Capital and income risk reconsidered: The case of non–expected utility

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9th May 2000

Abstract

We discuss an economy which is subject to aggregate productivity shocks affecting all factors of production. The additional presence of income risk shifts the margin where agents save out of precautionary motives downwards, such that it does no longer correspond to logarithmic preferences.

Keywords: capital risk, income risk, endogenous growth
JEL classification: D8, O4

1 Introduction

The effects of capital and income risk on intertemporal consumption choice of risk averse agents have originally been discussed by Sandmo (1970) within an expected utility setting. More recently, this issue has been addressed by Weil (1990, 1993) within the framework of non–expected utility preferences. But similar to the work of Obstfeld (1994a, b) or Turnovsky (1999), the author restricts his analysis to a single type of income. The new feature of the model presented here is that we discuss the effects of capital and income risk within one consistent framework. Our approach can be viewed as an extension to Obstfeld (1994b) or Smith (1996). We explicitly take account of non–diversifiable labor income risk and demonstrate that this leads to new insights with respect to the parameter ranges for the intertemporal elasticity of substitution where phenomena as certainty equivalence or precautionary saving occur.

2 The model

We assume an economy populated by a continuum $[0, 1]$ of identical, infinitely–lived agents. Each individual selects the optimal amount of consumption and savings in order to maximize intertemporal welfare $V(t)$. Households have iso–elastic, recursive preferences as in Obstfeld (1994b) which allows for a separate analysis of risk aversion and intertemporal substitution

$$G[(1-\rho)V(t)] = \frac{1-\rho}{1-1/\varepsilon} C(t)^{1-1/\varepsilon} h + e^{-\beta h} G[(1-\rho)\, \mathrm{E}_t V(t+h)]. \tag{1}$$

Here, $C(t)$ is the intertemporal consumption flow which we assume to be instantaneously deterministic. The parameter $\beta > 0$ is the subjective rate of time preference. E_t is the expectations operator conditional on time–t information. The parameter $\rho > 0$ is the coefficient of relative risk aversion and $\varepsilon > 0$ denotes the intertemporal elasticity of substitution. Without loss of generality, we do not discuss the limiting cases where $\rho = 1$ and / or $\varepsilon = 1$. The function $G(\centerdot)$ is given by

$$G(\centerdot) = \frac{1-\rho}{1-1/\varepsilon} \cdot [(1-\rho)V(t)]^{\frac{1-1/\varepsilon}{1-\rho}}. \tag{2}$$

Individual production is stochastic. At each increment of time the economy is subject to an aggregate productivity shock. The representative firm produces a homogeneous good according to the following stochastic Cobb–Douglas–technology

$$dY(t) = K(t)^{\alpha} L(t)^{\gamma} A(t)^{1-\alpha} (dt + \sigma\, dz(t)). \tag{3}$$

We apply the learning–by–doing setting developed by Romer (1986). The instantaneous output flow $dY(t)$ is assumed to be generated from physical capital $K(t)$ and labor $L(t)$. The factors income shares are given by α and γ. We assume $\alpha, \gamma \in (0,1)$ and $\alpha + \gamma \leqslant 1$ to assure the existence of a competitive equilibrium. For analytical convenience, we take labor to be inelastically supplied, and the labor force is normalized to unity. Nevertheless, the household receives an income from both factors of production. In terms of Sandmo (1970), the agent is exposed to *capital risk* and *income risk*. In (2), $dz(t)$ is the increment to a standard Wiener process $z(t)$ with zero mean and the instantaneous variance of production $\sigma^2\, dt$.

The production function (2) exhibits human capital externalities. The stock of technical knowledge $A(t)$ acts as a Harrod–neutral growth parameter and is enhanced by investment in privately owned capital. In macroeconomic equilibrium $A(t)$ equals $K(t)$ and aggregate production is linear in capital. Hence, the requirements for ongoing growth of per capita incomes are met. This assumption together with the assumption stated on the nature of the random disturbance implies that the economy evolves according to a stochastic trend.

Individuals save by investing in risky physical capital. The representative agent is endowed with the initial capital stock $K(0) > 0$. In each time increment t he receives capital and labor incomes. His flow budget constraint is given by

$$dK(t) = [rK(t) + wL(t) - C(t)]\, dt + dk(t), \tag{4}$$

where w is the wage rate and r is the rate of return to physical capital. In equilibrium r equals the private marginal product of capital and falls short of the social return. The stochastic process of capital is defined as $dk(t) = [rK(t) + wL(t)]\sigma dz(t)$ with the corresponding variance of physical capital given by $\sigma_K^2 = \mathrm{E}(dk)^2/dt$.

The consumer's problem is to select his rate of consumption in order to maximize his objective function $V(t)$ as specified by (1) and (2) subject to his budget constraint (4), taking prices as given. The solution conjecture usually applied for isoelastic preferences is that the propensity to consume out of capital $\mu = C/K$ is constant in macroeconomic equilibrium and that investment in capital is determined by constant relative risk aversion. This guess together with the equilibrium values of factor returns $r = \alpha[dt + \sigma dz]$ and $w = \gamma K[dt + \sigma dz]$ which can be obtained by the usual marginal productivity conditions, determines the optimal policy

$$\mu = \varepsilon(\beta - \alpha) + (\alpha + \gamma)\left[1 + \rho\sigma^2(\alpha + \gamma)\left(\frac{\varepsilon\alpha}{\alpha + \gamma} - \frac{\varepsilon + 1}{2}\right)\right]. \tag{5}$$

Substitution of the equilibrium value of the propensity to consume (5) into the budget constraint (4) leads to a closed–form expression for the expected growth rate of the economy

$$\psi = \varepsilon(\alpha - \beta) + (\alpha + \gamma)^2\rho\sigma^2\left(\frac{\varepsilon + 1}{2} - \frac{\varepsilon\alpha}{(\alpha + \gamma)}\right). \tag{6}$$

Equation (5) and (6) show that the propensity to consume out of capital as well as the expected growth rate of the economy are the sum of a drift and a diffusion component, the latter reflecting the agent's optimal response to technological risk.

3 Growth effects of capital and income risk

The optimal values of the propensity to consume (5) and the expected growth rate (6) show that one important characteristic of this class of recursive preferences is maintained. The impact of the coefficient of relative risk aversion ρ on long–run growth is unaffected by the additional presence of income risk. As well known from Weil (1990), the degree of risk aversion just has a size effect on macroeconomic growth.

The main difference between our model and the contributions of Obstfeld (1994b), Smith (1996) and Weil (1990) becomes apparent if we focus our attention on the sign of diffusion term. Whether or not the expected growth rate of the risky environment exceeds the deterministic one is not solely determined by the intertemporal elasticity of substitution. In addition it depends on the factor income distribution. It is easy to verify that the results of Obstfeld (1994b) can be obtained for $\alpha = 1, \gamma = 0$, and the results of Smith (1996), who also accounts for externalities in human capital accumulation, correspond to the case of $\gamma = 0$. Both authors discuss a framework where agents receive only capital incomes. In this case the question of dominance of intertemporal income and substitution effects depends entirely on whether the intertemporal elasticity of substitution exceeds or falls below unity, whereas certainty equivalence can be obtained for $\varepsilon = 1$. If the elasticity of substitution is sufficiently low, that is $\varepsilon < 1$, the agent tries to self–insure against the uncertainty of future income flows. Following Leland (1968) and Sandmo (1970), he has a motive for precautionary savings.

In our model the income shares of both factors of production enter into the diffusion term. For instance, with a change in the variance of the technological shock, the optimal response does not solely depend on $\varepsilon \gtrless 1$ but moreover on the size of α and γ. Let us now turn towards the question, as to what extent the factor income distribution affects growth. We discuss a variation of α and γ respectively, while holding the other income share fixed.[1] The derivatives can be obtained as follows

$$\frac{\partial\psi}{\partial\alpha} = \varepsilon + \rho\sigma^2 \left[\alpha\left(1-\varepsilon\right)+\gamma\right] \tag{7}$$

$$\text{and} \quad \frac{\partial\psi}{\partial\gamma} = \rho\sigma^2 \left[\alpha + \gamma(\varepsilon+1)\right] > 0. \tag{8}$$

An increase in the marginal product of capital α raises both, the mean and the volatility of future income flows. The effect on the drift term is positive, which is not surprising and well known from the deterministic setting. As expected, the effect on the diffusion term is of ambiguous sign. From (7) can be seen, that the growth rate increases with a rise α for every $\varepsilon < 1$. This result corresponds to the findings of Weil (1990) and Obstfeld (1994b) for the case of a pure capital risk. The positive intertemporal income effect outweighs the negative intertemporal substitution effect. But additionally, in the presence of income risk, the intertemporal effects do not offset for $\varepsilon = 1$. Even in this case the growth enhancing effect from an increase in capital returns prevails. The expected growth rate in total declines with an increase of α only if the certainty equivalent to capital return $i = \alpha(1-\rho\sigma^2)$ is sufficiently negative.[2]

Equation (8) shows that in contrast to the deterministic model, here an increase in the labor income share in fact has a positive impact on long–run growth. A ceteris paribus rise in the labor income share leads to an increase in future expected income. Similar to the deterministic setting, labor income does not enter into the drift term of the expected growth rate but, what is more important, its riskiness affects the diffusion term. Within the preference class considered here, the representative agent has a motive for precautionary saving out of income risk for any size of the elasticity of intertemporal substitution $\varepsilon > 0$.

[1] It is understood that with constant returns to scale in production an increase in the capital income share is to be accompanied by a proportional decrease in the labor income share and vice versa in order to sustain a competitive equilibrium. Clemens (1999) demonstrates that the results of this paper hold for the standard expected utility setting where $\varepsilon = 1/\rho$ and for $\gamma = 1-\alpha$.

[2] This effect is discussed by Smith (1996). In general it is not compatible with feasible solutions of the model because the transversality condition is violated in case of a negative certainty equivalent to capital return (see Clemens and Soretz, 1997).

Hence, the results match findings of Weil (1993) for the case of a pure income risk and CIES–CARA preferences and Sandmo (1970) for the expected utility case.

Combination of the growth effects from a change in the capital and the labor income share leads to the following conclusion: If the representative consumer is exposed to capital and income risk the margin of certainty equivalence is shifted downwards and does no longer correspond to a parameter value for the intertemporal elasticity of substitution of $\varepsilon = 1$. The higher the labor income share the earlier the household is inclined to save out of precautionary motives.[3]

4 Conclusions

This paper discussed the effects of a simultaneous appearance of capital and income risk on long–run expected growth. Within a CIES–CRRA framework, we found that (a) the higher the labor income share the earlier agents develop a motive to save out of precautionary motives, and that (b) the case of certainty equivalence does not correspond to logarithmic intertemporal preferences. Our analysis thus extends the results known from contributions which focus solely on either one of the risky income sources.

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[3] Of course, we have to restrict our attention to feasible solutions of the model, especially to positive growth rates.

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Appendix

We assume the value function $J[K(t),t] = \max V(t)$ to represent the maximized value of lifetime utility. We apply Itô's Lemma to derive the stochastic differential of $V(t)$ to get the following Lagrangian

$$\max_{C,K} \mathscr{L} = \frac{1-\rho}{1-1/\varepsilon} C^{1-1/\varepsilon} - \beta G(\cdot) + (1-\rho)\, G'(\cdot) \left[J'(K)\,(rK + w - C) + \frac{1}{2} J''(K)\, \sigma_K^2 \right]. \tag{A.1}$$

In the framework considered here the continuous–time stochastic Bellman equation cannot be employed because of the increasing returns to scale technology. So in addition to taking the derivative of (A.1) with respect to C, we have to differentiate (A.1) with respect to K which leads to the following first–order conditions (for details see Clemens, 1999)

$$0 = C^{-1/\varepsilon} - G'(\cdot)\, J'(K) \tag{A.2}$$

$$\begin{aligned} 0 = G'(\cdot) & \left[J'(K)\,(r-\beta) + J''(K) \left[rK + w - C + \frac{1}{2}\frac{\partial \sigma_K^2}{\partial K} \right] + \frac{1}{2} J'''(K)\, \sigma_K^2 \right] + \\ & + (1-\rho)\, G''(\cdot)\, J'(K) \left[J'(K)\,(rK + w - C) + \frac{1}{2} J''(K)\, \sigma_K^2 \right]. \end{aligned} \tag{A.3}$$

The solution conjecture stated above in the text implies the following derivatives of the value function

$$\begin{aligned} J'(K) &= \mu^{\frac{1-\rho\varepsilon}{1-\varepsilon}} C^{-\rho} > 0 \\ J''(K) &= -\rho\, \mu\, \mu^{\frac{1-\rho\varepsilon}{1-\varepsilon}}\, C^{-(\rho+1)} < 0 \\ J'''(K) &= \rho\,(\rho+1)\, \mu^2\, \mu^{\frac{1-\rho\varepsilon}{1-\varepsilon}}\, C^{-(\rho+2)} > 0. \end{aligned} \tag{A.4}$$

Substitution of (A.2) and (A.4) into (A.3) leads to the expression for the consumption–capital ratio (5) of the text.