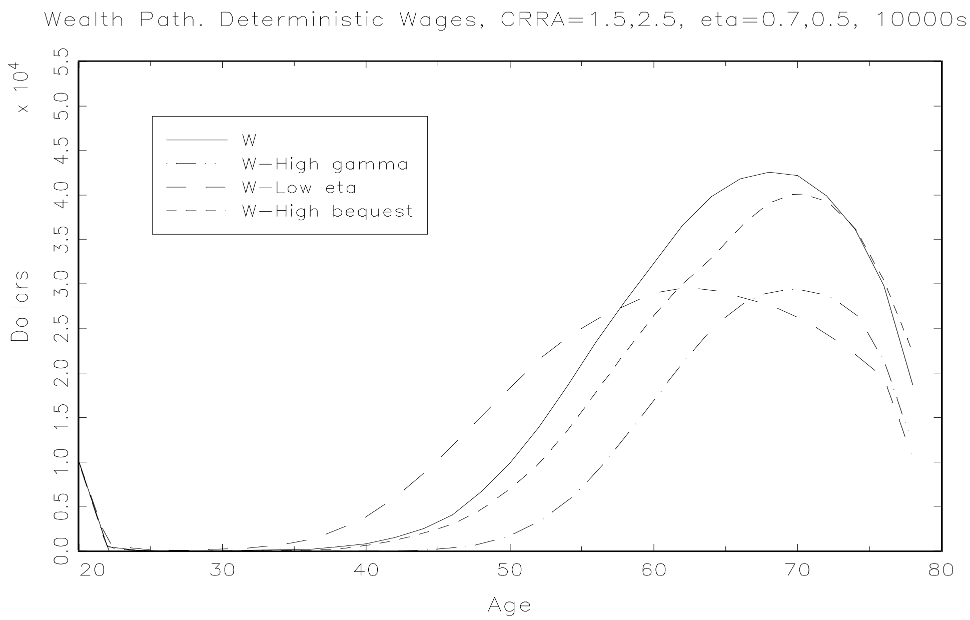


Figure 10: Simulated Consumption. Stochastic Wages

Figure 11: Simulated Labor Supply. Stochastic Wages

Figure 12: Simulated Wealth. Stochastic Wages

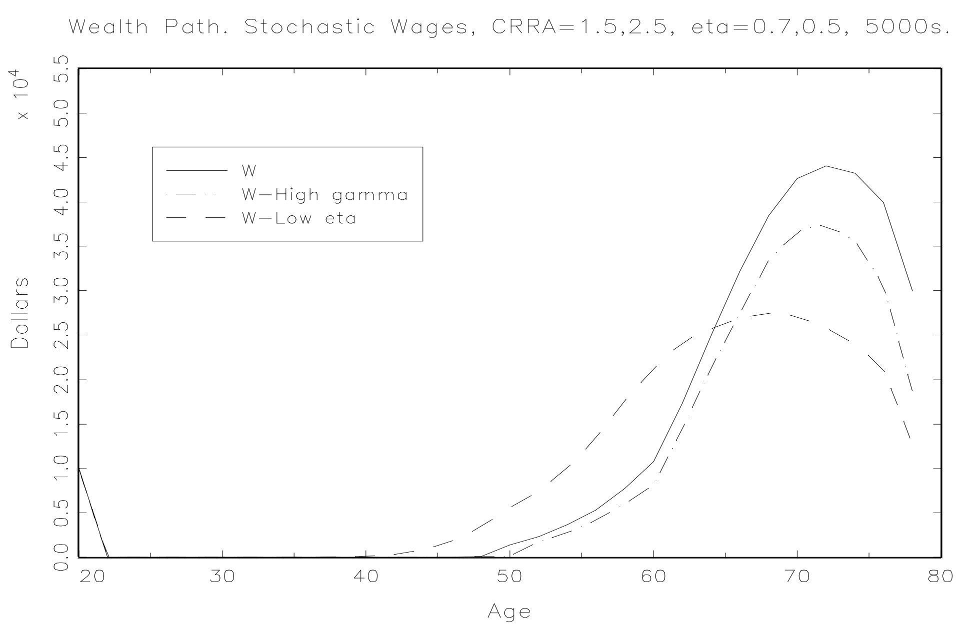


Figure 13: Simulated Consumption. Serially Correlated Wages

Figure 14: Simulated Labor Supply. Serially Correlated Wages

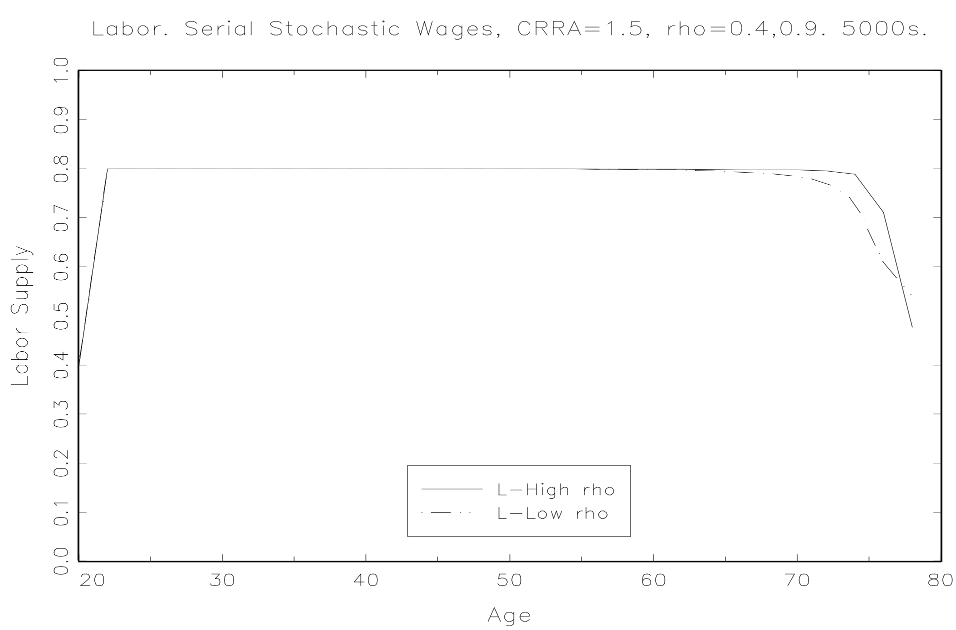


Figure 15: Simulated Wealth. Serially Correlated Wages

Figure 16: Simulated Consumption. C/S Problem. CRRA Utility

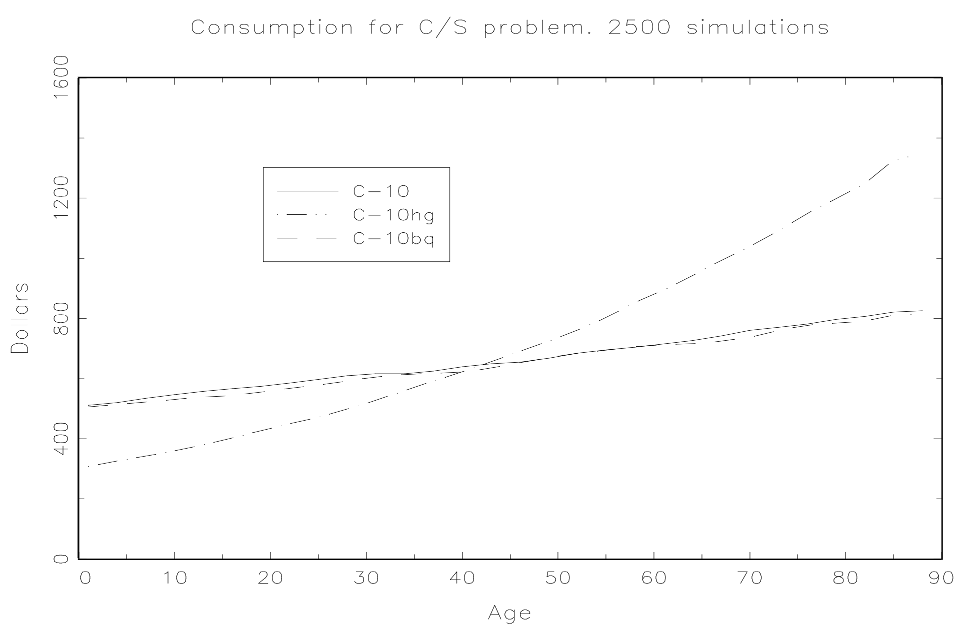


Figure 17: Simulated Consumption and Annuities. C/S Problem. CRRA utility

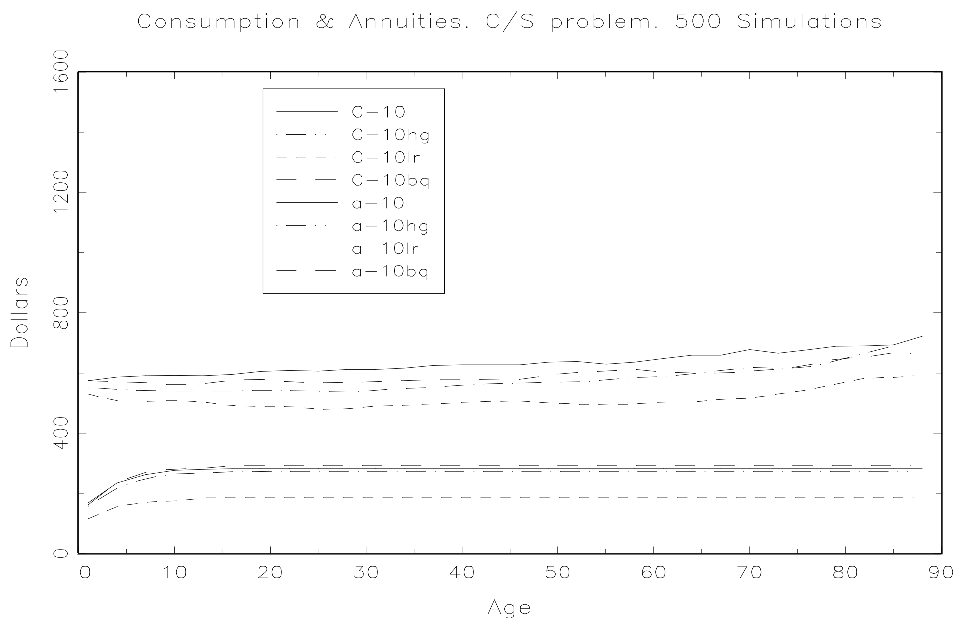


Figure 18: Simulated Wealth Accumulation. C/S Problem. CRRA Utility

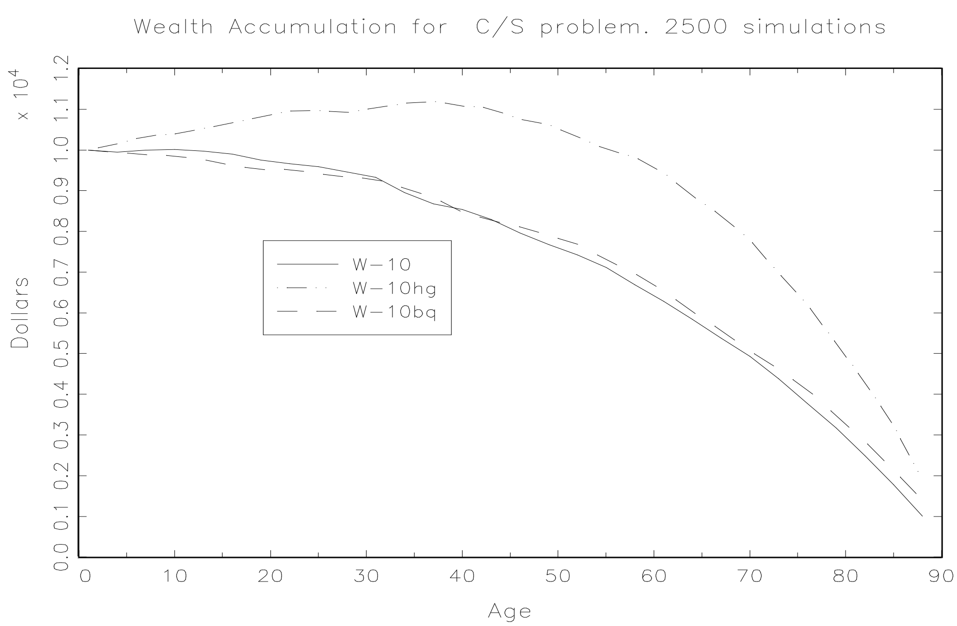


Figure 19: Simulated Wealth and Annuity Costs. C/S Problem. CRRA utility

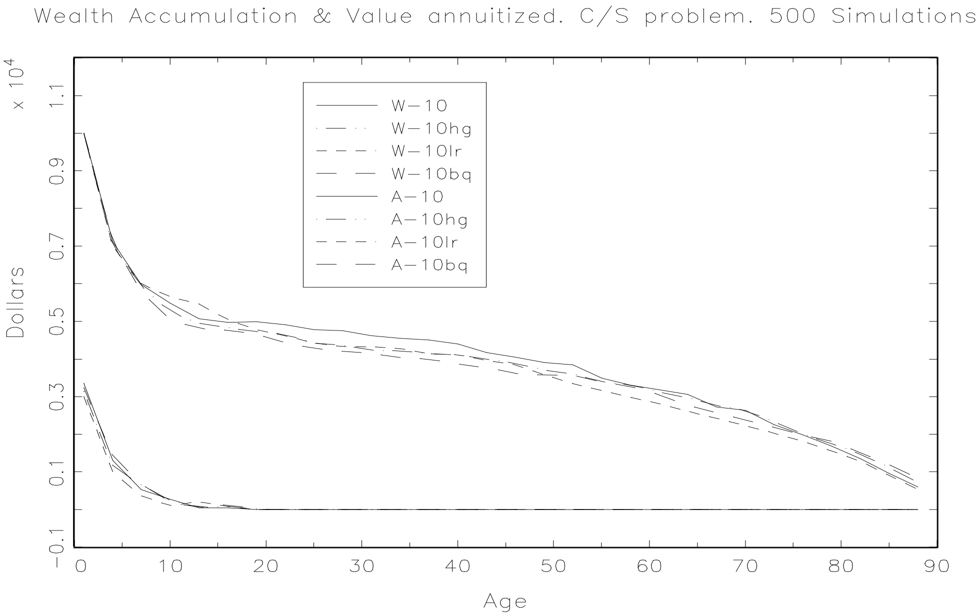


Figure 20: C/S Problem with Actuarially Fair Annuities. CRRA Utility

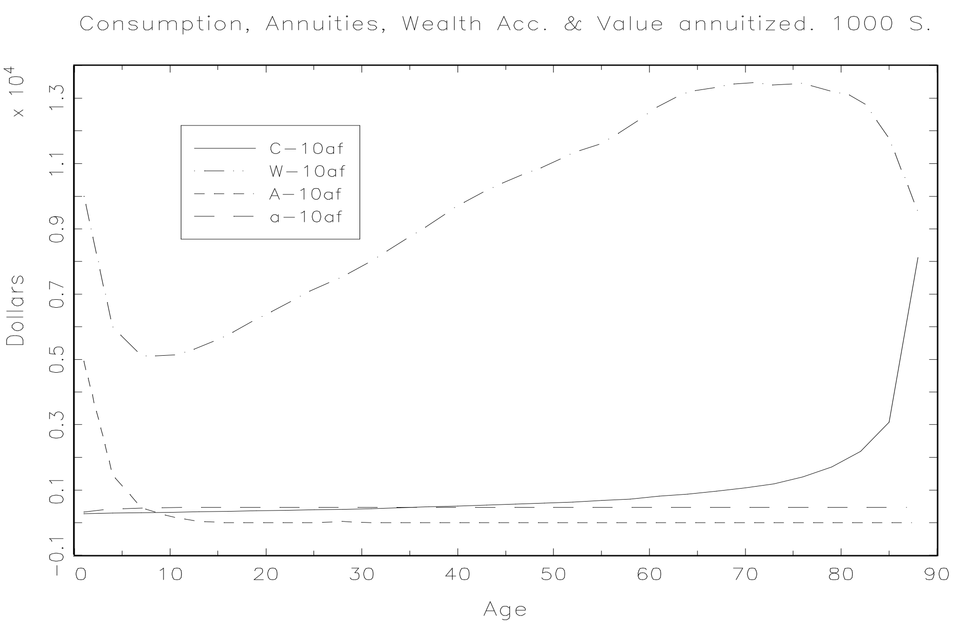


Figure 21: Simulated Consumption and Annuities. Full Model.

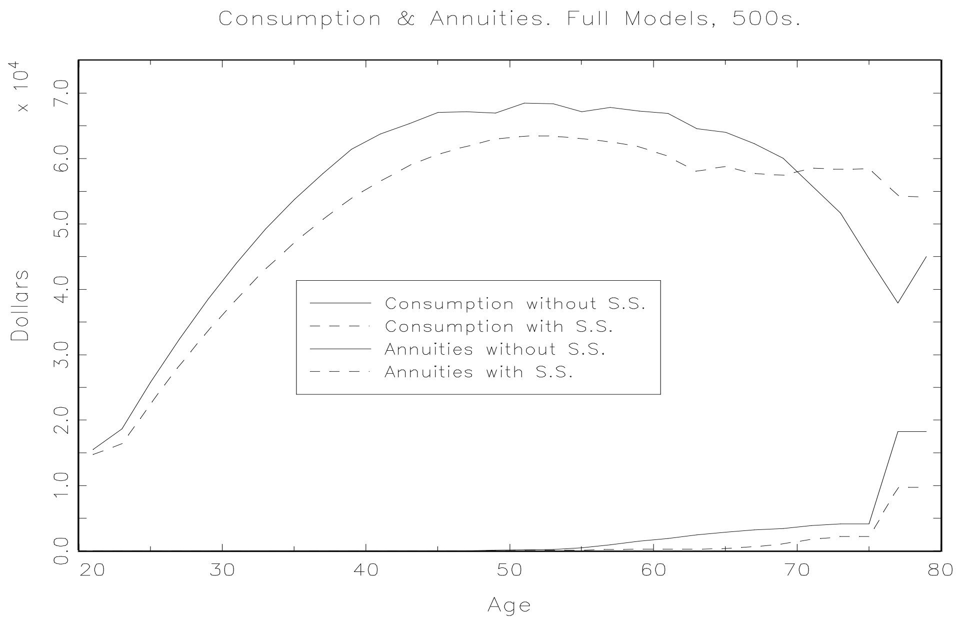
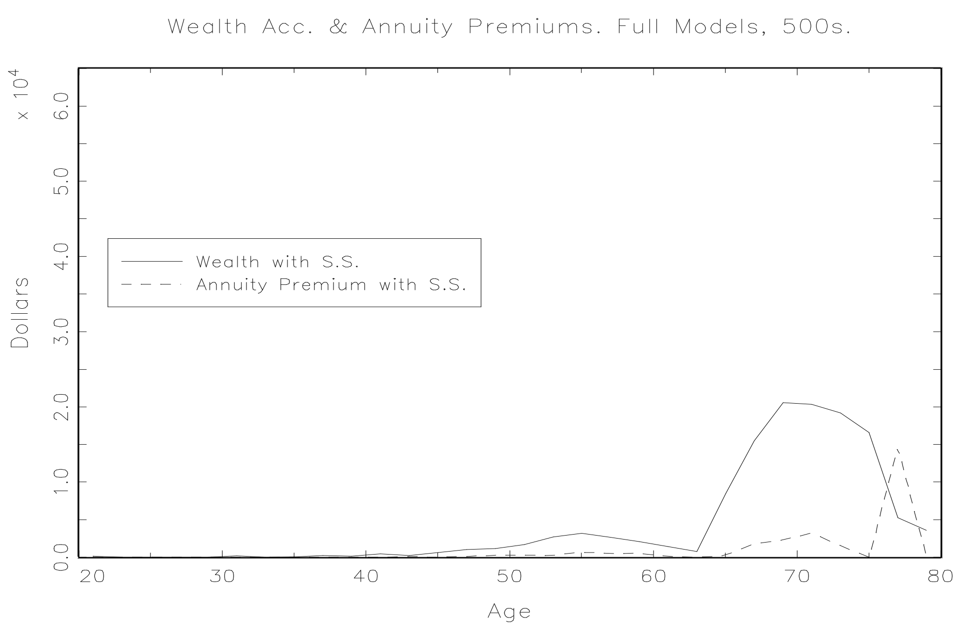


Figure 22: Simulated Labor Supply. Full Model.

Figure 23: Simulated Wealth, without S.S. Full Model.

Figure 24: Simulated Wealth, with S.S. Full Model.



Appendix

In this Appendix we derive the closed form solution of the finite horizon version of Phelps (1962) consumption/saving problem assuming a CRRA utility function. Our derivation is also close in nature to the one performed in Levhari and Srinivasan (1969). We can again solve this problem relying on Dynamic Programming and Bellman's principle of optimality, using backward induction. In the last period of life agents solve

$$V_T(w) \;=\; \max_{0 \leq c \leq w} \frac{c^{1-\gamma}}{1-\gamma} + K\,\frac{(w-c)^{1-\gamma}}{1-\gamma}\;,$$

where γ is the coefficient of relative risk aversion and K is the bequest factor, characterized as a number between zero and one.[39] By deriving the first order condition with respect to consumption it is straightforward to show that

$$c_T \;=\; \frac{w}{1+K^{\frac{1}{\gamma}}}\;,$$

we can then write the analytical expression for the last period value function:

$$V_T(w) \;=\; \frac{\left(\frac{w}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} + K\,\frac{\left(\frac{wK^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma}\;.$$

Then the problem that agents solve in the next to last period of life is:

$$V_{T-1}(w) \;=\; \max_{0 \leq c \leq w} \frac{c^{1-\gamma}}{1-\gamma} + \beta\,E\,V_T(w-c)\;.$$

Using the previous results we can write

$$V_{T-1}(w) \;=\; \max_{0 \leq c \leq w} \frac{c^{1-\gamma}}{1-\gamma} + \beta\,E\left[\frac{\left(\frac{\tilde{r}(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma} + K\left[\frac{\left(\frac{\tilde{r}(w-c)K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{1-\gamma}}{1-\gamma}\right]\right]\;.$$

Here in order to derive the first order condition with respect to consumption we assume, as in Lavhari and Srinivasan (1969), that the value function is differentiable and that the differential and expected value operators can be interchanged. The $f.o.c$ is then,

$$c^{-\gamma} - \beta\,E\left(\tilde{r}^{1-\gamma}\right)\left[\left(\frac{(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma}\frac{1}{1+K^{\frac{1}{\gamma}}} + K\left[\left(\frac{(w-c)K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma}\frac{K^{\frac{1}{\gamma}}}{1+K^{\frac{1}{\gamma}}}\right]\right] \;=\; 0\;.$$

[39] We also follow in this case the "egoistic" model of bequests.

Then some algebraic manipulation allows us to write the $f.o.c$ as

$$c^{-\gamma} \;=\; \beta\, E\left(\tilde{r}^{1-\gamma}\right) \left(\frac{(w-c)}{1+K^{\frac{1}{\gamma}}}\right)^{-\gamma} .$$

Some more tedious algebra leads to the following expression for the decision rule in the next to last period

$$c_{T-1} \;=\; \frac{w}{1+\beta^{\frac{1}{\gamma}}\,\left[E\left(\tilde{r}^{1-\gamma}\right)\right]^{\frac{1}{\gamma}}\left[1+K^{\frac{1}{\gamma}}\right]} ,$$

that can be rewritten as

$$c_{T-1} \;=\; \frac{w}{1+\beta^{\frac{1}{\gamma}}\,\left[E\left(\tilde{r}^{1-\gamma}\right)\right]^{\frac{1}{\gamma}}+\beta^{\frac{1}{\gamma}}\,\left[E\left(\tilde{r}^{1-\gamma}\right)\right]^{\frac{1}{\gamma}}\,K^{\frac{1}{\gamma}}} .$$

Assuming next that the interest rate, $\tilde{r}$, follows a log-normal distribution with mean μ and variance σ^2, then given that $E(\tilde{r}) = e^{\,\mu+\frac{\sigma^2}{2}}$ and denoting $E(\tilde{r})$ as $\overline{r}$ we can write

$$E\left(\tilde{r}^{1-\gamma}\right) \;=\; \overline{r}^{1-\gamma} e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}} .$$

We then substitute back in the formula for c_{T-1} and obtain

$$c_{T-1} \;=\; \frac{w}{1+\beta^{\frac{1}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}+\beta^{\frac{1}{\gamma}}\,K^{\frac{1}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}} ,$$

given the similarity with expression (8) in the text it is easy to see how backward induction would lead us to the decision rules for the rest of the periods, for example we can write c_{T-k} as

$$c_{T-k} \;=\; \frac{w}{1+\beta^{\frac{1}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}+\beta^{\frac{2}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}+\ldots+\beta^{\frac{k}{\gamma}}\,K^{\frac{1}{\gamma}}\,\left(\overline{r}^{1-\gamma}\,e^{\,-\gamma(1-\gamma)\frac{\sigma^2}{2}}\right)^{\frac{1}{\gamma}}} .$$

We can also see that if γ is equal to 1 we are back to the logarithmic utility case and the expression for c_{T-1} above is equivalent to (8), which is a special case of the expression above. It is also important to emphasize that this expression is the finite horizon counterpart to the one obtained in Levhari and Srinivasan (1969) once a bequest motive is introduced, and that their results regarding the effects of uncertainty (decreasing proportion of wealth consumed as the uncertainty grows if $\gamma > 1$) go through in this case.

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