The Blanchard and Kahn's conditions in macro-econometric models with perfect foresight[1]

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Abstract: Many recent macro-econometric models assume perfect foresight. This choice was made possible by the development of simulation algorithms, which are powerful and easy to use. However, the existence and the uniqueness of a solution for these models are not warranted *a priori.* Blanchard and Kahn established local conditions for these properties, which are easy to check, in terms of eigenvalues computed at the steady state of the model. However, these conditions can only be used on linear models, with coefficients independent of time, and with exogenous variables taking constant values after some date. Unfortunately, macro econometric models are non-linear, their linear approximation has coefficients which change over time, in the long run many variables grow at positive and different rates, and these models may present an hysteresis. This paper explains how to overcome these difficulties, and apply the Blanchard and Kahn's conditions on this kind of models. Our results can also be applied to the study of the stability of more traditional macro-econometric models, which assume adaptive expectations, and where the current state of the economy does not depend on the future states foreseen by the model.

Classification JEL: C3

0. Introduction

Many recent macro-econometric models assume perfect foresight, for instance the multinational model of the IMF, Multimod Mark 3, or the model of the European Commission, Quest 2. This choice was made possible by the development of simulation algorithms which are at the same time powerful and easy to use. For instance, an efficient relaxation algorithm was implemented by Juillard (1996) in the software Dynare which works under Gauss, and by Juillard and Hollinger in the command Stacks of Troll. However, the existence and the uniqueness of a solution for such models are not *a priori* warranted

Blanchard and Kahn (1980) established local conditions for the existence and uniqueness of a solution, which are especially easy to check in terms of eigenvalues computed at the steady state of the model. However, these conditions only apply to linear models, the coefficients of which do not depend on time, and such that the exogenous variables can be assumed to be constant after some time.

Thus, we are very far from the features of large macro-econometric models. To be able to use the results by Blanchard and Kahn we must first require that the model determines a balanced growth path. It is not necessary that this path represents a realistic approximation of a recent past of future, and we know that industrialized and developing

[1] The ideas developed in this paper came progressively to us while we were working to the building of the multinational model of Cepii and Cepremap, Marmotte. Thus, they have much benefited from discussions with my colleagues of Cepii: Loïc Cadiou, Stéphane Dées and Stéphanie Guichard. The comments by Antoine d'Autume, Michel Juillard and Pierre Malgrange on a preliminary version were extremely helpful. This paper benefited of a grant from the Commissariat Général du Plan. We want to thank all these persons and institution, while assuming all responsibilities for the weaknesses of our research.
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economies do not grow in the neighbourhood of a balanced growth path[3]. We just want, for the model to be economically consistent, that he can generate after an horizon which may be in a distant future, a reasonable balanced growth path[4].

When this condition is satisfied, we can compute the linear approximation of the model around its balanced growth path, and we can require that the solution of the model tends to this path when time increases indefinitely. This condition will be called stability in the absolute difference. However, some of the coefficients of the linear approximation appear as geometric functions of time, and we cannot apply the results by Blanchard and Kahn. However, we can apply them if we put all the variables on a common trend. If this trend has a zero growth rate, we will say that the model has been written in reduced variables, and that we require its stability in the relative difference. If the growth rate of the common trend is the highest balanced growth rate present in the model, we will say that we have written the linear approximation of the model in expanded variables, and that we require its stability in the expanded difference. In both cases, we can apply the results by Blanchard and Kahn. If they are satisfied for the model written in reduced variables and for its linear approximation written in expanded variables, then the model determines a unique solution stable in the absolute difference.

The first section presents the results by Blanchard and Kahn, with its extension to the case of hysteresis, which was developed by Giavazzi and Wyplosz (1986). In the second section we present a very simple example, which allows us to investigate some difficulties which are met in macro-econometric models, but which were not considered by Blanchard and Kahn. The third section uses a richer example: the endogenous growth model by Lucas, which includes all the difficulties we want to face in this paper. After these two examples we can give a complete theoretical treatment of the problem in section 4.

1. The Blanchard et Kahn's conditions: A reminder[5]

A dynamic linear model with perfect foresight and with coefficients independent of time can always be written in the following form:

(1) $C_1 y_{t-1}^1 + C_0 y_t + C_{-1} y_{t+1}^2 = U_t$, $t \geq 1$, with C_0 invertible.

To get this form we may have to introduce artificial variables to eliminate variables appearing with a lag or a lead greater than one, and to prevent a given variable from appearing simultaneously with leads and with lags. The endogenous variables belong to one of the three mutually exclusive classes which follow: n_1 variables appear in a contemporary or lagged form; They are denoted as predetermined. n_2 other variables

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[3] For example, Maddison (1996) shows that for all industrialized countries over the period 1973-1991, productive capital increased at a faster rate than production. Foreign trade also increased faster than output, and the various sectors of the economy exhibited very contrasted trends.

[4] The horizon after which we can reasonably assume that the economy follows a balanced growth path is high, let us say 20 years for a model poor in demographic variables, much more for a model richer in this respect. Thus, the economists who consider that a model must be Keynesian in the short term and neo-classical in the long-run, and who interpret the long-run as the system of equations which defines the balanced growth path, may create a misunderstanding by their readers. The neo-classical long-run, that is the horizon when all the nominal rigidities can be neglected, is a much nearer horizon, a few years, and it has no reason to exhibit smooth dynamics. Of course, this *a fortiori* implies that the balanced growth path of the model must be neo-classical.

[5] We followed in this section the presentation of our paper of 1990.

appear in a contemporary or led form; They are denoted as anticipated. The n_3 last variables only appear in a contemporary form; They are denoted as static. These three categories of variables constitute at time t the column vectors y_t^1, y_t^2 and y_t^3. The piling up of these vectors in the order: y_t^2, y_t^3 and y_t^1 defines the vector of the endogenous variables y_t of dimension: $n = n_1 + n_2 + n_3$.

Let us denote by: C_0^2, C_0^3 and C_0^1 the three matrices with respectively n_2, n_3 and n_1 columns. The concatenation of these matrices gives the matrix $C_0 : C_0 = (C_0^2 \mid C_0^3 \mid C_0^1)$. Let us make the change of variables: $x_t^1 = y_t^1$, $x_t^2 = y_{t+1}^2$, $x_t^3 = y_t^3$, and let us denote the piling up of these vectors in the order: x_t^3, x_t^1, x_t^2, by x_t. Then, the model can be rewritten:

(2) $C_1 x_{t-1}^1 + C_0^2 x_{t-1}^2 + (C_0^3 \mid C_0^1 \mid C_{-1}) x_t = U_t$.

In general, for x_{t-1}^1 and x_{t-1}^2 given, equation (2) does not determine a unique value for x_t: To get this absence of uniqueness it is sufficient that some anticipated variables appear in a led form always in the same linear combination. However, it is possible to make a series of eliminations and transformations of anticipated variables to put the model in the case where the uniqueness of x_t is warranted[6]. Thus, it is not restrictive to consider, in the rest of the section, the system, which could *a priori* look more specific:

(3) $x_t = A x_{t-1} + h_t$, $t \geq 1$.

We assume that all the static variables were eliminated; Then, the dimension of vectors x_t and h_t is $n_1 + n_2$, and A is a square matrix with the same dimension. The difficulty with system (3) is that, if it is justified to assume that the initial value of the predetermined endogenous variables: $x_0^1 = y_0^1$ is given, we cannot make the same assumption for the initial values of the anticipated endogenous variables: $x_0^2 = y_1^2$. However, it seems justified to require that if h_t is permanently fixed at a constant value $\bar{h}$, then there exists a unique path for the endogenous variables which tends to a finite value, which is the steady state of system (3), let be: $\bar{x} = (I - A)^{-1} \bar{h}$. We will see that this condition, which we will call stability of the model, implicitly defines x_0^2.

We will assume now that matrix A can be reduced to a diagonal form[7]. We will denote by Λ_1 the diagonal matrix of the $n_1^{'}$ eigenvalues of absolute values less than or equal to 1,

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[6] Sims (1997) and Juillard (1999) give a solution to this problem. It rests upon the computation of generalized eigenvalues based on a generalized decomposition of Schur. Then, the variables with a lead which can be eliminated are as many as there are infinite eigenvalues. Juillard used this method in its Gauss software, Dynare (Juillard, 1996), and with Hollinger in the command Lkroot of Troll.

[7] This means that if we have a multiple eigenvalue, we can associate to it the same number of eigen vectors as its order of multiplicity. This assumption can be removed at the cost of a heavier presentation, which we have preferred avoiding (see the paper by Blanchard and Kahn).

and by W_1 the matrix of dimension $n_1^{'}(n_1+n_2)$ of the associated left eigen vectors[8]. We can deduce from (3):

(4) $W_1 x_t = \Lambda_1^{'} W_1 x_0 + \sum_{j=1}^{t} \Lambda_1^{t-j} W_1 \overline{h}$.

We make assumption H_1 that $\overline{h}$ is not orthogonal to any of the rows of W_1. Then, for $W_1 x_t$ to be bounded, there must not exist any eigenvalues of absolute value equal to 1 in Λ_1. We define Λ_2 and W_2 in a similar way for the eigenvalues with absolute values larger than 1, of number $n_2^{'}$. We make assumption H_2 that W_{22}, which is the matrix built with the $n_2^{'}$ last columns of W_2, is regular, and we denote by W_{21} the matrix built with the other columns of W_2. We deduce from (3):

(5) $W_2 x_t = \Lambda_2^{t} (W_2 x_0 + \sum_{j=1}^{t} \Lambda_2^{-j} W_2 \overline{h})$.

For $W_2 x_t$ to be bounded, the expression between brackets must tend to zero when t increases indefinitely, let be:

(6) $W_2 x_0 = -\sum_{j=1}^{\infty} \Lambda_2^{-j} W_2 \overline{h}$.

This condition is also sufficient, because in this case $W_2 x_t$ can be written:

(7) $W_2 x_t = -\sum_{i=1}^{\infty} \Lambda_2^{-i} W_2 \overline{h}$.

The value of x_0^1 and relation (6) constraint the initial state of the economy x_0. For a value of this vector verifying these restrictions, equations (4) and (5) determine a unique path converging toward the steady state of model (3) (the square matrix got by piling up W_1 and W_2 is regular and it can be easily checked that (3) is satisfied). For x_0 to exist and be unique, it is necessary and sufficient that: $n_2 = n_2^{'}$.

Proposition 1 (Blanchard et Kahn (1980)). Under assumptions H_1 and H_2, the necessary and sufficient condition for equation (3) to determine a unique and stable path, is that matrix A has as many eigenvalues of absolute values smaller than 1 as there exist predetermined endogenous variables, and as many eigenvalues of absolute values larger than 1 as there are anticipated variables.

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[8] An eigenvalue of multiple order appears in Λ_1 as many times as its order of multiplicity, and a basis of the space generated by its eigen vectors appears in W_1. The same rule will be used later with Λ_2 and W_2 .

We can make a more general assumption than H_1, which is that there can exist eigenvalues of absolute value equal to 1, if $\bar{h}$ is orthogonal to the eigen vectors related to these eigenvalues. In general, this property results from exclusion relations in the structural form of the model. So, we can require that it is satisfied for all the vectors $\bar{h}$ having an economic meaning. Then, Proposition 1 stays valid, if we consider that n_1 represents the number of eigenvalues of absolute values less than *or equal to* 1. Let us consider in Λ_1 and W_1 a higher part Λ_1^* and W_1^* and a lower part Λ_1^{**} and W_1^{**}, respectively related to the eigenvalues of absolute values equal or smaller than 1. Their respective dimensions are n_1^* and n_1^{**}. Then, $W_1^{**}x_t$ and $W_2 x_t$ still tend to $W_1^{**}\bar{x}$ and $W_2\bar{x}$ when t increases indefinitely. However, this property does not hold for: $W_1^* x_t = \Lambda_1^{*t} W_1^* x_0 = W_1^* x_0$. This permanent dependency of the path relatively to its initial conditions is called hysteresis, and characterizes some economic mechanisms[9]. When there does not exist any eigenvalue equal to 1, $\bar{x}$ is unique. Otherwise, it belongs to a linear variety with dimension n_1^*. This problem was investigated by Giavazzi and Wyploz (1986). More precisely, we get the result:

Proposition 2. Under assumption $H_1^{'}$, that $\bar{h}$ is not orthogonal to any of the rows of W_1^{**} and is orthogonal to W_1^*, and assumption H_2, the necessary and sufficient condition for equation (3) to define a unique and stable path is that matrix A has as many eigenvalues of absolute values less or equal to 1 as there exists predetermined endogenous variables, and as many eigenvalues of absolute values larger than 1 as there exists anticipated variables.

The previous results cannot be directly applied to the case of macroeconometric models with perfect foresight. Actually, these models present the following properties: a) There does not exist a steady state for the endogenous and the exogenous variables, but a balanced growth path; b) On this path the various variables do not have the same growth rate; c) The linear approximation of the model in the neighbourhood of a balanced growth path has coefficients which change over time; d) Hysteresis does not manifest by the identical reproduction of initial conditions over time for some variables or combinations of variables, but by an expansion at a geometric rate endogenous to the model.

2. A simple example

In this example the model includes only one equation:

(8) $y_t = \lambda y_{t-1} + x_t$, y_0 given, $t \geq 1$.

y_t is the endogenous variable, x_t is the exogenous variable, λ is an adjustement parameter and t represents time. We will assume that x_t grows geometrically at rate g

[9] We can notice that in the case of hysteresis, stability does not refer to the endogenous variables of the model, but to the product of the eigen vectors non related to the unitary eigenvalue by the vector of the endogenous variables. As most economists, we give in this paper to the term hysteresis a meaning which is very different from the one given by mathematicians inspired by the theory of magnetism. The mathematical concept can have useful applications in economics, for instance in the case of temporary interventions by the Government which tries to solve problems of co-ordination failure or to act on irreversible decisions by private agents. These points are discussed by Amable, Henry, Lordon and Topol (1995). The concept of pseudo-hysteresis, which will be introduced later, is our own.

larger or equal to 1. We will also assume, until paragraph 2.3 which will investigate the question of hysteresis, that λ differs from g. We have:

(9) $x_t = x_0 g^t$,

where x_0 is the initial variable of the exogenous variable. We easily get the expression of the solution of the model:

(10) $y_t = x_0 g^t [1-(\lambda/g)^t]/(1-\lambda/g) + y_0 \lambda^t$.

Thus, this example is limited to the case where there do not exist anticipated variables. We can notice that Proposition 1 can be applied to this case: When there are no anticipated variables, the convergence of model (3) to its steady state, *that is* its stability, requires that all its eigenvalues are of absolute values less than 1. The specificity of this case is that it is easy to compute an unstable solution, by a succession of recursive resolutions from present to future. On the other hand when some variables are anticipated, the concept of unstable solution becomes complex, and such solutions, if we succeed in defining them, cannot be easily computed by the usual softwares. This example will allow us to investigate in a simple way the concept of stability in an economy where variables grow permanently.

2.1 A definition of stability in absolute difference

We will start by defining a balanced growth path of rate g, Y_t, by using the steady-state model:

(11) $Y_t = \lambda Y_t / g + x_0 g^t$, $t \geq 0$

The solution of this equation is:

(12) $Y_t = x_0 g^t /(1-\lambda/g)$.

The difference between the path effectively followed by the economy and the balanced growth paths is:

(13) $y_t - Y_t = -x_0 \lambda^t /(1-\lambda/g) + y_0 \lambda^t$.

We will say the model to be stable in the absolute difference when this difference tends to zero when time increases indefinitely. Thus, a necessary and sufficient condition of stability is: $-1 < \lambda < 1$.

An interesting case is when the eigenvalue of the model λ is exactly equal to 1, with: $g > 1$. Then, the balanced growth path of the economy Y_t exists and is defined without ambiguity. However, the effective path y_t differs from it by an amount which remains constant over time and which depends on the initial state of the economy: $y_0 - x_0/(1-1/g) = y_0 - Y_0$. As the balanced growth path of the economy exists and is unique, and as the effective path of the economy differs from it by an amount which becomes *relatively* smaller and smaller over time, we will not call this property *hysteresis*. However, we can define *pseudo-hysteresis* as the existence of unitary eigenvalues in the model written with its original variables.

2.2. A definition of stability in relative difference

To introduce this second definition we must start by rewriting the model in reduced variables. To do that we define the endogenous reduced variable:
(14) $y_t^{'} = y_t / g^t$.

The exogenous reduced variable is x_0. The model in reduced variables can be written:

(15) $y_t^{'} = (\lambda / g) y_{t-1}^{'} + x_0$, $y_0^{'} = y_0$ given, $t \geq 1$.

The steady state of the model is Y_0 defined by equation (11) applied at time zero. We will say the model to be stable in relative difference if $y_t^{'} - Y_0$ tends to zero when time increases indefinitely. Thus, a necessary and sufficient condition for stability is: $-1 < \lambda / g < 1$. This condition can be expressed as the requirement for the eigenvalue of the transition matrix of the model written in reduced variables, to be of absolute values smaller than 1.

The second condition of stability is less strict than the first. It is also the one which is usually applied to theoretical models of growth where it is frequent to work with models written in reduced variables. The case when: $1 < \lambda < g$, corresponds to a model where the absolute difference between the effective path and the balanced growth path increases indefinitely over time, but where the relative difference tends to zero. If: $\lambda = 1$, the absolute difference remains constant, but the relative difference tends to zero. Thus, we will extend our previous definition and call the situation when the model is stable in relative difference and unstable in absolute difference, *that is* the case when: $1 \leq \lambda < g$, *pseudo-hysteresis.*

We can notice that the stability in the absolute difference requires that: $y_t - Y_t$ tends to zero when time increases indefinitely. This stability in the relative difference requires that it is to $(y_t - Y_t) g^{-t}$ to have this property. Then, we can notice two things. First, we could define an intermediary stability where it is: $(y_t - Y_t) g^{'-t}$, with : $0 < g' < g$, which tends to zero. We could express this requirement by the fact that $y_t - Y_t$ must be of an order smaller than $g^{'t}$ when t tends to infinity. Secondly, the stability in the relative difference *reduces* the variables to a zero trend, which comes to put them on a common trend when they follow different trends in their original forms. In this last case, the stability in the absolute difference does not use this adjustment to a common trend and concerns variables which grow at different rates. Then, it is natural to consider another fixation to a common trend which is the highest balanced growth path present in the model. Then, we *expand* the variables having a smaller long run growth rate. The stability of the model written in this last way will be called stability in the expanded difference. The convergence in the relative difference is then a necessary condition for the convergence in the absolute difference, and the convergence in the expanded difference is a sufficient condition.

When the model is stable in the relative difference and unstable in the expanded difference, we will say that it presents a pseudo-hysteresis.

2.3. The case of hysteresis

Until now, we have assumed that the adjustment parameter λ was different from the growth rate of the exogenous variables g. Now, let us assume that these two parameters are equal. Then, the model can be written:

(16) $y_t = gy_{t-1} + x_0 g^t$.

The solution of this equation is:

(17) $y_t = tx_0 g^t + y_0 g^t$.

The steady-state model can be written:

(18) $Y_t = Y_t + x_0 g^t$.

In the case where x_0 is non zero, this model has no solution. In the case where x_0 is zero, this model has an infinity of solutions. Then, the steady-state model does not define a reference balanced growth path relatively to which a concept of stability could be given. The model written in reduced variables has for solution:

(19) $y_t^{'} = tx_0 + y_0$.

For x_0 non zero, the model diverges linearly, and the effect of the initial value of the endogenous variable y_0 stays eternally present. If x_0 is zero we have:

(20) $y_t^{'} = y_0$, and: $y_t = y_0 g^t = y_0 \lambda^t$.

Then, the model written in its original form has an eigenvalue equal to g. In its reduced form it has an eigenvalue equal to 1, and we find again the definition of hysteresis given in section 1.

3. A complete example: The Lucas model

3.1. The model

The model of endogenous growth by Lucas[10] is much richer than the previous model, has the advantage of having an economic meaning and includes all the technical ingredients which are met in large macro-econometric models: Different balanced growth rates for the different variables, predetermined and anticipated variables, hysteresis. Certainly, the economic mechanisms which are included are fairly different from those present in large macro-econometric models. In general, these last models assume exogenous real growth

[10] A good reference for this model is chapter 5 of the book by Barro and Sala-I-Martin (1995). These two authors present the model in its time-continuous version, which has the advantage of simplifying some computations.

rates and the hysteresis property they may have is relative to prices and not to the level of activity. However, these differences are secondary for our purpose, which is to give an example richer than the previous one. The two basic equations of the model are:

(21) $C_t + K_t = K_{t-1} + AK_{t-1}^{\alpha}(g^t u_t H_{t-1})^{1-\alpha}$

(22) $H_t = H_{t-1} + B(1-u_t)H_{t-1}$

There are two production factors: Physical capital K_t and human capital H_t, the depreciation rates of which are assumed to be zero. The output of physical commodity is determined by a Cobb-Douglas production function, and is allocated between the accumulation of physical capital and consumption C_t. We will differ of the original model by Lucas by introducing an exogenous technical progress in the production function of physical commodity, specified by the trend g^t, with : $g > 1$. Human capital is produced only with human capital. Its quantity available at period t is allocated between the productions of physical commodity and of human capital, in the proportions u_t and $1-u_t$. The optimization criterion of the households is: $\sum_{t=0}^{\infty}(1+\rho)^t \log(C_t)$. As there are neither distortions nor externalities on markets, the economic equilibrium is identical to the optimum, and it suffices to maximize the households' criterion under the constraints of equations (21) and (22). We get the first order conditions:

(23) $(1+\rho)C_{t+1}/C_t = 1 + \alpha A(g^{t+1}u_{t+1}H_t/K_t)^{1-\alpha}$

(24) $(1+\rho)C_{t+1}/C_t = (1+B)g^{1-\alpha}[(K_t/u_{t+1}H_t)/(K_{t-1}/u_t H_{t-1})]^{1-\alpha}$

The model has four equations and four endogenous variables: C_{t+1}, H_t, K_t, u_{t+1}. H_t and K_t are predetermined variables, the initial values of which are given. C_{t+1} and u_{t+1} are anticipated variables. Then, we notice that, with a change of variables, we can decompose this dynamic system into two blocks, which can be solved successively. To do that we introduce the variables deflated by human capital:

(25) $k_t = K_t/H_t \quad c_t = C_t/H_t$,

The first block of dynamic equations is:

(26) $[1+B(1-u_t)](c_t + k_t) = k_{t-1} + Ak_{t-1}^{\alpha}(g^t u_t)^{1-\alpha}$

(27) $[1+B(1-u_t)](1+\rho)c_{t+1}/c_t = 1 + \alpha A k_t^{\alpha}(g^{t+1}u_{t+1})^{1-\alpha}$

(28) $[1+B(1-u_t)](1+\rho)c_{t+1}/c_t = (1+B)g^{1-\alpha}[(k_t/u_{t+1})/(k_{t-1}/u_t)]^{\alpha}$

This block determines the dynamics of variables c_{t+1}, k_t and u_{t+1}. k_t is a predetermined variable the initial value of which is known, c_{t+1} et u_{t+1} are anticipated variables. The second block is limited to equation:

(29) $H_t / H_{t-1} = 1 + B(1 - u_t)$

3.2. Analysis of the first block of equations

The first step of the analysis is the computation of the balanced growth path generated by this block: $c_{t+1} = \bar{c} g^{t+1}$, $k_t = \bar{k} g^t$, $u_{t+1} = \bar{u}$. We notice that the two first variables grow at a geometrical rate, and that the third remains constant. This diversity in the trends followed by the various variables is a characteristic of large macro-econometric models, which was lacking in the previous example. We easily get the three expressions:

(30) $\bar{u} = \dfrac{\rho(1+B)}{B(1+\rho)}$

(31) $\bar{k} = g \dfrac{\rho(1+B)}{B(1+\rho)} \left[\dfrac{\alpha A}{g - 1 + Bg} \right]^{1/(1-\alpha)}$

(32) $\bar{c} = (\bar{k} / \alpha)[1 + \rho - \alpha - \dfrac{(1+\rho)(1-\alpha)}{(1+B)g}]$

To have: $0 \le \bar{u} \le 1$, we must assume: $\rho \le B$. The second step is the computation of a linear approximation of the model in the neighbourhood of the balanced growth path. We denote by a ^ the difference between the effective value taken by a variable and its balanced growth path at the same time. We get:

(33) $$\begin{aligned} & -Bg(\bar{c}/\bar{k} + 1)\hat{u}_t + [(1+B)/(1+\rho)](g/\bar{k})[g^{-t}\hat{c}_t + g^{-t}\hat{k}_t - (\bar{c}/\bar{k} + 1)g^{-t+1}\hat{k}_{t-1}] = \\ & (1-\alpha)A(g\bar{u}/\bar{k})^{1-\alpha}(\hat{u}_t/\bar{u} - g^{-t+1}\hat{k}_{t-1}/\bar{k}) \end{aligned}$$

(34) $$\begin{aligned} & (g^{-t}\hat{k}_t/\bar{k} - g^{-t+1}\hat{k}_{t-1}/\bar{k} + \hat{u}_t/\bar{u} - \hat{u}_{t+1}/\bar{u})(1+B)g = \\ & (1-\alpha)A(g\bar{u}/\bar{k})^{1-\alpha}(\hat{u}_{t+1}/\bar{u} - g^{-t}\hat{k}_t/\bar{k}) \end{aligned}$$

(35) $$\begin{aligned} & -[B(1+\rho)/(1+B)]\hat{u}_t + g^{-t-1}\hat{c}_{t+1}/\bar{c} - g^{-t}\hat{c}_t/\bar{c} = \\ & \alpha(g^{-t}\hat{k}_t/\bar{k} - g^{-t+1}\hat{k}_{t-1}/\bar{k} + \hat{u}_t/\bar{u} - \hat{u}_{t+1}/\bar{u}) \end{aligned}$$

We notice that we can define several concepts of stability of the model. The most natural is to require that each variable tends to its balanced growth path value when time increases indefinitely, that is: $\hat{c}_{t+1}, \hat{k}_t, \hat{u}_{t+1} \to 0$ when $t \to \infty$. For the two first variables, the convergence is to a path which grows at the geometrical rate: $g > 1$, but for the third variable the convergence is to a fixed value. In the previous section we called this property, stability in the absolute difference. The problem with this concept is that the dynamic linear system of the equations (33), (34) et (35) is non autonomous relatively to time, or more precisely that it has some coefficients which decrease geometrically over time. Thus, we cannot use the results by Blanchard and Kahn and get local conditions for the existence and uniqueness of a solution in terms of eigenvalues computed at an equilibrium state. There exists at least two natural ways to overcome this difficulty.

The first consists in requiring that the variables, corrected of their balanced growth trend, that is the reduced variables, tend to the initial value of their balanced growth path. The reduced variables of the linear approximation of the model are: $\hat{c}'_{t+1} = g^{-t-1}\hat{c}_{t+1}, \hat{k}'_t = g^{-t}\hat{k}_t, \hat{u}_{t+1}$, and we require that they all tend to 0 when t increases indefinitely. The system, rewritten in reduced variables is:

(36) $$-Bg(\bar{c}/\bar{k}+1)\hat{u}_t + [(1-\delta+B)/(1+\rho)](g/\bar{k})[\hat{c}'_t + \hat{k}'_t - (\bar{c}/\bar{k}+1)\hat{k}'_{t-1}] = (1-\alpha)A(g\bar{u}/\bar{k})^{1-\alpha}(\hat{u}_t/\bar{u} - \hat{k}'_{t-1}/\bar{k})$$

(37) $$(\hat{k}'_t/\bar{k} - \Delta\hat{k}'_{t-1}/\bar{k} + \hat{u}_t/\bar{u} - \hat{u}_{t+1}/\bar{u})(1-\delta+B)g = (1-\alpha)A(g\bar{u}/\bar{k})^{1-\alpha}(\hat{u}_{t+1}/\bar{u} - \hat{k}'_t/\bar{k})$$

(38) $$-[B(1+\rho)/(1-\delta+B)]\hat{u}_t + \hat{c}'_{t+1}/\bar{c} - \hat{c}'_t/\bar{c} = \alpha(\hat{k}'_t/\bar{k} - \hat{k}'_{t-1}/\bar{k} + \hat{u}_t/\bar{u} - \hat{u}_{t+1}/\bar{u})$$

The second solution consists in *expanding* the variables of the model, which means giving them as common trend the highest balanced growth rate of the model g. Then, we require that these expanded variables tend to their balanced growth path. The expanded variables of the linear approximation of the model are: $\hat{c}_{t+1}, \hat{k}_t, \hat{u}''_{t+1} = g^{t+1}\hat{u}_{t+1}$, and we require that they tend to zero when t increases indefinitely. The system rewritten in expanded variables is:

(39) $$-Bg(\bar{c}/\bar{k}+1)\hat{u}''_t + [(1+B)/(1+\rho)](g/\bar{k})[\hat{c}_t + \hat{k}_t - (\bar{c}/\bar{k}+1)g\hat{k}_{t-1}] = (1-\alpha)A(g\bar{u}/\bar{k})^{1-\alpha}(\hat{u}''_t/\bar{u} - g\hat{k}_{t-1}/\bar{k})$$

(40) $$(\Delta k_t/\bar{k} - g\Delta k_{t-1}/\bar{k} + \Delta u''_t/\bar{u} - g^{-1}\Delta u''_{t+1}/\bar{u})(1+B)g = (1-\alpha)A(g\bar{u}/\bar{k})^{1-\alpha}(g^{-1}\Delta u''_{t+1}/\bar{u} - \Delta k_t/\bar{k})$$

(41) $$-[B(1+\rho)/(1+B)]\hat{u}''_t + g^{-1}\hat{c}_{t+1}/\bar{c} - \hat{c}_t/\bar{c} = \alpha(\hat{k}_t/\bar{k} - g\hat{k}_{t-1}/\bar{k} + \hat{u}''_t/\bar{u} - g^{-1}\hat{u}''_{t+1}/\bar{u})$$

The second stability condition is stricter than the first, and this explains why the two dynamic systems which are related to them are different. For the system written in reduced variables, we get a first eigenvalue equal to: $1-\frac{(1-\alpha)(g-1+Bg)}{g-1+\alpha+Bg}$. We easily see that this value is included between 0 and 1. The two other eigenvalues are equal to 1 plus the roots of the equation in λ: $\lambda^2 - [\rho(\bar{c}/\bar{k}+1) + \bar{c}/\bar{k}] + \rho\bar{c}/\bar{k} = 0$. We can easily see that these roots are real and positive. More precisely, their product is equal to $\rho\bar{c}/\bar{k}$ and their sum to $\rho(\bar{c}/\bar{k}+1) + \bar{c}/\bar{k}$. As there are as many eigenvalues larger (smaller) than 1 as there exist anticipated (predetermined) variables, the Blanchard and Kahn's conditions are satisfied and the Lucas's model has a unique solution (according to the criteria of the stability in the relative difference). We can easily show that the eigenvalues of the system written in expanded variables are equal to the previous eigenvalues time g. The growth rate g should take a very high value to have the three eigenvalues higher than 1, and to reach the conclusion that the Lucas's model has no solution according to the criteria of the stability in the expanded difference. We recall that this situation was denoted *pseudo-hysteresis.*

To define a stability which can be associated with an autonomous linear dynamic system to which we can apply the results by Blanchard and Kahn, we adjusted all the variables of the model to a common trend. In the first case this trend had zero growth rate, in the second case its growth rate was the highest balanced growth rate present in the model. Of course, we can choose a rate intermediary between these two last values.

We must point here to a tricky element of our results. The eigenvalues associated with the stability in the expanded difference are greater than the eigenvalues associated with the stability in the relative difference. The Blanchard and Kahn's conditions require the first of these eigenvalues to be smaller than 1, and the two others to be greater than 1. This last requirement is severer for the stability in the relative difference than for the stability in the expanded difference, that is for the less strict stability criteria. The solution to this apparent contradiction is simple. If the conditions of Blanchard and Kahn are satisfied for the model written in expanded variables, the solution of the model stable in the expanded difference *a fortiori* satisfies the stability in the absolute difference. However, it is possible that other solution paths of the model satisfy this less strict stability condition. *A fortiori* all these solutions satisfy the stability in the relative difference. Thus, if for the model written in reduced variables the Blanchard and Kahn's conditions are still satisfied, there can only exist one solution path of the model which is stable in the absolute difference. In conclusion, the satisfaction of the Blanchard and Kahn's conditions for the model written in reduced and in expanded variables is a sufficient condition for the existence and the uniqueness of a solution stable in the absolute difference. In heuristic terms, the stability in the expanded difference warrants the existence of a solution and the stability in the reduced difference implies its uniqueness.

More generally, if in macro-econometric models the Blanchard and Kahn's conditions are satisfied by the model written in expanded variables but not in reduced variables, this implies that there exists an infinity of solutions, stable in relative differences. If the Blanchard and Kahn's conditions are verified for the model written in reduced variables, but not in expanded variables, this implies that there does not exist any solution stable in the expanded difference. We could continue to call pseudo-hysteresis these two situations. However, the meaning of the concept would become a bit fuzzy then.

3.3. Analysis of the second block of equations and conclusion on hysteresis

This block is limited to equation:

(29) $H_t / H_{t-1} = 1 + B(1 - u_t)$

In the long run u_t tends to its steady state $\bar{u}$, and the right-hand side of equation (29) tends to: $(1+B)/(1+\rho)$, which is the last eigenvalue of the model. As we have assumed condition: $\rho \leq B$, we deduce that the growth rate of human capital is non negative in the long run. We can notice that we are in a situation of hysteresis: Human capital eternally depends on its initial value. This dependency is transmitted to other variables of the model written in its initial form, more precisely to physical capital and consumption. What is new is that human capital, which is the variable at the origin of hysteresis, has a positive long run growth rate which is determined by the steady state of the first block of equations of the model, which means that it is endogenous. In a more elaborated model we could make it sensitive to economic policy.

We could have analysed the Lucas's model in a more direct way, by computing the balanced growth rates of all the original variables in the model. These rates would of course have depended on the solution of the steady-state model. Then we would have rewritten the model in reduced variables. We would have got an eigenvalue equal to 1, resulting from the property of hysteresis, and we could have applied Proposition 2: Two eigenvalues smaller or equal to 1, and as many predetermined variables. We should have had to check the orthogonality of the eigen vector associated with the unitary eigenvalue, with the right-hand side of equation (3). If we had worked with expanded variables, the previous unitary eigenvalue would have become equal to the highest balanced growth rate in the model g.

Most of the endogenous growth model can be analysed in a way similar to the one presented here. However, large macro-econometric models generally assume that growth is exogenous. However, they often present an hysteresis which has many similarities with the one which was investigated here. A model of a closed economy has an inflation rate which is determined in the long run by the monetary rule followed by the Central Bank. Quite frequently, this is the case for the rules put by Taylor and Furher and Moore, it links a nominal short-term interest rate to inflation rates and to indicators of the level of activity. Thus, the steady state of the model determines the inflation rate, but leaves undetermined the price level[11]. Over time this last variable will remain dependent on its initial value. Thus, we are exactly in the situation of the Lucas's model if we substitute the endogenous real growth rate by the inflation rate, and human capital by the price level. Then, the first block of equations determines variables expressed in volumes and the inflation rate. The second block determines the price level and the variables expressed in value. A model of an open economy will have the same property if the exchange rate is flexible, which will prevent the price level from being determined in the steady state by exogenous foreign prices. A multinational model where the rules of monetary policy would be relative to rates and not to levels, would have the same property of hysteresis: The number of price levels and consequently of exchange rates minus 1, undetermined in the steady state, would be equal to the number of monetary policy rules expressed in rates[12].

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[11] Giavazzi and Wyplosz (1985) give an example of this kind of hysteresis. The result that the price level is undetermined when monetary policy sets the nominal interest rate, comes back to Wicksell at the beginning of the century. Juillard (1999) investigates in a systematic way this kind of hysteresis for several monetary policy rules.

[12] Béraud (1998) gives another example of hysteresis. She considers a neo-classical economy of two countries. Each country has specific utility and production functions. Both of them have the same discount rate and the international capital market is perfect. A first block of equations of the model determines the path followed by all the world

4. Theoretical analysis

A perfect foresight model can be written[13]:

(42) $F(y_t, y_{t+1}, y_{t-1}, x_t) = 0$, y_0 given.

F is a vector of n equations, y is the column vector of the n endogenous variables, x is the column vector of the m exogenous variables (which can include time). At time t, the model determines the current values of the endogenous variables y_t in function of the values of these variables which are anticipated for the future or which were observed in the past, y_{t+1} and y_{t-1}, and of the values of the exogenous variables x_t.

We will assume that this model can determine a balanced growth path. To get this property we assume that there exists two diagonal matrices g and h, with respective dimensions n and m, such that for all vector $\bar{x}$ belonging to a subset Ω of R^m, there exists (at least) a vector $\bar{y}$ of R^n, satisfying:

(43) $F(g^t \bar{y}, g^{t+1}\bar{y}, g^{t-1}\bar{y}, h^t \bar{x}) = 0$,

g and h are the vectors of the growth rates of, respectively, the endogenous variables and the exogenous variables. These rates are assumed to be equal or larger than 1. The initial values of the balanced growth paths of the endogenous and exogenous variables are related by:

(44) $F(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x}) = 0$.

$(\bar{x}, \bar{y})$ will be called a steady state of the model[14] and (x_t^s, y_t^s) = $(h^t \bar{x}, g^t \bar{y})$ will be a balanced growth path. We can see that the existence of a balanced growth path requires that equation (44) has a solution for all $\bar{x} \in \Omega$. We will assume that this solution $\bar{y}$ belongs to a subset Φ of R^n. But it is also necessary for function F to exhibit a homogeneity property implying that relation (43) is satisfied when equation (44) is verified. This property can be written:

(45) $F(g^t \bar{y}, g^{t+1}\bar{y}, g^{t-1}\bar{y}, h^t \bar{x}) \equiv F(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x})$

for all $\bar{x} \in \Omega$ and $\bar{y}$ solution of (44).

---

aggregated variables and the allocation of production between the two countries. Hysteresis appears in a second block of equations which determines the distribution of the ownership of the world capital between the two countries, and consequently, the distribution of consumption. These distributions will permanently depend on the initial state of the economy. This hysteresis is absent from multinational models Quest 2 and Multimod Mark 3, but we do not know which features of these models are responsible for the disappearance of this property, which seems to us rather robust.

[13] The result that we can write under this form a very general rational expectations model, when the random shocks are small enough so we can approximate the expected value of a function by the function of the expected values of its variables, was proven by Broze, Gouriéroux and Safarz (1989) and Laffargue (1990).

[14] In the case of hysteresis there exist several solutions. We choose one of them in an arbitrary way.

Let us denote by $F_1^{'}$, $F_2^{'}$ and $F_3^{'}$ the matrices of the partial derivatives of F relatively to the vectors of the endogenous variables appearing respectively without lead and lag, with a lead and with a lag. More precisely, $F_1^{'}$ represents a square matrix with dimension n, the rows of which refer to equations and the columns to the contemporaneous endogenous variables relatively to which the derivation was computed. $F_2^{'}$ and $F_3^{'}$ are defined in the same way, but for the variables appearing respectively with a lead or with a lag. Let us now compute a linear approximation of model (42) in the neighbourhood of a balanced growth path:

$$
\begin{aligned}
&F(y_t, y_{t+1}, y_{t-1}, x_t^s) = F(y_t^s, y_{t+1}^s, y_{t-1}^s, x_t^s) + \\
&F_1^{'}(y_t^s, y_{t+1}^s, y_{t-1}^s, x_t^s)(y_t - y_t^s) + F_2^{'}(y_t^s, y_{t+1}^s, y_{t-1}^s, x_t^s)(y_{t+1} - y_{t+1}^s) \\
(46)\quad &F_3^{'}(y_t^s, y_{t+1}^s, y_{t-1}^s, x_t^s)(y_{t-1} - y_{t-1}^s) = \\
&F_1^{'}(y_t^s, y_{t+1}^s, y_{t-1}^s, x_t^s)(y_t - y_t^s) + F_2^{'}(y_t^s, y_{t+1}^s, y_{t-1}^s, x_t^s)(y_{t+1} - y_{t+1}^s) \\
&F_3^{'}(y_t^s, y_{t+1}^s, y_{t-1}^s, x_t^s)(y_{t-1} - y_{t-1}^s) = 0
\end{aligned}
$$

We deduce from this expression a first definition: The stability in the absolute difference is the tendency of every endogenous variable of the model to its balanced growth path value when time increases indefinitely:

$y_t - y_t^s \rightarrow 0$, when $t \rightarrow \infty$

This definition gives the justification of the most natural way to simulate the model. Model (42) represents a system of finite difference equations with initial conditions (on the predetermined variables) and with final conditions (on the anticipated variables). We select a time horizon long enough and we choose as terminal conditions at this horizon the equality of the anticipated variables to their balanced growth values. Then, we can use the usual algorithms developed to solve this kind of mathematical problem.

However, in linear approximation (46), the matrices of the coefficients depend on time, so we cannot use the results by Blanchard and Kahn[15]. To be able to use these results we must first strengthen the homogeneity property (45), in a way which does not look restrictive from a practical point of view. Then, we impose condition:

(47) $F(g^t y_1, g^{t+1} y_2, g^{t-1} y_3, h^t x_1) \equiv k^t F(y_1, g\ \ y_2, g^{-1} y_3, x_1)$,

$\forall t \geq 1$, $\forall y_1, y_2, y_3 \in \Phi$ and $\forall x_1 \in \Omega$,

where k represents a diagonal matrix of dimension n.

[15] Malgrange (1981) had already noticed this difficulty. More precisely, he proved that the dynamic system (46) has eigenvalues which remain constant over time, and he shows on an example that these values can give very mistaken indications for stability. He also proved that the requirement of the tendency to zero of the relative differences of the variables to their values of balanced growth, is equivalent to the stability in the relative difference which will be defined later.

This property of homogeneity has interesting implications. Let us differentiate identity (47) relatively to y_1, y_2 and y_3, for: $t = 1$. We get identities:

(48) $F_1^{'}(g^t y_1, g^{t+1} y_2, g^{t-1} y_3, h^t x_1) g^t \equiv k^t F_1^{'}(y_1, g\ \ y_2, g^{-1} y_3, x_1)$

(49) $F_2^{'}(g^t y_1, g^{t+1} y_2, g^{t-1} y_3, h^t x_1) g^t \equiv k^t F_2^{'}(y_1, g\ \ y_2, g^{-1} y_3, x_1)$

(50) $F_3^{'}(g^t y_1, g^{t+1} y_2, g^{t-1} y_3, h^t x_1) g^t \equiv k^t F_3^{'}(y_1, g\ \ y_2, g^{-1} y_3, x_1)$

Then, the linear approximation (46) can be rewritten, after simplifying by k^t:

(51) $$\begin{aligned} &F_1^{'}(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x}) g^{-t} (y_t - y_t^s) + F_2^{'}(\bar{y}, g\bar{y}, g^{-1}\bar{y}, x) g^{-t} (y_{t+1} - y_{t+1}^s) \\ &F_3^{'}(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x}) g^{-t} (y_{t-1} - y_{t-1}^s) = 0 \end{aligned}$$

We can deduce from this expression new definitions of stability. For the first, in the relative difference, we define the vector of the reduced endogenous variables by: $y_t^{'} = g^{-t} y_t$. Then, equation (51) can be rewritten in reduced variables:

(52) $$\begin{aligned} &F_1^{'}(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x})(y_t^{'} - \bar{y}) + F_2^{'}(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x}) g (y_{t+1}^{'} - \bar{y}) \\ &F_3^{'}(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x}) g^{-1} (y_{t-1}^{'} - \bar{y}) = 0 \end{aligned}$$

Stability is defined as the convergence of the vector of the reduced endogenous variables to the steady state. The matrix of the coefficients does not depend on time any more, so we can apply the results by Blanchard and Kahn. A remark, which has interesting practical consequences, is that relation (51), which is the writing in reduced variables of the linear approximation of the model computed with its original variables, can also be obtained as the linear approximation of the model directly written in reduced variables. To show that let us define the vector of the reduced exogenous variables by: $x_t^{'} = h^{-t} x_t$. Then, model (42) can be rewritten:

(53) $F(g^t y_t^{'}, g^{t+1} y_{t+1}^{'}, g^{t-1} y_{t-1}^{'}, h^t x_t^{'}) = 0$,

and, if we use the homogeneity condition (47):

(54) $F(y_t^{'}, g y_{t+1}^{'}, g^{-1} y_{t-1}^{'}, x_t^{'}) = 0$.

The linear approximation of this equation is identical to equation (51).

For another definition of stability, in expanded difference, we must first introduce the diagonal matrix of dimension n, with generic element the highest balanced growth rate among those appearing in matrix g. We denote this matrix as $g_{\max}$. We multiply equation (51) by $g_{\max}^t$, and we define the vector of expanded endogenous variables by: $y_t^{''} = g_{\max}^t g^{-t} y_t = g_{\max}^t y_t^{'}$. Stability requires that these expanded endogenous variables converge to their balanced growth path $g_{\max}^{t}\bar{y}$. We see that this condition is stricter than the previous one, and we recall that we called pseudo-hysteresis the case when the model had a unique solution with the stability in the reduced difference, and no solution with the stability in the absolute difference. Then, the linear approximation of the model can be written:

$$(55)\quad \begin{aligned} &F_1^{'}(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x})(y_t^{"} - g_{\max}^{t}\bar{y}) + F_2^{'}(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x})g_{\max}^{-1}g(y_{t+1}^{"} - g_{\max}^{t+1}\bar{y}) \\ &F_3^{'}(\bar{y}, g\bar{y}, g^{-1}\bar{y}, \bar{x})g_{\max}g^{-1}(y_{t-1}^{"} - g_{\max}^{t-1}\bar{y}) = 0 \end{aligned}$$

This time yet, the matrix of the coefficients does not depend on time, and it is possible to use Blanchard and Kahn's results. It is interesting to notice that equation (55) cannot be considered as the linear approximation of the model written in expanded variables, except if we strengthen the homogeneity condition (47).

We show in the Appendix that the eigenvalues of the linear approximation of the model written in expanded variables are equal to $g_{\max}$ times the eigenvalues of the model written in reduced variables. Thus, we can observe that the satisfaction of the Blanchard and Kahn's conditions is not easier for one or the other kind of stability. When we go from the stability in the relative difference to the stability in the expanded difference we reduce the possibility of the non existence of a solution path, but we increase the possibility of the existence of an infinity of solutions. As we saw in the previous section, if the Blanchard and Kahn's conditions are satisfied for these two stabilities, the properties of existence and uniqueness is warranted for the case of the stability in absolute difference[16].

The possibility of hysteresis raises a new problem at the level of the simulation of the model. In this case, the balanced growth path is no more unique, and we have an infinity of available terminal conditions for some of the anticipated variables. In general, only one of these terminal conditions is compatible with the initial conditions of the economy, but we do not know which. We can overcome this difficulty at the level of the writing of the model. Let us assume, for example, that the model builder considers as a possibility an hysteresis on the prices level, but not on the inflation rate. This means that the balanced growth path of prices is undetermined, but not the one of the inflation rate. Then, we can substitute all the led prices variables by the product of their current values by led inflation rates. By selecting at random a balanced growth path among all those which are possible, we do not introduce any error on the terminal conditions of the anticipated variables which are present in the model, and we can use the standard algorithms which were developed to solve the systems of finite difference equations with initial and final conditions.

An interest of the stability in the relative difference is that hysteresis is characterized by eigenvalues equal to 1. We will still have to check that the number of associated eigen vectors is equal to the order of multiplicity of the unitary eigenvalue, and are orthogonal to the right-hand side of the model written as equation (3). Then, we can apply Proposition 2. In the case of the stability in the expanded difference, things are a little more complicated. We first have to write the model such that any variable which could present an hysteresis

[16] We have met a situation where the linear approximation of the model written in reduced variables has less eigenvalues larger than 1 than lead variables, and where the linear approximation of the model written in expanded variables has more eigenvalues larger than 1 than lead variables. In these situations, sometimes we could compute a unique solution path for the model written with its original variables, sometimes there did not exist any solution or there existed an infinity of them.

does not appear with a lead. Then, we can apply Proposition1, but without taking into account the eigenvalues equal to $g_{\max}$.

The multinational model Quest 2 is written in reduced variables. So, it is very easy to compute its eigenvalues with the Lkroot command of Troll. We get the result that the model does not present any hysteresis and that the Blanchard and Kahn's conditions of Proposition 1 are satisfied. Things are more complicated with model Multimod Mark 3. This model is written with variables in their original form, and it is solved under the condition of stability in the absolute difference. The checking of Blanchard and Kahn's conditions for the stability in relative and expanded differences are sufficient conditions for the existence and uniqueness of a solution path. However, the checking of these conditions requires the rewriting of the equations of the model in reduced variables, what we have not done. However, we were able to prove that for the model written in reduced variables, we have an eigenvalue equal to 1, with multiplicity equal to the number of countries, let be 9. This hysteresis concerns the price level of these countries, and consequently their exchange rates.

5. Conclusion

In this paper we have explained how we can use the local conditions of Blanchard and Kahn to investigate the existence and the uniqueness of the solution of macro-econometric models of large size. To do that we have had to overcome the following difficulties: The model is non linear, its linear approximation gives coefficients which change over time, in the long run many variables grow at positive rates which differ between them, and finally the model may present an hysteresis. We have introduced the notion of stability in the absolute difference, which is most natural but which does not allow the application of Blanchard and Kahn's conditions. Then, we have defined two other notions of stability which are consistent with the application of the results by Blanchard and Kahn: The stability in the relative difference and the stability in the expanded difference. If the Blanchard and Kahn's conditions are satisfied for these two stabilities, then the model has a unique solution under the condition of stability in the absolute difference. Our results can also be applied to the analysis of the stability of more traditional macro-econometric models where expectations are of an adaptive kind, and where the current state of the economy does not depend on its future states which are foreseen by the model.

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Appendix

We consider the case where there does not exist any static variable and we use the notations of section 1. The model written in reduced variables is

(A1) $C_1 g_1^{-1} y_{t-1}^1 + (C_0^2 | C_0^1)[\begin{matrix} y_t^2 \\ y_t^1 \end{matrix}] + C_{-1} g_2 y_{t+1}^2 = 0$,

where g_1 et g_2 represent the diagonal matrices of the balanced growth rates of respectively the predetermined and the anticipated variables. Let us put:

(A2) $x_t^1 = y_t^1$, $x_t^2 = y_{t+1}^2$, $x_t = [\begin{matrix} x_t^1 \\ x_t^2 \end{matrix}]$

Then, the model in reduced variables can be written:

(A3) $(C_1 g_1^{-1} | C_0^2) x_{t-1} + (C_0^1 | C_{-1} g_2) x_t = 0$

The model in expanded variables is:

(A4) $C_1 g_{\max} g_1^{-1} y_{t-1}^1 + (C_0^2 | C_0^1)[\begin{matrix} y_t^2 \\ y_t^1 \end{matrix}] + C_{-1} g_{\max}^{-1} g_2 y_{t+1}^2 = 0$

It can be written:

(A5) $(C_1 g_{\max} g_1^{-1} | C_0^2) x_{t-1} + (C_0^1 | C_{-1} g_{\max}^{-1} g_2) x_t = 0$

Let us denote by λ an eigenvalue of system (A3) and by $[\begin{matrix} V_1 \\ V_2 \end{matrix}]$ the associated eigen vector, the two components of which respectively correspond to x_t^1 et x_t^2. Then, we can easily show that we can associate to λ the eigenvalue $g_{\max}\lambda$ and the eigen vector $[\begin{matrix} V_1 \\ g_{\max}V_2 \end{matrix}]$ of system (A3).