Global Dynamics in Macroeconomics: An Overlapping Generations Example*

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Abstract

In this paper we present two techniques used in the dynamical systems literature that let us compute the shape of the stable and unstable manifolds of a given dynamical system. These techniques can be used to study how an economy behaves as it moves far away from the steady state. As a result, we can quantify the difference between a local and a global analysis. In order to illustrate these techniques, we present a general equilibrium model under two different policy regimes demonstrating that the local and global dynamics of an economic system can be substantially different.

1 Introduction

In the early seventies Smale raised some concerns regarding the static nature of the equilibrium concept resulting from Debreu's *Theory of Value* [5]. The theory did not attempt

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to answer the question “how is equilibrium reached?” Instead, dynamic considerations were necessary to resolve this question. As Smale said “sometimes static theories pose paradoxes whose resolution lies in a dynamic perspective” [14]. In particular, the study of macroeconomics can not be completely understood without the use of a dynamic process. Many macroeconomic concepts rely on intertemporal trade offs, thus inherently embracing the concepts and methodology of dynamical systems. Therefore, many of the phenomena that macroeconomists study intrinsically involve a dynamic process. As Costas Azariadis points out, “macroeconomics is about *human interactions over time*” [2].

In the representative agent framework we often encounter problems where it is impossible to attain closed form solutions while iterating the Bellman equation. As a result we need to resort to numerical methods to characterize the equilibria. The approximations used to solve these problems tend to be local, although there is an increasing interest in nonlocal approximations. In this spirit, Gaspar and Judd (1997) address the problem of computing equilibria of large rational expectations models that go beyond the standard linearization method which uses perturbation and projection methods to multidimensional dynamic models. Gaspar *et al* (1997) show that rational expectation models of moderate size can be quickly solved using nonlocal approximation methods. Their research hints that computationally speaking the dominant approach to large models with Euler equation formulations will be fixed-point iteration solution methods combined with complete polynomial bases. The authors suggest that these techniques will outperform Newton-based methods and monotone- and contraction-operator methods [8]. As they argue, “future work should examine the quality of the approximation in various specific contexts, in particular competitive equilibria”. In the same spirit, Guu and Judd (1997) show the gain of a Taylor series expansion approach while studying the global properties in an aggregate growth model [9]. Finally, there is an increasing number of articles that address the importance of complex dynamics and global analysis in non representative agent frameworks. For example, Yokoo (2000) investigates the global dynamics using a perturbation method identifying conditions under which homoclinic points to the golden rule are generated with an overlapping generations model [16]. Similarly, Bischi, Gardini and Kopel (2000) analyze global bifurcations in a marketing model of market share attraction [3].

The purpose of this paper is similar in spirit to Gaspar and Judd (1997) and Guu and Judd (1997). In this paper we characterize the dynamic behavior of a particular economy as it moves away from the steady state. We treat the dynamical system as given as is the case in most overlapping generation models. In order to investigate the global dynamics of an economy, we introduce some mathematical techniques used in the dynamical systems literature. We explore global properties of a dynamical system using successive local approximations.[1]

2 Importance of global dynamics in macroeconomics

The pioneering modern macroeconomic models tended to be one dimensional systems. Nowadays, many macroeconomic models are described by multidimensional systems. In the one dimensional world the phase diagram can describe the most important aspects of any dynamical system.[2] Once we move away from the one dimensional world the study of global dynamics becomes more complicated. For instance, higher dimensional systems have the possibility of both expansion and contraction of the same invariant set, which gives rise to new nonstationary equilibria.

A first step towards a better understanding of the global behavior of a given macroeconomic system or, in general, a dynamical system, is to identify its invariant objects, such as fixed points, periodic orbits or their associated invariant manifolds. The steady states of an economic system are interpreted as descriptions of the long run behavior of the economy; i.e, the fixed points of the dynamical system or if there are fluctuations, the corresponding periodic orbits. But if one wants to study the transition dynamics of the economy, one needs to characterize how the dynamical system evolves through time. One may classify these steady states by studying the temporal evolution of a point that is near the steady state; i.e, its stability. In particular, we can classify a steady state as *stable* if all orbits that start near it stay near it, and *asymptotically stable* if all orbits that start near it converge to the fixed point. On the other hand, we classify a steady state as *unstable* if all orbits that start near it move away from it. Finally, as we move from the one dimensional world a new type of stability arises, *saddle path stability.* In this situation an orbit that starts near the steady state stays near it only for a given subset of initial conditions.[3] With this classification in mind, one can then describe the short run effect of different policy regimes.

[1]This methodology is similar to Judd's perturbation and projection methods.

[2]A classic example in macroeconomics is Solow's (1956) unidimensional law of motion for capital accumulation. We note that even with one dimensional systems, one can find very complicated dynamics. A classic example of complex behavior is the logistic map which can give rise to cycles of any period.

[3]This latter type of situation is the most frequent in economics.

Among the invariant objects, the invariant manifolds of codimension 1 are very important because they split the phase space into non connected regions. The standard practice in economics is to consider the linear approximations of these manifolds.[4] But when we contemplate the nonlinear properties of these manifolds their associated non connected regions can behave quite differently. For example, when we consider global properties we may find transport phenomena and resonances. Furthermore, these non linear manifolds of codimension 1 can also introduce *barriers* to the dynamical system. In other words, they can impose some physical obstructions to the existence of invariant curves when studying area preserving maps. It is therefore convenient to have fast computational methods that enable us to study these non linear manifolds which may shed some light on some economic puzzles.

Finally, there are several reasons why being able to describe the properties of a dynamical system is of paramount interest. One of the most important is the fact that once we determine the characteristics of the dynamical system, there is the possibility of *controlling* the system. If economics was like physics, with its controlled experiments, one could, in principle, choose *preferred paths* for the economy by setting appropriate controls and initial conditions. Unfortunately, economics does not have as many *degrees of freedom* as in the natural sciences.[5] In particular, in the physical sciences one is able to chose the initial conditions. In economics, on the other hand, the initial conditions are always given.

The fact that we can not choose initial conditions and the fact that most economists are interested in how exogenous shocks are transmitted through the economy strongly suggests the study of global dynamics. Furthermore, if shocks to the economy are sufficiently large and the change in policies substantially alters the laws of motion of our economy, then a local analysis may be somewhat uninformative. Lastly, the advantages of considering a global analysis is that we can determine the quality of the local approximation. Furthermore, a global analysis can also capture new dynamical phenomena not observed when performing linear analysis. For instance, wandering cycles which can predict cyclic patterns of different periods, and homoclinic points which can yield chaotic behavior. These new phenomena may be able to shed some light on some economic puzzles like the changing periodicity of

[4]These models investigate how the economy evolves as we perturb the economy slightly away from the steady state. This local analysis can be considered as a first approximation to the dynamical problem.

[5]In a world with multiple equilibria that can be Pareto ranked initial conditions become quite relevant. In such a world, the policy maker, in principle, could pick initial conditions to rule out *inferior* equilibria.

business cycles or the different rates of convergence.

3 Global analysis

In this section we present some mathematical techniques used in the dynamical systems literature that enable us to study the behavior of a system away from the steady state.[6] In particular, we primarily focus our attention on techniques that let us learn more about the shape of the stable and unstable manifolds of a given dynamical system.

In order to clarify ideas, consider a generic dynamical system. The dynamical system that we consider is a discrete planar system, which is given by the following smooth map $F = (f, g) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$. The motion of this dynamical system is given by a sequence of points $X_t = (x_t, y_t)$ such that

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} f(x_t, y_t) \\ g(x_t, y_t) \end{pmatrix}. \tag{1}$$

Once the steady states are determined, we can study how the system evolves through time as we move away from the steady state. For convenience, we choose the coordinates so that the steady state is at the origin. If our fixed point is hyperbolic, the corresponding Jacobian matrix of the fixed point has no eigenvalue with modulus equal to one, then local behavior around a hyperbolic fixed point is like the linearized system in terms of continuity (but not differentiability necessarily), as asserted by the Hartman-Grobman theorem. In many instances one can get differential or analytic equivalence using the normal forms theorems of Poincare, Dulac and Siegel (see [1] or [11] for a detailed description of the theorems).

We will further assume that the fixed point has real eigenvalues. One eigenvalue, λ_s, is inside the unit circle and the other one, λ_u, is outside of it.[7] The corresponding eigenvectors for λ_s and λ_u are v^s and v^u respectively.

In order to clarify ideas let's assume that the phase diagram corresponding to the dynamical system is described by Figure 1; where the linear stable and unstable subspaces $E^s = < v^s >$ and $E^u = < v^u >$ become the stable and unstable manifolds $\mathcal{W}^s$ and $\mathcal{W}^u$ when the

[6] In the dynamical system literature, local analysis refers to a linear description of the system and global analysis refers to a nonlinear description.

[7] As is the case in most economic applications.

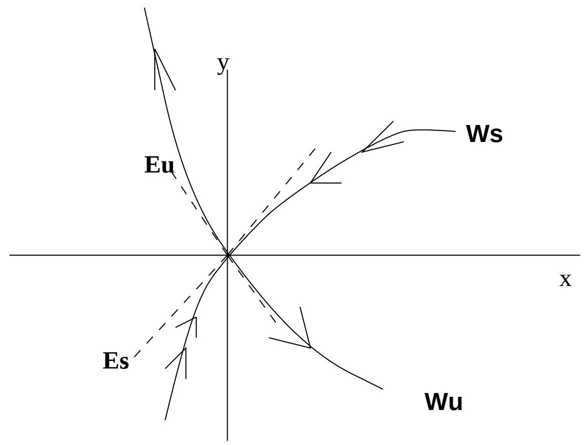

Figure 1: Phase diagram of our dynamical system.

entire dynamical system is considered. Moreover, the degree of differentiability of these manifolds is of the same order as the original map [1].

Our goal is then to determine the shape of these manifolds away from the steady state. In particular, we already know that near the steady state these manifolds can be described by straight lines as is suggested by Figure 1. Therefore, our problem is then reduced to the description of a manifold in $\mathbb{R}^2$. We can think the manifold in terms of a graph in $\mathbb{R}^2$. Or we can interpret the manifold as being defined as a result of a particular parameterization in $\mathbb{R}^2$.

The techniques presented in the next section can be generalized to dynamical systems of higher dimensions. The major drawback is the increased computational complexity, although there is an extensive literature on dynamical systems that focuses on higher dimensional systems [13].

3.1 The graphical approach

The basic idea of this approach is to appropriately characterize the manifolds in terms of a graph in the (x, y) plane. First, we can consider a linear change of coordinates such that $v^s = (1, 0)$ and $v^u = (0, 1)$. As a result, we can greatly simplify the calculations. Then, the stable manifold can be thought in terms of a graph as follows

$$\mathcal{W}^s : \quad y = \Phi(x).$$

Similarly for the unstable manifold

$$\mathcal{W}^u: \quad x = \Psi(y).$$

Without loss of generality, we focus our attention on the construction of the stable manifold. Since the stable manifold is tangent to the corresponding linear space at the steady state, we know that the following conditions have to be verified: $\Phi(0)$=0 and $\Phi'(0)$=0.[8] Furthermore, since the stable manifold is invariant, any point on it moves to another point on the manifold. In other words,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ \Phi(x) \end{pmatrix} \longrightarrow \begin{pmatrix} f(x,\Phi(x)) \\ g(x,\Phi(x)) \end{pmatrix}. \tag{2}$$

The resulting *invariance equation* for this dynamical system is then

$$g(x,\Phi(x)) = \Phi(f(x,\Phi(x))). \tag{3}$$

Notice that since x determines the position on the graph, the evolution of this variable which is given by

$$x \rightarrow f(x,\Phi(x))$$

completely determines the dynamics on $\mathcal{W}^s$.

In order to further characterize the global properties of the stable manifold, we Taylor expand this unknown function, $\Phi(x)$, around the steady state $(0,0)$ which is given by

$$\Phi(x) = \phi_2 x^2 + \phi_3 x^3 + \ldots,$$

where the '...' represent terms of higher order.

We then perform a similar expansion for the dynamical system

$$f(x,y) = \lambda_s x + f_2(x,y) + \ldots$$
$$g(x,y) = \lambda_u y + g_2(x,y) + \ldots,$$

where f_k, g_k are homogeneous polynomials of order k.[9] Note that due to the initial linear transformation the resulting Taylor expansion is greatly simplified. From now on, we shall use the notation $f_{\leq k}(x,y)$=$f_2(x,y) + \ldots + f_k(x,y)$ to denote a function up to order k.

[8] Note that if $v^s = (v^s_x, v^s_y)$ we need $v^s_x \neq 0$ because $\Phi'(0)$=$\frac{v^s_y}{v^s_x}$.

[9] A homogeneous quadratic polynomial is denoted by $f_2(x,y) = f_{xx}x^2 + f_{xy}xy + f_{yy}y^2$.

In contrast to the known coefficients corresponding to the polynomials f_k, g_k describing the dynamical system, we need to find the unknown coefficients ϕ_k corresponding to the stable manifold. The main idea of this graphical approach is to find these unknown coefficients by matching polynomials of the same order. Notice that we already know the first two terms since $\Phi(0)=0$ and $\Phi'(0)=0$. In order to obtain the k^{th} order term with $k\geq 2$, we impose the invariance condition up to order k. In other words, we need to compose the different functions, as suggested by equation (3), keeping track of the coefficients of the same order and cutting off the terms of order greater than k. We can then find closed form solutions for these unknown coefficients, ϕ_k.[10]

Algebraically speaking, the calculation of these coefficients ϕ_k is as follows. Assuming that the coefficients corresponding to $\Phi_{<k}$ have already been computed, we describe a recurrence method. In what follows the '...' represent terms of order higher that k. The left hand side term of the invariance equation can then be written as follows:

$$g(x, \Phi(x)) \quad = \quad \lambda_u \left(\Phi_{<k}(x) + \phi_k x^k\right) + g_{\leq k}(x, \Phi_{<k}(x)) + \ldots,$$

Similarly, we have the following expression for the right hand side term of the invariant equation,

$$\Phi(f(x, \Phi(x)) \quad = \quad \Phi_{<k}(\lambda_s x + f_{<k}(x, \Phi_{<k})) + \phi_k(\lambda_s x)^k + \ldots.$$

Thus, the k^{th} coefficient that we need to solve for our invariance condition is given by the following linear equation, the *homology equation*,

$$\begin{aligned}(\lambda_u - \lambda_s^k)\phi_k x^k &= (\Phi_{<k}(\lambda_s x + f_{<k}(x, \Phi_{<k})))_{\leq k} - \lambda_u \Phi_{<k}(x) - (g_{\leq k}(x, \Phi_{<k}(x)))_{\leq k} \\ &= c_k x^k.\end{aligned}$$

Notice that the terms of order lower than k vanish because $\Phi_{<k}$ satisfies the invariance equation up to the order k-1. As a result, the only remaining term is $c_k x^k$. Therefore, we can easily solve for ϕ_k which is given by

$$\phi_k = \frac{c_k}{\lambda_u - \lambda_s^k}.$$

As we can see, whenever we have a saddle fixed point we can always compute all the coefficients corresponding to the Taylor expansion of the manifold. Notice that this method

[10] This approach is similar in spirit to Judd's perturbation method [10].

can be generalized to other cases, although resonances may appear. Suppose, for instance, that the two eigenvalues satisfy the following condition, $0<\lambda_f<\lambda_s<1$ where 'f' and 's' stand for fast and slow, respectively.[11] In this situation the fixed point is stable, attracting all points in a neighbourhood around the steady state. When solving the homology equation for the 'fast' invariant manifold, $\phi_k(\lambda_s - \lambda_f^k) = c_k$, we realize that ϕ_k is well defined $\forall k$. However, if we want to compute the 'slow' invariant manifold a resonance may appear; that is, there may exist a value of k such that $\lambda_f{=}\lambda_s^k$. Under these circumstances, we can not solve the homology equation as long as $c_k \neq 0$. The 'slow' manifold is dynamically interesting because the dynamics around the fixed point collapses into it since the 'fast' component approaches the fixed point very rapidly. See Cabré *et al* (1999) for a detailed description of this sort of phenomena [4].[12]

3.2 The parameterization approach

The basic goal of the parameterization approach is to describe the manifolds through an appropriate parameterization in the (x, y) plane. In particular, for the stable manifold we may chose the following parameterization W^s: $\tau \rightarrow \Omega(\tau)$ and the dynamics on the manifold is given by $\tau \rightarrow \lambda_s \tau$. As we can see, this approach ensures that the iterated point is just a "translation" on the manifold. In other words, we are fixing the dynamics on the manifold.[13]

The resulting *invariance equation* is then given by the following expression

$$F(\Omega(\tau)) = \Omega(\lambda_s \tau), \tag{4}$$

where we recall that $F = (f, g) : \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}\times\mathbb{R}$ with $\Omega(0) = (0, 0)$. Now taking derivatives on the both sides of the equation and setting $\tau = 0$, we have the following condition

$$\mathrm{D}F(0, 0)\mathrm{D}\Omega(0) = \lambda_s \mathrm{D}\Omega(0),$$

where $\mathrm{D}\Omega(0)$ must be an eigenvector of the linearized system whose eigenvalue is λ_s.

In order to gather information about the global properties of the stable manifold we proceed as in the previous case. In other words, we Taylor expand around the steady state

[11] Let v_f and v_s denote the corresponding eigenvectors.

[12] This class of techniques has been recently used to study chemical processes and reactions [7].

[13] This approach is similar in spirit to Judd's projection method [10].

$$\Omega(\tau) = \begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} = \begin{pmatrix} x_1\tau + x_2\tau^2 + \dots \\ y_1\tau + y_2\tau^2 + \dots, \end{pmatrix} = \omega_1\tau + \omega_2\tau^2 + \dots,$$

keeping in mind that our goal is to find the unknown vectors ω_2, $\omega_3, \dots$; where $\omega_1 = (v_x^s, v_y^s)$.

Let A=D$F(0,0)$ denote the linear part of our system. Suppose we already know $\omega_2, \dots, \omega_{k-1}$ and we want to obtain ω_k. The left-hand side of the invariance equation up to order k is given by

$$F(\Omega(\tau)) = A\ (\Omega_{<k}(\tau) + \omega_k\tau^k) + F_{\leq k}(\Omega_{<k}(\tau)) + \dots$$

Similarly for the right-hand side of the invariant equation,

$$\Omega(\lambda_s\tau) = \Omega_{<k}(\lambda_s\tau) + \omega_k\lambda_s^k\tau^k + \dots.$$

Thus, the k^{th} vector from the invariance condition, equation (4), can be represented by the following linear equation

$$(A - \lambda_s^k I)\tau^k\omega_k = \Omega_{<k}(\lambda_s\tau) - A\Omega_{<k}(\tau) - (F_{<=k}(\Omega_{<k}(\tau)))_{\leq k} = z_k\tau^k,$$

where z_k is a known vector. We can easily solve this linear system and solve for ω_k, which is given by

$$\omega_k = (A - \lambda_s^k I)^{-1} z_k.$$

As we saw earlier, if the fixed point is a saddle point we can not find resonances. This technique can also be generalized to higher dimensions and other types of stability. Finally, we may find resonances whenever det(A-$\lambda_s^k I$) = 0.

3.3 Numerical considerations

In the previous sections we have examined two different approaches to compute the manifolds of a given dynamical system. We now explore the numerical advantages and disadvantages of their implementation. Since both methods rely on the invariant condition, we can use this invariant property to find any possible programming errors. Note that at every step k we can see if the computation is satisfactory because the invariance condition up to order k-1 has to be satisfied.[14]

[14]The k^{th} order term gives rise to the homology equation.

There are some computational drawbacks that we need to consider. From a geometrical point of view, the parametric approach seems to be more suitable. In principle, the graphical approach can only be used when the underlying dynamical system yields well-behaved manifolds, that is, they *"do not bend backwards"*. This problem can be avoided, however, by taking a small enough 'fundamental' domain of the curve. We can then recover the entire manifold by iterating this fundamental domain.

The Taylor expansions used in the previous sections give an approximation of the manifold that is better than its linear approximation. But even in the analytic case, where we can compute 'all' the terms corresponding to a particular expansion, we may have a finite radius of convergence. For instance, the function $y=1-\frac{1}{1+x^2}$ is well defined over the real numbers, but its Taylor expansion around 0, $y = x^2 - x^4 + x^6 - \ldots$, is not defined for $|x| \geq 1$. As a result, the corresponding manifolds of the planar dynamical system may not be defined $\forall\ \mathbb{R}^2$, which should be taken into account when programming so we do not consider these regions of our domain.

4 An example: An overlapping generations model

In this section we consider a monetary general equilibrium model under two different policy rules taken from Schreft and Smith (1997) [12]. This paper addresses the implications for monetary policy when there is an exogenous decreasing demand for cash. The economy consists of two identical islands with limited communication. These islands are inhabited by an infinite sequence of two period lived overlapping generations. Agents have an endowment of the single nonstorable good when young and nothing when old. In addition, agents derive utility from consumption only when old. To generate a demand for cash transactions, the model incorporates spatial separation and limited communication along the lines of Townsend (1987) [15]. To generate a role for banks, the model includes shocks to liquidity needs resembling those in Diamond and Dybvig (1983) [6]. Furthermore, the underlying assets in this economy are government-issued fiat currency and government bonds. It is assumed that fiat money is dominated in the rate of return. Therefore, banks will hold the minimum amount of cash possible so that they can meet the liquidity needs of their depositors. These features imply a derived demand for base money that depends on the need for currency in payments and the demand by banks for cash reserves.

With this environment in mind, we explore how the economy dynamically evolves under different policy rules, targeting inflation and having a constant money growth rate. Furthermore, we compare the corresponding manifolds that one obtains using a local versus a global approach.

4.1 Targeting inflation

Under this scenario the Central Bank targets the inflation rate. The dynamical system corresponding to this policy rule is given by the following system of equations

$$P = (I_t - (I_t - 1)\, \gamma(I_t, \pi_t)) \tag{5}$$

$$\pi_{t+1} - \bar{\pi} = \mu(\pi_t - \bar{\pi}) \tag{6}$$

where I_t represents the gross nominal interest rate for bond holdings, π_t denotes the fraction of households that require cash in order to transact, $\mu(<1)$ represents the speed at which the money demand decreases, P is the targeted inflation rate and $\gamma(I_t, \pi_t)$ is the optimal cash reserves held by banks. For this particular model, the optimal cash reserves is given by

$$\gamma(I_t, \pi_t) = \frac{1}{1 + \frac{1-\pi_t}{\pi_t} I_t^{\frac{1-\rho}{\rho}}} \tag{7}$$

where ρ is the coefficient of relative risk aversion.

As we can see, this dynamical system is a decoupled system; i.e, the dynamics of I_t are inherited by the dynamic behavior of π_t. Furthermore, since the dynamics of π_t are determined by a linear system, the local and global analysis are exactly the same. Therefore, the local dynamics corresponding to this particular policy regime does not introduce any approximation error nor new dynamical phenomena. As a result, the corresponding manifold is just a straight line.

4.2 Constant money growth rate

Under this policy rule the Central Bank keeps printing fiat money at a constant rate. In other words,

$$M_{t+1} = \sigma M_t \tag{8}$$

where M_t is the total money supply at time t and σ is the constant money growth rate. The dynamical system corresponding to this policy rule is given by the following system of equations

$$I_{t+1} = \left(\frac{\pi_{t+1}}{\sigma(1-\pi_{t+1})}\left(\frac{1-\pi_t}{\pi_t}I_t^{\frac{1}{\rho}} - (\sigma-1)\right)\right)^{\frac{\rho}{1-\rho}} \tag{9}$$

$$\pi_{t+1} - \bar{\pi} = \mu(\pi_t - \bar{\pi}). \tag{10}$$

The dynamics of the gross nominal interest rate are much more complicated than the previuos example. The effect of a decreasing demand for money now influences how the nominal interest rate changes over time. As a result, we may find very different local and global dynamics.

4.2.1 A numerical example

In order to explore the difference between a local versus a global approach we numerically computed the shape of the manifolds. As a simplification, we are going to consider economies with ρ=0.5 in order to avoid rational exponents. As a result, we can compute as many coefficients corresponding to the manifold as desired. The resulting economy evolves as follows,

$$I_{t+1} = \frac{\pi_{t+1}}{\sigma(1-\pi_{t+1})}\left(\frac{1-\pi_t}{\pi_t}I_t^2 - (\sigma-1)\right) \tag{11}$$

$$\pi_{t+1} - \bar{\pi} = \mu(\pi_t - \bar{\pi}). \tag{12}$$

We next compute the stable and unstable manifolds corresponding to an economy with the following properties μ=0.8, σ=1.1 and $\bar{\pi}$=0.2, see Figure 2.

As we can see, the stable and unstable manifolds do a better job of describing the dynamics as we move farther away from the steady state than do their linear counterparts. In this case, the linear unstable space coincides with the unstable manifold. On the other hand, the linear stable space and the stable manifold are not the same. The stable manifold is computed using the parameterization method up to order 50. The convergence radius of the series is approximately 0.363.[15]

In order to emphasize the error associated with the local approximation, we have performed two of numerical exercises. When studying the phase diagram of a planar system,

[15]Throughout our numerical exercise we used a Pentium III processor and the time required for our program to compute all the manifolds was approximately one quarter of a second.

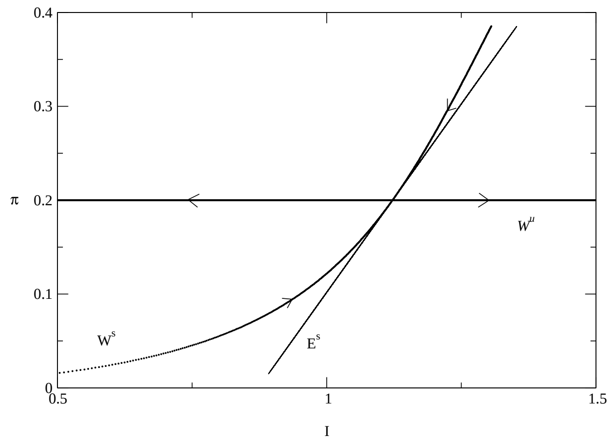

Figure 2: The stable and unstable manifolds and their linear counterparts.

we divide the phase space into four regions. These regions then determine the trajectory of a point if it were to fall into that region. In order to see how different these trajectories may be, we compute the image of the linear stable manifold. By doing so we can determine how invariant the linear approximation really is, see Figure 3.

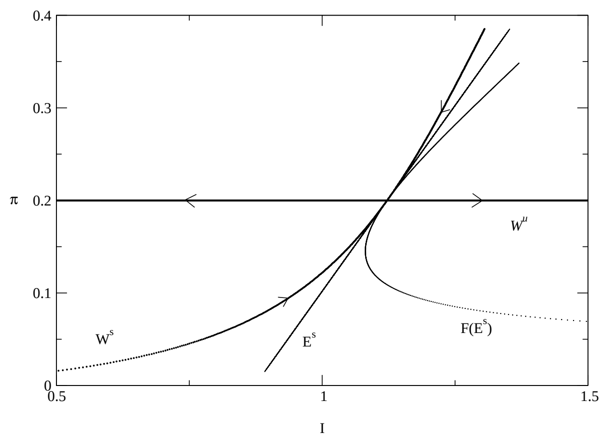

Figure 3: The manifolds corresponding to ρ=0.5, μ=0.8, σ=1.1 and $\bar{\pi}$=0.2.

The trajectories corresponding to the linear manifold are different from the nonlinear manifold. As a result, we find that the predictions corresponding to the linear manifold can be substantially misleading since they allow trajectories that are otherwise not possible.

We also consider how the dynamical system evolves as we increase the money growth rate. As shown in Figure 4, the shape of the stable manifold becomes more linear near the steady state as the money growth rate increases. This is because the stable eigenvalue is a decreasing function of the money growth rate.

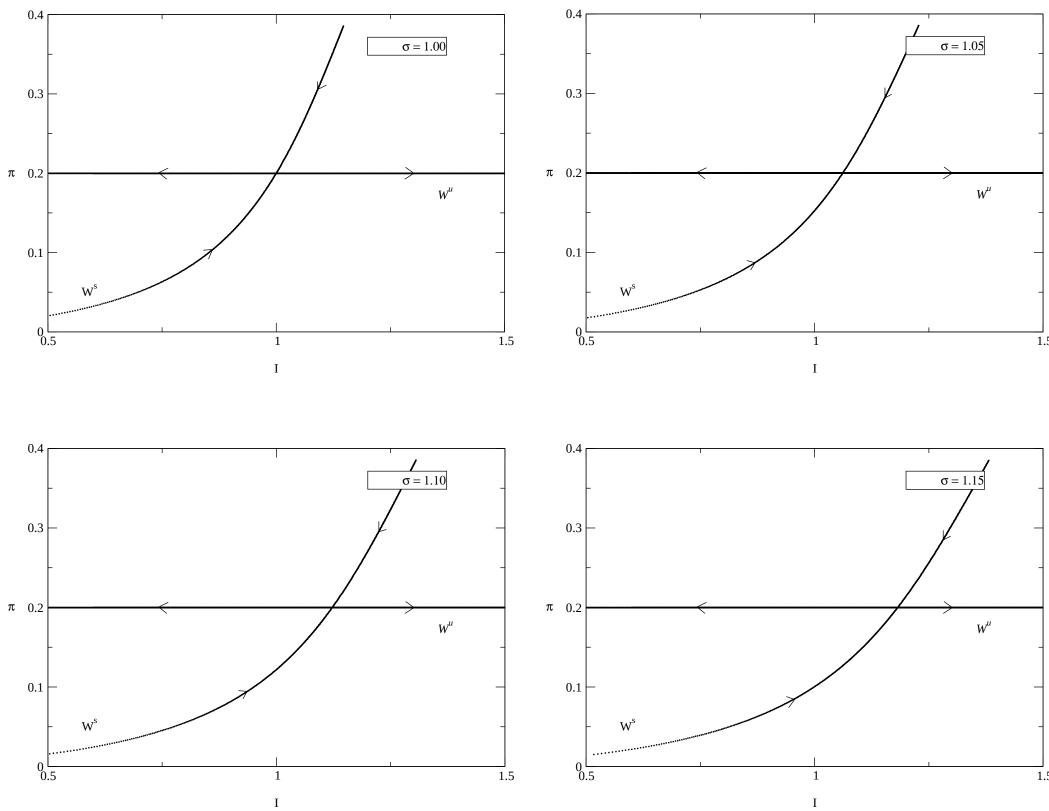

Figure 4: The manifolds corresponding to ρ=0.5, μ=0.8, and $\bar{\pi}$=0.2.

Finally, we explore how the linear error evolves as the money growth rate changes; see Figure 5. We define the error, ϵ, as the distance between a point on the nonlinear manifold and the corresponding point on the linear manifold, and d as the distance between the steady state and a point on the linear manifold. The error decreases as the money growth rate increases. As a result, the stable manifold becomes more linear since the dynamical system is attracting points with greater "strength".

As we can see, the error associated with the local analysis is a decreasing function of the money growth rate. Therefore, as we take into account the global properties of the dynamical system the number of possible patterns corresponding to the predicted time series increases. Finally, we note that the error function is not symmetric which emphasazies the importance of how we approach the steady state. As a result, different initial conditions may result in different speeds of convergence.

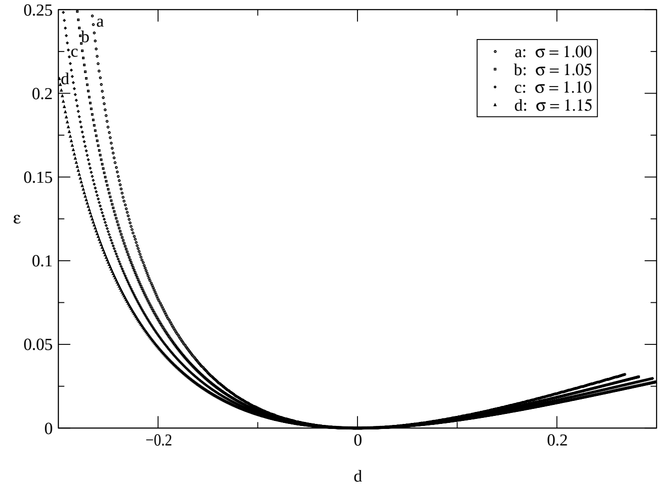

Figure 5: The error functions for different σs when ρ=0.5, μ=0.8, and $\bar{\pi}$=0.2.

5 Conclusions

In this paper we present two techniques used in the dynamical systems literature that let us compute the shape of the stable and unstable manifolds of a given dynamical system. These techniques can be used to study how an economy behaves as it moves far away from the steady state. These techniques let us quantify the difference between the local analysis and the global analysis. Furthermore, one can also determine the existence of new dynamical phenomena like wandering cycles that can not be found when performing a local analysis.

Finally, this paper presents an application of these techniques. We consider an overlapping generation model under two different policy regimes and study how the corresponding manifolds evolve as we move away from the steady state. We find that the resulting dynamical properties of a local and global analysis can be quite different depending on what kind of target is been used. Therefore, as we consider more complicated models with increased heterogenity and imperfect competition the errors associated with a local analysis will probably increase as well as the possibility of finding complex dynamics; thus suggesting the importance of incorporating global analysis into our models.

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