Computational Tools for the Analysis of Market Risk

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Abstract

The estimation and management of risk is an important and complex task with which market regulators and financial institutions are faced. It has become apparent that accurate and reliable quantitative measures of risk are needed in order to avert, or at least minimize, the undesirable effects on a given portfolio of large fluctuations in the conditions of the market. In order to accomplish this task, a series of computational tools have been designed, implemented and incorporated into MatRisk, an integrated environment for risk assessment developed in MATLAB. Besides standard measures, such as Value at Risk (VaR), the application MatRisk allows the calculation of other more sophisticated risk measures. These novel risk measures (Shortfall, Max-VaR, conditional VaR, etc.) have been introduced by a number of authors in order to address the insufficiencies of VaR to properly characterize the structure of risk. Conditional risk measures can also be estimated by a module of the application devoted to the analysis of the heteroskedastic structure of time series with autoregressive models.

Keywords: Risk analysis, Value-at-Risk, Shortfall, MaxVaR, heteroskedasticity, autoregressive processes, mixture models.

1 Introduction.

The financial portfolio of an institution is composed of a number of products (including bonds, assets, derivatives, etc.) that are openly traded in financial markets. The prices of these products depend on the values of a number of risk factors: Interest rates, asset prices, volatility, etc. Market risk analysis consists in the estimation of the response of the portfolio to fluctuations of the values of these risk factors. Portfolio managers may use this analysis to adopt different strategies: In situations where the goal is securing the value of a portfolio, risk analysis can be used to minimize the effects of unexpected fluctuations in the conditions of the market. Alternatively, a portfolio manager may intend to acquire a competitive advantage through a calculated and carefully controlled exposure to risk. On the other hand, market regulators are interested in enforcing restrictions to discourage an excessive or uncontrolled exposure to risk, which may lead to severe disruptions in the economic system. These restraints should be also supple enough not to have a negative effect on the functioning of the markets.

There are different elements that make the problem of risk assessment a difficult one. A non-exhaustive list includes the complexity of the portfolio itself, whose value may depend on thousands of risk factors, the correlation structure between the values of the different assets that make up the portfolio, and finally, the synthesis of all these inputs into a small set of numbers (such as VaR or expected Shortfall) that capture in a manageable form the information about the risk profile of the portfolio.

In this work we assume that in the near future the portfolio exhibits the same behavior, from a statistical viewpoint, as in its recent history. With this premise, the analysis focuses on a synthetic time series composed of the reconstructed values in the recent past of a hypothetical portfolio whose composition is constant over time and is identical to that of the actual portfolio. The reconstruction process is a complex task and may involve some portfolio compression (where the portfolio is replaced by a few components that account for a large fraction of risk), replication and other manipulations. Given this time series, the most common measure of risk is the Value at Risk, VaR [1]. The objective is to estimate the worst trading losses of a fixed portfolio during a certain period (T, the time horizon), for a given probability level P. Assuming the time horizon is chosen to be one day, the histogram of daily returns (relative daily changes in the value of the portfolio) is constructed from the historical data. This empirical distribution is then fitted to a parametric model from which a percentile at the level P is obtained. The usual assumption is that the daily returns are random independent normally distributed variables.

This procedure has the advantage of a fast implementation, but has recently come under a great deal of criticism (see, for instance [2, 3]). As a matter of fact, there seems to be a consensus in the financial community that current Value at Risk measures fail to capture some of the essential features of actual market risks. It is an empirical observation that the tails of the distribution of returns on financial products have in general more weight than what would be predicted by a fit to a normal distribution. In particular, the assumption of normality is known to fail for large fluctuations in the value of the portfolio [4]. These extreme fluctuations are precisely the ones that a risk measure such as VaR is trying to capture. The failure to correctly represent the behavior of the portfolio in worst-case scenarios has prompted several suggestions of alternative, more sophisticated models that may correctly reflect the probability of extreme events. One of the extensions of the classical VaR methodology is to drop the assumption of normality of the daily losses and to posit a different distribution with a positive kurtosis and possibly skewness (eg. hyperbolic distributions [5] , mixture of Gaussians [4], stable distributions [6], etc.), which would provide a more accurate model of the behavior of the market. The parameters of these non-normal distributions can be determined by maximum likelihood estimation. Finally, a measure of risk analogous to the normal VaR is extracted by calculation of percentiles. Extreme Value Theory [7, 3, 8] also provides a framework for the definition of other non-standard risk measures such as MaxVaR and Shortfall. In order to extract these parameters, the analysis focuses on the distribution of extreme events itself.

The risk measures described in the previous paragraphs take the unconditional distribution of returns as the basic object for the analysis. This corresponds to a long-term view of risk management, where we are interested in the losses one may incur in a long period of time (say a month). If on the contrary, we are concerned by the daily changes in the value of a portfolio, it may be more useful to focus on conditional distributions of returns. The question to be answered in this case is the following: Given the recent changes in the value of my portfolio, what is the range of changes I should expect and with what probability will they occur? In order to give an answer to this question, the time series is assumed to follow an autoregressive model [9], where X_t, the value of the return at time t depends on its recent history

$$X_t = f(X_{t-1}, X_{t-2}, \ldots) + u_t. \quad (1)$$

The first term in Eq. (1) reflects the trends in the time series. The function f is generally parameterized as a linear model. The innovations u_t are random variables of zero mean and independent of one another. They are assumed to be generated by a normal distribution with a time-dependent variance that also depends on the previous values of the innovations (ARCH models [10]) or on both the innovations and the variance itself (GARCH models [11]). Piecewise linear autoregressive processes (MixARCH, MixGARCH) can be constructed by considering mixtures of linear models [12]. Besides the increased representation flexibility allowed by the nonlinearity of the mixture models, these can also account for the leptokurtosis generally observed in the distribution of the innovations.

In this work we present a series of computational tools that can be used to calculate these different risk measures. These tools have been integrated into MatRisk, an application developed in MATLAB with a graphical user interface. MATLAB has been chosen because it provides a well-suited programming environment, where both numerical and interface design challenges can be met with a reduced development effort. Throughout this article, we illustrate the different capabilities of the application by analyzing the time series for the IBEX35 stock index corresponding to the period between 4/1/93 and 23/12/97. A stock index can be thought of as a reference portfolio composed by a fixed proportion of assets traded in a given stock market. The series used contains the 1296 daily closing values of the index during a period of about 5 years (see Fig. (1)).

The paper is organized as follows: Section 2 describes the modules of the application MatRisk that facilitate the calculation unconditional risk measures. The tools provided are a module for fitting a parametric model (normal, mixture of normals, hyperbolic distribution) to the full probability distribution of portfolio returns, a module to carry out a fit to the tail of the distribution, assuming Pareto behavior, and, finally, a module to fit the maxima of the time series to a generalized extreme value distribution. A summary of the results of a risk analysis for the IBEX35 data is also presented. Section 3 is devoted to the modules of the application for the estimation of conditional risk measures. The tools integrated in MatRisk permit the selection of several autoregressive models: ARCH, MixARCH, GARCH, and MixGARCH.

2 Unconditional risk measures

A first module of the application allows the computation of various risk measures based on the unconditional probability distribution of historical returns. It is assumed that we have at our disposal a reconstructed series of the values of the negative

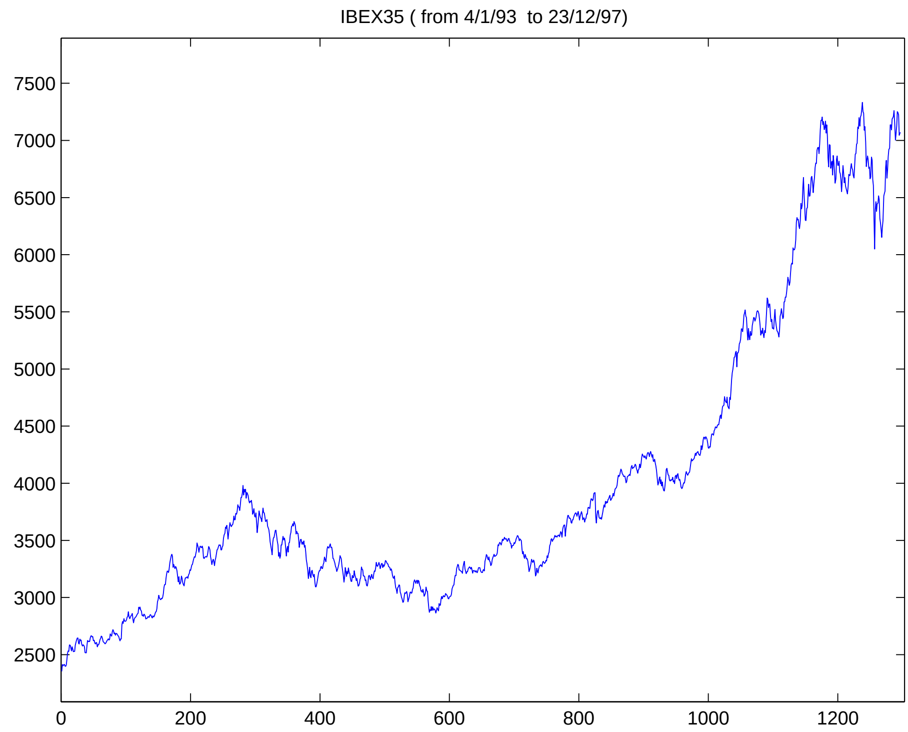

Figure 1: Daily values for the IBEX35 stock index.

of the log returns

$$r_t = -\log \frac{S_{t+1}}{S_t} \approx -\frac{S_{t+1} - S_t}{S_t}, \tag{2}$$

where S_t are the values of the hypothetical portfolio with the same constant composition as the actual portfolio. The approximation of log-returns by relative returns can only be made if the time horizon (here normalized to unity, to simplify the notation) is short. The approximation is sufficient for daily returns. The negative sign is included so that losses appear to the right-hand side of the probability distributions. As an illustration, the values of the relative daily returns for the IBEX35 are plotted in Fig. (2). A histogram displaying the number of observations in non-overlapping segments covering the whole range of observed returns gives a graphical representation of the empirical probability density distribution for the relative returns (see Fig. 3). The unconditional risk measures that can be derived from the empirical

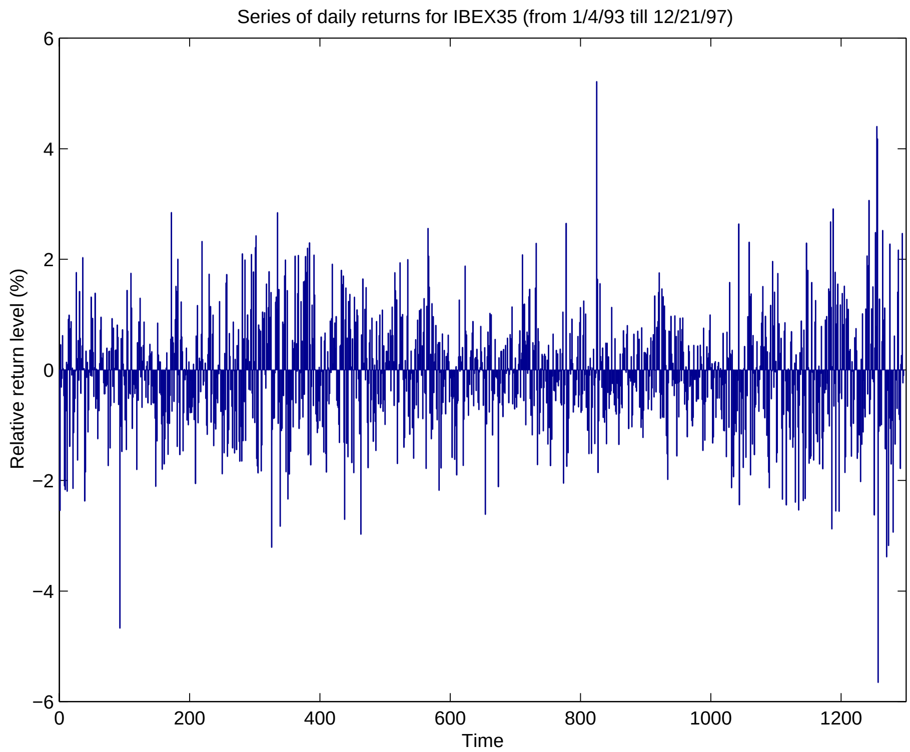

Figure 2: Daily relative returns for the IBEX35 stock index.

data with the help of the application can be categorized into three different classes: Value-at-Risk, Shortfall and MaxVaR. We now proceed to describe in detail the modules of the application for the computation of these risk measures.

2.1 Modeling the distribution: Parametric VaR

The usual risk measure, VaR (Value-at-Risk), is a percentile of the profit-loss distribution at a given probability level, P (expressed as a percentage) for a given time horizon T. Intuitively, if the time horizon is one day, this quantity represents the minimum relative loss a portfolio will have in at least P out a hundred days, assuming the composition of the portfolio remains unaltered during that period. The value of this percentile can in principle be obtained from the empirical probability density distribution function (pdf). Due to the limited amount of data available,

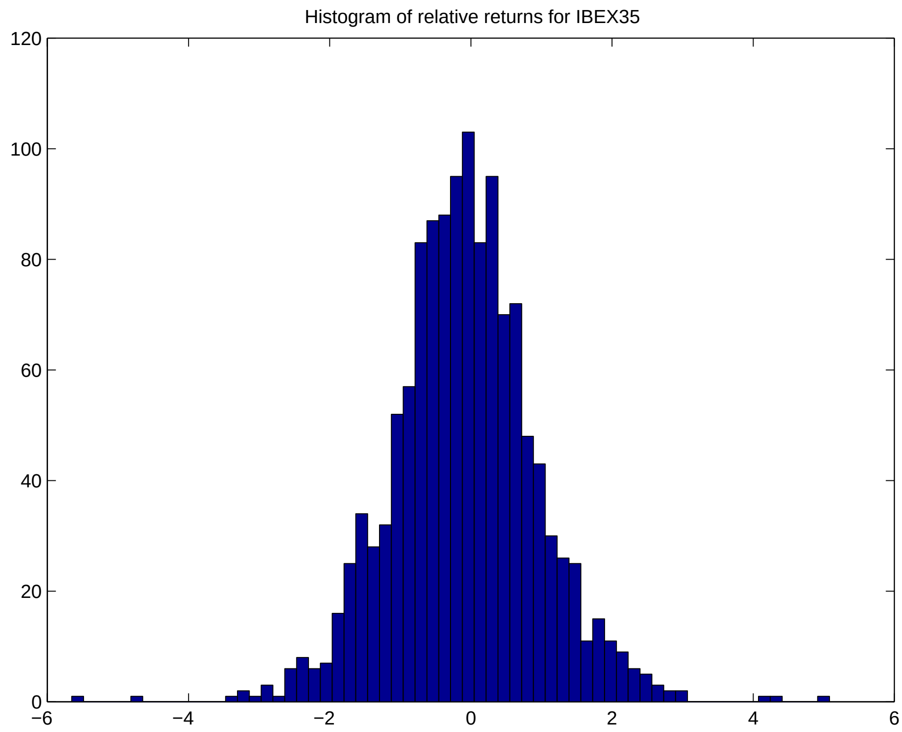

Figure 3: Daily relative returns for the IBEX35 stock index.

this measure is not very robust. In particular, the VaR estimations obtained are very sensitive to sample fluctuations. This is specially true in the tail of the distribution, in the region where extreme losses may occur. Furthermore no out-of-sample VaR estimates can be given by this procedure. This fact may become crucial for situations with small quantities of data available, such as portfolios based on new markets or new economies products.

A more robust estimate is obtained if we assume a parametric form for the profit-loss distribution and then find the parameters by likelihood maximization. The value of VaR can then be obtained as a percentile of the fitted distribution. The problem of selecting a model from the data is a difficult one, for which no general solution can be given, except if a model for the market can be explicitly formulated. In the absence of such model, one can assume different parametric forms. Various suggestions have been made in the literature. MatRisk incorporates the possibility of carrying out parametric fits with a normal distribution, mixtures of normals, and hyperbolic distributions.

2.1.1 Normal VaR

It is commonly assumed that the portfolio returns behave as an asset in the Black-Scholes model, and that the relative returns are normally distributed. Figure (4) depicts the main results of a risk analysis with a time horizon of 1 day and a probability level of 99% using MatRisk. The empirical distribution and corresponding normal fit appear on the left hand side of the application window. The results of two different tests to assess the quality of the fit (the Kolmogorov-Smirnov statistical test and the quantile-quantile plot) are presented on the right-hand side of the figure. The Kolmogorov-Smirnov statistical test [13] gives a measure of the likelihood of the hypothesis that a set of data is an empirical sampling of a given model pdf with certain values of the parameters. The test defines a sample statistic as

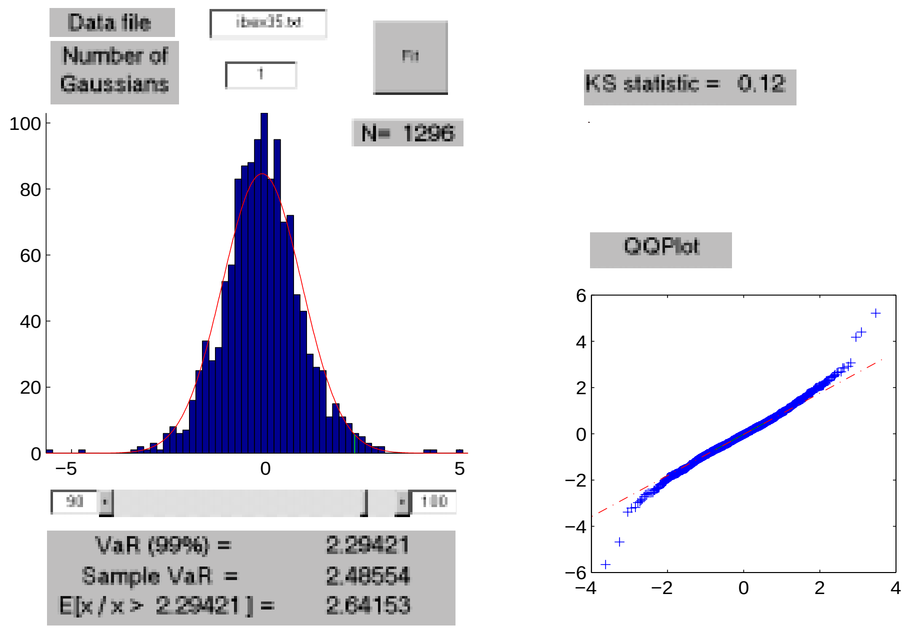

Figure 4: Risk measures for the IBEX35 using a normal model. The parameters of the fit are $\mu = -0.0901$, $\sigma = 1.0249$.

the maximum distance between the empirical and the model cumulative probability distribution. An analytic form for the probability distribution of this statistic can be given if the parameters of the model distribution are independent of the data. even though the parameters are obtained by maximum likelihood optimization from the sample itself, the application makes use of this analytical form. Nonetheless, it is expected that low values of the likelihood of the hypothesis as measured through the Kolmogorov-Smirnov statistic indicate a low quality fit.

The quantile-quantile plot gives the empirical quantiles as a function of the model quantiles. It also provides a way to assess the quality of the parametric fit by inspection. If the fit is good, the quantile-quantile plot should be a straight line with slope one. Deviations from this behavior are easy to detect by inspection in this type of plot.

Besides the value of VaR at a certain probability level P, the application also calculates the shortfall for the data

$$\text{Shortfall} \equiv E\left[X \mid X > VaR(P)\right]. \tag{3}$$

Shortfall is a measure of the average loss a portfolio will have, given that the loss is above a certain threshold. It has been proposed by several authors as an alternative measure of risk with the desired properties of subadditivity and coherence [14].

There are several indications of the failure of the normal model to reflect the distribution of extreme events. On one hand the Gaussian fit severely underestimates the magnitude of the tails. The low value of the test on the likelihood for the Kolmogorov-Smirnov (KS) statistic indicates that we should be wary of the quality of the fit obtained. The quantile-quantile plot reveals that the main discrepancies occur at the tails of the distribution. It can also be seen that the VaR level predicted by the normal fit is much lower than the VaR derived directly obtained from the empirical distribution (labeled as sample VaR in the figures).

2.1.2 VaR with mixture models

Amongst the possible generalizations of the normal model for the portfolio of returns, the Gaussian Mixture (GM) model has the appeal of greatly expanding the range of phenomena that can be accounted for within the model, while remaining close to the Gaussian paradigm. The mixture of normals model has been shown to be flexible enough to capture features that are commonly observed in actual financial time series. One particularly important example is the fact that extreme events in the financial world occur more frequently than the classical normal models would predict. This leads to distribution functions of returns on a financial asset that have tails of greater weight than those of a normal distribution. This observation has implications for the pricing of derivative products, for hedging and for the estimation of risk measures. In 1997 Hull and White [15], introduced two new Greeks, Ψ and χ, to reflect the sensibility of a portfolio composed of derivatives to the kurtosis and skewness, respectively, in the distribution of returns. They also showed how a simple model based on a mixture of normals was sufficient to capture this effect and correctly model the smile observed in the implicit volatility in, amongst others, foreign exchange and equity options. The parameters of the mixture models are estimated by maximization of a modified likelihood function with an Expectation Maximization algorithm [16].

Let us examine the IBEX35 data when the distribution of returns is modeled by a mixture of two Gaussians (Fig. 5). The fit is more plausible, as indicated by the value of KS statistic test, which is now close to one, and by the quantile-quantile plot. The values of the model and the sample VaR are now close to each other. Given that we are obtaining within-sample values of VaR, this is an indication that the model produces estimates consistent with the observations.

The mixture model is a simple yet flexible extension of the normal model that allows to model features that are observed in actual financial data. In particular, it can be used to carry out a reliable and robust risk analysis, at least up to the sample edges. Note however that the decay of the tails of a mixture distribution is still Gaussian-like. It is unclear at this point whether the actual tail decay is algebraic or exponential. Consequently, out-of-sample extrapolations made with mixture of Gaussians models should be regarded with utmost caution. One should also try to limit the number of components that enter the mixture, lest some amount of overfitting should occur. Generally two or three components provide a reasonably appropriate fit for the data.

2.1.3 VaR with hyperbolic models

Another proposal encountered in the literature to account for heavy tails is to model the portfolio returns by a hyperbolic distribution. [5, 17]. The hyperbolic family depends on four parameters

$$P_{Hyp}(x;\alpha,\beta,\delta,\mu) = \frac{\sqrt{\alpha^2-\beta^2}}{2\,\alpha\,\delta\,\mathcal{K}_1(\delta\sqrt{\alpha^2-\beta^2})} \exp\left(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\right), \quad (4)$$
$$\delta \geq 0; \quad \alpha > |\beta| \geq 0,$$

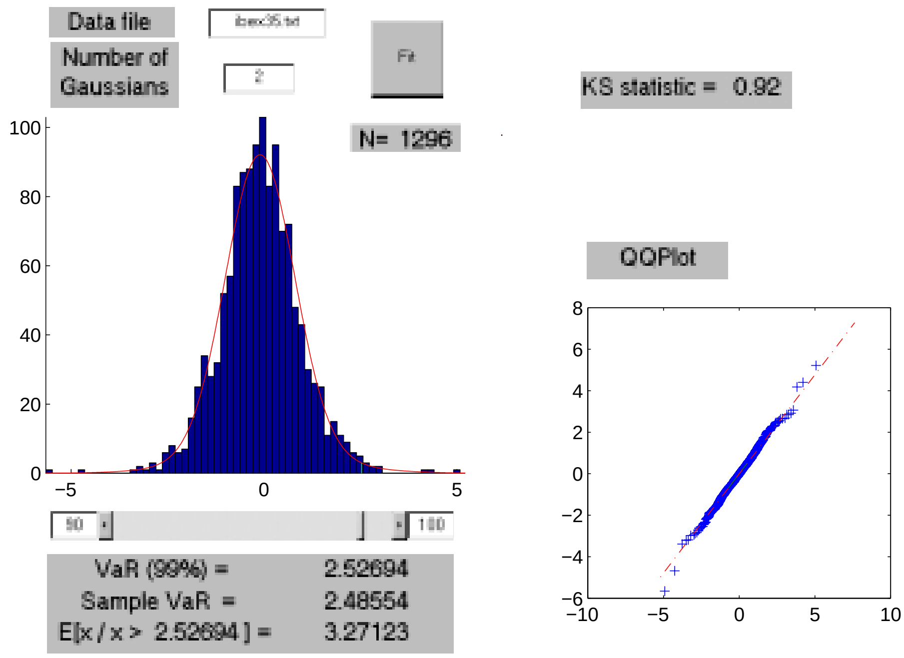

Figure 5: Risk measures for the IBEX35 using a mixture of two normal distributions. The parameters of the fit are $p_1 = 0.8988, \mu_1 = -0.1052, \sigma_1 = 0.8934$ for the first normal in the mixture and $p_2 = 0.1012, \mu_2 = 0.0438, \sigma_2 = 1.8053$ for the second normal.

where $\mathcal{K}_1(x)$ is a modified Bessel function the third kind with index 1. The quantities μ and δ are the location and scale parameters, whereas α, β determine the shape of the distribution. Several other distributions (eg. normal, symmetric and asymmetric Laplace, exponential, generalized inverse Gaussian) appear as limiting cases of this distribution. One advantage of the hyperbolic distribution is that it can be seen as a mixture of an infinite number of Gaussians where the weights are given by a generalized inverse Gaussian distribution [5]. Furthermore, the hyperbolic distribution appears as a stationary distribution of a continuous time Markov process described by a particular stochastic differential equation [5]. This observation implies that it can be used as a model to price options.

Figure 6 displays the fit obtained with IBEX35 data when the distribution of returns is modeled by a hyperbolic distribution. The resulting fit is quite reasonable, except possibly at the tails, which still decay exponentially.

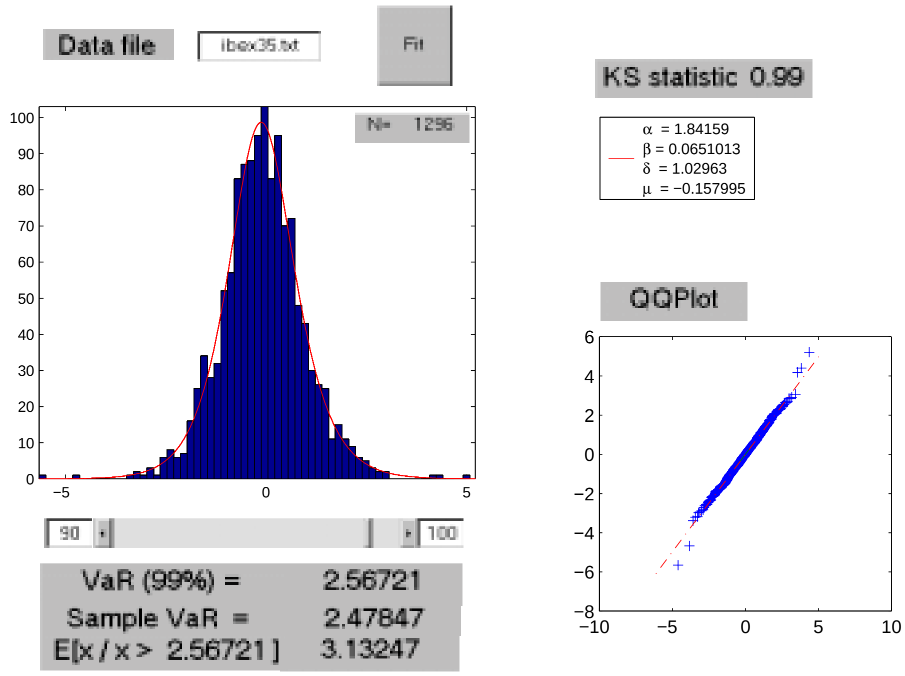

Figure 6: Risk measures for the IBEX35 using a hyperbolic distribution.

2.2 Modeling the tails: Pareto VAR and Shortfall

In the estimation of risk measures we are interested in events that occur in the tails of the distribution. In contrast to the parametric approach exposed in the previous section, where one attempts to model the full pdf of the portfolio fluctuations, one can focus on finding an appropriate model for the tail of the distribution alone [7]. Assuming that the tail has a regular behavior, the distribution should asymptotically behave like a Generalized Pareto distribution

$$P_{tail}(X) = \frac{1}{\beta}\left(1 + \frac{\xi}{\beta}(X-u)_+\right)^{\left(1+\frac{1}{\xi}\right)}, \quad X > u. \tag{5}$$

The method is known in the literature on extreme events as the Peaks Over Threshold (POT) method [7]. Note that the predicted decay is algebraic except if the tail index ξ becomes close to zero, which corresponds to exponential decay. The parameter u should be chosen sufficiently large, so that the asymptotic behavior predicted by Eq. 5 obtains, but as small as possible so that there is a sufficient amount of data on the tail from which to produce reliable estimates for the remaining parameters (ξ, β).

By default the application sets the value of u arbitrarily equal to 1. The user is given the choice of modifying the value of this threshold using as a guide the mean excess plot that appears at the right-hand corner of the window for the tail analysis. The mean excess over a threshold of a random variable is defined as the conditional expectation of that variable, provided that it exceed the selected threshold, minus the value of the threshold. For an empirical realization with N data samples, the mean excess is

$$E\left[X-\mu \mid X>\mu\right]=\frac{1}{N_{X>\mu}}\sum_{j=1}^{N}(X_j-\mu)_+, \tag{6}$$

where $N_{X>u}$ is the number of instances above the threshold in the sample. If for $X>u$ the distribution of this variable can be approximated by a Generalized Pareto (Eq. (5)), the mean excess is linear in the threshold

$$E\left[X-\mu \mid X>\mu\right]=\frac{\beta}{1-\xi}+\frac{\xi}{1-\xi}(\mu-u). \tag{7}$$

In order to select parameter u of the Generalized Pareto (GP) distribution, the user should inspect the mean excess plot and identify regions for which the plot is linear. Then the threshold u is chosen as small as possible, but within the linear region.

For the IBEX35 data, we present the results with different choices of the threshold. Figure (7) displays the results with $u=1$, corresponding to the flat region ($\xi \approx 0$, exponential decay) in the mean excess plot Even though data is scarce, it is possible to identify a separate linear region for values $X>2$. The corresponding Pareto Fit for a threshold $u=2.25$ is displayed in Fig. (8) In contrast to the previous choice of the threshold, which predicts an exponential decay, for this second choice the decay predicted is algebraic. This question is of importance shortfall becomes very sensitive to the type decay for edge-of-sample probability levels (around and above 99.9%). It is possible that the true asymptotic decay is algebraic, as predicted focusing on points beyond the threshold $u=2.25$. One could also conclude that the decay is exponential and the algebraic-like behavior observed for $u \geq 2.25$ comes from sample fluctuations. With the data available is difficult to decide which of the two hypothesis is closer to reality.

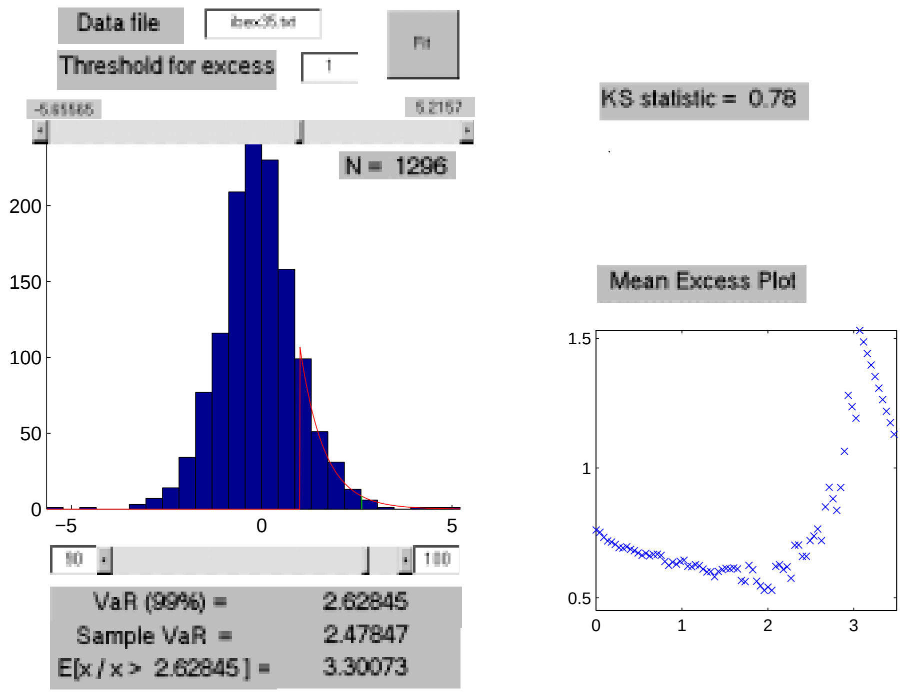

Figure 7: Risk measures for the IBEX35 data using a Generalized Pareto Fit $u = 1$. The fit parameters are $\beta = 0.6397, \xi = 0.0141$.

2.3 Modeling extreme events: MaxVaR

The evolution of the value of a portfolio in response to large fluctuations in the risk factors is one of the important elements in risk management. It is thus useful to develop models for distributions of extremal events [7]. Under certain relatively weak conditions (identical distribution of events, independence of the maxima, etc.), it can be shown that the maxima of a time series follow asymptotically a Generalized Extreme Value (GEV) distribution

$$P(x) = \frac{1}{\psi}\left(1 + \frac{\chi}{\psi}(x-\mu)\right)^{\left(1+\frac{1}{\xi}\right)} \exp\left\{-\left(1 + \frac{\chi}{\psi}(x-\mu)\right)^{\left(1+\frac{1}{\xi}\right)}\right\}. \qquad (8)$$

Incidentally, parameter ξ is the tail index introduced in the previous section, and should be similar to the one estimated from the Generalized Pareto fit. The anal-

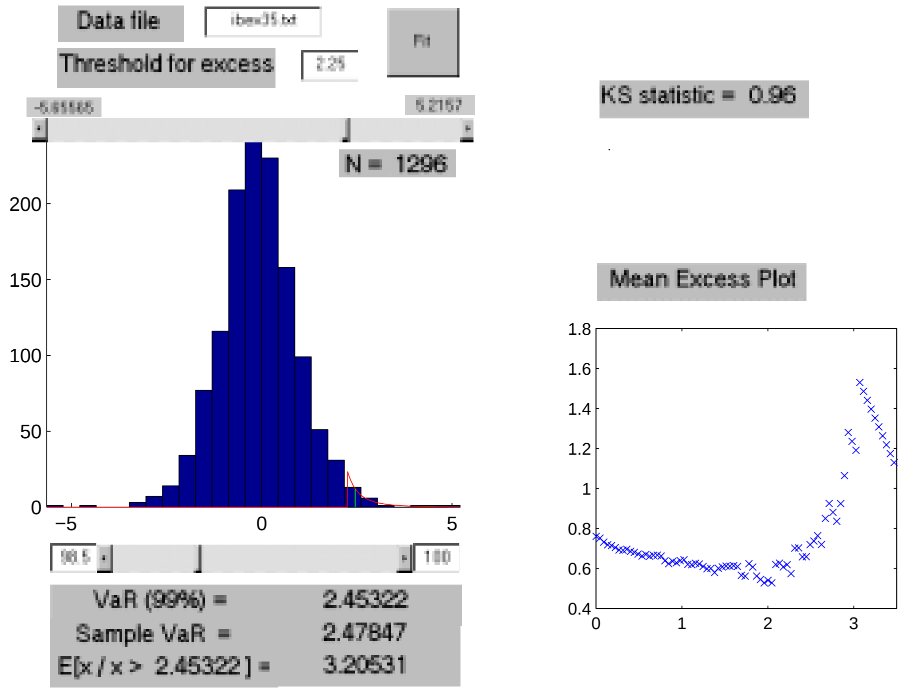

Figure 8: Risk measures for the IBEX35 data using a Generalized Pareto Fit $u = 2.25$. The fit parameters are $\beta = 0.3830, \xi = 0.3864$.

ysis proceeds as follows: We choose a time lapse, and partition the data into non-overlapping segments of the selected length. A new series is formed with the maximum values for the losses within each of the segments. The size of the boxes is chosen through a compromise between selecting sufficiently large boxes, so that the independence and asymptotic conditions are met, and having a sufficient amount of data to produce reliable estimates of the GEV parameters. The application allows the user to chose the length of the periods from which maxima are to be extracted. A minimum of 25 points is fixed by the application to estimate the GEV parameters by maximization of the likelihood function. Focusing on maxima has the advantage that these should have smaller correlations than the original returns, being further separated in time. This fact possibly means that risk measures based on the analysis of maxima are more reliable.

Let us consider the MaxVaR analysis for the IBEX35 data. We focus on weekly maxima (Fig. (9)) and maxima over periods of 2 weeks (Fig. (10)). The novel

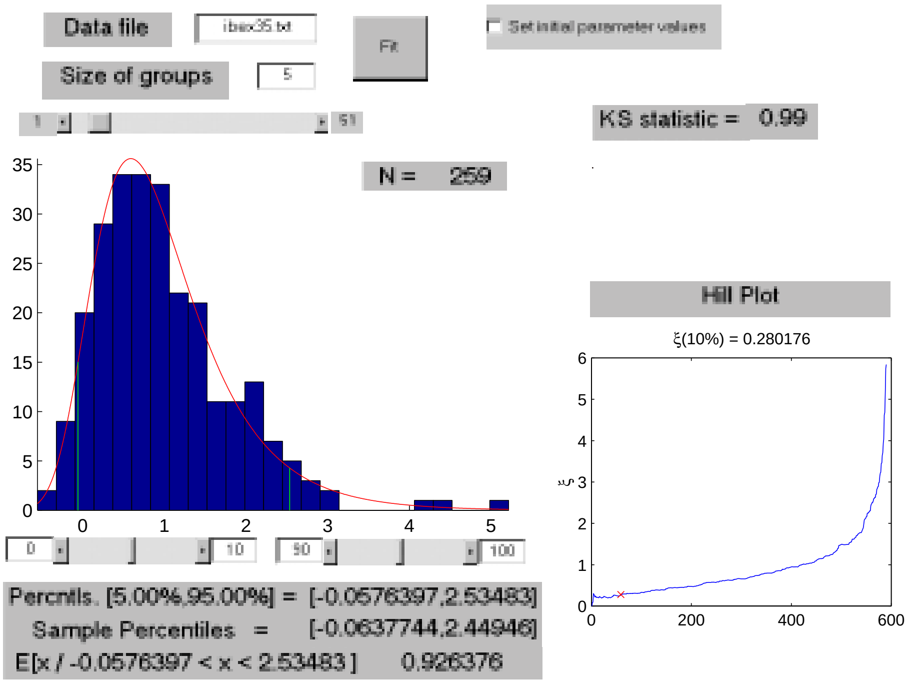

Figure 9: Risk measures for the weekly maxima of IBEX35 data. The parameters of the GEV distribution fitted are $\mu = 0.6084, \psi = 0.6179, \xi = 0.0324$.

elements in the interface are the Hill plot and the module for risk analysis based on maxima. The Hill plot permits the estimation of the parameter ξ, when it is positive. The points in the time series $\{X_1, X_2, \ldots, X_T\}$, are ordered according to their magnitude

$$X_{T;T} \leq X_{T-1;T} \leq \ldots \leq X_{2;T} \leq X_{1;T} \tag{9}$$

The Hill plot gives the value of the Hill estimator for the tail index

$$\hat{\xi}^{(H)} = \frac{1}{k} \sum_{j=1}^{k} \log \frac{X_{j;T}}{X_{k;T}} \tag{10}$$

as a function of the integer $k = 1, 2 \ldots T$. The point selected in the graphic by a cross is the 10% Hill estimator (i.e. $k = 0.1\,T$). We observe that this estimate is close to the tail index found by a fit to a GP with a threshold $u = 2.25$, which indicates an algebraic decay in the distribution tails. However, the maximum likelihood estimate

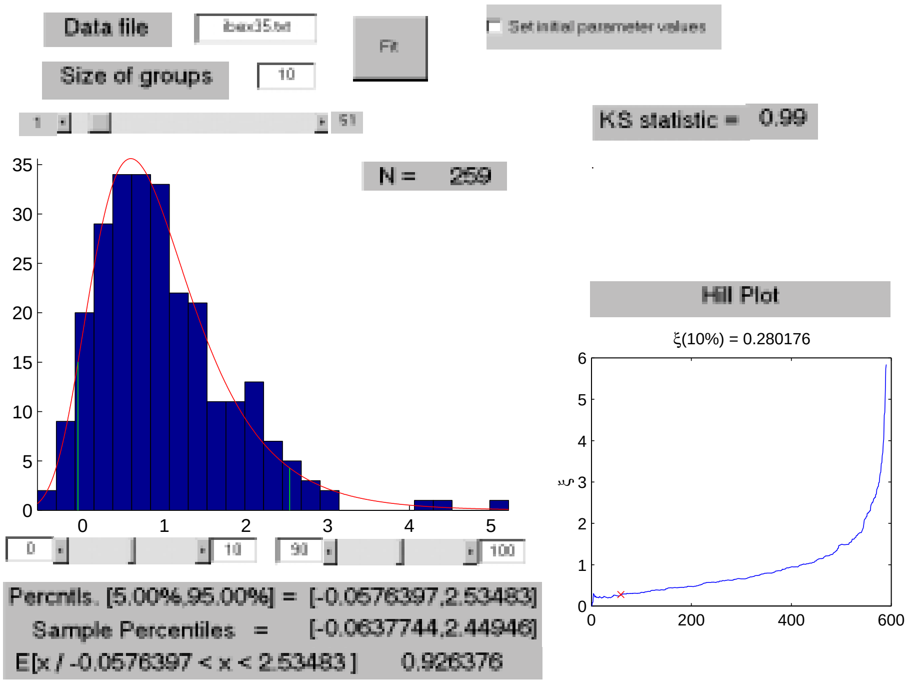

Figure 10: Risk measures for the biweekly maxima of IBEX35 data. The parameters of the GEV distribution fitted are $\mu = 0.9666, \psi = 0.5864, \xi = 0.0774$.

for the distribution of weekly and biweekly maxima predicts a tail index close to 0 (exponential decay of the tails), in agreement with the Pareto fit with a threshold $u = 1$.

The module for the risk analysis permits the selection of two probability levels $P_l < P_u$. By calculating the corresponding percentiles of the GEV distribution, one obtains a range where the corresponding maxima can be found with a certain probability $[Max_l, Max_u]$. The expected value for the maxima conditioned to the maxima being in the range derived is also reported in a text window

$$E\left[Max | Max_l \leq Max \leq Max_u\right]. \quad (11)$$

2.4 Summary of results for IBEX35

The main results obtained for the Value-at-Risk and shortfall for the IBEX35 data are summarized in tables 2.4 and 2.4. Table 2.4 displays the Value at Risk , and Table 2.4 the Shortfall for different probability levels and with different parametric fits. The rows are labeled according to the model used to compute the values: empirical distribution VaR (sample), normal (GM1), mixture models (GM2, GM3, mixtures with 2 and 3 Gaussians, respectively), hyperbolic distribution (HYP), and Generalize Pareto distributions (GP, with the chosen threshold between parentheses). The second column displays the result of the Kolmogorov Smirnov statistical test (KS test). The rest of the columns are labeled according to a probability level at which the risk measures are computed.

We observe that within-sample risk measures (95%) are fairly insensitive to the model selected. For the probability level $P = 99\%$, all parametric fits basically predict the same risk measures, except for the normal VaR, which begins to exhibit its shortcomings; namely, it severely underestimates both VaR and Shortfall. The tendency becomes more marked for higher probability levels. Beyond this probability level, data are scarce, and sample-derived measures should not be trusted.

For an edge-of-sample probability level (P = 99.9 %), and beyond sample measures (P = 99.99 %), whether the asymptotic decay is algebraic or exponential is of some consequence. Shortfall is specially sensitive to this issue. We observe that the model with a mixture of two Gaussians and the Pareto Fit with $u = 1$ predict fairly consistent risk measures. The hyperbolic model, with fast decaying exponential tails yields an excellent agreement for the body of the distribution, but clearly predicts much thinner tails than observed. Much higher values for the risk measures are predicted by the Pareto Fit with $u = 2.25$, with slow algebraic decay. The model fit with 3 Gaussians lies somewhere in-between.

3 Conditional risk measures: Autoregressive models

Classical risk measures generally focus on the unconditional distribution of portfolio returns. This approach cannot account for the autocorrelations that often appear in financial time series, or for the time-dependent structure in the volatility [18]. From the graphic presented in Fig. (11) it is apparent that extreme events (of either sign) seem to cluster in periods of high volatility. A more quantitative manner of unveiling these serial dependences is to plot the autocorrelation function of the

Table 1: Comparison between different measures of VaR

	KS test	95 %	99 %	99.9 %	99.99 %
Sample		1.58	2.49	4.40	5.22
GM1	0.12	1.60	2.29	3.36	3.72
GM2	0.92	1.54	2.53	4.25	5.63
GM3	0.99	1.58	2.46	4.41	6.86
HYP	0.99	1.59	2.57	3.91	5.30
GP ($u = 1$)	0.78	1.57	2.63	4.18	5.78
GP ($u = 2.25$)	0.76	*	2.45	4.16	8.34

Table 2: Comparison between different measures of Shortfall

	KS test	95 %	99 %	99.9 %	99.99 %
Sample		2.19	3.16	4.81	5.22
GM1	0.12	2.02	2.64	3.07	3.97
GM2	0.92	2.16	3.27	4.86	6.13
GM3	0.99	2.17	3.21	5.51	7.69
HYP	0.99	2.20	3.15	4.49	5.80
GP ($u = 1$)	0.78	2.23	3.30	4.87	6.50
GP ($u = 2.25$)	0.76	*	3.20	5.98	12.79

portfolio returns. Consider the time series,

$$X_1, X_2, \dots X_t, \dots, X_T, \tag{12}$$

which we assume to be of zero mean and stationary in the weak sense, the autocovariance can be estimated form the data through the formula

$$C(\tau) = \frac{1}{T-\tau} \sum_{t=1}^{T-\tau} X_t X_{t+\tau}. \tag{13}$$

If we normalize this expression by the variance, as estimated from the data, we obtain the function of autocorrelations

$$\rho(\tau) = \frac{\frac{1}{T-\tau}\sum_{t=1}^{T-\tau} X_t X_{t+\tau}}{\frac{1}{T}\sum_{t=1}^{T} X_t X_t}. \tag{14}$$

The autocorrelation functions of the returns and the absolute value of the returns for the IBEX35 data are plotted in Fig. (12). The two lines parallel to the time axis give the 95% band associated to sampling errors. The autocorrelation functions exhibit two of the common features of financial time series: First, the correlations between returns are short-lived. In this case, correlations disappear after approximately a day. However, there exist long-term correlations among the absolute value of the returns. Intuitively, this means that large relative changes in the value of a portfolio tend to be followed by changes which are also large, but which can be of either sign.

Conditional risk measures rely on the analysis of the conditional probability distribution of portfolio returns in an attempt to capture trends in the market volatility [19].

In the problem of estimating conditional risk measures, we consider the series of daily portfolio returns. We then make the hypothesis that the behavior of the variables that describe the financial system is determined by the values that these same variables (or a subset thereof) have taken in the recent past, and select an autoregressive model to carry out the fit [9]. Assuming that the innovations in the series exhibit some definite statistical properties and temporal structure, there are a range of technical tools available, which permit the estimation of the model parameters. Standard autoregressive models for the temporal structure of the volatility are ARCH (Autoregressive conditional heteroskedasticity [10]) and GARCH (generalized ARCH, [11]). MatRisk also incorporates the possibility of selecting mixtures of autoregressive models (MixARCH, MixGARCH). These models have been introduced by the author as an extension of the MixAR models of Zeevi et al. [12], in order to account for both the heteroskedastic structure of financial time series, and the presence of heavy tails. This generalization is a natural way of introducing in the estimation of conditional risk the mixtures of normals paradigm, which has proved quite useful in the estimation of unconditional risk measures. We now proceed to give a detailed descriptions of these autoregressive models and their application to the IBEX35 data. The interface permitting the analysis of the time series with different models is displayed in Fig. (13).

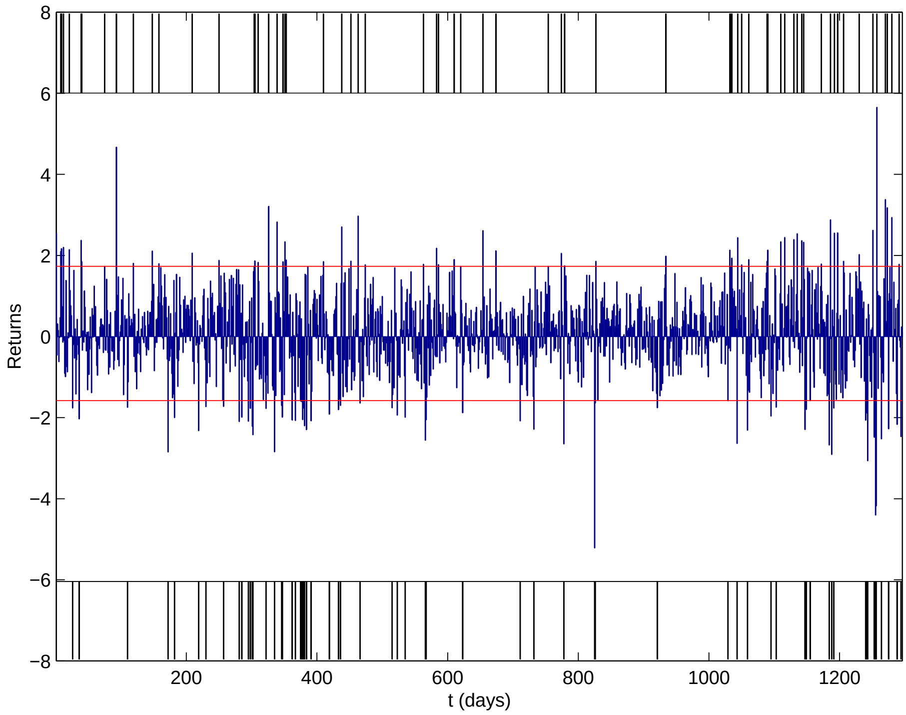

Figure 11: Unconditional Risk measures for the IBEX35. The daily VaR at a probability level of 95% corresponds to the straight line on the lower part of the plot. Days in which the loss is below this level are indicated by the bars at the bottom. A similar measure is given for the side corresponding to profits (positive part of the plot).

3.1 The ARCH model

Consider the historical series of portfolio returns

$$X_1, X_2, \dots, X_t, \dots, X_T. \tag{15}$$

Without loss of generality we can work with the zero-mean series,

$$\hat{X}_1, \hat{X}_2, \dots, \hat{X}_t, \dots, \hat{X}_T, \qquad \hat{X}_t = X_t - \langle X_t \rangle \,, \tag{16}$$

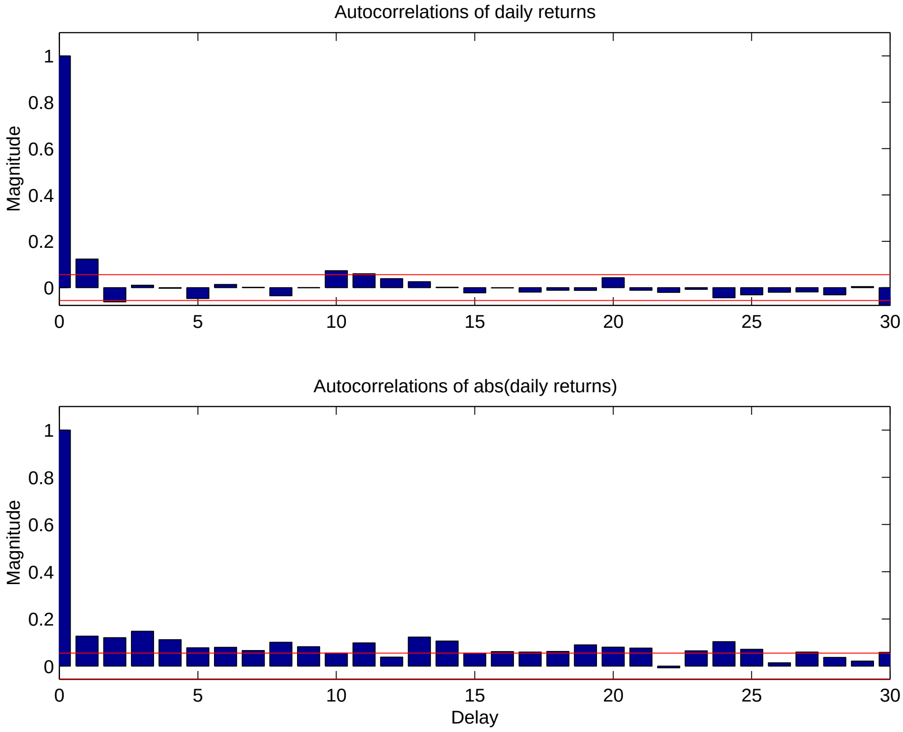

Figure 12: Normalized autocorrelations for the IBEX35 returns time series.

which is obtained by subtracting from the original return the unconditional mean, estimated from the data

$$\langle X_t \rangle = \frac{1}{T} \sum_{t=1}^{T} X_t. \tag{17}$$

We assume weak stationarity for this time-series, and posit the model

$$\hat{X}_t = \boldsymbol{\phi}^{\dagger} \cdot \hat{\boldsymbol{X}}_t^{(m)} + u_t \tag{18}$$

$$u_t = \sigma_t \ \epsilon_t \tag{19}$$

$$\sigma_t^2 = \kappa + \boldsymbol{\alpha}^{\dagger} \cdot \left[\boldsymbol{u}_t^2 \right]^{(q)}, \tag{20}$$

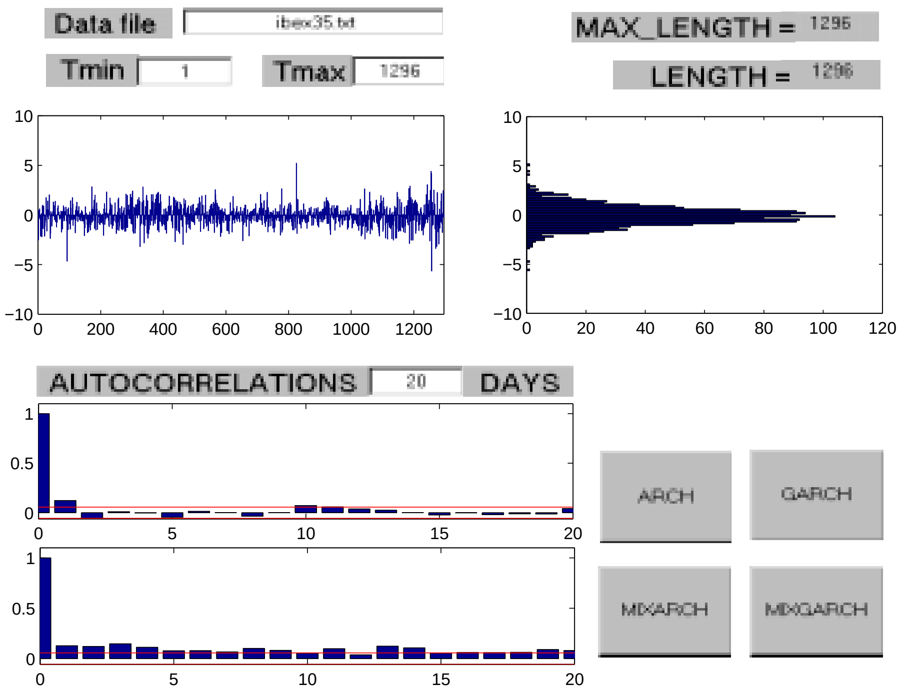

Figure 13: MatRisk Interface for time series analysis.

where ϵ_t is Gaussian white noise with zero mean and unit variance. The delay vectors for the time-series values and for the innovations are

$$\begin{aligned} \hat{\boldsymbol{X}}_t^{(m)} &= \left(\hat{X}_{t-1}\ \hat{X}_{t-2}\ \ldots\ \hat{X}_{t-m}\right) \\ \left(\left[\boldsymbol{u}_t^2\right]^{(q)}\right)^{\dagger} &= \left(u_{t-1}^2\ u_{t-2}^2\ \ldots\ u_{t-q}^2\right), \end{aligned}$$

respectively.

In summary, the portfolio returns are modeled as an autoregressive process of order m. The non-negativity condition in the ARCH(q) model for time-dependent variance is satisfied provided that the parameters satisfy the conditions

$$\begin{aligned} \kappa &> 0; \\ \alpha_i &\geq 0; \end{aligned} \tag{21}$$

With these assumptions, the requirement that the process be co-variance stationary leads to the inequality

$$\sum_{i=1}^{q} \alpha_i < 1. \tag{22}$$

In terms of these parameters, the unconditional volatility is

$$\sigma^2 = E\left[u_t^2\right] = \frac{\kappa}{1 - \sum_{i=1}^{q} \alpha_i}, \tag{23}$$

The conditional volatility, which is the relevant quantity for conditional risk measures, is

$$E\left(u_t^2 \mid u_{t-1}u_{t-2}\dots u_{t-q}\right) = \kappa + \boldsymbol{\alpha}^{\dagger} \cdot \left[\boldsymbol{u}_t^2\right]^{(q)}. \tag{24}$$

The application we present contains a module to model the time series by an ARCH model. Once the order of the model is selected by the user, the parameters are estimated by maximization of the likelihood function. Then independence hypothesis is tested by plotting the correlograms for the residuals and for the absolute value of the residuals. The normality assumption can be checked by a quantile plot of the residuals against computer-generated Gaussian random numbers. Figure (14) presents the results for the IBEX35 data, modeled by an AR(1)/ARCH(1) model. The conditional volatility is presented in Fig.(15). The resulting ARCH process is

$$\hat{X}_t = 0.1129\hat{X}_{t-1} + \sigma_t \epsilon_t \tag{25}$$

$$\sigma_t^2 = 0.9097 + 0.1118(\hat{X}_{t-1} - 0.1129\hat{X}_{t-2})^2. \tag{26}$$

Results for the IBEX35 data show that the ARCH(1) model is insufficient to account for either correlations or for heavy tails. The correlation functions displayed in Fig. (14) show that, in spite of the fact that the one-day correlations are negligible, the magnitude of the correlations with longer delays is only slightly lowered. It is clear that a more sophisticated model (eg. a GARCH process) is needed to account for the correlation structure of the time series. There is another deficiency in the fit obtained. The normality hypothesis for the distribution of residuals is not fulfilled, as shown by the quantile-quantile plot.

3.2 The MixARCH model

In this section we consider the natural extension of the mixture model introduced for unconditional distributions in order to account for the heteroskedastic structure

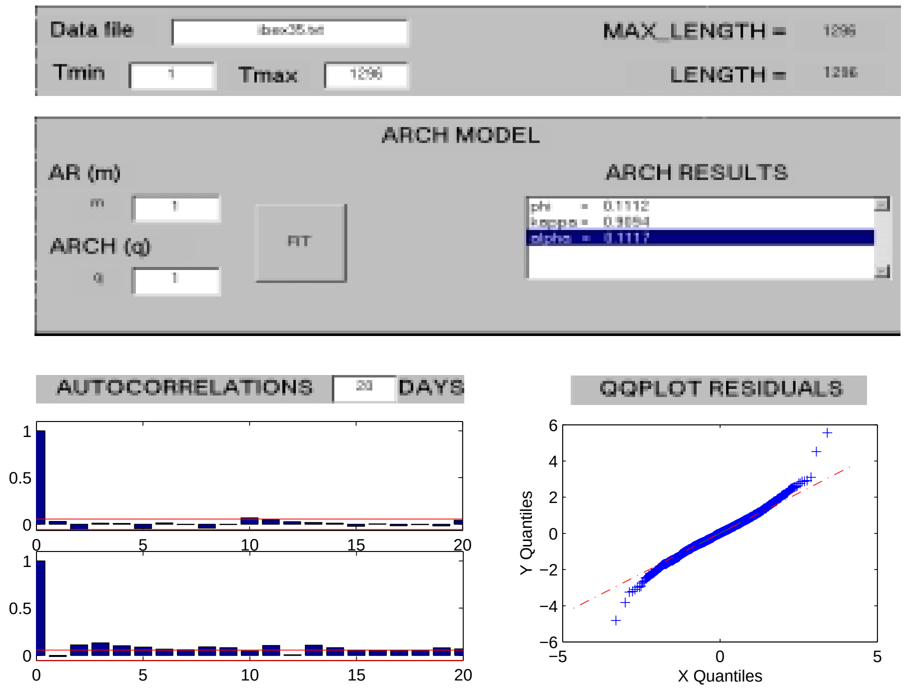

Figure 14: ARCH model for the IBEX35 time series.

of a time series. Our final goal is to use this model to carry out a estimation of conditional risk measures for financial time series.

In a MixARCH model we assume that the heteroskedastic time series is generated from a probabilistic mixture of AR(m) models

$$\hat{X}_t = \boldsymbol{\phi}^{\dagger}_{[i]} \cdot \hat{\mathbf{X}}^{(m)}_t + u_{[i]}(t), \qquad \text{with probability } g_{[i]}(\mathbf{X}^{(d)}_t, \boldsymbol{\theta}^{\dagger}_{[i]}), \quad i = 1, 2, \ldots, J. \quad (27)$$

The probabilities are given in terms of sigmoidal functions depending on the vector of delays

$$g_{[i]}(\mathbf{X}^{(d)}_t, \boldsymbol{\theta}^{\dagger}_{[i]}) = \frac{\exp\left\{c_{[i]}\left(\hat{X}_{t-1} - (b_{[i]} + \mathbf{a}^{\dagger}_{[i]} \cdot \hat{\mathbf{X}}^{(d-1)}_{t-1})\right)\right\}}{\sum_{j=1}^{J} \exp\left\{c_{[j]}\left(\hat{X}_{t-1} - (b_{[j]} + \mathbf{a}^{\dagger}_{[j]} \cdot \hat{\mathbf{X}}^{(d-1)}_{t-1})\right)\right\}}. \quad (28)$$

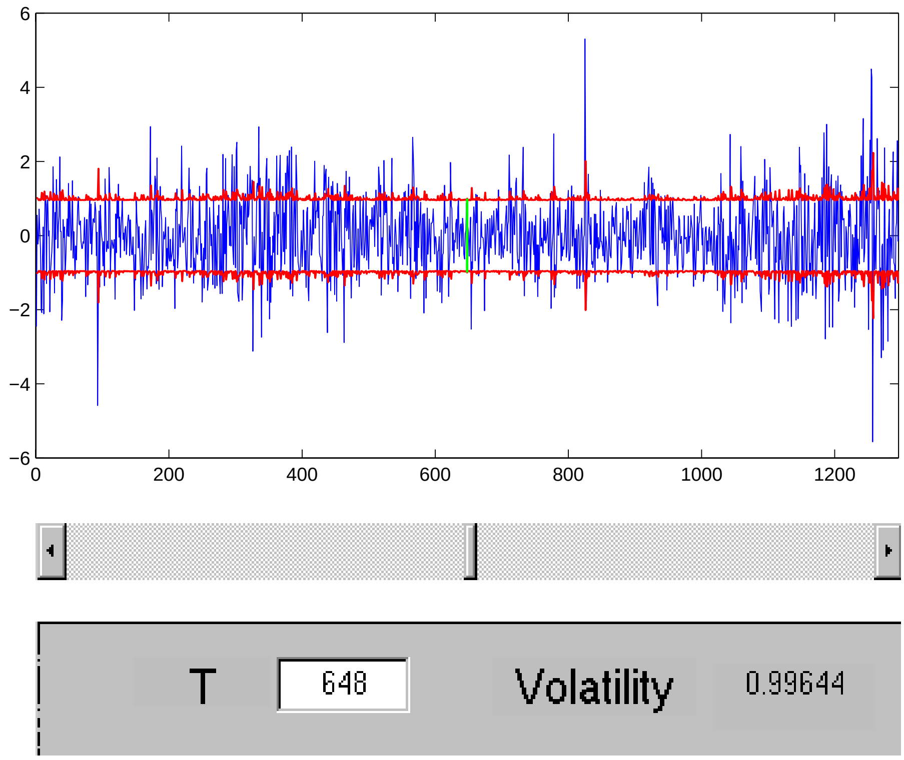

Figure 15: Conditional volatility of the ARCH model for the IBEX35 time series. The bars correspond to the residuals of the ARCH process. The continuous lines delimit the one σ_t band.

The model for the innovations is

$$u_{[i]}(t) = \sigma_{[i]}(t)\epsilon_t. \tag{29}$$

The quantities $\{\epsilon_t; t = 0, 1, 2, \ldots\}$ are assumed to be identically distributed independent random variables drawn from a time-independent normal distribution with zero mean and unit variance.

The variance of the process σ_t^2 is assumed to follow the model

$$\sigma^2_{[i]}(t) = \kappa_{[i]} + \boldsymbol{\alpha}^{\dagger}_{[i]} \cdot [\mathbf{u}^2_{[i]}]^{(q)}(t) \tag{30}$$

for each of the different components in the mixture. The procedure specified is a natural way of introducing the use mixtures in the estimation of conditional volatilities.

The determination of the parameters of the MixARCH model is made through the maximization of the likelihood function

$$\mathcal{L}(\boldsymbol{\phi}, \boldsymbol{\alpha}, \boldsymbol{\theta}; \{X_t\}_{t_0}^{T}) = \prod_{t=t_0}^{T} \sum_{i=1}^{m} g_{[i]}(\mathbf{X}_t^{(d)}, \boldsymbol{\theta}_{[i]}^{\dagger}) \frac{1}{\sqrt{2\pi\sigma_{[i]}^2(t)}} \exp\{-\frac{\left(\hat{X}_t - \boldsymbol{\phi}_{[i]}^{\dagger} \cdot \hat{\mathbf{X}}_t^{(m)}\right)^2}{2\sigma_{[i]}^2(t)}\}.$$

The optimization problem can be solved by the Expectation Maximization algorithm, or, as done in MatRisk, by a constrained optimization algorithm.

At this point, the application MatRisk only allows for an analysis with a probabilistic mixture of two AR(1)/ARCH(1) models. In Fig. (16) we can examine the results for the IBEX35 dataset. It can be seen that the correlation structure for the residuals is not much improved by the use of a mixture model. However, the quantile-quantile plot shows that the MixARCH process yields a better approximation for the probability of extreme events occurring in the tails of the distribution. The fit parameters are

$$\begin{aligned}
\text{Model 1} \quad \hat{X}_t &= 0.0559\hat{X}_{t-1} + \sigma_t\epsilon_t && (31)\\
\sigma_t^2 &= 2.2194 + 0.1976(\hat{X}_{t-1} - 0.0559\hat{X}_{t-2})^2 && (32)\\
\text{Model 2} \quad \hat{X}_t &= 0.1380\hat{X}_{t-1} + \sigma_t\epsilon_t && (33)\\
\sigma_t^2 &= 0.6820 + 0.0381(\hat{X}_{t-1} - 0.1380\hat{X}_{t-2})^2 && (34)
\end{aligned}$$

The probabilities for the mixture are

$$g_{[1]}(X_{t-1}) = \frac{1}{1 + e^{-0.6839(\hat{X}_{t-1} - 2.5155)}}; \qquad g_{[2]}(X_{t-1}) = 1 - g_{[1]}(X_{t-1}) \qquad (35)$$

3.3 The GARCH model

The GARCH(p,q) model for a time series has the following structure

$$\begin{aligned}
\hat{X}_t &= \boldsymbol{\phi}^{\dagger} \cdot \hat{\boldsymbol{X}}_t^{(m)} + u_t && (36)\\
u_t &= \sigma_t \; \epsilon_t && (37)\\
\sigma_t^2 &= \kappa + \boldsymbol{\alpha}^{\dagger} \cdot \left[\boldsymbol{u}_t^2\right]^{(q)} + \boldsymbol{\beta}^{\dagger} \cdot \left[\boldsymbol{\sigma}_t^2\right]^{(p)}, && (38)
\end{aligned}$$

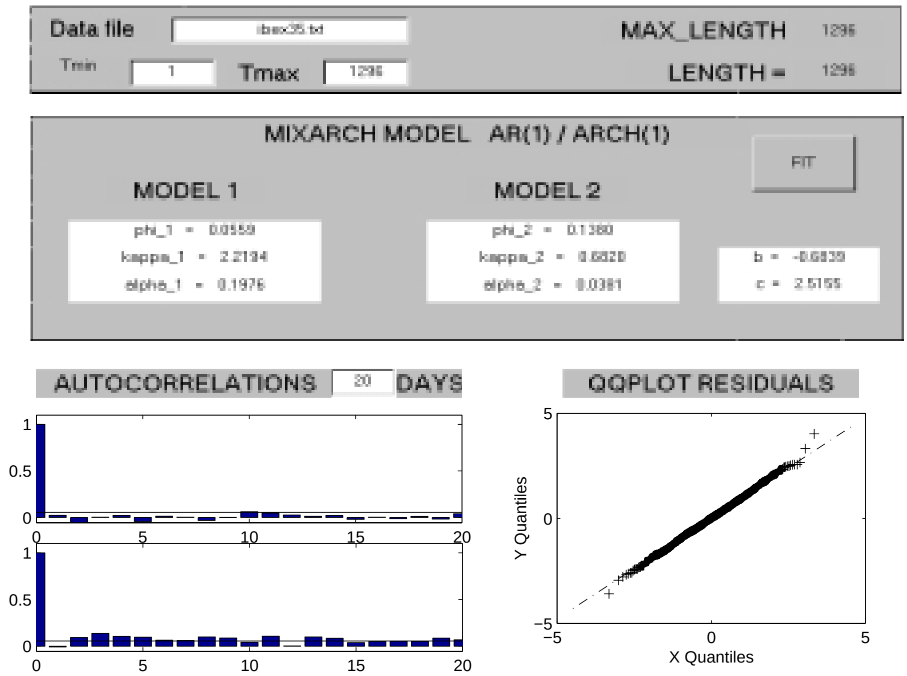

Figure 16: MixARCH model for the IBEX35 time series.

where ϵ_t is Gaussian white noise with a standard deviation equal to 1, the delay vectors are defined in Eq. (21), except that for the variance, which is

$$\left(\left[\boldsymbol{\sigma}_t^2\right]^{(p)}\right)^\dagger = \left(\sigma_{t-1}^2 \; \sigma_{t-2}^2 \; \ldots \; \sigma_{t-p}^2\right). \tag{39}$$

The conditions specified in Eq. (21) should be complemented with the requirement that all components of $\boldsymbol{\beta}$ be non-negative. The requirement of covariance-stationarity is now

$$\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i < 1. \tag{40}$$

In terms of these parameters, the unconditional volatility is

$$\sigma^2 = E\left[u_t^2\right] = \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{i=1}^{p} \beta_i}, \tag{41}$$

The conditional volatility, which is the relevant quantity for conditional risk measures, is

$$E\left(u_t^2 \mid X_{t-1}X_{t-2}\ldots\right) = \kappa + \boldsymbol{\alpha}^\dagger \cdot \left[\boldsymbol{u}_t^2\right]^{(q)} + \boldsymbol{\beta}^\dagger \cdot \left[\boldsymbol{\sigma}_t^2\right]^{(p)}. \tag{42}$$

The interface for the GARCH fit is similar to the one described for the ARCH model. Figure (17) displays the results for the IBEX35 data with a AR(1) / GARCH(1,1) model. The conditional volatility is displayed in Fig.(18). The resulting process is

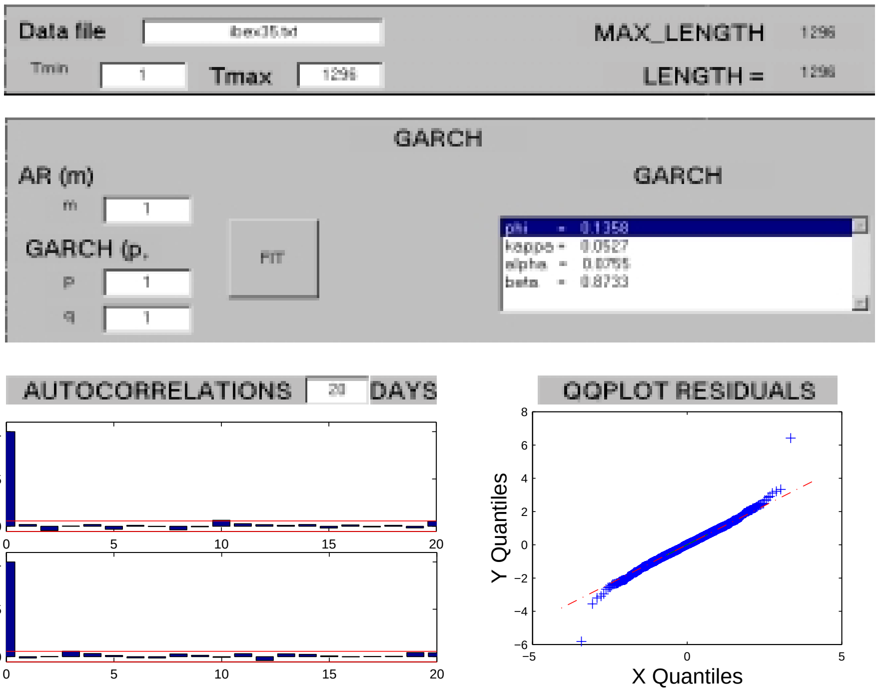

Figure 17: GARCH model for the IBEX35 time series.

$$\hat{X}_t = 0.1358\hat{X}_{t-1} + \sigma_t\epsilon_t \tag{43}$$

$$\sigma_t^2 = 0.0527 + 0.0755(\hat{X}_{t-1} - 0.1358\hat{X}_{t-2})^2 + 0.8733\sigma_{t-1}^2. \tag{44}$$

Note that in this model the correlations between the residuals are negligible. The normality hypothesis is less convincingly supported by the quantile plot.

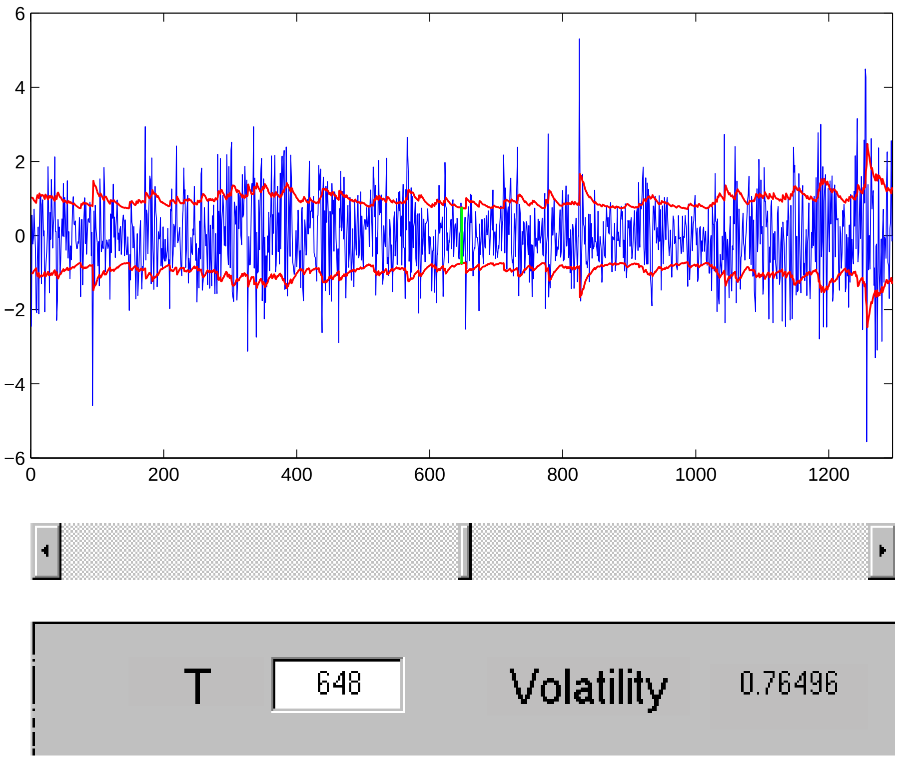

Figure 18: Conditional volatility of the GARCH model for the IBEX35 time series. The bars correspond to the residuals of the ARCH process. The continuous lines delimit the one σ_t band.

3.4 The MixGARCH model

A MixGARCH model is probabilistic mixture of GARCH processes. The formulas for MixARCH can be easily extended to include terms dependent on the delayed average volatilities for the prediction of the actual conditional volatility.

The analysis with MatRisk of the IBEX35 data with a probabilistic mixture of two AR(1)/GARCH(1,1) models is presented in Fig. (19). The resulting model is

$$\text{Model 1} \quad \hat{X}_t = 0.1678\hat{X}_{t-1} + \sigma_t \epsilon_t \tag{45}$$

$$\sigma_t^2 = 1.5041 + 0.0(\hat{X}_{t-1} - 0.1678\hat{X}_{t-2})^2 + 0.0228\sigma_{t-1}^2 \tag{46}$$

$$\text{Model 2} \quad \hat{X}_t = 0.1313\hat{X}_{t-1} + \sigma_t \epsilon_t \tag{47}$$

$$\sigma_t^2 = 0.0084 + 0.0978(\hat{X}_{t-1} - 0.1313\hat{X}_{t-2})^2 + 0.8794\sigma_{t-1}^2 \quad (48)$$

The probabilities for the mixture are

$$g_{[1]}(X_{t-1}) = \frac{1}{1 + e^{0.3054(\hat{X}_{t-1}+4.8489)}}; \qquad g_{[2]}(X_{t-1}) = 1 - g_{[1]}(X_{t-1}) \quad (49)$$

The MixGARCH model seems to be able to account well for both the correlations

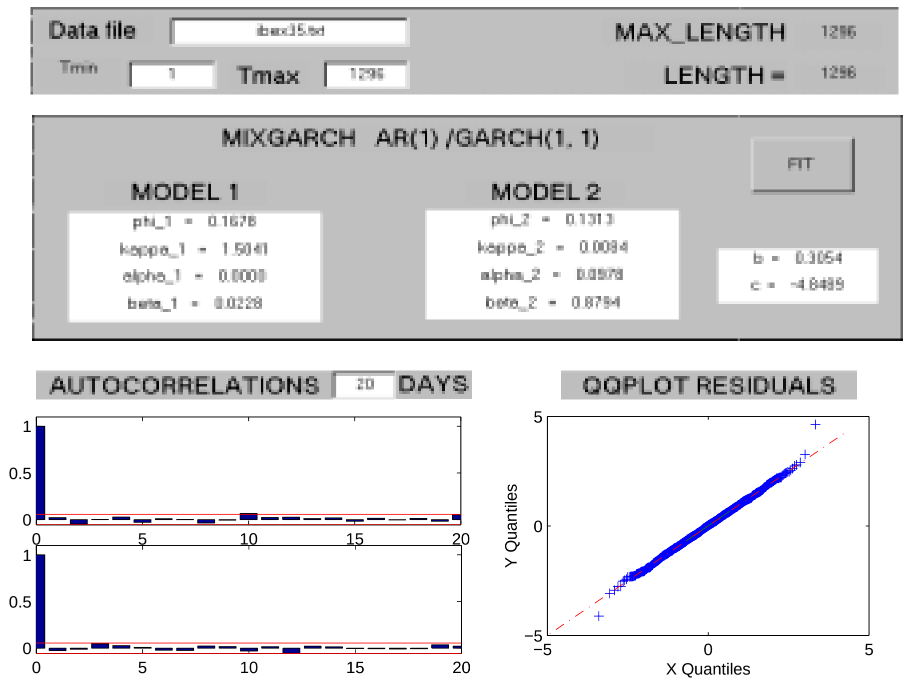

Figure 19: MixGARCH model for the IBEX35 time series.

and for heavy tails in the data.

4 Summary.

There exist two different but complementary approaches to the analysis of the risk of a portfolio. A risk manager may take a long-term view and proceed to estimate the probability of occurrence of an adverse extreme, irrespective of the recent history of the behavior of the portfolio. Alternatively, if one is interested in a more local measure of risk, one's attention should veer to quantities related to the conditional volatility. These conditional risk measures constitute an attempt to uncover the risk structure of the portfolio using information about recent fluctuations in its value.

Both types of analysis can be carried out with the help of MatRisk, the application presented in this work. For unconditional risk analysis the application integrates the following tools

- Parametric fit with a choice of various distributions (normal distribution, mixture of normals, hyperbolic distribution).
- Pareto fit to the tails of the distribution.
- Generalized Extreme Value Fit, for the maximal losses within a given period.

Besides traditional risk measures, such as VaR, novel risk measures such as Shortfall and MaxVaR can also be calculated. This permits an integral view of the risk structure of the portfolio and can be a helpful tool to adopt risk management strategies from a more informed perspective.

For conditional risk measures, besides the classical linear autoregressive models for the analysis of time series (ARCH, GARCH), MatRISK integrates a limited vocabulary of mixture models. These models are an attempt to analyze the implications of deviations from normality in the estimation of conditional risk measures is still reduced.

5 Acknowledgments

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