A Dynamic Non-Tâtonnement Macroeconomic Model with Stochastic Rationing*

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Abstract

In this paper we present a dynamic macroeconomic model with stochastic quantity rationing. Trades take place in each period even when prices are not at their Walrasian level. Moving from one period to the next, prices and wages are adjusted according to the intensity of rationing, a reliable measure of which is obtained by means of stochastic rationing. A complete characterization of the typology of equilibria is given and dynamic adjustment equations are derived. From this it is evident that structural parameters such as the adjustment speed of prices, as well as government policy parameters, are decisive for the type of dynamics that emerges. In particular there is a tendency for nominal wage stickiness to stabilise the economy whereas high wage flexibility favours cyclical and irregular behaviour.

JEL classification: D45, D50, E32, E37

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1. Introduction

Starting from the mid seventies the unemployment rate in the European Community experienced a substantial increase. Unemployment rates over 10% of the labor force since 1983 have been the rule in many countries even up to now. These high levels of unemployment are posing problems in almost all the countries of the European Union. Moreover unemployment is only one of several disequilibrium phenomena that seem to characterize many advanced economies for relevant periods of time. Capacity underutilization for ten years before the second world war and high inflation levels in the eighties are other examples.[1]

In spite of the importance of these issues, economic theory has struggled in providing satisfactory theoretical accounts. Many of the existing dynamic macroeconomic models, even though giving rise to valuable insights, show a number of weaknesses with respect to these facts. Models belonging to the competitive business cycle approach (e.g. Lucas-Prescott [1974], Lucas [1975]) study intertemporal allocation problems in a market clearing context assuming a representative agent maximizing his utility over an infinite horizon. Their structure impedes being able to account for prices which are not at their Walrasian level in each period. On the other hand dynamic models based on Keynesian-type assumptions (e.g. Kaldor [1940], Dana-Malgrange [1984]) may generate rich and complex dynamic behavior, but usually they are not built on explicit micro foundations.

In the present paper we suggest an overlapping generations macroeconomic model which aims to avoid the above drawbacks. Its distinctive feature is that, although trades take place in every period, prices may not be at their Walrasian level. This non-tâtonnement approach makes it possible to investigate disequilibrium phenomena like underemployment, inflation, excess productive capacities and, moreover, to provide a framework for the study of the effectiveness of economic policies. More precisely, disequilibrium situations may arise because the adjustment of prices to market imbalances is not instantaneous but proceeds with finite speed only; thus their functioning as an allocation device is imperfect, though not nil. As a consequence, quantity adjustments have to take place which complement prices in their task of making trades feasible. This appears to reflect well the effective functioning of many markets. Labor markets, at least in Europe, are the foremost example, but also product markets often show a high degree of price inertia and stickiness (for empirical accounts see Carlton [1986] and Bean [1994])[2].

[1] For evidence of some of these facts see e.g. Drèze [1991], Figure 1, page 199.

[2] On theoretical grounds, price stickiness has been explained by (menu) costs of adjustments of prices (Akerlof and Yellen [1985]), strategic interactions among oligopolists and coordination failure among monopolistic competitors (Ball and Romer [1991]), by certain characteristics of the

The topic of this paper has been on the agenda of the fixed-price literature as developed in the seventies (Bénassy [1975], Drèze [1975], Malinvaud [1977]), too. However, while it is true that that literature has obtained valuable results like the distinction of Keynesian from classical unemployment and the importance of feedback effects of rationing constraints between different markets, it has not succeeded in producing a convincing modelling of a dynamic economy with Keynesian features. Its major drawback is that it has not been able to extend its analysis in a convincing way beyond the short-run static set-up. That would have required price adjustment but, as we shall argue, this is not possible to do satisfactorily in the standard fixed-price approach. In contrast, we do offer a dynamic analysis, building on the fact that, in many markets, quantity adjustments occur faster than price adjustments (Greenwald and Stiglitz [1989]). To account formally for this asymmetry, and be able to work out its consequences most clearly, it is convenient to assume that, while quantities adjust within any period so as to produce a feasible allocation, prices are fixed during each period but are adjusted when the economy moves from one period to the next.

Although the idea just outlined is neither new nor complicated, it gives rise to two problems: how to find a consistent allocation when prices are not at their market clearing levels and how to define a reasonable mechanism for the adjustment of prices. Regarding the first, it can be solved using the concept of temporary equilibrium with quantity rationing, developed for the case of deterministic rationing mainly by Drèze [1975] and Bénassy [1975].

As far as the adjustment of prices is concerned, a natural idea is to relate it to the size of the dissatisfaction of agents with their (foregone) trades. At first sight it might appear that a measure for this could easily be obtained from a comparison of the desired trades in presence of rationing constraints, called effective demands, with actual trades. In fact, Bénassy [1976, 1978] and Laroque [1981], among others, have studied models in which the value of the aggregate excess demand, arising from Benassy's concept of effective demand, is assumed as such a measure.[3] But, as has been pointed out by Grandmont [1977], Green [1980]

production function and of the demand function, by imperfect and asymmetric information and by risk aversion of order one (Weinrich [1997]). Regarding wage rigidity, it has been derived for instance from insider-outsider arguments (Shaked and Sutton [1984]), fairness (Hahn and Solow [1995, ch. 5]), efficiency wages (Salop [1979], Solow [1979], Shapiro and Stiglitz [1984], Weiss [1991]) and uncertainty combined with imperfectly competitive markets (Holmes and Hutton [1996]). See also the recent contribution by Bewley [2000] who found by means of interviews of a large number of businessmen that in recessions for firms it is preferable to lay off workers rather than to lower wages.

[3] Bénassy's concept of effective demand is based on the dual decision hypothesis (Clower [1965]) and assumes that, in determining his effective demand for any good, the agent takes into account all rationing constraints for all goods except that for the very good he is formulating

and Svensson [1980], among others, this measure can hardly be considered reliable. In fact, their criticism applies to any measure derived from effective demands based on deterministic rationing because in that case only non manipulable rationing is compatible with the existence of an equilibrium with rationing. But then agents do not have an incentive to express demands that exceed their expected transactions (which in equilibrium equal actual trades). Stochastic rationing, on the other hand, reconciles manipulable rationing with the existence of equilibrium (Weinrich [1984]) and thus provides an incentive for rationed agents to express demands that exceed trades (see also Gale [1979, 1981] , Honkapohja and Ito [1985] and Weinrich [1988]).[4]

The structure of the model then is the following. In any period t prices and wages are fixed. An allocation is obtained by means of an equilibrium with stochastic manipulable rationing and exchange takes place. Then the economy moves on to period $t+1$, prices are adjusted according to the excesses of effective demands and supplies over actual trades in period t, prices remain fixed during period $t+1$, an allocation takes place and so on. In this way, starting from any initial situation, we obtain a sequence of allocations and prices which represents the dynamic behavior of the economy. Such a sequence may converge, diverge, cycle or produce chaotic attractors. Moreover, even when it converges, the resulting limit allocation is not necessarily a Walrasian steady state but may be a quasi-stationary state with permanent rationing (underemployment or capacity underutilization) and nominal price adjustment (but constant relative prices).

The remainder of the paper is organized as follows. In section two we present the model and derive the behavior of consumers, producers and government under stochastic manipulable rationing. Section three introduces temporary equilibrium with rationing and the different equilibrium regimes (Keynesian and classical unemployment, inflation and underconsumption) whereas in section four we derive the partitioning of the price and wage space according to the type of equilibrium with rationing that any price-wage combination gives rise to. This prepares the field for the dynamic analysis to which we turn in section five and where we define

the demand for. This means that, if there are n goods, he has to solve n optimization problems.

[4]More precisely, manipulable and deterministic rationing is not compatible with the existence of an equilibrium because, in presence of manipulable rationing constraints, all agents have an incentive to exaggerate their intended transactions in such a way so as to realize exactly their desired transactions. But since this is impossible, rationing has to become more and more restrictive and transaction proposals grow unbounded. This is not the case with stochastic rationing because, due to uncertainty, an agent who makes too large a transaction offer cannot be sure that he will not be "taken by his word" and required to realize the demand or supply that he expressed. But this may be extremely unpleasant which is why, with stochastic rationing, transaction offers remain bounded. This point will be taken up again and illustrated in section 3.

the respective adjustment equations. In section six we present simulation results and section seven contains concluding remarks. Four appendices contain proofs of the more technical results and the equations of the dynamic system on which the simulation program is based.

2. The Model

We consider an economy in which there are both a private sector (composed of consumers and firms) and a public sector (government). The consumption sector has an overlapping generations structure in which there are $2n$ identical agents living for two periods (n young and n old) who offer labor when young and consume the produced good in both periods. The production sector is represented by n' identical firms, characterized by an atemporal production function whose only input is labor. The government levies a proportional tax on firms' profits to finance its expenditure for goods. Nevertheless, budget deficits and surpluses may arise and are made possible by means of money creation or destruction.

2.1. *Timing of the Model*

The time structure of the model is depicted in Figure 2.1. In period $t-1$ producers obtain an aggregate profit of Π_{t-1}, distributed at the beginning of period t in part as tax to the government ($tax\Pi_{t-1}$) and in part to young consumers ($(1-tax)\,\Pi_{t-1}$), where $0 \leq tax \leq 1$. Also at the beginning of period t old consumers hold a quantity of money M_t, consisting of savings generated in period $t-1$. In other words, young households use money as a means of transfer of purchasing power between periods. Since consumption choices of young consumers may be rationed, they are possibly forced to hold unwanted savings in the form of money.

Let X_t denote the quantity of the good purchased by young consumers in period t, p_t its price, w_t the nominal wage and L_t the quantity of labor. Then

$$M_{t+1} = (1-tax)\,\Pi_{t-1} + w_t L_t - p_t X_t.$$

Denoting with G the quantity of goods purchased by the government and taking into account that old households want to consume all their money holdings in period t, the aggregate consumption of young and old households and the government is $Y_t = X_t + \frac{M_t}{p_t} + G$. Using that $\Pi_t = p_t Y_t - w_t L_t$, considering $\Pi_t - \Pi_{t-1} = \Delta M_t^P$ as the variation in the money stock held by producers and denoting with ΔM_t^C the one referring to consumers, the following standard accounting identity obtains:

$$\Delta M_t^C + \Delta M_t^P = (M_{t+1} - M_t) + (\Pi_t - \Pi_{t-1}) = p_t G - tax\Pi_{t-1} = \text{ budget deficit}$$

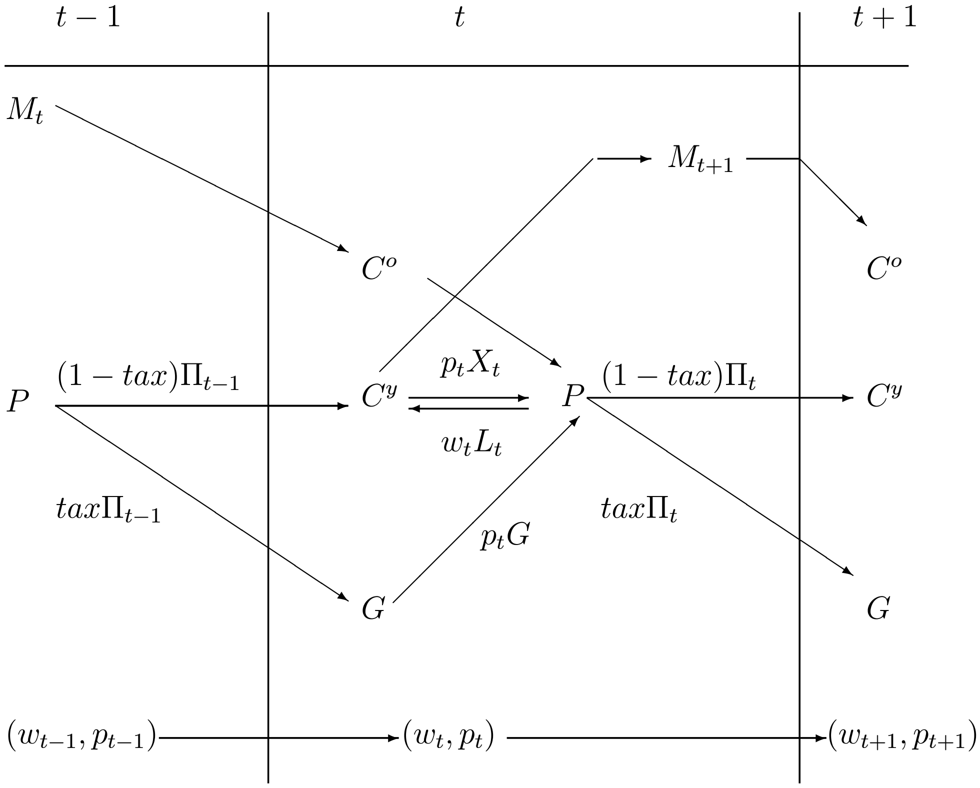

Figure 2.1: The time structure of the model

Setting $m_t = M_t/p_t$, $\theta_t = p_{t+1}/p_t$ and $\pi_t = \Pi_{t-1}/p_t$, we obtain the following dynamic equation in the real money stock and real profits:

$$m_{t+1} = \frac{1}{\theta_t}\left[m_t + G + (1 - tax)\,\pi_t\right] - \pi_{t+1.} \tag{2.1}$$

2.2. *The Consumption Sector*

In his first period of life each consumer born at t is endowed with labor ℓ^s and the amount of money $(1 - tax)\,\Pi_{t-1}/n$. His utility arises from consumption in both periods and is given by $u\left(x_t, x_{t+1}\right) = x_t^h x_{t+1}^{1-h}, 0 < h < 1$. In taking any decision the young household has to meet the following constraints:

$$0 \le x_t \le \omega_t^i, 0 \le x_{t+1} \le \left(\omega_t^i - x_t\right)\frac{p_t}{p_{t+1}}, i = 0, 1$$

where

$$\omega_t^1 = \frac{1 - tax}{p_t}\frac{\Pi_{t-1}}{n} + \frac{w_t}{p_t}\ell^s \quad \text{and} \quad \omega_t^0 = \frac{1 - tax}{p_t}\frac{\Pi_{t-1}}{n}$$

denote the consumer's real wealth when he is employed and unemployed, respectively. Implicit in this formulation is that rationing on the labor market is of type all-or-nothing and that the labor market is visited before the goods market.

On the goods market the young household may be rationed according to the following stochastic rule:

$$x_t = \begin{cases} x_t^d & \text{with prob.} \quad \gamma_t^d \\ 0 & \text{with prob. } 1 - \gamma_t^d \end{cases}$$

where x_t^d is the quantity demanded and $\gamma_t^d \in [0, 1]$ a rationing coefficient which the household perceives as given but which will be determined in equilibrium. From this follows that the expected value of x_t is $\gamma_t^d x_t^d$. In particular this means that rationing is manipulable.

Denoting with θ_t^e the expected relative price for consumption in period $t + 1$, x_t^{di}, $i = 0, 1$, is obtained by maximizing the agent's expected utility $\gamma_t^d x_t^h \left(\left(\omega_t^i - x_t\right)/\theta_t^e\right)^{1-h}$. The solution is $x_t^{di} = h\omega_t^i$. In particular, the young consumer's effective demand is independent both of γ_t^d and of θ_t^e. It does depend, however, on the real income ω_t^i and hence on whether the consumer has been employed.[5]

[5] Since there is 0/1-rationing on the goods market, the Cobb-Douglas form of the utility function implies that a young consumer rationed on the goods market derives no utility from consuming in the second period. This could be easily changed by assuming $u\left(x_t, x_{t+1}\right) = \left(x_t + \underline{x}\right)^h x_{t+1}^{1-h}$ where $\underline{x} \le G/n$ is a small fixed subsistance consumption provided for free by the government. Then expected utility would be $\gamma_t^d \left(x_t + \underline{x}\right)^h \left(\left(\omega_t^i - x_t\right)/\theta_t^e\right)^{1-h} + \left(1 - \gamma_t^d\right)\underline{x}^h \left(\omega_t^i/\theta_t^e\right)^{1-h}$ and therefore $x_t^{di} = h\omega_t^i - (1 - h)\,\underline{x}$ which, as h and $\underline{x}$ are fixed, would not change anything substantial.

The aggregate supply of labor is $L^s = n\ell^s$. Denoting with L_t^d the aggregate demand of labor and with $\lambda_t^s = \min\left\{\frac{L_t^d}{L^s}, 1\right\}$ the fraction of young consumers that will be employed, the aggregate demand of goods of young consumers is

$$X_t^d = \lambda_t^s n x_t^{d1} + (1 - \lambda_t^s)\, n x_t^{d0} = X^d\left(\lambda_t^s; \frac{w_t}{p_t}, \frac{(1 - tax)\,\Pi_{t-1}}{p_t}\right).$$

The total effective aggregate demand of the consumption sector is then obtained by adding old consumers' aggregate demand m_t and government demand G:

$$Y_t^d = X^d\left(\lambda_t^s; \alpha_t, (1 - tax)\,\pi_t\right) + m_t + G$$

where $\alpha_t = w_t/p_t$ denotes the real wage.

In order to provide a geometric characterization which will be crucial in studying the temporary equilibrium regimes we define the locus

$$H = \left\{\left(L^s, X^d(\lambda^s)\right) \mid \lambda^s \in [0,1]\right\}$$

and, following Honkapohja-Ito [1985], refer to it as the young consumers' *offer curve.*[6] It represents the aggregate transaction offers of the young consumers for different values of λ^s (holding all other arguments in the function $X^d(\cdot)$ fixed). From this locus we obtain, for any given m and G, the consumption sector's offer curve as $\left\{\left(L^s, X^d(\lambda^s) + m + G\right) \mid \lambda^s \in [0,1]\right\} = H + \{(0, m+G)\}$.[7]

Similarly, the locus of young consumers' expected aggregate transactions is

$$\overline{H} \equiv \left\{\left(\lambda^s L^s, \gamma^d X^d(\lambda^s)\right) \mid \left(\lambda^s, \gamma^d\right) \in [0,1]^2\right\}$$

and it can be partitioned in the following way: $\overline{H}^K \equiv \overline{H}\mid_{\gamma^d=1,\lambda^s<1}$, $\overline{H}^I \equiv \overline{H}\mid_{\gamma^d<1,\lambda^s=1}$, $\overline{H}^C \equiv \overline{H}\mid_{\gamma^d<1,\lambda^s<1}$, $\overline{H}^U \equiv \overline{H}\mid_{\gamma^d=1,\lambda^s=1}$. We will refer to these loci as the young consumers' *trade curves* whereas the consumption sector's trade curves are given by $\overline{H}^T + \{(0, m+G)\}$, $T \in \{K, I, C, U\}$. They are shown in Figure 2.2.[8]

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[6] We use the terminology *offer curve* because it has been established so in the literature. Of course, that does not mean that households act as suppliers on all markets.

[7] For two sets A and B, $x \in A + B$ iff $x = a + b$ for some $a \in A$ and $b \in B$.

[8] Notice that the aggregate offer and trade curves are deterministic concepts although, at the individuel level, rationing is stochastic. This is not a contradiction but can be easily understood thinking of, for example, a situation in which two workers apply for the same job at a firm. The "transaction level" will be one worker and is thus deterministic, but any individual worker may think to have a 50% chance of being assumed.

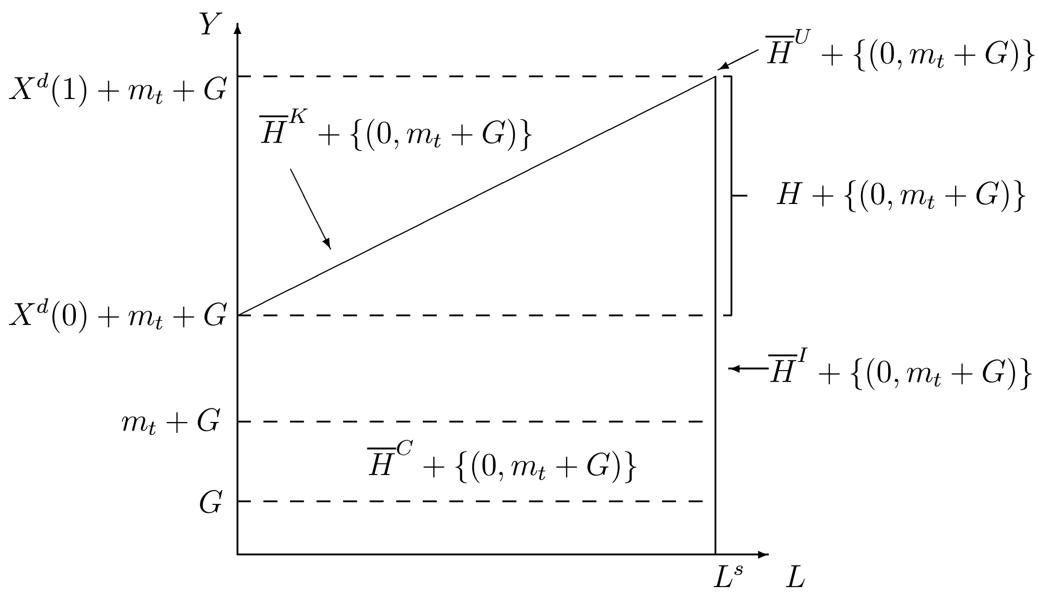

Figure 2.2: Consumption sector's trade curves.

2.3. *The Production Sector*

Each of the n' identical firms uses an atemporal production function $y_t = f(\ell_t)$. As with consumers, firms too may be rationed, by means of a rationing mechanism analogue to that assumed for the consumption sector.

Denoting the single firm's effective demand of labor by ℓ_t^d, the quantity of labor effectively transacted is

$$\ell_t = \begin{cases} \ell_t^d, & \text{with prob. } \lambda_t^d \\ 0, & \text{with prob. } 1 - \lambda_t^d \end{cases}$$

where $\lambda_t^d \in [0,1]$. It is obvious that $E\ell_t = \lambda_t^d \ell_t^d$. On the goods market the rationing rule is assumed to be

$$y_t = \begin{cases} y_t^s, & \text{with prob. } \sigma\gamma_t^s \\ d_t y_t^s, & \text{with prob. } 1 - \sigma\gamma_t^s \end{cases},$$

where $\sigma \in (0,1)$, $\gamma_t^s \in [0,1]$ and $d_t = (\gamma_t^s - \sigma\gamma_t^s) / (1 - \sigma\gamma_t^s)$. σ is a fixed parameter of the mechanism whereas λ_t^d and γ_t^s are perceived rationing coefficients taken as given by the firm the effective value of which will be determined in equilibrium. The definition of d_t implies that $Ey_t = \gamma_t^s y_t^s$; in particular it is independent of σ. It is obvious that $E\ell_t = \lambda_t^d \ell_t^d$.

The effective demand $\ell_t^d = \ell^d(\gamma_t^s; \alpha_t)$ is obtained from the firm's expected profit maximization problem

$$\max_{\ell_t^d} \gamma_t^s f\left(\ell_t^d\right) - \alpha_t \ell_t^d \quad \text{subject to} \quad 0 \leq \ell_t^d \leq \frac{d_t}{\alpha_t} f\left(\ell_t^d\right)$$

while its effective supply is $y_t^s = f\left(\ell_t^d\right)$. The upper bound on labor demand reflects the fact that the firm must be prepared to finance labor service purchases even if rationed on the goods market (since the labor market is visited first it will know whether it is rationed on the goods market only after it has hired labor). In general the solution depends on this constraint but if we assume that $f(\ell) = a\ell^b$, $a > 0$, $0 \leq b \leq (1-\sigma)$, then it is not binding (Appendix 1, Lemma 1).

The aggregate labor demand is $L_t^d = n'\ell_t^d(\gamma_t^s; \alpha_t) \equiv L^d(\gamma_t^s; \alpha_t)$ and, because only a fraction λ_t^d of firms can hire workers, the aggregate supply of goods is $Y_t^s = \lambda_t^d n' f\left(\ell^d(\gamma_t^s; \alpha_t)\right) \equiv Y^s\left(\lambda_t^d, \gamma_t^s; \alpha_t\right)$.

As in the case of the consumption sector we can define the producers' *offer curve* as

$$F = \left\{\left(L^d(\gamma^s)\right), Y^s\left(\lambda^d, \gamma^s\right) \mid \left(\lambda^d, \gamma^s\right) \in [0,1]^2\right\}$$

and partition it into $F^K \equiv F \mid_{\lambda^d=1, \gamma^s<1}$, $F^I \equiv F \mid_{\lambda^d<1, \gamma^s=1}$, $F^C \equiv F \mid_{\lambda^d=1, \gamma^s=1}$ and $F^U \equiv F \mid_{\lambda^d<1, \gamma^s<1}$. Similarly, setting

$$\overline{F} \equiv \left\{\left(\lambda^d L^d(\gamma^s), \gamma^s Y^s\left(\lambda^d, \gamma^s\right)\right) \mid \left(\lambda^d, \gamma^s\right) \in [0,1]^2\right\}$$

we obtain the producers' *trade curves* as $\overline{F}^K \equiv \overline{F} \mid_{\lambda^d=1, \gamma^s<1}$, $\overline{F}^I \equiv \overline{F} \mid_{\lambda^d<1, \gamma^s=1}$, $\overline{F}^C \equiv \overline{F} \mid_{\lambda^d=1, \gamma^s=1}$ and $\overline{F}^U \equiv \overline{F} \mid_{\lambda^d<1, \gamma^s<1}$. As we show in Appendix 1 (Lemma 2), under the assumptions made the loci $\overline{F}^K$, $\overline{F}^I$ and $\overline{F}^U$ coincide and have a particularly simple form, namely:

$$\overline{F}^K = \overline{F}^I = \overline{F}^U = \left\{\left(L, \frac{\alpha_t}{b} L\right) \mid 0 \leq L < L^d(1; \alpha_t)\right\} \tag{2.2}$$

Moreover,

$$\overline{F}^C = \left\{\left(L^d(1; \alpha_t), \frac{\alpha_t}{b} L^d(1; \alpha_t)\right)\right\}$$

These curves are shown in Figure 2.3.

3. Temporary Equilibrium Allocations

For any given period t we can now describe a feasible allocation as a temporary equilibrium with rationing as follows.

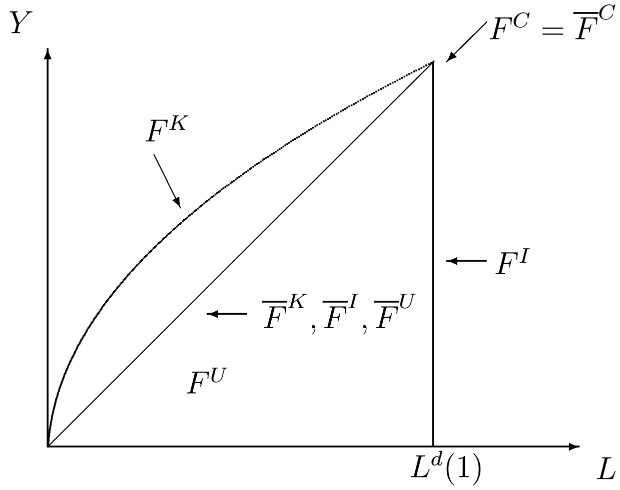

Figure 2.3: Producers' trade curves.

Definition 3.1. *: Given a real wage α_t, a real profit level π_t, real money balances m_t, a level of public expenditure G and a tax rate tax, a list of rationing coefficients $\left(\gamma_t^d, \gamma_t^s, \lambda_t^d, \lambda_t^s, \delta_t, \varepsilon_t\right) \in [0,1]^6$ and an aggregate allocation $\left(\overline{L}_t, \overline{Y}_t\right)$ constitute* a temporary equilibrium *if the following conditions are fulfilled:*

(1) $\overline{L}_t = \lambda_t^s L^s = \lambda_t^d L^d\left(\gamma_t^s; \alpha_t\right);$

(2) $\overline{Y}_t = \gamma_t^s Y^s\left(\lambda_t^d, \gamma_t^s; \alpha_t\right) = \gamma_t^d X^d\left(\lambda_t^s; \alpha_t, (1-tax)\,\pi_t\right) + \delta_t m_t + \varepsilon_t G;$

(3) $(1-\lambda_t^s)\left(1-\lambda_t^d\right) = 0; (1-\gamma_t^s)\left(1-\gamma_t^d\right) = 0;$

(4) $\gamma_t^d(1-\delta_t) = 0;\ \delta_t(1-\varepsilon_t) = 0.$

Conditions (1) and (2) require that expected aggregate transactions balance. This means that all agents have correct perceptions of the rationing coefficients. Equations (3) formalize the short-side rule according to which at most one side on each market is rationed. The meaning of the coefficients δ_t and ε_t is that also old households and/or the government can be rationed. However, according to condition (4) this may occur only after young households have been rationed (to zero).

In Appendix 2 we show the existence and uniqueness of equilibrium for any given $(\alpha_t, \pi_t, m_t, G, tax)$. This implies that the equilibrium levels on the labor and the goods market are well-defined functions $\mathcal{L}(\alpha_t, \pi_t, m_t, G, tax)$ and $\mathcal{Y}(\alpha_t, \pi_t, m_t, G, tax)$.

As shown in the table below it is possible to distinguish different types of equilibrium according to which market sides are rationed: excess supply on both markets is called *Keynesian Unemployment* $[K]$, excess demand on both markets *Repressed Inflation* $[I]$, excess supply on the labor market and excess demand on the goods market *Classical Unemployment* $[C]$ and excess demand on the labor market with excess supply on the goods market *Underconsumption* $[U]$.

	K	I	C	U
λ_t^s	<1	$=1$	<1	$=1$
λ_t^d	$=1$	<1	$=1$	<1
γ_t^s	<1	$=1$	$=1$	<1
γ_t^d	$=1$	<1	<1	$=1$
δ_t	$=1$	≤ 1	≤ 1	$=1$
ε_t	$=1$	≤ 1	≤ 1	$=1$

Of course there are further intermediate cases which, however, can be considered as limiting cases of the above ones. In particular, when all the rationing coefficients are equal to one, we are in a Walrasian Equilibrium. Using the consumption and the production sectors' trade and offer curves it is possible to analyze the various equilibrium regimes in more detail. We do this here for the case of Keynesian Unemployment only. This type of equilibrium involves rationing of households on the labor market and of firms on the goods market. It is given by a pair $(\lambda_t^s, \gamma_t^s)$ such that

$$\overline{L}_t = \lambda_t^s L^s = L^d\left(\gamma_t^s\right)$$
$$\overline{Y}_t = \gamma_t^s Y^s\left(1, \gamma_t^s\right) = X^d\left(\lambda_t^s\right) + m_t + G$$

(where we have suppressed all arguments that are not rationing coefficients). Recalling the definition of the trade curves $\overline{H}^K$ and $\overline{F}^K$ the pair $\left(\overline{L}_t, \overline{Y}_t\right)$ is a Keynesian equilibrium allocation if

$$\begin{aligned}
\left(\overline{L}_t, \overline{Y}_t\right) \;\in\;& \left\{\left(\lambda^s L^s, X^d\left(\lambda^s\right) + m_t + G\right) \mid \lambda^s \in [0,1)\right\} \\
& \cap \left\{\left(L^d\left(\gamma^s\right), \gamma^s Y^s\left(1, \gamma^s\right)\right) \mid \gamma^s \in [0,1)\right\} \\
=\;& \left[\overline{H}_t^K + \{(0, m_t + G)\}\right] \cap \overline{F}_t^K .
\end{aligned}$$

Thus $\left(\overline{L}_t, \overline{Y}_t\right)$ is given by the intersection of the trade curves $\overline{H}_t^K + \{(0, m_t + G)\}$ and $\overline{F}_t^K$, as shown in Figure 3.1. The consumption sector supplies the amount of labor $L^s > \overline{L}_t$ and demands the quantity of goods $Y_t^d = \overline{Y}_t$ whereas firms demand

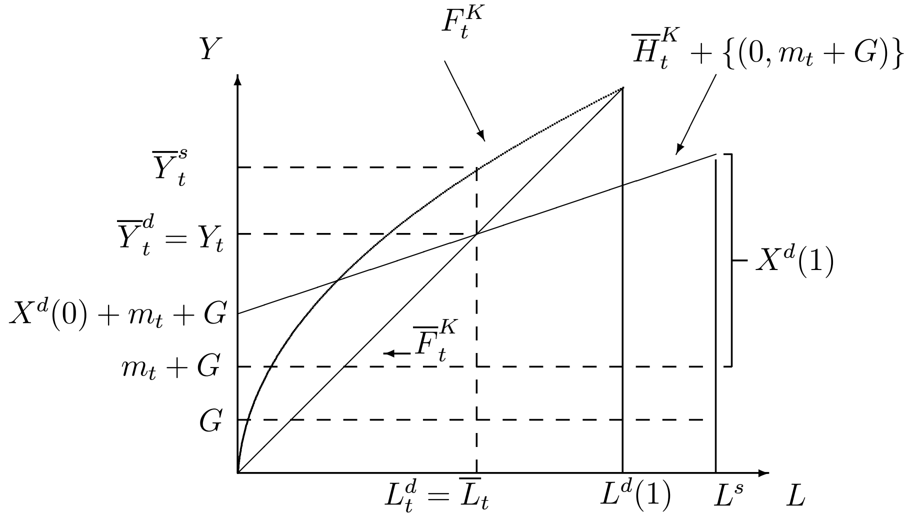

Figure 3.1: Keynesian Unemployment Equilibrium

labor $L_t^d = \overline{L}_t$ and supply $Y_t^s > \overline{Y}_t$ of goods. It follows that $\lambda_t^s = \overline{L}_t / L^s, \gamma_t^s = \overline{Y}_t / Y_t^s$ and $\lambda_t^d = \gamma_t^d = 1 \; (= \delta_t = \varepsilon_t)$, which are just the values that led households and firms to express their respective transaction offers. Thus their expectations regarding these rationing coefficients are confirmed. Nevertheless, due to the randomness in rationing at an individual agent's level, effective aggregate demands and supplies of rationed agents exceed their actual transactions.

Moreover, as indicated earlier, these excesses can be used to get an indicator of the strength of rationing. Since there is zero-one rationing on the labor market, $1 - \lambda_t^s = (L^s - \overline{L}_t)/L^s$ is the ratio of the number of unemployed workers and the total number of young households. Regarding the goods market, in a K-equilibrium $\overline{Y}_t - \gamma_t^s Y^s(1, \gamma_t^s) = 0$, and therefore

$$\frac{d\left(1 - \gamma_t^s\right)}{d\overline{Y}_t} = -\frac{1}{Y_t^s + \gamma_t^s \frac{\partial Y_t^s}{\partial \gamma_t^s}} < 0$$

since $\frac{\partial Y_t^s}{\partial \gamma_t^s}(1, \gamma_t^s) = n' f'\left(\ell^d\left(\gamma_t^s\right)\right) \frac{d\ell_t^d}{d\gamma_t^s} > 0$. So a decrease in $\overline{Y}_t$ (for example due to a reduction of government spending), and thus an aggravation of the shortage of aggregate demand for firms' goods, is unambiguously related to an increase in $1 - \gamma_t^s$ which can therefore be interpreted as a measure of the strength of rationing on the goods market. A similar reasoning justifies the use as rationing measures

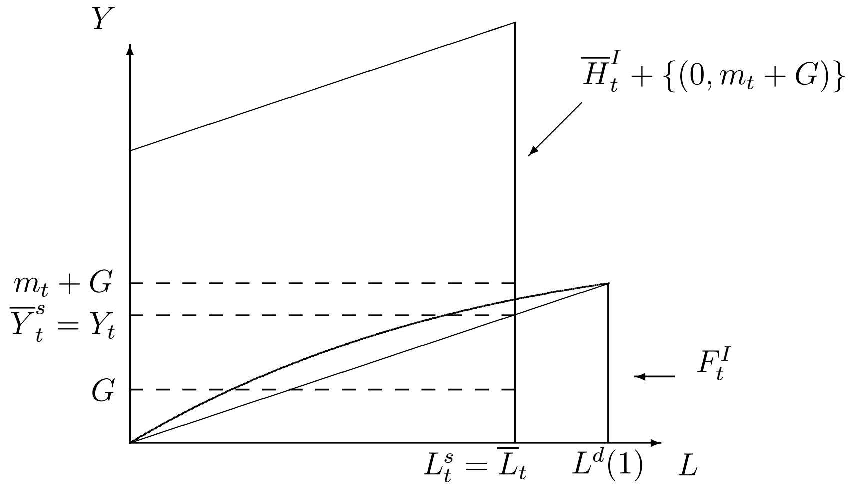

Figure 3.2: Repressed Inflation Equilibrium

of the terms $1 - \lambda_t^d$ and $1 - \gamma_t^d$ in the other equilibrium regimes.

The illustration of the other temporary equilibrium regimes works similarly except for the fact that under repressed inflation and classical unemployment old agents and/or the government may be rationed, too. This is shown in Figure 3.2 for the case of repressed inflation and rationing of the old agents.

4. Representation of Equilibrium Regimes

Given the existence and uniqueness of temporary equilibrium (see Appendix 2) we can, holding all other variables fixed, partition the set R_+^3 of all combinations of real wage α_t, real profits π_t and real money stock m_t according to the type of equilibrium they give rise to. Formally, we have a map $(\alpha_t, \pi_t, m_t) \mapsto T \in \{K, I, C, U\}$. Holding also nominal money M and nominal profits Π parametrically fixed, we can furthermore derive from this a map

$$(p_t, w_t) \mapsto (w_t/p_t, \Pi/p_t, M/p_t) \mapsto T$$

which is illustrated in Figure 4.1 and shows the partitioning of $p_t - w_t-$plane in different regimes of types of equilibrium.[9] From this diagram, in principle familiar

[9] The arrows in Fig. 4.1 will be explained later.

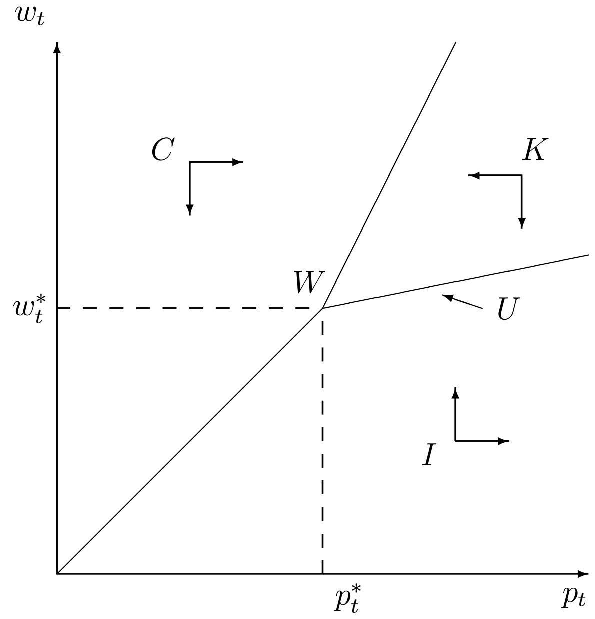

Figure 4.1: Temporary Equilibrium Regimes in the $p_t - w_t$ plane.

from the literature,[10] it can be seen that too high a goods price and a nominal wage give rise to a state of Keynesian unemployment and hence excess supply on both markets, even if the real wage is at its Walrasian level. If the real wage is too high, Classical unemployment occurs whereas in the opposite case a situation of repressed inflation obtains. Finally note that the Underconsumption regime (U) is degenerate: it coincides with the borderline between K and I.[11]

The figure differs from what is shown in the literature with respect to the slope of the borderline between regimes K and I: there it is negative whereas here it is positive. To see this, consider (p_t, w_t) such that $T(w_t/p_t, \Pi/p_t, M/p_t) = U = K^c \cap I^c$.[12] Then consumers are not rationed and, writing $(w_t/p_t, \Pi/p_t, M/p_t) = (\alpha, \pi, m)$, the corresponding equilibrium must satisfy the following conditions:

$$L^s = \lambda^d L^d(\gamma^s; \alpha); \; \gamma^s Y^s\left(\lambda^d, \gamma^s; \alpha\right) = X^d(1; \alpha, (1 - tax)\,\pi) + m + G; \; \lambda^d\gamma^s < 1.$$

By definition of $\overline{F}^U$, (2.2) and the fact that here $L = L^s \leq L^d(1)$, the left hand

[10]See for instance Malinvaud [1977] and Muellbauer and Portes [1978].

[11]This is due to the fact that the trade curve $\overline{F}^U$ is one-dimensional (Lemma 2 in Appendix 1) and derives from our choice of the production function, the assumption that the labor market opens before the goods market and the absence of an intertemporal optimization. Otherwise $\overline{F}^U$ might be an area and the equilibrium non-unique (see Honkapohja and Ito [1985]).

[12]S^c indicates the closure of the set S.

side of the second equation is $(\alpha/b)\, L^s$ whereas the right hand side can be written $h\,(1 - tax)\,\pi + h\alpha\lambda^s L^s + m + G$. This yields

$$h\,(1 - tax)\,\pi + \left(h - \frac{1}{b}\right)\alpha L^s + m + G = 0$$

or, equivalently,

$$\alpha = \frac{h\,(1 - tax)\,\pi + G}{L^s\left(\frac{1}{b} - h\right)} + \frac{1}{L^s\left(\frac{1}{b} - h\right)}m.$$

Multiplying by p_t we obtain the frontier $K - I$ as

$$w_t = \frac{h\,(1 - tax)\,\Pi + M}{L^s\left(\frac{1}{b} - h\right)} + \frac{G}{L^s\left(\frac{1}{b} - h\right)}p_t,\ p_t \geq p_t^* \ . \tag{4.1}$$

It has positive slope since $hb < 1$. The frontiers $K - C$ and $C - I$ can be derived analogously.

5. Dynamics

So far our analysis has been essentially static. For any given vector $(\alpha_t, \pi_t, m_t, G,$ $tax)$ we have described a feasible allocation in terms of a temporary equilibrium with rationing. Moreover we have found a way to measure, by means of the rationing coefficients $\lambda_t^s, \lambda_t^d, \gamma_t^s$ and γ_t^d associated to any equilibrium allocation, the strength of rationing on the various market sides. This allows us now to extend our framework to a dynamic analysis.

To this end we must link successive periods one to another. This link will of course be given by the adjustment of prices but also by the changes in the real stock of money and in real profits. Regarding the latter, this is automatic since by definition of these variables

$$m_{t+1} = \frac{1}{\theta_t}\left[\delta_t m_t + \varepsilon_t G + (1 - tax)\,\pi_t\right] - \pi_{t+1} \tag{5.1}$$

and

$$\pi_{t+1} = \frac{\left[\mathcal{Y}\left(\alpha_t, \pi_t, m_t, G, tax\right) - \alpha_t \mathcal{L}\left(\alpha_t, \pi_t, m_t, G, tax\right)\right]}{\theta_t}$$

hold.[13]

[13] Notice that, if δ_t is smaller than one, then, by the ranking of rationing assumed, $\overline{X}_t = 0$ and thus $M_{t+1} = (1 - tax)\,\Pi_{t-1} + w_t \overline{L}_t$. This is a very artificial situation but, to be consistent, we have to accept that the part $1 - \delta_t$ of M_t which the old households cannot spend is lost. Likewise $\varepsilon_t \leq 1$, which takes account of the possibility of government rationing, is included mainly for consistency reasons. This explains also the slight difference in equations (2.1) and (5.1).

Regarding the adjustment of the goods price and the nominal wage we follow the standard hypothesis that, whenever an excess of demand (supply) is observed, the corresponding price or wage rises (falls). In terms of the rationing coefficients observed in period t, this amounts to

$$p_{t+1} < p_t \Leftrightarrow \gamma_t^s < 1; \; p_{t+1} > p_t \Leftrightarrow \gamma_t^d < 1;$$

$$w_{t+1} < w_t \Leftrightarrow \lambda_t^s < 1; \; w_{t+1} > w_t \Leftrightarrow \lambda_t^d < 1$$

and is illustrated by the arrows in Figure 4.1. More precisely, in our simulation model we will write these adjustments as

$$p_{t+1} = (\gamma_t^s)^{\mu_1} p_t, \text{ if } \gamma_t^s < 1; \; p_{t+1} = \left(\frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)^{-\mu_2} p_t, \text{ if } \gamma_t^d < 1;$$

$$w_{t+1} = (\lambda_t^s)^{\nu_1} w_t, \text{ if } \lambda_t^s < 1; \; w_{t+1} = \left(\lambda_t^d\right)^{-\nu_2} w_t, \text{ if } \lambda_t^d < 1$$

where μ_1, μ_2, ν_1 and ν_2 are nonnegative parameters for the "speeds" of adjustment (for a general formulation of the adjustment mechanism as well as a further specific example of it see Appendix 3).[14] This formalizes that the size of price and wage adjustment depends on the strength of rationing and allows us to encompass a wide variety of circumstances. For example, wage flexibility upwards greater than downwards is obtained whenever $\nu_2 > \nu_1$ and wage rigidity downwards corresponds to $\nu_1 = 0$.[15]

From the adjustment of nominal prices we obtain the one for the real wage as

$$\alpha_{t+1} = \frac{(\lambda_t^s)^{\nu_1}}{(\gamma_t^s)^{\mu_1}} \alpha_t \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \cup U,$$

[14] We use "speed" of adjustment although, literally speaking, with a non linear mechanism, a speed in this discrete model would be the difference between p_{t+1} and p_t or between w_{t+1} and w_t. However, that difference and the size of the corresponding parameter μ_1, μ_2, ν_1 or ν_2 are positively related which is why we use for simplicity this terminology.

[15] A subtle point may be noted here. On the borderline $K - I$ we have $\lambda^s = \gamma^d = 1$, but $\lambda^d \gamma^s < 1$. However, λ^d and γ^s are not uniquely determined. The two limiting cases are $\lambda^d = 1$ and $\gamma^s < 1$, in which the price is adjusted whereas the wage is unaffected, and $\lambda^d < 1$ and $\gamma^s = 1$ which means that only the wage is adjusted. Only when both $\lambda^d < 1$ and $\gamma^s < 1$ price and wage change simultaneously. This indeterminacy is not so problematic, however, because, for any conceivable values of λ^d and γ^s, the price will never increase and the wage never decrease. This, and the positive slope of the borderline $K - I$ (in the $p - w$ plane), imply that unambiguously $(p_{t+1}, w_{t+1}) \in K$ for all $(p_t, w_t) \in K^c \cap I^c$, no matter what the choice of λ^d and γ^s is. Note also that this is not true for comparable models with deterministic rationing like Malinvaud [1977] and Muellbauer and Portes [1978], because there the frontier between K and I is downward sloping.

$$\alpha_{t+1} = \frac{\left(\lambda_t^d\right)^{-\nu_2}}{\left(\frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)^{-\mu_2}}\alpha_t \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in I,$$

$$\alpha_{t+1} = \frac{\left(\lambda_t^s\right)^{\nu_1}}{\left(\frac{\gamma_t^d+\delta_t+\varepsilon_t}{3}\right)^{-\mu_2}}\alpha_t \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in C.$$

whereas the growth factor of the price level $\theta_t = p_{t+1}/p_t$ is given by

$$\theta_t = (\gamma_t^s)^{\mu_1} \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in K \cup U,$$

$$\theta_t = \left(\frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)^{-\mu_2} \text{ if } \left(\overline{L}_t, \overline{Y}_t\right) \in I \cup C.$$

6. Numerical Analysis

The economic model introduced in the previous sections represents a non-linear dynamical system that cannot be studied with analytical tools only. This is due to the fact that the system is three-dimensional, with state variables α_t, m_t and π_t. Moreover, since there are three nondegenerate equilibrium regimes, the overall dynamic system can be viewed as being composed of three subsystems each of which may become effective through endogenous regime switching. (The complete equations of these systems are given in Appendix 4.)

In order to get some insights in these dynamics we are reporting numerical simulations using programs developed for this paper's purposes based on the packages GAUSS and MACRODYN [16]. The basic parameter set specifies values for the technological coefficients (a and b), the exponent of the utility function (h), the labor supply (L^s) and the total number of producers in the economy (n'), for the price adjustment speeds downward and upward (respectively μ_1 and μ_2) and the corresponding wage adjustment speeds (ν_1 and ν_2). We also have to specify initial values for the real wage, real money stock and real profit level (α_0, m_0 and π_0) and values for the government policy parameters (G and tax).

Starting from the following parameter values corresponding to a stationary Walrasian equilibrium

$$\begin{array}{lllll} a = 1 & b = 0.85 & h = 0.5 & L^s = 100 & n' = 100 \\ \alpha_0 = 0.85 & m_0 = 46.25 & \pi_0 = 15 & G = 7.5 & tax = 0.5 \end{array}$$

[16] MACRODYN has been developped at the University of Bielefeld. See Böhm,V., Lohmann, M. and U. Middelberg [1999], MACRODYN – a dynamical system's tool kit, version x99 and Böhm and Schenk-Hoppe' [1998].

we first address the question of the impact of changes in the downward speed of adjustment of the wage rate. To this end we consider a reduction in the initial money stock to $m_0 = 40$, allowing for an adjustment of the price in both directions ($\mu_1 = \mu_2 = 0.4$) but imposing downward wage rigidity ($\nu_1 = 0, \nu_2 = 0.4$). As can be seen from Figure 6.1, the restrictive money shock gives rise to a transitional phase of unemployment before the system returns to a Walrasian equilibrium with full employment. The unemployment phase can be shortened by allowing for downward wage flexibility (as was to be expected from textbook theory). This is shown in Figure 6.2 where ν_1 has been changed to 0.1. However, Figure 6.3, where $\nu_1 = 0.2$, suggests that further increasing this downward flexibility results in irregular behavior with frequently high unemployment rates.

To see which downward wage flexibility is "too much" we can consider the bifurcation diagram in Figure 6.4. It reveals that the system is stable with convergence to full employment for all ν_1 smaller than 0.14 whereas from that value on until 0.28 its behavior is irregular or cyclical . For speeds of adjustment still bigger there results divergence.

The attractor in Figure 6.5, drawn for $\nu_1 = 0.2$, confirms that, in addition to regular periodic behavior, the dynamical system can also produce very complicated patterns in the form of quasi-periodic or even chaotic behavior. Complex and strange geometric objects have been found by deviating from the standard parameter set in some of the relevant parameters (initial conditions, technological coefficients, government policy instruments). Moreover it emerges quite clearly that cycles of different order co-exist for the same parameter set, but for different initial conditions. Therefore, a minor variation in the initial state can drive the system to a completely different cycle or, in other words, there is sensitive dependence of the order of a cycle on initial conditions.

Figure 6.6 shows a further attractor in the diagram plotting the wage inflation rate against the unemployment rate. It in fact is a Phillips curve which, by very construction, is a long-run phenomenon. Notice, however, that it would be wrong to interprete this Phillips curve as a policy instrument in terms of a trade-off between unemployment and inflation. Any point on the curve is but one element of a trajectory of pairs of rates of unemployment and wage inflation, and successive points of this trajectory may lie far away one from the other. Thus, even if the government tried to select a specific point on the curve in one period, in the next period already the system may go to a very different point on the curve.

To understand better how the curve comes about, we can look at the diagrams in the second row of charts of Figure 6.3. The left-hand chart plots the price for periods 1 to 100. Until about period 50 the price does not change very much but then it starts to alternatingly increase and decrease quite substantially. The right-hand chart depicts the trajectory of price and wage couples. It starts out in

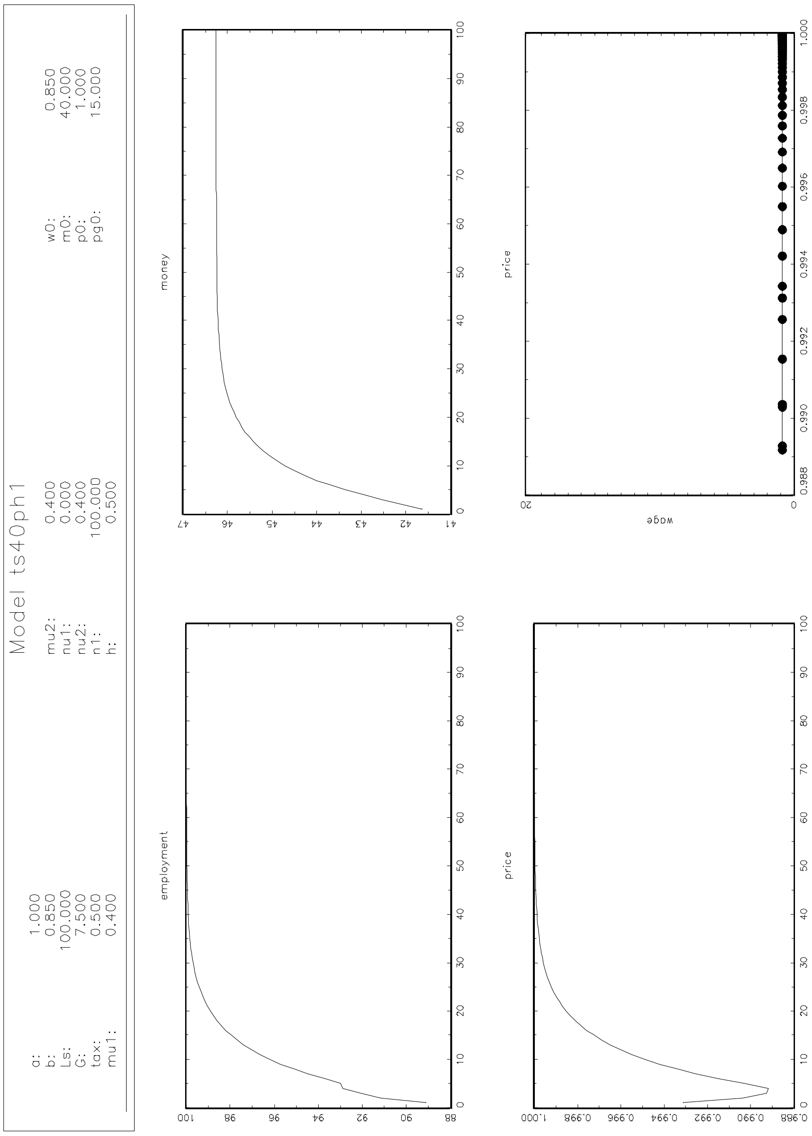

Figure 6.1: The time series in the figure show the emergence of transitional unemployment when the real stock of money is reduced from the Walrasian equilibrium level of 46.25 to 40 and the coefficients for the adjustment of the goods price and the nominal wage are respectively $\mu_1 = \mu_2 = 0.4$ and $\nu_1 = 0$, $\nu_2 = 0.4$.

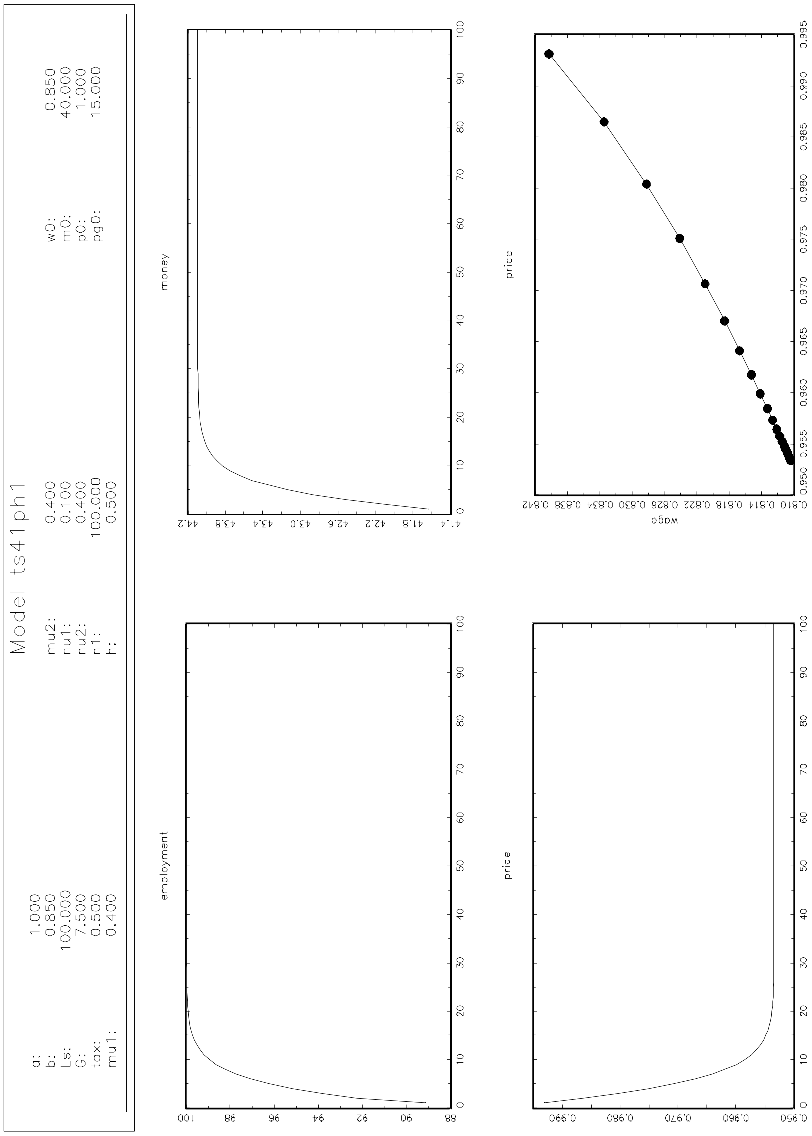

Figure 6.2: The unemployment phase is shortened when downward wage flexibility is allowed. The parameter set is the same considered in the previous figure, except that now it is $\nu_1 = 0.1$.

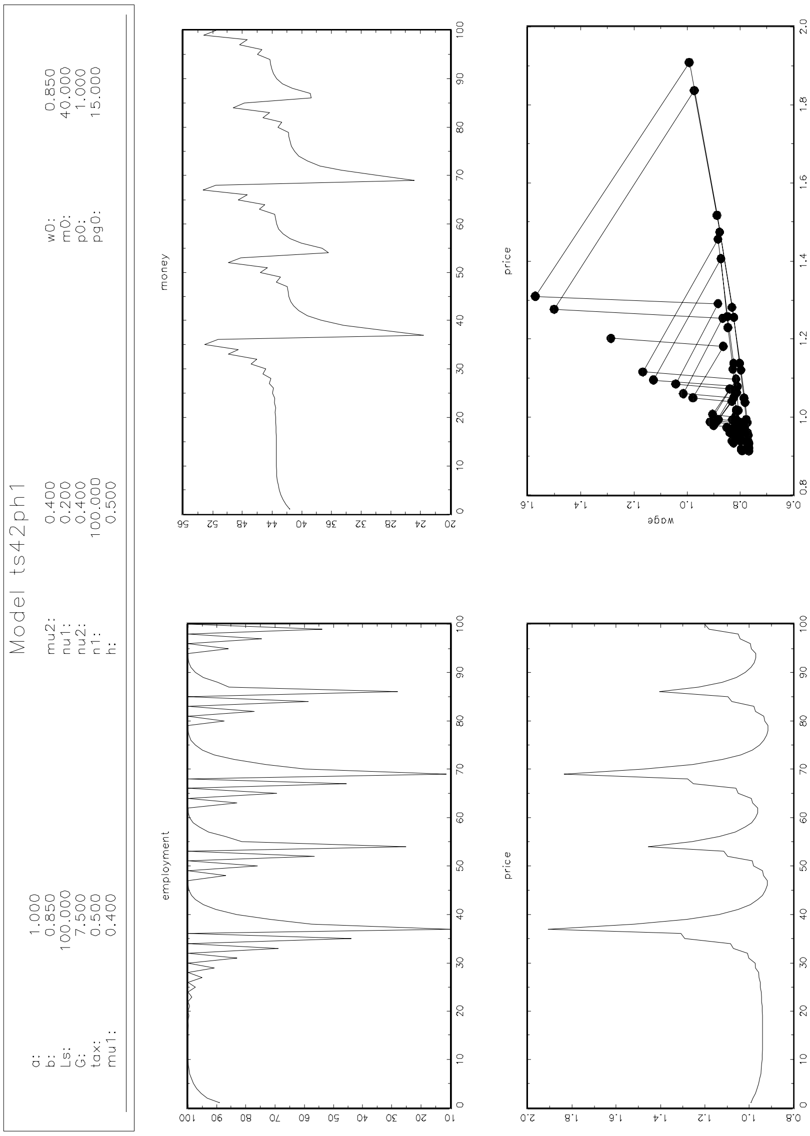

Figure 6.3: The behavior of the system becomes highly irregular when the downward wage flexibility increases. In the simulation represented here ν_1 is 0.2.

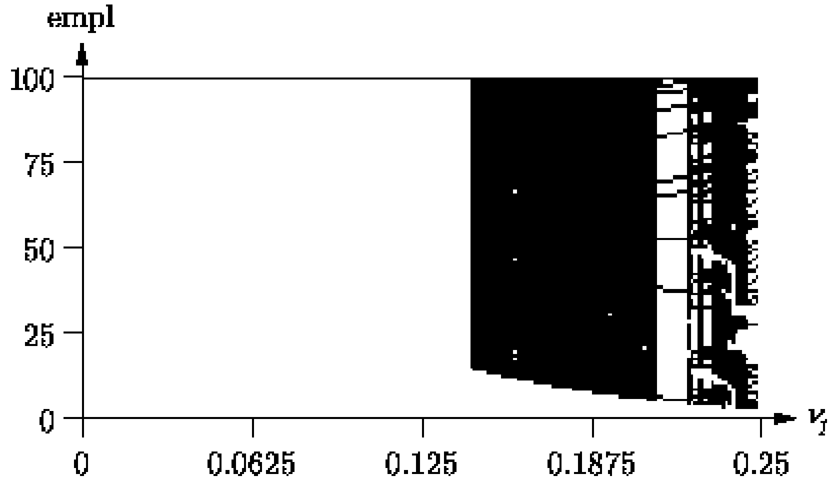

Figure 6.4: The bifurcation diagram shows that for values of ν_1 smaller than 0.14 the system converges, but for higher downward flexibility it displays a seemingly irregular behavior.

the lower left angle and then has a tendency to move upwards and to the right. If, in a given period, the economy finds itself in a state of Keynesian unemployment, both the price and the wage are decreased, whereas the opposite is true in a state of repressed inflation. Furthermore, in a state of classical unemployment the price increases but the wage diminishes. Therefore the chart displaying price and wage couples shows that the economy visits all three types of equilibria along its trajectory. From the chart displaying employment, on the other hand, it is obvious that unemployment rates may vary substantially and therefore the points on the Phillips curve may jump consirably from one period to the next.

Of particular interest in the context of our macroeconomic model is the impact of variations in the values of the government policy instruments G and tax. In investigating this we make use of a new technical tool, called c*yclogram*, and developed by Lohmann and Wenzelburger [1996]. It provides a concise visualization technique and permits to establish a relationship between the values of the relevant parameters and the structure of the resulting dynamics (although it is not able to distinguish between regular quasi-periodic behavior and "true" chaotic motion)[17].

[17]The logic behind cyclograms reminds of the so-called Mandelbrot plots, when the convergence to an attractor or different divergence speeds are plotted, and that of two dimensional

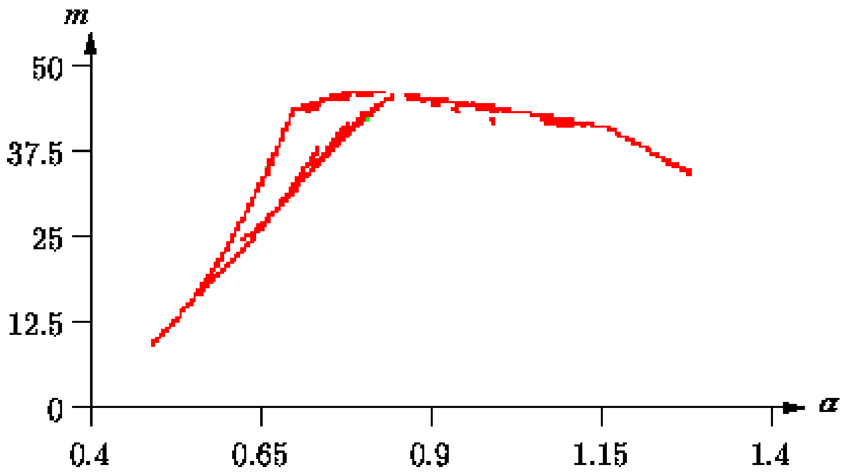

Figure 6.5: An attractor in the (α, m) space. The object represented in the figure is an attractor because the transient phase has been excluded from plotting and the geometric shape of the object does not change for a very high number of iterations.

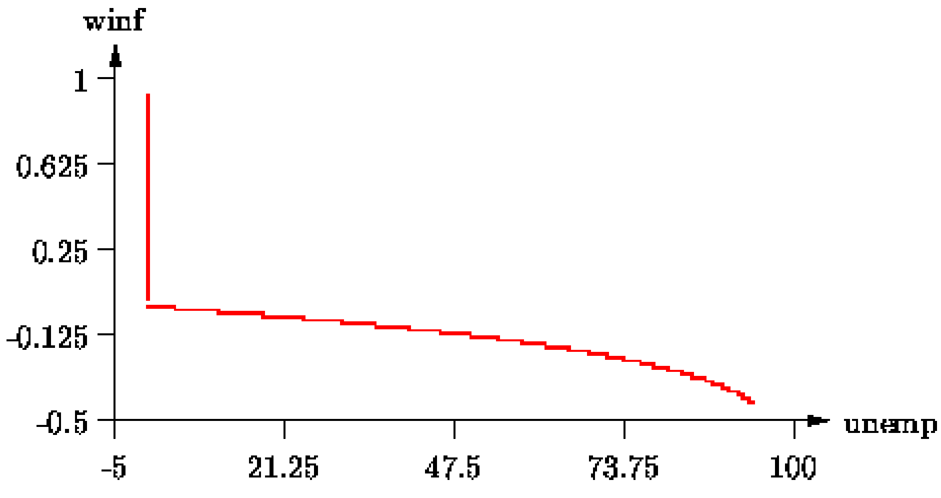

Figure 6.6: A long run Phillips-curve as an attractor, generated with the parameter values of Figure 6.3.

More precisely, Figure 6.7 displays two cyclograms for $\nu_1 = 0$ and $\nu_1 = 0.1$, respectively, where the color attached to each point (G, tax) in the rectangle $[0.75, 14.25] \times [0.05, 0.95]$ reflects a certain type of dynamics. Points on the diagonal correspond to stationary (but not necessarily stable) Walrasian states. All points above the diagonal give rise to convergence and more precisely to quasi-stationary states (i.e. the goods price and the nominal wage are adjusted but the real wage is constant) with Keynesian unemployment. Thus permanent unemployment is possible although there is price and wage flexibility.

To support our claim that the quasi-stationary states we find are of a Keynesian type we may look at the bifurcation diagram in Figure 6.8, obtained for the same parameter set as in the first cyclogram shown in Figure 6.7 by setting $tax = 0.5$. For values of G smaller than 7.5 (the Walrasian value), the economy converges to quasi-stationary states with unemployment. This is consistent with the convergent behaviour shown by the cyclogram for points above the diagonal. Both diagrams fit nicely with economic intuition. They show that, as expected from textbook theory, an expansionary fiscal policy (even holding constant the tax rate and changing only public expenditure) can help driving the economy towards full employment. However, a too expansionary fiscal policy can prove to be destabilizing and give rise to highly cyclical and irregular dynamic behaviour, and this is a feature that has to be added to the risk of inflation traditionally associated with expansionary policy measures.

Cyclograms in the $G - tax$ plane as above may potentially prove to be useful economic policy tools. It is conceivable that, in an extended model, it would be possible to calibrate the parameters of the model and to apply it to discuss the implications of different fiscal policies. However, some caution would be needed. As is clear from the discussion above, the behaviour of the economy may become highly irregular just past the Walrasian equilibria on the diagonal.Thus an attempt to "tune" the economy to full employment may result counterproductive and, on the contrary, to keep it at some small level of unemployment may be preferable!

Coming back to quasi-stationary Keynesian states, their existence can be explained economically as follows. In a state of Keynesian unemployment there is excess supply on both markets. Therefore both the goods price and the nominal wage diminish. However, if both decrease in the same measure, the real wage remains constant. If, in addition, there is a government balance surplus, the money stock decreases. If this decrease is proportional to the one in price and wage, the real stock of money held by consumers does not change. Thus, it is possible that in addition to the real wage also the real wealth of households remains constant or, in other words, there is no real-balance effect. In these circumstances, no agent has an incentive to modify his decisions relative to the ones taken in the previous

bifurcation diagrams, when the change in the behavior of state variables is plotted.

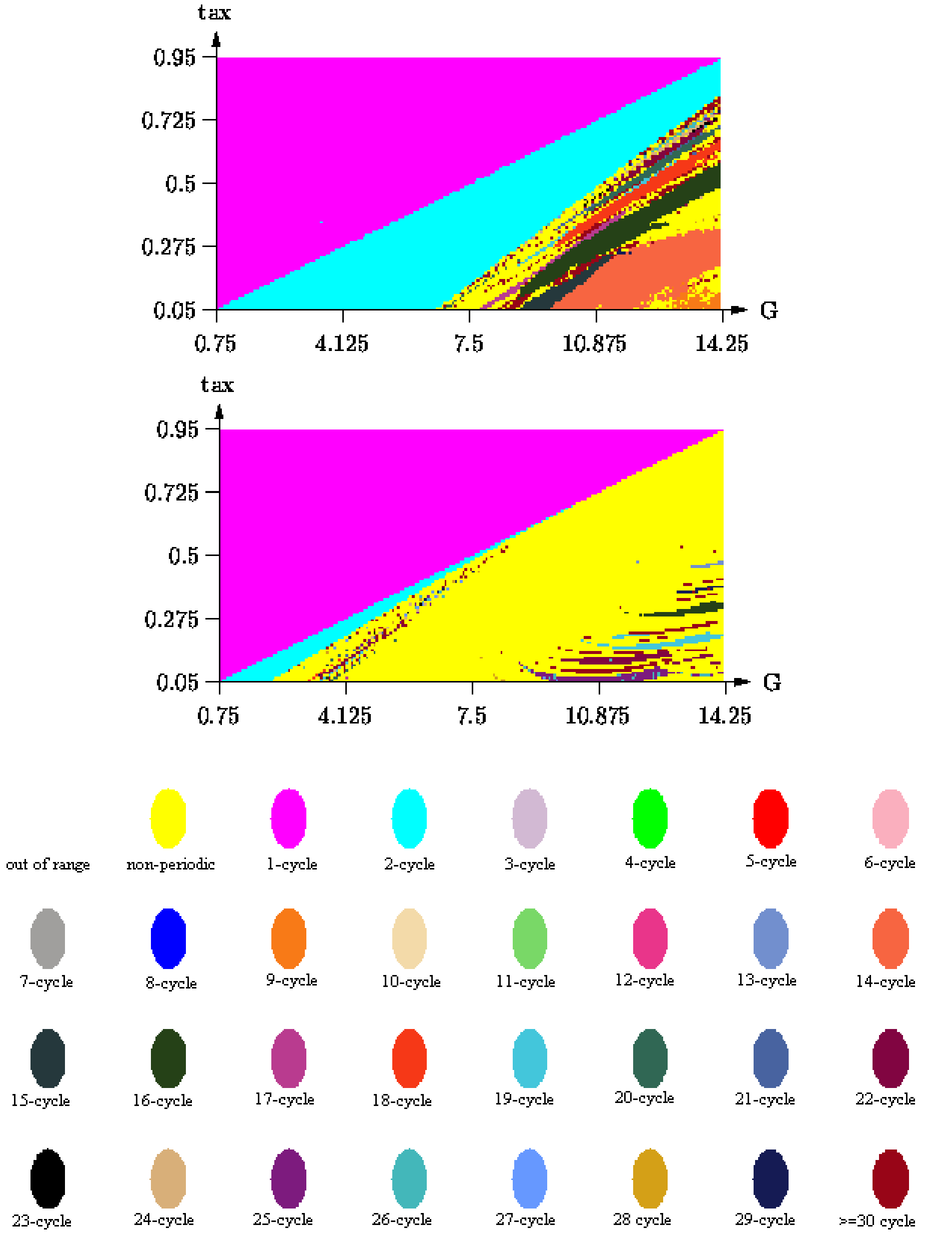

Figure 6.7: These cartograms show the dynamic behavior of the economy for $\nu_1 = 0$ (upper diagram) and $\nu_1 = 0.1$ when the values of the government policy instruments tax and G vary respectively in the range $(0,05;0,95)$ and $(0,75;14,25)$. Notice that downward wage flexibility favors irregular behavior.

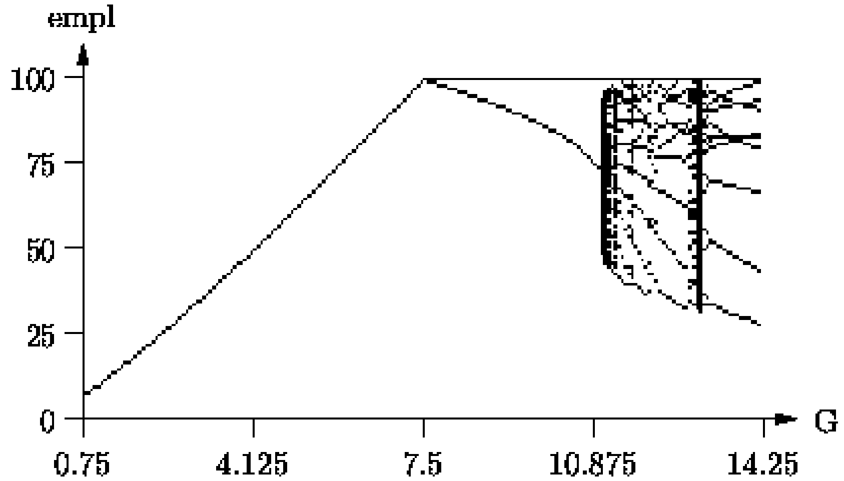

Figure 6.8: The bifurcation diagram shows that for values of G smaller than 7.5 the economy converges to a quasi-stationary Keynesian state with unemployment, whereas for G larger than 7.5 there is cyclical and irregular behaviour.

period.Therefore, the state of the economy is stationary with respect to its real variables, although the nominal ones do change. This shows that the flexibility of prices and wages *per se* is not sufficient to overcome situations of market imbalances; it is due to the spillover effects between the labour and the consumption goods market, and could not have been obtained in partial equilibrium models.

7. Concluding Remarks

In this paper we have presented a dynamic macroeconomic model possessing three main features: it is built on first principles like optimizing behavior of all agents, it allows agents to trade also when prices are not market clearing and it gives rise to complex dynamics. More precisely we have adopted a non-tâtonnement approach involving temporary equilibria with stochastic quantity rationing and price adjustment between successive equilibria. In this way we have obtained a process which allows us to describe consistent allocations in every period but which at the same time obeys a well defined dynamics.

While the resulting dynamic system is too complex to be completely understood by means of analytical tools only, we have been able to shed some light on it using simulations in the form of time series, bifurcation diagrams, attractors and cyclograms. From these it is clear that a large variety of dynamic phenomena can be obtained, as for example a Phillips curve as an attractor. On the one hand the model shows a behavior familiar from textbook economics. This tends to be the case for small speeds of the adjustment of prices and, in particular, of the nominal wage. For high speeds, on the other hand, there is typically divergence whereas for intermediate speeds it is not difficult to obtain irregular and possibly chaotic behavior. As a by-product, we have been able to generate a Phillips-curve as the image of an attractor. Unfortunately, for three-dimensional systems with endogenous regime switching as the one presented in this paper there do not yet appear to exist sufficient mathematical results to decide this issue analytically.[18]

In the absence of such analytical results cyclograms have shown to be useful tools as they indicate for any chosen parameter set the type of dynamics that is related to it. From this we have seen that there are parameter configurations for which the model's structural dynamic behavior is very sensitive to the choice of economic policy instruments. Moreover it is clear from the simulations that there exist parameter configurations for which the economic system gets stuck in quasi-stationary states that involve rationing - permanent unemployment or permanent capacity underutilization - although both the goods price and the nominal wage are flexible.

Appendix 1: Lemma 1 - 2

Lemma 1. *When the production function is $f(\ell) = a\ell^b$, with $a > 0$ and $0 < b \leq 1 - \sigma$, the solution to the firm's maximization problem is independent of the constraint $\ell_t^d \leq \frac{d_t}{\alpha_t} f\left(\ell_t^d\right)$.*

Proof. The first order condition for an interior solution of the firm's problem is

$$\gamma^s f'(\ell) = \alpha \Leftrightarrow \gamma^s \frac{bf(\ell)}{\ell} = \alpha \Leftrightarrow \ell = \gamma^s \frac{bf(\ell)}{\alpha}.$$

[18]Böhm, Lohmann and Lorenz [1994] have recently adopted, in a similar framework, a geometric approach to the determination of complex dynamic behavior according to which a dynamical system generates chaotic motion if it possesses horseshoes. This is based upon Smale [1963] who, investigating a 2D return map which can be embedded in higher dimensional systems, proved that the presence of horseshoes implies the existence of a Cantor set with fractal geometry. Such a definition may be powerful also in our case and easier to handle in comparison to the definitions based on the Li-Yorke criterion, the positivity of the largest Lyapunov exponent or Ruelle's strange attractor criterion. However, it is important to stress that in almost any case horseshoes can only be detected numerically and therefore they can not be considered an analytical substitute for numerical analysis.

Moreover the inequalities $\frac{1}{b} \geq \frac{1}{1-\sigma} \geq \frac{1-\gamma^s\sigma}{1-\sigma}$ yield $1 \leq \frac{1-\sigma}{b(1-\gamma^s\sigma)}$. From this follows

$$\ell \leq \frac{\gamma^s(1-\sigma)}{1-\gamma^s\sigma}\frac{1}{\gamma^s}\frac{1}{b}\ell = d\frac{1}{\gamma^s}\frac{1}{b}\ell = d\frac{1}{\gamma^s}\frac{1}{b}\gamma^s\frac{bf(\ell)}{\alpha} = \frac{d}{\alpha}f(\ell),$$

which proves our claim. ■

Lemma 2. *When the production function is* $f(\ell) = a\ell^b$, *with* $a > 0$ *and* $0 < b \leq 1-\sigma$, *the producers' trade curves are given by*

$$\overline{F}^K = \overline{F}^I = \overline{F}^U = \left\{\left(L, \frac{\alpha_t}{b}L\right) \mid 0 \leq L < L^d(1;\alpha_t)\right\}$$

and $\overline{F}^C = \left\{\left(L^d(1;\alpha_t), \frac{\alpha_t}{b}L^d(1;\alpha_t)\right)\right\}$.

Proof. From $f(\ell) = a\ell^b$ follows $f'(\ell) = b\frac{f(\ell)}{\ell}$, which implies $f(\ell) = \frac{1}{b}f'(\ell)\ell$ and

$$Y = \gamma^s Y^s(\lambda^d, \gamma^s) = \gamma^s\lambda^d n' f\left(\ell^d(\gamma^s;\alpha_t)\right) = \gamma^s\lambda^d n'\frac{1}{b}f'\left(\ell^d(\gamma^s;\alpha_t)\right)\ell^d(\gamma^{s;\alpha_t})$$

But $\gamma^s f'\left(\ell^d(\gamma^s;\alpha_t)\right) = \alpha_t$ from any producer's optimizing behavior, and thus

$$Y = \frac{\alpha_t}{b}\lambda^d L^d(\gamma^s;\alpha_t) = \frac{\alpha_t}{b}L \;■$$

Appendix 2: Existence and Uniqueness of Temporary Equilibrium

We show existence and uniqueness of a temporary equilibrium allocation for any given list of parameters $(\alpha_t, \pi_t, m_t, G, tax)$ by means of the trade curves introduced in section two. From their definition follows that a pair $(\overline{L}, \overline{Y}) \in R^2_+$ is a temporary equilibrium allocation if and only if it is an element of the set

$$Z = \left(\left(\overline{H}_0^K\right)^c \cap \left(\overline{F}^K\right)^c\right) \cup \left(\left(\overline{H}_0^I\right)^c \cap \left(\overline{F}^I\right)^c\right) \cup \left(\left(\overline{H}_0^C\right)^c \cap \left(\overline{F}^C\right)^c\right)$$

where $\overline{H}_0^K$ denotes the set $\overline{H}^K + \{(0, m_t + G)\}$ and

$$\overline{H}_0^I = \left\{\left(L^s, \gamma^d X^d(1) + m_t + G\right) \mid \gamma^d \in (0,1)\right\} \cup \{(L^s, \delta m_t + G) \mid \delta \in (0,1]\}$$
$$\cup \{(L^s, \varepsilon G) \mid \varepsilon \in [0,1]\},$$
$$\overline{H}_0^c = \left\{\left(\lambda^s L^s, \gamma^d X^d(\lambda^s) + m_t + G\right) \mid \left(\lambda^s, \gamma^d\right) \in [0,1) \times (0,1)\right\}$$

$$\cup\{(\lambda^s L^s, \delta m_t + G) \mid (\lambda^s, \delta) \in [0,1) \times (0,1]\} \cup \{(\lambda^s L^s, \varepsilon G) \mid (\lambda^s, \varepsilon) \in [0,1) \times [0,1]\}.$$

Here S^c indicates the closure of the set S. Note that equilibria of type U do not appear as they can be seen as limiting cases of K- as well as of I-type equilibria.

To show that Z is not empty we consider first the locus

$$\left(\overline{H}_0^K\right)^c = \left\{\left(\lambda_t^s L^s, X^d\left(\lambda_t^s\right) + m_t + G\right) \mid \lambda_t^s \in [0,1]\right\}$$

and recall that

$$X^d\left(\lambda_t^s\right) = nh\left(\lambda_t^s \omega_t^1 + \left(1 - \lambda_t^s\right)\omega_t^0\right) = h\left(1 - tax\right)\pi_t + h\alpha_t \lambda_t^s L^s.$$

Defining the functions

$$\Gamma_t(L) \equiv h(1 - tax)\pi_t + h\alpha_t L + m_t + G,\ \ L \geq 0,$$

and

$$\Delta_t(L) \equiv \frac{\alpha_t}{b} L,\ \ L \geq 0,$$

we see that $\left(\overline{H}_0^K\right)^c$ is the part of the graph of Γ_t for which $L \leq L^s$ whereas the locus $\left(\overline{F}^K\right)^c$ is the part of the graph of Δ_t for which $L \leq L^d(1)$ (cf. Figures 2.2, 2.3 and 3.1).

Next observe that the graphs of the functions Γ_t and Δ_t always intersect. Indeed, $\Gamma_t(L) = \Delta_t(L)$ if and only if

$$L = \frac{b}{\alpha_t(1 - hb)}\left[h(1 - tax)\pi_t + m_t + G\right] \equiv \widetilde{L}\left(\alpha_t, \pi_t, m_t, G, tax\right)$$

which is well defined and positive since $hb < 1$. Thus the equilibrium level of employment is

$$\overline{L}_t = \min\left\{\widetilde{L}(.), L^d(1, \alpha_t), L^s\right\} \equiv \mathcal{L}\left(\alpha_t, \pi_t, m_t, G, tax\right).$$

and the equilibrium level on the goods market is

$$\overline{Y}_t = \Delta_t\left(\overline{L}_t\right) \equiv \mathcal{Y}\left(\alpha_t, \pi_t, m_t, G, x\right).$$

More precisely, if $\min\{\cdot\} = \widetilde{L}(.)$, then $\left(\overline{L}_t, \overline{Y}_t\right) \in \left(\overline{H}_0^K\right)^c \cap \left(\overline{F}^K\right)^c$ and the resulting equilibrium is of type K or a limiting case of it. If $\min\{\cdot\} = L^d(1, \alpha_t)$, then $\left(\overline{L}_t, \overline{Y}_t\right) \in \left(\overline{H}_0^C\right)^c \cap \left(\overline{F}^C\right)^c$ and type C or a limiting case of it occurs. Finally, if $\min\{\cdot\} = L^s$, an equilibrium of type I or a limiting case results because then $\left(\overline{L}_t, \overline{Y}_t\right) \in \left(\overline{H}_0^I\right)^c \cap \left(\overline{F}^I\right)^c$.

Since the functions $\mathcal{L}(\cdot)$ and $\mathcal{Y}(\cdot)$ are well defined and positive, an equilibrium always exists. Uniqueness follows from the linearity of $\Gamma_t(\cdot)$ and $\Delta_t(\cdot)$.

Appendix 3: Price and wage adjustment mechanisms

Consider a nominal wage adjustment function of the form

$$w_{t+1} = \zeta_\nu\left(\lambda_t^s, \lambda_t^d\right) w_t$$

where $\nu = (\nu_1, \nu_2)$ represents the speeds of the downward and upward adjustments, respectively. Moreover,

$$\zeta_\nu(1,1) = 1 \forall \nu; \zeta_\nu\left(\lambda_t^s, \lambda_t^d\right) < 1 \Leftrightarrow \lambda_t^s < 1 \forall \nu_1 > 0; \zeta_\nu\left(\lambda_t^s, \lambda_t^d\right) > 1 \Leftrightarrow \lambda_t^d < 1 \forall \nu_2 > 0.$$

Similarly for the price adjustment mechanism,

$$p_{t+1} = \xi_\mu\left(\gamma_t^s, \gamma_t^d, \delta_t, \varepsilon_t\right) p_t$$

where $\mu = (\mu_1, \mu_2)$, $\gamma_t^d(1-\delta_t) = 0$, $\delta_t(1-\varepsilon_t) = 0$ (cf. condition (4) of Def. 3.1) and

$$\xi_\mu(1,1,1,1) = 1 \ \forall \mu;$$

$$\xi_\mu\left(\gamma_t^s, \gamma_t^d, \delta_t, \varepsilon_t\right) < 1 \Leftrightarrow \gamma_t^s < 1 \forall \mu_1 > 0; \xi_\mu\left(\gamma_t^s, \gamma_t^d, 1, 1\right) > 1 \Leftrightarrow 0 < \gamma_t^d < 1 \forall \mu_2 > 0;$$

$$\xi_\mu(\gamma_t^s, 0, \delta_t, 1) > 1 \Leftrightarrow 0 < \delta_t < 1 \forall \mu_2 > 0; \xi_\mu(\gamma_t^s, 0, 0, \varepsilon_t) > 1 \Leftrightarrow \varepsilon_t < 1 \forall \mu_2 > 0.$$

The families of functions $\{\zeta_\nu\}_{\nu \geq 0}$ and $\left\{\xi_\mu\right\}_{\mu \geq 0}$ may be assumed to have the following properties:

(I) ζ_ν and ξ_μ are differentiable and $\frac{\partial \zeta_\nu}{\partial \lambda_t^s} \geq 0$, $\frac{\partial \zeta_\nu}{\partial \lambda_t^d} \leq 0$, $\frac{\partial \xi_\mu}{\partial \gamma_t^s} \geq 0$, $\frac{\partial \xi_\mu}{\partial \gamma_t^d} \leq 0$, $\frac{\partial \xi_\mu}{\partial \delta_t} \leq 0$, $\frac{\partial \xi_\mu}{\partial \varepsilon_t} \leq 0$.

(II) $\nu_1' > \nu_1$ implies $\zeta_{\nu_1'}\left(\lambda_t^s, \lambda_t^d\right) < \zeta_{\nu_1}\left(\lambda_t^s, \lambda_t^d\right)$, $\forall \left(\lambda_t^s, \lambda_t^d\right)$ such that $\lambda_t^s < 1$, $\lambda_t^d = 1$ and $\nu_2' > \nu_2$ implies $\zeta_{\nu_2'}\left(\lambda_t^s, \lambda_t^d\right) > \zeta_{\nu_2}\left(\lambda_t^s, \lambda_t^d\right)$, $\forall \left(\lambda_t^s, \lambda_t^d\right)$ such that $\lambda_t^s = 1$, $\lambda_t^d < 1$;

$\mu_1' > \mu_1$ implies $\xi_{\mu_1'}\left(\gamma_t^s, \gamma_t^d, 1, 1\right) < \xi_{\mu_1}\left(\gamma_t^s, \gamma_t^d, 1, 1\right)$, $\forall \left(\gamma_t^s, \gamma_t^d, 1, 1\right)$ such that $\gamma_t^s < 1$, $\gamma_t^d = 1$ and $\mu_2' > \mu_2$ implies $\xi_{\mu_2'}\left(\gamma_t^s, \gamma_t^d, \delta_t, \varepsilon_t\right) < \xi_{\mu_2}\left(\gamma_t^s, \gamma_t^d, \delta_t, \varepsilon_t\right)$, $\forall \left(\gamma_t^s, \gamma_t^d, \delta_t, \varepsilon_t\right)$ such that $\gamma_t^s = 1$, $\gamma_t^d < 1$.

(III) $\xi_0(\cdot) \equiv \zeta_0(\cdot) \equiv 1$.

This formulation includes the specification in the main text as special case. Other cases are of course possible, for example the linear rule

$$\frac{w_{t+1} - w_t}{w_t} = -\nu_1 \left(1 - \lambda_t^s\right)$$

which can equivalently be expressed as $w_{t+1} = \left(1 - \nu_1 \left(1 - \lambda_t^s\right)\right) w_t$ (and which is in fact a linearization of the one chosen in the text at $\lambda_t^s = 1$). However, simulations have shown that this rule does not give rise to essential differences; as with the previous one, many different dynamic phenomena emerge and, in particular, our finding of the existence of long run Phillips curves is robust with respect to this change in the adjustment rules.

Appendix 4: The complete dynamic system

The dynamic system is given by three different subsystems, one for each of the equilibrium types K, I and C, and endogenous regime switching. For given (G, tax), any list (α_t, π_t, m_t) gives rise to a uniquely determined equilibrium allocation $\left(\overline{L}_t, \overline{Y}_t\right)$ being of one of the above types (or of an intermediate one). This type is determined according to the procedure described in Appendix 2. More precisely,

$$\overline{L}_t = \min\left\{\widetilde{L}\left(\alpha_t, \pi_t, m_t, G, tax\right), L^d\left(1, \alpha_t\right), L^s\right\}$$

with

$$\widetilde{L}\left(\alpha_t, \pi_t, m_t, G, tax\right) = \frac{b}{\alpha_t\left(1 - hb\right)}\left[h\left(1 - tax\right)\pi_t + m_t + G\right],$$

$$L^d\left(1, \alpha_t\right) = n'\left(\frac{\alpha_t}{ab}\right)^{\frac{1}{b-1}}.$$

When $\overline{L}_t = \widetilde{L}(\cdot)$, the K-subsystem applies whereas when $\overline{L}_t = L^d(\cdot)$ or L^s, the systems associated to the types C or I, respectively, are the ones to be used. Regime switching may occur because $\left(\overline{L}_t, \overline{Y}_t\right)$ may be of type $T \in \{K, I, C\}$ and $\left(\overline{L}_{t+1}, \overline{Y}_{t+1}\right)$ of type $T' \neq T$. Regarding the subsystems, they are the following.

Keynesian unemployment system

Employment level: $\overline{L}_t = \widetilde{L}(\alpha_t, \pi_t, m_t, G, tax)$; output level: $\overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t$; rationing coefficients: $\lambda_t^s = \frac{\overline{L}_t}{L^s}, \lambda_t^d = 1, \gamma_t^s = \left[\frac{\overline{Y}_t}{n'a}\left(\frac{\alpha_t}{ab}\right)^{\frac{b}{1-b}}\right]^{1-b}, \gamma_t^d = 1, \delta_t = \varepsilon_t = 1$; price inflation: $\theta_t = (\gamma_t^s)^{\mu_1}$; real wage adjustment: $\alpha_{t+1} = \frac{(\lambda_t^s)^{\nu_1}}{(\gamma_t^s)^{\mu_1}}\alpha_t$; real profit: $\pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t\overline{L}_t\right)$; real money stock: $m_{t+1} = \frac{1}{\theta_t}\left[m_t + G + (1-tax)\pi_t\right] - \pi_{t+1}$.

Repressed inflation system

$\overline{L}_t = L^s; \overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t; \lambda_t^s = 1, \lambda_t^d = \frac{L^s}{L^d(1,\alpha_t)}; \gamma_t^s = 1;$
if $\overline{Y}_t \geq G + m_t$, then $\gamma_t^d = \frac{\overline{Y}_t - m_t - G}{h(1-tax)\pi_t + h\alpha_t\overline{L}_t}, \delta_t = \varepsilon_t = 1;$
if $G + m_t > \overline{Y}_t \geq G$, then $\gamma_t^d = 0, \delta_t = \frac{\overline{Y}_t - G}{m_t}, \varepsilon_t = 1;$
if $\overline{Y}_t < G$, then $\gamma_t^d = \delta_t = 0, \varepsilon_t = \frac{\overline{Y}_t}{G};$
$\theta_t = \left(\frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)^{-\mu_2}; \alpha_{t+1} = \frac{\left(\frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)^{\mu_2}}{\left(\lambda_t^d\right)^{\nu_2}}\alpha_t; \pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t\overline{L}_t\right); m_{t+1} = \frac{1}{\theta_t}\left[\delta_t m_t + \varepsilon_t G + (1-tax)\pi_t\right] - \pi_{t+1}.$

Classical Unemployment System

$\overline{L}_t = L^d(1, \alpha_t); \overline{Y}_t = \frac{\alpha_t}{b}\overline{L}_t; \lambda_t^s = \frac{\overline{L}_t}{L^s}, \lambda_t^d = 1, \gamma_t^s = 1;$
if $\overline{Y}_t \geq G + m_t$, then $\gamma_t^d = \frac{\overline{Y}_t - m_t - G}{h(1-tax)\pi_t + h\alpha_t\overline{L}_t}, \delta_t = \varepsilon_t = 1;$
if $G + m_t > \overline{Y}_t \geq G$, then $\gamma_t^d = 0, \delta_t = \frac{\overline{Y}_t - G}{m_t}, \varepsilon_t = 1;$
if $\overline{Y}_t < G$, then $\gamma_t^d = \delta_t = 0, \varepsilon_t = \frac{\overline{Y}_t}{G};$
$\theta_t = \left(\frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)^{-\mu_2}; \alpha_{t+1} = \left(\lambda_t^s\right)^{\nu_1}\left(\frac{\gamma_t^d + \delta_t + \varepsilon_t}{3}\right)^{\mu_2}\alpha_t; \pi_{t+1} = \frac{1}{\theta_t}\left(\overline{Y}_t - \alpha_t\overline{L}_t\right); m_{t+1} = \frac{1}{\theta_t}\left[\delta_t m_t + \varepsilon_t G + (1-tax)\pi_t\right] - \pi_{t+1}.$

The underconsumption case is not represented with an own dynamical system because choosing $\gamma_t^s = \gamma_t^s\lambda_t^d$ and $\lambda_t^d = 1$ it can be treated as a special case of the Keynesian case.

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