On a Complex Microeconomical Model for the Optimal Control of a Concern

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Abstract. A state constrained optimal control problem in economics with four linear control variables is discussed. First of all, the complex model of a concern is introduced and a suitable choice of the model functions and the model parameters is investigated. This means the adjustment of initial data as well as the storage charges and the introduction of a price function depending on the trade cycle. To solve the optimal control problem, direct and indirect methods can be used. With the help of a direct approach it is possible to solve the problem fast and easily, without any knowledge of the necessary conditions of optimal control theory. Moreover, the direct method provides good initial estimations which can be improved w.r.t. accuracy and reliability by an indirect method. Furthermore, a brief outline about the theoretical analysis of control or state constrained optimal control problems - including the derivation of necessary conditions - and the numerical solution by means of the indirect method is represented. The complex switching structure of the optimal controls, which results from the appearance of singular subarcs and overlapping boundary arcs of two active state constraints, is remarkable.

Keywords. Microeconomic model, model improvement, optimal control, linear control, singular subarc, state constraint, necessary condition, direct collocation method, indirect multiple shooting method.

1 Introduction

Mathematical models of microeconomic as well as macroecnomic models are very important for many purposes. Considering this, we should recognize additionally, that the numerical methods and computational capabilities for practical applications of optimal control theory are permanently improved. Therefore, increasingly sophisticated as well as even more complex and realistic economical models can be investigated. There, as much knowledge about the real world as possible can be used. For example, these models can help to explain economic phenomena, help to improve the management of a concern or even enable getting a survey of a whole planning horizon. The development of concern models and their formulation as optimal control problems are well-known in literature, cf. Refs. 1–4. In fact, conventional models are simple and they unfortunately

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lack in expressiveness. Developing more complex models means, amongst others, designing realistic economic model functions and model parameters. In Ref. 5 and Ref. 6, a complex model of a concern is introduced. It consists of seven state variables, four linear controls and several control and state constraints. However, also this concern model can still be improved.
This paper, which is based on Ref. 7, mainly deals with the improvement of this concern model, i.e., the adjustment of the initial data as well as the storage charges and the introduction of a price function depending on the trade cycle. To solve the corresponding optimal control problem, we use direct methods as well as indirect methods, combined in a so-called hybrid approach.
Regarding the solution of the concern model provided by the direct method, only two of the given state constraints become active during the whole time interval on overlapping subintervals. These state constraints are replaced by stronger, suitably designed control constraints (see Ref. 6). Thus, the solution with the indirect method is simplified. The results confirm also, that concerning the loss of optimality, it is more than enough to use the control constraints instead of the state constraints. Nevertheless, it would be of mathematical interest to investigate precisely the original state constrained problem and to formulate the corresponding well-defined multipoint boundary value problem. Details regarding the necessary conditions of the state constrained problem can be read up in Ref. 7.
Although there are no more state constraints, further complexity is still caused by singular subarcs which occur in addition to the two simultaneously active control constraints. One difficulty is determining their entry points.
The numerical solutions of the improved optimal control problems are presented and discussed, also from an economical point of view.

2 Concern Model

In the following section the corresponding optimal control problem of a complex concern model is introduced. Here, the vector of the state variables is determined by

$$x^T = (S(t), L(t), Y(t), X(t), X_m(t), X_r(t), d(t))$$

with the stock S, the number of employees L, the loan capital Y, the equity capital X, the remaining part in alternative investment X_m, the risk premium X_r and the discounting position d. Furthermore, the vector of the control variables is given by

$$u^T = (S_c(t), L_c(t), Y_c(t), I(t)) \ ,$$

where S_c controls the stock, L_c the staff, Y_c and I, resp., control the loan and the equity capital of the concern.
The problem is to find the vector of control variables u so as to maximize the total profit of the capital owners, i.e. to minimize the performance index

$$Z[u] = \phi(x(t_f)) = -\left(X(t_f) + X_m(t_f) + (1-\tau)\ p \cdot \frac{S(t_f)}{d(t_f)} \right) \tag{1}$$

for given initial conditions $x(t_0)$ subject to the differential equations

$$\begin{array}{lcl} \dot{S} & = & S_c \ , \\ \dot{L} & = & L_c \ , \\ \dot{Y} & = & Y_c \ , \\ \dot{X} & = & +I + (1-\tau)\ \ (P(x,u) - \rho_r\ X)\ \ , \\ \dot{X}_m & = & -I + (1-\tau)\ \rho_m\ X_m\ \ , \\ \dot{X}_r & = & (1-\tau)\ \rho_r\ X\ \ , \\ \dot{d} & = & -d\ \ln(1+i) \end{array} \tag{2}$$

as well as the following state and control constraints.
The state constraints read $S \leq S_{\max},\ 0 \leq L,\ 0 \leq Y,\ 0 \leq X_m$ and

$$Z_1 = \quad Y - \kappa X \ \leq\ 0, \tag{3}$$

$$Z_2 = -S + S_{\min} \ \leq\ 0\,. \tag{4}$$

Due to former calculations, we know that only two of them, the constraints (3) and (4) become active during the whole planning horizon. Thus, for the sake of simplicity, concerning the numerical solution of the problem, the two state constraints (3) and (4) are replaced by stronger, suitably designed control constraints (cf. Ref. 6)

$$V_{\min} := -\beta_2(\kappa X\ -\ Y)\ \leq\ \kappa\dot{X} - \dot{Y} \overset{(2)}{\Rightarrow} Y_c \leq \kappa\dot{X} - V_{\min} =: \tilde{Y}_{c\,\max}(S_c, I), \tag{5}$$

$$\tilde{S}_{c\,\min} := -\beta_1(S - S_{\min})\ \leq\ \dot{S} = S_c\,, \tag{6}$$

which guarantee the fulfillment of the state constraints. The greater the parameters $\beta_1, \beta_2 > 0$ are, the better the correspondence with the original state constraints (3) and (4) is. Here, $\beta_1 = \beta_2 = 10$ seems to be a good choice. If these control constraints are active, the optimal controls $Y_c^* = \tilde{Y}_{c\,\max}(S_c^*, I^*)$ and $S_c^* = \tilde{S}_{c\,\min}$ are determined by Eqs. (5) and (6). The four controls are bounded by

$$S_c \in [S_{c\,\min}, S_{c\,\max}]\ ,\ L_c \in [L_{c\,\min}, L_{c\,\max}]\ ,\ Y_c \in [Y_{c\,\min}, Y_{c\,\max}]\ ,\ I \in [I_{\min}, I_{\max}] \tag{7}$$

and, in addition, S_c has to obey the inequality constraint

$$0 \leq F - S_c. \tag{8}$$

The time t is the independent variable. The initial time t_0 and the terminal time t_f are fixed. In the following $t_0 = 0$ and $t_f = 10$ $[years]$ hold. The model functions and model parameters appearing in (1) to (8) are explained in Table 1. Note, that the selection of their values has a great effect on the trajectories of the solution, see Refs. 5, 8, 9 for a realistic design.

Remark 2.1 (Regarding k_p, cf. Table 1)
In Ref. 9 the variable k_p is treated as a state variable subject to the differential equation $k_p' = 2\pi/k_l$ and the initial value $k_p(0) = \pi/2$. This differential equation is independent of the other state variables. Hence, it can be integrated and with its solution $k_p(t) = \frac{\pi}{2} + \frac{2\pi}{k_l} \cdot t$ the dimension of the vector of state variables can be reduced.

Abbr.	Formula / Value	Meaning
$P(x,u)$	$\frac{1}{d}[p(F-S_c)-\sigma S-\omega L]-\rho_K Y-\delta K$	profit (of the concern)
$K(x)$	$X+Y$	capital
$F(x)$	$\alpha K^{\alpha_K} L^{\alpha_L}$	output of the concern
α	100	suitable parameters of the output
α_K	0.35	
α_L	0.5	
$\rho_K(t)$	$0.110+0.030\cdot\sin k_p(t)$	loan interest rate
$\rho_m(t)$	$0.074+0.018\cdot\sin k_p(t)$	current yield
$i(t)$	$0.019+0.029\cdot\sin k_p(t)$	inflation rate
$\rho_r(t)$	$\rho_K(t)-0.05$	high risk premium rate
$k_p(t)$	$\frac{\pi}{2}+\frac{2\pi}{k_l}\cdot t$	position in an economic cycle
k_l	8	duration of the economic cycle
p	0.05	selling price
τ	0.5	tax rate
σ	0.01	storage charges
ω	2	labor cost
κ	0.8	rate of maximal borrowing
δ	0.322	depreciation rate

Table 1: Model functions and model parameters.

Remark 2.2 (Regarding Eq. (5))
Solving Eq. (5), $V_{\min} \leq \kappa\dot{X}-\dot{Y}$, with the help of the differential equations (2), we could analogously obtain a control constraint for I:

$$I \geq \tilde{I}_{\min}(S_c, Y_c)\,. \tag{9}$$

3 Theory for optimal control problems

Let us consider the following problem: The performance index

$$Z[u] := \phi(x(t_f),t_f) + \int_{t_0}^{t_f} L(x(t),u(t))\,dt \tag{10}$$

has to be minimized with respect to the class of functions

$$C_k^p := \{u : [t_0,t_f] \to U \subset \mathbb{R}^k,\ u \text{ piecewise continuous}\}.$$

The state trajectory $x(t)$, $x : [t_0,t_f] \to \mathbb{R}^n$, satifies a vector differential equation

$$\dot{x} = f(x,u)\,, \quad f : \mathbb{R}^{n+k} \to \mathbb{R}^n\,, \tag{11}$$

with given initial conditions and terminal constraints,

$$\begin{aligned} x(t_0) &= x_0\,, \quad x_0 \in \mathbb{R}^n \ \text{ and} && (12)\\ \Psi(x(t_f),t_f) &= 0\,, \quad \Psi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{q_f}\,. && (13) \end{aligned}$$

Furthermore, some applications include interior point conditions, control constraints

$$C(x(t), u(t)) \le 0\,, \quad C : \mathbb{R}^{n+k} \to \mathbb{R}^{l}\,, \tag{14}$$

or state constraints

$$S(x(t)) \le 0\,, \quad S : \mathbb{R}^{n} \to \mathbb{R}^{\bar{l}}\,. \tag{15}$$

The functions appearing in Eqs. (10) to (15) are supposed to be sufficiently often continuously differentiable with respect to all their arguments.
Considering the problem given by Eqs. (10) to (13) and defining the Hamiltonian

$$H(x, u, \lambda) := L(x, u) + \lambda^T \, f(x, u) \tag{16}$$

the following necessary conditions (see Refs. 10,11) are obtained:

- differential equations of Euler-Lagrange

$$\dot{x} = H_\lambda = f(x, u) \tag{17}$$

$$\dot{\lambda}^T = -H_x \tag{18}$$

- minimum principle

$$H(x^*, u^*, \lambda) = \min_{u \in U} \; H(x^*, u, \lambda)\,, \tag{19}$$

  where U denotes the set of admissible control values, and

- transversality conditions

$$\lambda^T(t_f) = \Phi_x|_{t=t_f}\,, \tag{20}$$

$$(\Phi_t + H)|_{t=t_f} = 0\,, \tag{21}$$

  with $\Phi(x, t, \nu) := \phi(x, t) + \nu^T \Psi(x(t), t)$.

If u appears linearly in H, Eq. (16) can be written in the form

$$H(x, u, \lambda) = u^T \, \sigma(x, \lambda) + R(x, \lambda)\,. \tag{22}$$

$(\sigma)_i$, $i = 1, \ldots, k$, is called the switching function associated with the control variable $u_i \in [u_{i\,\min}, u_{i\,\max}]$. The minimum principle leads to bang-bang controls

$$u_i = \begin{cases} u_{i\,\max}, & \text{if} \quad (\sigma)_i < 0\,, \\ u_{i\,\min}, & \text{if} \quad (\sigma)_i > 0\,. \end{cases} \tag{23}$$

If $(\sigma)_i$ vanishes on a subinterval, the control variable u_i has a singular subarc. In addition, all derivatives of $(\sigma)_i$ then vanish on this subinterval.

Considering now the problem of minimizing the cost functional (10) subject to the side conditions (11) – (13) and additionally subject to the state constraint (15), the necessary conditions can be deduced in two different ways. This means, the Hamiltonian can be augmented by the state constraint directly (cf. Refs. 12–14) or by the derivative of the state constraint (see Refs. 15,16). Both ways and the resulting necessary conditions are summed up in Refs. 7,17.

The necessary conditions for both kinds of problems result in a multipoint boundary value problem which can be solved using the multiple shooting method.

4 Improvement of the concern model

In the following, a short description of the adjustment of initial data, the modification of the price and a brief outline about using storage charges to rationalize is represented. To solve the corresponding optimal control problem of the concern model, a so-called hybrid approach is used. This means, a direct method and an indirect method are combined, to use the advantages of both methods. The direct method, e.g. DIRCOL (Ref. 18) or NUDOCCCS (Ref. 19), quickly provides a good initial estimation, which can be improved w.r.t. accuracy and reliability by an indirect method, e.g. the multiple shooting method MUMUS (Ref. 20). First of all, we want to give an overview about the initial solution (cf. Refs. 5–7) of the optimal control problem and the needed preparatory works.

4.1 Initial Solution

Using the indirect multiple shooting method MUMUS the time interval, with $t \in [0, t_f]$, should be transformed into $s \in [0, 1]$ with:

$\dot{x} = f(x, u) \Rightarrow dx/ds = x' = (f(x, u) \cdot t_f, t_f)^T$ with $x^T = (S, L, Y, X, X_m, X_r, d, t)$

to obtain a better convergence in case of parameter variations in t_f (note, $t(1) = t_f$).
The control variables S_c, L_c, Y_c and I are appearing linearly in the differential equations (2). Thus, the Hamiltonian reads

$$H(x, u, \lambda) = \sigma_{S_c} \cdot S_c + \sigma_{L_c} \cdot L_c + \sigma_{Y_c} \cdot Y_c + \sigma_I \cdot I + R(x, \lambda) \tag{24}$$

with the switching functions depending on x and λ:

$$\begin{aligned} \sigma_{S_c} &= \lambda_S - (1-\tau)\, p \cdot \tfrac{1}{d} \cdot \lambda_X \quad , \\ \sigma_{L_c} &= \lambda_L \quad , \\ \sigma_{Y_c} &= \lambda_Y \quad \text{and} \\ \sigma_I &= \lambda_X - \lambda_{X_m} \; . \end{aligned}$$

If constraint (5) is active, the switching functions of S_c and I are changing into

$$\tilde{\sigma}_{S_c} = \sigma_{S_c} + \sigma_{Y_c} \frac{\partial \tilde{Y}_{c\,\max}}{\partial S_c} = \lambda_S - (1-\tau)\; p \cdot \frac{1}{d} \cdot (\kappa\, \lambda_Y + \lambda_X) \tag{25}$$

$$\tilde{\sigma}_I = \sigma_I + \sigma_{Y_c} \frac{\partial \tilde{Y}_{c\,\max}}{\partial I} = \kappa\, \lambda_Y + \lambda_X - \lambda_{X_m} \; . \tag{26}$$

Remark 4.1
If constraint (9) holds, instead of Eq. (5), the switching function of Y_c changes into $\tilde{\sigma}_{Y_c}$ and σ_I holds further on. Because of $\kappa \cdot \tilde{\sigma}_{Y_c} = \tilde{\sigma}_I$, the control variables Y_c and I are singular always at same time.

Regarding the described model, it was possible to verify the switching structure, with the switching points $0 < s_2 < \cdots < s_8 < 1$, specified in Table 2. The borrowing limit (5) becomes active at s_3, the stock constraint (6) at s_4.
With the help of the switching structure (Table 2 and cf. Ref. 5) the piecewise defined adjoint differential equations can be stated:

	$[0, s_2]$	$]s_2, s_3]$	$]s_3, s_4]$	$]s_4, s_5]$	$]s_5, s_6]$	$]s_6, s_7]$	$]s_7, s_8]$	$]s_8, 1]$
${S_c}^*$	$S_{c\,\min}$	$S_{c\,\min}$	$S_{c\,\min}$	$\tilde{S}_{c\,\min}$	$\tilde{S}_{c\,\min}$	$\tilde{S}_{c\,\min}$	$\tilde{S}_{c\,\min}$	$\tilde{S}_{c\,\min}$
${L_c}^*$	$L_{c\,\max}$	$L_{c\,\max}$	$L_{c\,\max}$	$L_{c\,\max}$	$L_{c\,\max}$	$L_{c\,\min}$	singular	$L_{c\,\max}$
${Y_c}^*$	$Y_{c\,\min}$	$Y_{c\,\max}$	$\tilde{Y}_{c\,\max}(S_c, I)$	$\tilde{Y}_{c\,\max}(S_c, I)$	singular	singular	singular	singular
I^*	$I_{\min}$	$I_{\min}$	$I_{\min}$	$I_{\min}$	singular	singular	singular	singular

Table 2: Switching structure.

- On $s \in [0, s_3]$ all controls are independent of the state variables:

$$\lambda'_{x_i} = -\frac{\partial H}{\partial x_i}(x, u^*, \lambda) \ . \tag{27}$$

- On $s \in]s_3, 1]$ the control variable $Y_c = \tilde{Y}_{c\,\max}$ depends on the state variables:

$$\lambda'_{x_i} = -\frac{\partial H}{\partial x_i} - \sigma_{Y_c} \cdot \frac{\partial \tilde{Y}_{c\,\max}}{\partial x_i} \ . \tag{28}$$

- On $s \in]s_4, 1]$ the stock constraint (6) is additionally active. Compared to (28) only λ'_S changes, because the control $S_c = \tilde{S}_{c\,\min}$ depends only on the state variable S:

$$\lambda'_S = -\frac{\partial H}{\partial S} - \sigma_{Y_c} \cdot \frac{\partial \tilde{Y}_{c\,\max}}{\partial S} - \tilde{\sigma}_{S_c} \cdot \frac{\partial \tilde{S}_{c\,\min}}{\partial S} \ . \tag{29}$$

On singular subarcs, the optimal controls are functions depending on state variables and adjoint variables. Regarding the theoretical section, the switching functions and all their derivatives vanish on these subintervals. Here, the control variables appear again linearly in the second derivatives of the switching functions:

$$\begin{array}{rclcll} \sigma''_{L_c} & = & \lambda''_L & = & 0 & s \in]s_7, s_8]\,, \\ \tilde{\sigma}''_I & = & \kappa\lambda''_Y + \lambda''_X - \lambda''_{X_m} & = & 0 & s \in]s_5, 1]\,. \end{array} \tag{30}$$

On the subinterval $]s_7, s_8]$ a linear system of equations has to be solved, cf. Ref. 5. Elsewhere, only a linear equation has to be solved. In any case the singular control Y_c is determined by the constraint (5).

4.2 Adjustment of initial data

In order to increase the sensitivity of the problem, especially of the performance index, with respect to parameters and initial values, the remaining part in alternative investment is reduced from $X_m(0) = 450$ MU to 100 MU (=monetary

units) at the initial point.
This causes no changes apart from the trajectory of X_m, cf. Eq. (2) and Fig. 1.

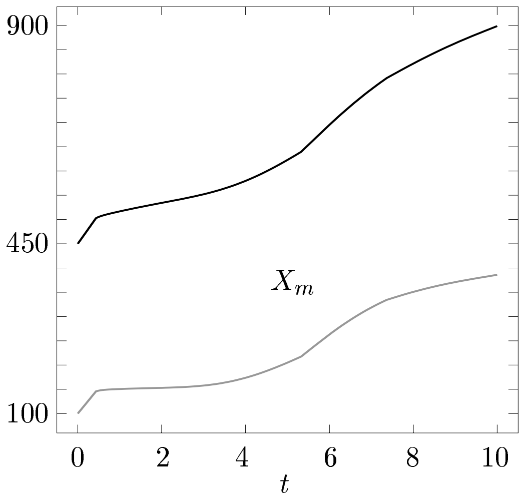

Figure 1: Solution of the state variable X_m with initial data $X_m(0) = 450$ (black) and $X_m(0) = 100$ (grey).

As supposed the terminal value of the performance index decreases:

$X_m(0)$	450	100
$Z[u]$	-1117.62	-605.07

4.3 Modifying the price

Emanating from the constant price $p = p_{const}$ the price is modified by a cyclic function, i.e. a sinus cycle, which is displaced by $\frac{\pi}{2}$ because of the economic trend,

$$p(k_p(s)) = 0.05 + 0.01 \cdot \sin(k_p(s) - \pi/2)\,.$$

Hereby the delayed reaction on supply and demand is reflected (cf. Refs. 21–23). The economic trend can be classified into four sections, named

I the phase of contraction or recession,

II the phase of depression,

III the phase of expansion, and

IV the boom.

Phase II is ended by the lower reversal point (minimum) and Phase IV by the upper reversal point (maximum).
As a consequence of the boom the price is raised at the beginning of the planning horizon, i.e. at the upper reversal point. The recession even starts while raising the price for 2 years. The reaction on the phase of depression is also delayed, cf. Fig. 2.

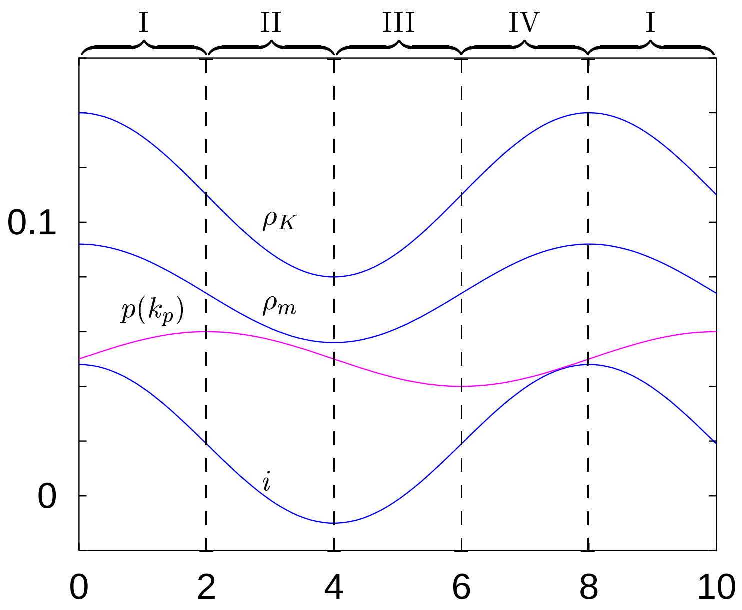

Figure 2: Functions depending on the economic trend.

In the following sections, we want to investigate what kind of changes are caused by using that price function.

4.3.1 Investigation of the switching structure

With the aforementioned hybrid approach, a direct method is used to solve the optimal control problem in order to realise the effects of modelling the price as a function. Especially the changes in the switching structure are interesting. The switching structure can be investigated with the help of the approximations of the control variables computed with DIRCOL or NUDOCCCS . However, often the switching structure can not be determined precisely with the help of these informations. In such cases the switching functions have to be additionally considered.
Considering the histories of the control variables L_c and S_c, however, their switching structure can be determined clearly:

- Firstly, L_c is maximal (nearly 4 years), then minimal (more than 2 years) and at the end again maximal. This means, at the beginning of the planning horizon as many employees as possible are engaged as long as the number of employees is maximally reduced within the recession. From nearly the end of the phase of expansion to the terminal time, the maximal amount of employess is engaged.
- The control S_c is minimal, i.e. the stock is reduced as fast as possible, as long as the stock constraint becomes active (after 0.4 years). The stock remains unchanged till the end of the planning horizon.

Regarding the approximations of Y_c and I only the bang-bang structure can be detected well. To determine the entry points of the constraints as well as the singular arcs, the switching functions σ_{Y_c}, σ_I, $\tilde{\sigma}_{Y_c}$ and the histories of the approximations of both constraints (5) and (6) have to be considered. Note, the data of the graphs in Figs. 3–5 arise from the approximations obtained by the direct method.

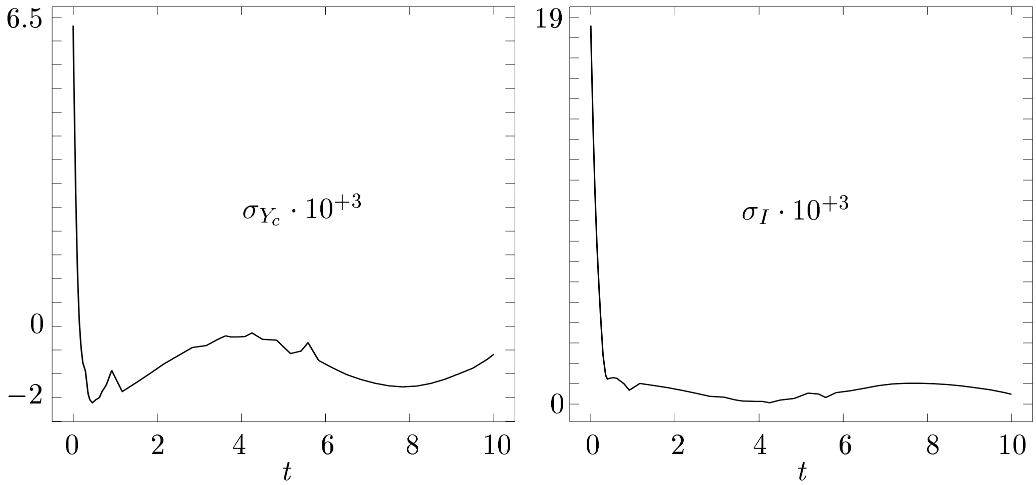

Figure 3: The switching functions σ_{Y_c} (left) and σ_I (right).

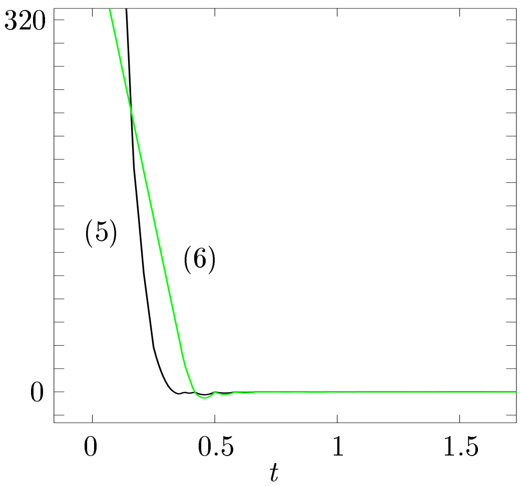

Figure 4: The critical part of the stock constraint (6) (grey) and the borrowing limit (5) (black).

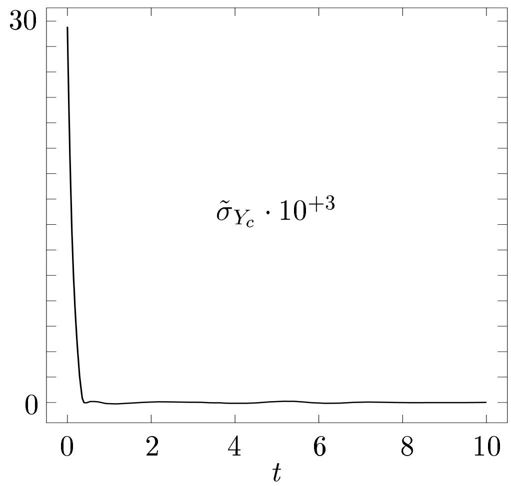

Figure 5: The switching function $\tilde{\sigma}_{Y_c}$.

By means of Fig. 3 (left picture) we can see clearly, that the switching function σ_{Y_c} changes its sign, thus the control Y_c is firstly minimal and then maximal. That means, that within the boom at the beginning of the planning horizon (which is rather the upper reversal point), the borrowing of loan capital is as low as possible. After a very short time the borrowing becomes maximal. Regarding Fig. 4, we can see that the borrowing limit becomes active at about $t = 0.32$ years (note, then $\tilde{\sigma}_{Y_c}$ holds). The stock constraint is not active at that time. I is minimal up to $\tilde{\sigma}_{Y_c}$ vanishes (Fig. 5). Then I becomes, together with the control Y_c (cf. Remark 4.1), singular at about $t = 0.4$ years.

Summarizing the problems investigating the switching structure:

The problem is, that the stock constraint (6) becomes active and in addition Y_c and I become singular nearly at the same time, at about $t = 0.4$ years. Thus, we have to take into account these two possibilities for the switching structure:

	Possibility 1	Possibility 2
Eq. (5) becomes active	s_3	s_3
Y_c and I singular	s_4	s_5
Eq. (6) becomes active	s_5	s_4

Table 3: Two possibilities for the switching structure.

4.3.2 Investigation of the necessary conditions

To solve the problem with the help of the multiple shooting method MUMUS, a so-called homotopy parameter $HOM \in [0,1]$ is introduced:

$$p(k_p) = 0.05 + HOM \cdot 0.01 \cdot \sin(k_p - \frac{\pi}{2}). \tag{31}$$

Thus, starting from the solution for $p = 0.05$ with $HOM = 0$ (cf. Section 4.1) the homotopy parameter is increased up to $HOM = 1$.
First of all, the formulation of the problem has to be modified, so that the multipoint boundary value problem is again well-defined.

Adjoint differential equations and boundary conditions

Neither in the adjoint differential equations nor in the boundary conditions any changes occur. Nevertheless, the adjoint differential equations of the original optimal control problem (cf. Section 4.1 and Ref. 5) are denoted by $\lambda'_{x_i\,old}$ in the following.

Remark 4.2
Since k_p is treated as a state variable (cf. Remark 2.1), the adjoint differential equation $\lambda'_{k_p\,old}$ and the final condition, i.e. the transversality condition for λ_{k_p}, are to be modified.

Modification in computing the optimal controls

Now, the control L_c does not become singular anymore. That means we have to solve only a linear equation (cf.(30)) on $s \in]s_4, 1]$ or $s \in]s_5, 1]$ for I:

$$\tilde{\sigma}''_I = \kappa\lambda''_Y + \lambda''_X - \lambda''_{X_m} = 0\ . \tag{32}$$

Regarding the second derivatives of the adjoint variables λ_Y, λ_X and λ_{X_m}, we realise, that λ''_{X_m} remains unchanged, but $\lambda''_{Y\,old}$ and $\lambda''_{X\,old}$ are changing into

$$\lambda''_Y = \lambda''_{Y\,old} - t_f \cdot (1-\tau) \cdot h_1\,, \tag{33}$$
$$\lambda''_X = \lambda''_{X\,old} - t_f \cdot (1-\tau) \cdot h_1\,, \tag{34}$$

with

$$h_1 = \frac{1}{d} \cdot (\lambda_X + \kappa \cdot \lambda_Y) \cdot F \cdot \frac{\alpha_K}{X+Y} \cdot 0.01 \cdot \cos(k_p - \frac{\pi}{2}) \cdot t_f \cdot \frac{2\pi}{k_l}\,.$$

Now, we have to solve, in an analogous way to Ref. 5, Section 6.3.4, the equation $a \cdot I = b$ to get the singular control I. No further control variables occur in the additional term in (33) and (34). Hence, there are no changes in a:

$$a = -(\lambda_X + \kappa \cdot \lambda_Y) \cdot \frac{p}{d} \cdot \frac{\partial F}{\partial K} \cdot \frac{\alpha_K - 1}{K} \cdot (\kappa + 1)^2 \,. \tag{35}$$

But the right-hand sight b has to be modified:

$$b = b_{old} + h_1 \cdot (1 + \kappa) \tag{36}$$

with

$$\begin{aligned} b_{old} = & -(1-\tau) \cdot M \cdot \Big(M \cdot (\lambda_X + \kappa \cdot \lambda_Y) + \rho_r \cdot \lambda_{X_r} \Big) + (1+\kappa) \cdot (\lambda_X + \kappa \cdot \lambda_Y) \\ & \cdot \Bigg(\frac{p}{d \cdot K} \cdot \Big[(\alpha_K - 1) \cdot \frac{\partial F}{\partial K} \cdot \big((1-\tau) \cdot (P(x,u) - \rho_r \cdot X) + h_2 \big) \\ & \quad + \alpha_K \cdot \frac{\partial F}{\partial L} \cdot L_c \Big] + \frac{p}{d} \frac{\partial F}{\partial K} \cdot \log(1+i) - \frac{2\pi}{k_l} \cdot 0.03 \cdot \cos(k_p) \Bigg) \\ & + \cos(k_p) \cdot \frac{2\pi}{k_l} \cdot (0.03 \cdot \lambda_{X_r} - 0.018 \cdot \lambda_{X_m}) + (1-\tau) \cdot \rho_m^2 \cdot \lambda_{X_m} \,. \end{aligned}$$

and the auxiliary variables

$$\frac{\partial F}{\partial K} = \frac{\alpha_K \cdot F}{X + Y} \,, \quad \frac{\partial F}{\partial L} = \frac{\alpha_L \cdot F}{L} \,,$$

and

$$h_2 = \kappa \cdot (1-\tau) \cdot (P(x,u) - \rho_r X) + \beta_2 \cdot (\kappa \cdot X - Y) \,, \tag{37}$$

$$M = (\kappa + 1) \cdot (\frac{p}{d} \cdot \frac{\partial F}{\partial K} - \delta) - \kappa \cdot \rho_K - \rho_r \,. \tag{38}$$

4.3.3 Numerical Solution with the indirect method MUMUS

It has been tried to solve the optimal control problem with the indirect method using the two switching structures stated in Table 3. During the homotopy it has been recognized that "Possibility 1" is correct. Detecting the switching structures was not the only problem in solving the corresponding multipoint boundary value problem. Note, that the changes in the history of the price function p has a tremendous effect not only on the switching structure, but also on the problem formulation itself. Even a homotopy method could not find a remedy. By starting with a smaller planning horizon, e.g. $t_f = 5$, the homotopy (cf. Eq. (31)) can be realized. Then, the final time could be increased step by step until $t_f = 10$ is reached.
Figures 6 and 7 compare the solutions of the state and control variables, obtained by solving the problem, with the indirect method MUMUS, using the constant price and the price depending on the economic trend, resp..

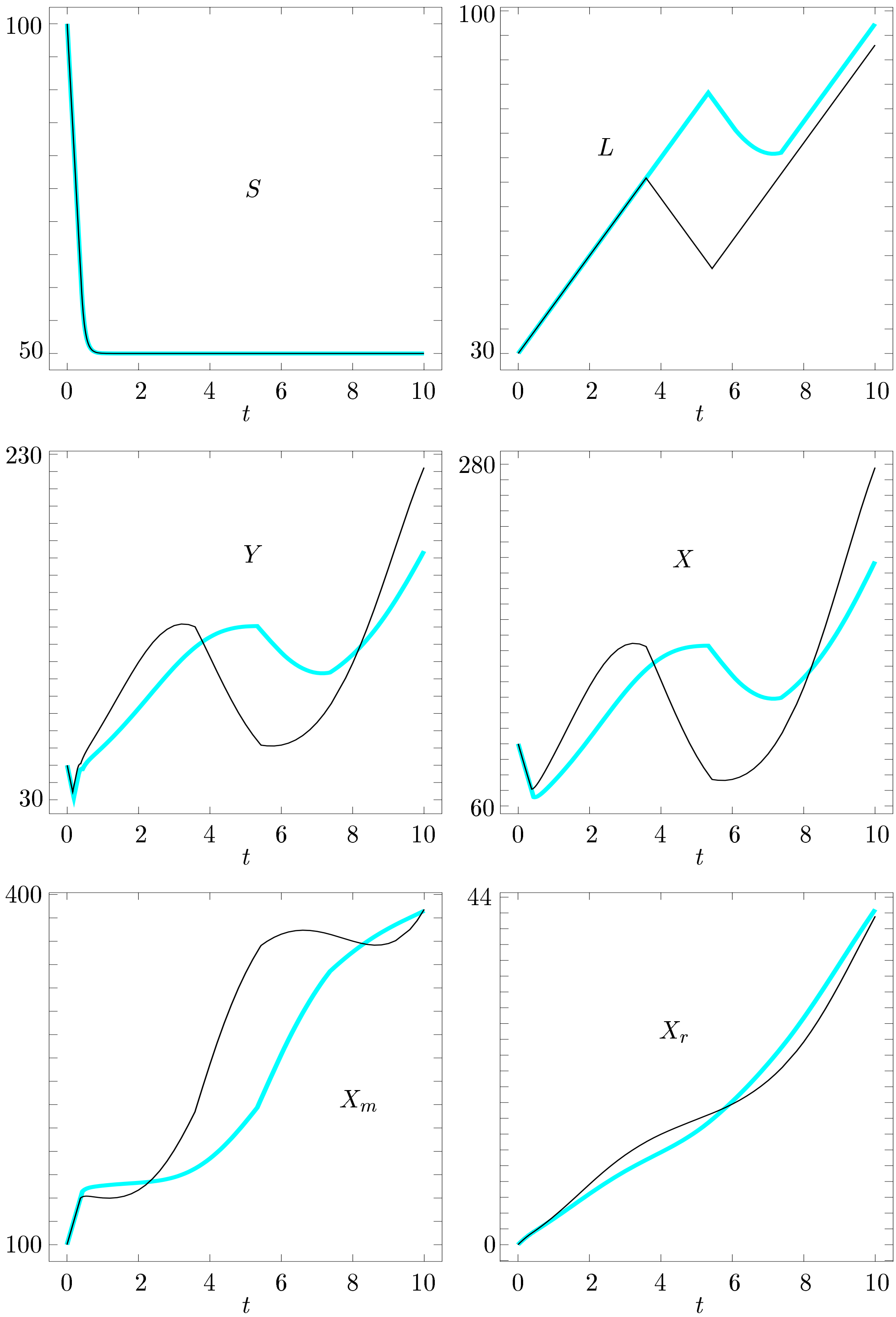

Figure 6: Solutions of the state variables S, L, Y, X, X_m and X_r with price $p = 0.05$ (grey) and $p = p(k_p)$ (black).

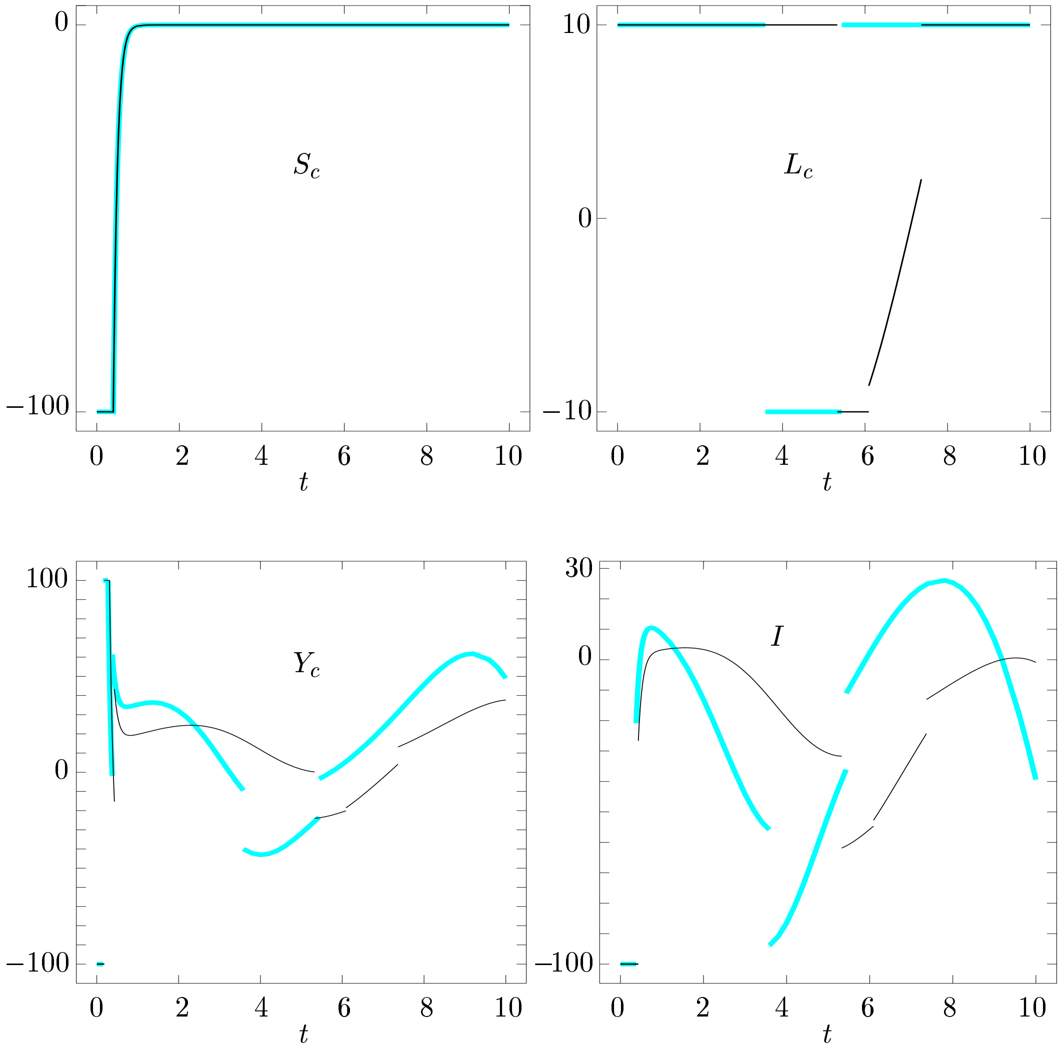

Figure 7: Solutions of the control variables S_c, L_c, Y_c and I with price $p = 0.05$ (grey) and $p = p(k_p)$ (black).

Altogether, Tables 4 and 5 show the changes referring to the switching structure as well as the positions of the switching points.

	$[0, s_2]$	$]s_2, s_3]$	$]s_3, s_4]$	$]s_4, s_5]$	$]s_5, s_6]$	$]s_6, s_7]$	$]s_7, 1]$
${S_c}^*$	$S_{c\,\min}$	$S_{c\,\min}$	$S_{c\,\min}$	$S_{c\,\min}$	$\tilde{S}_{c\,\min}$	$\tilde{S}_{c\,\min}$	$\tilde{S}_{c\,\min}$
${L_c}^*$	$L_{c\,\max}$	$L_{c\,\max}$	$L_{c\,\max}$	$L_{c\,\max}$	$L_{c\,\max}$	$L_{c\,\min}$	$L_{c\,\max}$
${Y_c}^*$	$Y_{c\,\min}$	$Y_{c\,\max}$	$\tilde{Y}_{c\,\max}(S_c, I)$	singular	singular	singular	singular
I^*	$I_{\min}$	$I_{\min}$	$I_{\min}$	singular	singular	singular	singular

Table 4: Switching structure with $X_m(0) = 100$, $p = p(k_p)$ and $t_f = 10$.

	$p = 0.05$	$p = p(k_p)$
$Y_{c\,\min} \to Y_{c\,\max}$	$s_2 = 0.01851616239$	$s_2 = 0.01541157099$
Eq. (3) active	$s_3 = 0.03093418080$	$s_3 = 0.02752838254$
Eq. (4) active	$s_4 = 0.04000000000$	$s_5 = 0.04000000000$
Y_c, I singular	$s_5 = 0.04309544934$	$s_4 = 0.03780647755$
$L_{c\,\max} \to L_{c\,\min}$	$s_6 = 0.53261860857$	$s_6 = 0.35845196178$
L_c singular	$s_7 = 0.60913202729$	
$L_c \to L_{c\,\max}$	$s_8 = 0.73646847059$	
$L_{c\,\min} \to L_{c\,\max}$		$s_7 = 0.54337053706$

Table 5: Variation of the switching points caused by modifying the price function.

Note the price p varies within a cycle in the interval $[0.04, 0.06]$. Since, at the final time, $p(k_p(1)) = 0.06$ is greater than the constant price $p_{const} = 0.05$ prescribed previously, the total profit of the capital owner (cf. (1)) increases, i.e. , the performance index decreases:

p	0.05	$0.05 + 0.01 \cdot \sin(k_p - \frac{\pi}{2})$
$Z[u]$	-605.07	-668.88

Finally it is necessary to have a look at the Hamiltonian to verify whether the corresponding multipoint boundary value problem is well-defined. Because the stated problem is autonomous, the Hamiltonian has to be nearly constant, within computational accuracy. Figure 8 proves that.

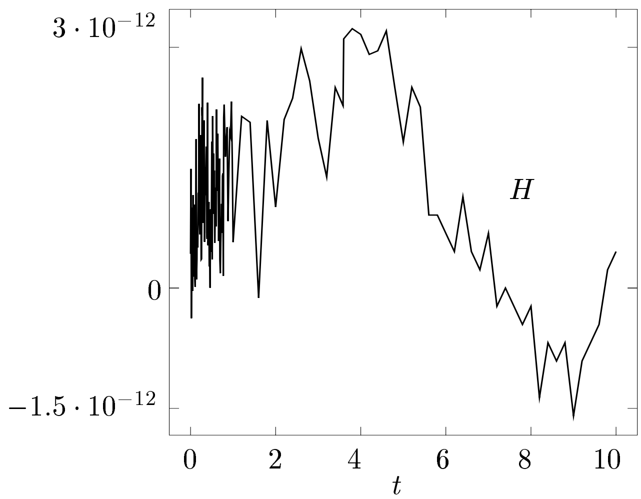

Figure 8: The Hamiltonian.

Generally, the optimality of a solution obtained by an indirect method is checked using the minimum principle, cf. Eq. (19). Since the concern model is linear w.r.t. the controls, the optimal controls satisfying (19) are the solutions of a linear programming problem and have to fulfill the Karush-Kuhn-Tucker conditions (cf. Refs. 6, 24, 25).

The vector of control constraints is called $B = (B_i)_{i=1,\ldots,10}$ and consists of (5),

(6) and (7). Easily it can be proven, that the Karush-Kuhn-Tucker conditions

$$\langle -\mathrm{grad}\, H, -\mathrm{grad}\, B_i \rangle > 0\,, \quad \text{with } \mathrm{grad}\, H = \frac{\partial H}{\partial u}\,, \ \mathrm{grad}\, B_i = \frac{\partial B_i}{\partial u}$$

hold. Hence, there is no contradiction to the minimum principle.

4.4 Using the storage charges to rationalize

Up to now, the storage charges σ has been very high. This has resulted in an unrealistic behaviour of the stock, i.e. the stock has been reduced, as far as possible, but has not been refilled anymore. Now, because of their importance for rationalization, the storage charges are reduced from $\sigma = 0.01$ to $\sigma = 0.005$, hoping to observe fluctuation in the stock, especially an increase of the stock.
For the purpose of model verification a fast direct method is preferred for obtaining a solution and for deciding if the reduction is great enough for rationalization.
Figure 9 confirms the assumption, that σ affects the control variable S_c. Regarding the history of S_c, the economic cycle of duration $k_l = 8$ years can be observed. This results in an even more complex switching structure. Solving that problem with an indirect method, too, would result in even greater efforts.
Now, the consequence of rationalizing can be considered. The following table shows, that the benefit of the lower storage charges is on the total profit of the capital owner, see $|Z[u]|$, but not on the profit of the concern, cf. $P(x,u)$:

	$\sigma = 0.01$	$\sigma = 0.005$
$\|Z[u]\|$	666.97	669.08
$P(x,u)$	218.06	216.92

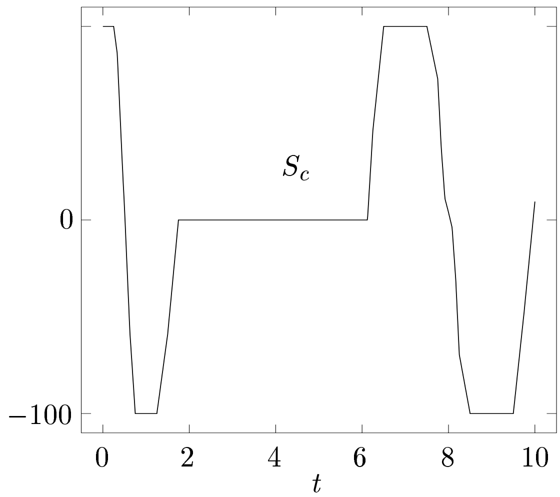

Figure 9: The control S_c, with storage charges $\sigma = 0.005$.

5 Conclusions

A complex model of a concern and its corresponding optimal control problem with four linear controls, seven state variables and several control and state constraints has been considered. An improvement of the concern model, concerning initial data as well as storage charges and the introduction of a price function depending on the trade cycle, has been realized. With the help of a so-called hybrid approach, i.e. combination of a direct and an indirect method, the numerical solution of the optimal control problem has been obtained.
Singular subarcs occuring in addition to two simultaneously active constraints have caused further complexity. Within this paper, the optimal control problem has been solved with stronger, suitably designed control constraints, replacing the two active state constraints. The results, obtained in Ref. 17 showed that the investigation of the necessary conditions of the optimal control problem with state constraints is only of mathematical interest. Concerning the loss of optimality, it is more than enough to use the control constraints.
Our future aim is to develop even more sophisticated and realistic models, e.g. designing more realistic model functions depending on the trade cycle. Besides, it should be noted, that the optimal control problem presented in this paper is one of the rare problems published in literature that has four control variables appearing linearly in the dynamical system. For such problems, second-order sufficiency conditions are not yet known, which play an important role in the development of real-time methods on the basis of a sensitivity analysis of the underlying optimal control problem. Although, the demand for real-time solutions of economical problems are not cogent, the problem may nevertheless serve as test problem for the development of real-time methods.

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