Recovering Local Volatility Functions of Forward LIBOR Rates

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Abstract

This paper investigates the implied pricing methods in the interest rate market. We assume the local volatility is a function of the time and the underlying simple forward rate. We propose a method to approximate the local volatility function of forward LIBOR rates as a natural tensor product of cubic splines. The spline functional approximation is applied within the framework of forward LIBOR rate model so that local drifts need not to be computed. We back out the local volatility functioncs of forward LIBOR rates from market caplet prices so that the volatility skew and volatility term structure are matched. We give Two computation examples. In the first, the caplet prices are simulated with the analytical formula from extended forward LIBOR rate model. It shows that the method is able to accurately recover a constant elasticity variance volatility structure. In the second example, the method is applied to market values of three months GBP LIBOR caps. The recovered local volatility surface appears non-linear in both time and forward LIBOR rates.

1 Introduction

Implied volatilities of standard European options vary with strike levels and expiration dates. The former is usually referred to as the volatility skew and the latter is the volatility term structure. The observed volatility skew and volatility term structure contradict the assumption that the underlying asset is log-normally distributed with constant volatility.

Several approaches are suggested to relax this assumption, starting with the constant elasticity variance model [14], and including the stochastic volatility models [18], and jump diffusion models [25]. Derman and Kani [16], Dupire [15] and Rubinstein [32] use implied binomial or trinomial trees to price OTC products so that pricing is consistent with the volatility skew and term structure of European options.

In addition to implied trees, many other implied pricing methods have been developed. The idea is to recover the dynamics of the underlying asset from market prices of liquid options prices and use the information to price and hedge less liquid products. In other words, the volatility skew and volatility term structure observed in the market are taken as input to back out the implied probability density or process of the underlying asset.

The implied methods developed so far focus mainly on the application in the equity market and foreign exchange market. In the interest rate market, a volatility skew and term structure are also observed. The main difficulties of implementing implied methods to interest rate options mainly are

1. The discount factors are stochastic in the pricing of interest rate options while in the equity market and foreign exchange market, the discount factors are usually assumed to be deterministic.

2. Generally, in the interest rate models, both the drift and the volatility coefficients of the interest rate process are unknown under the pricing measure. However, in the equity and foreign exchange market markets, the drift of the underlying asset process under the accumulator measure is the risk-free rate and the only unknown is the volatility coefficient.

In this paper, the implied pricing methodology is implemented within the framework of forward LIBOR rate models developed by Miltersen, Sandmann and Sondermann [26] and Musiela and Rutkowski [27]. We use spline functional approach suggested by Coleman, Li and Verma [12] to recover the local volatility surfaces of forward LIBOR rates from caplet prices. Within the framework of forward LIBOR rate model, only the local volatility surfaces need to be approximated. Besides, given the local volatility surfaces, the local drifts of forward LIBOR rate under the spot LIBOR measure (or alternatively for one particular forward measure) can be easily obtained for the one factor model.

The paper is organized as follows. Sections 2 gives an overview to implied pricing methods developed in both equity market and interest rate market. In section 3, we describe the numerical procedure to recover local volatilities

and discuss the consistent pricing of bond options given recovered local volatilities. Section 4 includes two computation examples. One simulates the market caplet prices using the extended forward LIBOR model developed by Andersen and Andreasen [5]. The other implements the numerical procedure on market prices of caps on three month GBP LIBOR. The conclusion is given in section 5.

2 The Implied Pricing Methods

There is a considerable literature on the recovery of underlying asset distributions and processes implied from market options prices. Most of these methods are applied in the equity market. They include implied trees, the approximation of implied densities, and the recovery of local volatility functions. Relatively few works focus on implied pricing for the interest rate market. In this section, we will give an overview of the above methods and discuss the difficulty of implementing implied pricing to interest rate options.

2.1 Methods Developed for the Equity Market Products

Consider a non-dividend paying share S_t following one factor diffusion process under the objective measure Q^*,

$$\mathrm{d}S = \mu^{Q^*}(S,t)S\mathrm{d}t + \sigma(S,t)S\mathrm{d}z^{Q^*},$$

for some drift $\mu^{Q^*}(S,t)$ and volatility function $\sigma(S,t)$, where z^{Q^*} is a standard Brownian motion under Q^*. We refer $\mu(S,t)$ and $\sigma(S,t)$ as the local drift and volatility of S_t. Under the accumulator measure Q, where the money market account is taken as the numeraire, S has the process

$$\mathrm{d}S = rS\mathrm{d}t + \sigma(S,t)S\mathrm{d}z^{Q},$$

where z^Q is a standard Brownian motion under Q. The risk-free interest rate r is assumed to be deterministic. The value of the European call options $V_t(K,T)$ with strike level K and expiration date T is

$$
\begin{aligned}
V_t(K,T) &= \mathbf{E}^Q\left[\exp\left(-\int_t^T r_u\mathrm{d}u\right)V_T \mid \mathcal{F}_t\right] \\
&= \exp\left(-\int_t^T r_u\mathrm{d}u\right)\int_0^\infty (S_T - K)_+\, \phi^Q(S_T)\,\mathrm{d}S_T,
\end{aligned}
$$

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where $V_T = (S_T - K)_+$ is the payoff to the call at maturity. The probability density $\phi^Q(S_T)$ is often referred as the risk-neutral probability density or implied distribution of S_T at time T under accumulator measure. V_t satisfies the following backward PDE

$$\frac{1}{2}S^2\sigma^2(S,t)\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} = rV \tag{1}$$

If the local volatility $\sigma(S,t)$ is assumed to be a constant so that the probability density of S_t is log-normal, the solution to (1) is the Black-Schole's formula for the call option. However, the problem of volatility skew and volatility term structure observed from market prices shows the violation of the constant volatility assumption.

Dupire [15], Derman and Kani [16] and Rubinstein [32] suggest to use implied binomial and trinomial tree methods. The transition probabilities of the tree are obtained from European option prices. This approach recovers the underlying asset process on a discrete lattice.

One can also choose a functional form for the implied probability density $\phi^Q(S_T)$ and choose parameter values to match market options prices. Bahra [8], and Melick and Thomas [24] approximate implied underlying asset probability density with the mixture of log-normal distributions. Rubinstein [31] uses Edgeworth expansion to approximate the implied probability density. A truncated Hermitian polynomial expansion is also used by Madan and Milne [23]. If a complete set of European call prices for all strikes nad maturities is available, theoretically, the implied probability density can be obtained by twice differentiating the call price by the exercise price.

Under objective measure Q^*, options can be priced with a pricing kernel. Therefore the pricing kernel can also, in practice, be recovered from option data. Denote the pricing kernel as the function $\Psi^{Q^*}(S)$. The call option $V_t(K,T)$ can be written as

$$V_t(K,T) = \mathbf{E}^{Q^*}\left[(S_T - K)_+\Psi^{Q^*}(S_T)\right].$$

Ait-Sahalia and Lo [2] use kernel regression to approximate pricing kernel on the long term average. Rosenberg and Engle [30] estimate the empirical pricing kernel on a monthly basis to measure the time variation in risk preference.

Another approach to achieve pricing consistent with market prices of European call is to back out coefficients in the underlying asset process. In the equity market, the only unknown coefficient is the local volatility function $\sigma(S,t)$. If option prices for a complete continuous range of strikes are known,

local volatilities can be recovered directly from implied volatilities using the formula given by, among others, Andreasen [6]. Unfortunately the formula gives poor results if it is applied without regularization. Andersen [4] uses finite difference methods to discretize the adjoint equation,

$$\frac{\partial V}{\partial T} = \frac{1}{2}\sigma^2(K,T)K^2\frac{\partial^2 V}{\partial K^2} - rK\frac{\partial V}{\partial K},$$

from which local volatilities can be found.

To apply this implied grid approach, it is again necessary to assume that a complete set of European option prices with different strikes and maturities are available. However, in practice only a small number of liquid option prices can be observed in the market. Therefore, for the implied methods mentioned above, regularization by interpolatiing and extrapolating from market prices, or by optimizing some smoothness measure is inevitable. Although the implied grid approach is able to obtain local volatilities by inverting a linear tri-diagonal system, the low magnitude of Arrow-Debreu securities near the upper and lower boundaries may cause numerical instability in the linear system. The inverse problem of recovering the local volatilities from insufficient market option prices is ill posed.

To overcome this ill-posed problem, various approaches have been suggested. Regularization techniques to alleviate the numerical instability include Lagnado and Osher [20], Avellaneda, Friedman, Holmes and Samperi [7], and Lavin [21]. The idea is to find a smooth surface for the local volatility $\sigma(S,t)$ such that option prices obtained via equation (1) fit the market prices as closely as possible. Finding the surface involves a multi-dimensional optimization problem where the dimensions increase with the size of the grid (in the finite difference methods). In the equity case, Coleman, Li and Verma [12] use natural tensor product splines to represent the local volatility surface. They minimize the difference between the theoretical option prices from equation (1) and the market option prices by adjusting the local volatility parameters at spline knots. The advantage of the spline functional approach is that it reduces the dimension of the optimization problem from the number of discrete points in the grid to the number of spline knots. In this paper, we apply this method to recover the local volatility functions of forward LIBOR rates.

2.2   Implied Pricing for The Interest Rate Market

The implied pricing methods mentioned in the previous section are mainly applied to the equity market, where only the local volatility function needs to

be estimated. In the interest rate market, the estimation of parameters in the processes of the state variables forms an important part of empirical studies in interest rate modelling. For the models of the instantaneous short rate, the time series of short term interest rates are commonly used to calibrate the models. For example, Chan, Karolyi, Longstaff and Sanders [11] use the general method of moments to estimate the parameters in the short rate process. Another more general approach to model interest rates is to specify the dynamics of instantaneous forward rates suggested by Heath, Jarrow and Merton [17]. The estimation of forward rate models focus on the specification of the local volatility functions of instantaneous forward rates. Amin and Morton [3] examine six different volatility functions of forward rates using Eurodollar futures and future options in the context of one-factor HJM model. The volatility functions they examine can be expressed in the following general form :

$$\sigma(t, T, f(t, T)) = \alpha_0 + \alpha_1(T - t)\exp(-\alpha_2(T - t))f(t, T)^{\alpha_3},$$

where $f(t, T)$ is the instantaneous forward rate. Brace and Musiela [10] use cap prices and options on bank bill futures to estimate the volatility function in the Gaussian HJM model. The volatility function is assumed to be piecewise constant for some period of time.

To avoid the misspecification of the drift and volatility functions, some non-parametric methods have been developed. Ait-Sahalia [1] non-parametrically estimates the volatility function of short rate process but restricts the drift function to be mean-reverting. Stanton [33] approximates drift, volatility and market price of risk with a Taylor expansion of conditional expectation. The time series of short term interest rates is used in above non-parametric approaches to estimate the parameters in the short rate process. To price interest rate derivatives, it requires one to estimate market price of risk by matching cross-sectional data.

Estimating a pricing kernel is another way to calibrate interest rate models. Constantinides [13] specifies the pricing kernel for a positive interest rate and estimates the pricing kernel for the one-state variable model with 10 years bond price. The consumption-based pricing kernel from Eurodollar future options are used by Rosenburg [29] to test different utility functions in equilibrium interest rate models. These studies are more closed to the spirit of implied pricing methods. However, the pricing kernels in these studies are specified parametrically.

If we consider implied pricing in the short rate models, where the short

rate process under accumulator measure Q is assumed to be

$$dr = \mu^Q(r, t)dt + \sigma(r, t)dz^Q, \tag{2}$$

where μ^Q is the local drift, $\sigma(r, t)$ is the local volatility functions under Q and z^Q is a standard Brownian motion under Q. The problem is that both μ^Q and $\sigma(r, t)$ are unknown. Recovering both μ^Q and $\sigma(r, t)$ non-parametrically without restricting their relationship with each other may cause insufficient ranks while solving the linear system in implied grid approach. It also causes a difficulty in obtaining the gradients for objective function in regularization techniques. If we consider practical application and the underlying of many interest rate options are market rates such as LIBOR rates or swap rates, it appears that working within the framework of market models may facilitate the numerical procedures.

The approach to model market observed interest rates such as forward LIBOR rates or swap rates is suggested by Miltersen, Sandmann and Sondermann [26], Musiela and Rutkowski [27] and Jamshindian [19]. In the forward LIBOR rate model, the process of forward LIBOR rates is specified as log-normal to obtain a Black-like formula for caplets. Therefore, the local volatility function of forward LIBOR rates is assumed to be deterministic. To relax this assumption, we use the spline functional approach to approximate the local volatility functions of forward LIBOR rates. It is assumed to be a function of time and the forward LIBOR rate. In the next section, we will give a brief review for the forward LIBOR rate model and discuss how to fit caplet volatility skew and smile without assuming forward LIBOR rates are log-normally distributed.

3 The Numerical Procedure

3.1 The Review of the Forward LIBOR Rate Model

Assume a finite set of expiration dates is given at current time t^*: $t^* = T_0 < T_1 < T_2, \ldots, < T_N$, which is referred to as the tenor structure. Denote $\delta = T_j - T_{j-1}$, for $j = 1, \ldots, N$. [1] Given a finite number of bond prices $B(t, T_j)$, for $j = 0, \ldots, N$, at time t. The forward LIBOR rate is defined by the market convention, for $j = 0, \ldots, N - 1$:

$$L(t, T_j) = L(t, T_j, T_{j+1}) = \frac{B(t, T_j) - B(t, T_{j+1})}{\delta B(t, T_{j+1})}, \forall t \in [T_0, T_j]. \tag{3}$$

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[1] For the simplicity, we set the tenor δ as a constant.

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The model constructs a family of forward LIBOR rates, $L(t, T_j)$, for $j = 0, \ldots, N-1$, a collection of mutually equivalent forward measure P_{T_j}, for $j = 1, \ldots, N$ and a family of $z^{P_{T_j}}$, for $j = 1, \ldots, N$, such that

1. for any $j = 1, \ldots, N$, the process $z^{P_{T_j}}$ follows a one dimensional standard Brownian motion under P_{T_j} forward measure and

2. for any $j = 1, \ldots, N-1$, the forward LIBOR rate $L(t, T_j)$ satisfies

$$\mathrm{d}L(t, T_j) = L(t, T_j)\sigma(L(t, T_j), t)\mathrm{d}z^{P_{T_{j+1}}}, \forall t \in [T_0, T_j], \tag{4}$$

with

$$L(t^*, T_j) = \frac{B(t^*, T_j) - B(t^*, T_{j+1})}{\delta B(t^*, T_{j+1})}.$$

Under the forward measure $P_{T_{j+1}}$, the price of a caplet settled in arrear with expiration date T_j and strike K_i, for $i = 1, \ldots, M$ at time t is

$$c_t(K_i, T_j) = \delta B(t, T_{j+1})\mathbf{E}^{P_{T_{j+1}}}\left[\left(L(T_j, T_j) - K_i\right)_+ \mid \mathcal{F}_t\right]. \tag{5}$$

Define the forward price of $c_t(K_i, T_j)$ as

$$\tilde{c}_t(K_i, T_j) = \frac{c_t(K_i, T_j)}{B(t, T_{j+1})}.$$

Under forward measure $P_{T_{j+1}}$, $\tilde{c}_t(K_i, T_j)$ is a martingale and satisfies the PDE [2]

$$\frac{\partial \tilde{c}}{\partial t} + \frac{1}{2}L(t, T_j)^2 \sigma^2(L(t, T_j), t)\frac{\partial^2 \tilde{c}}{\partial L(t, T_j)^2} = 0, \tag{6}$$

with boundary condition :

$$\tilde{c}_{T_j}(K_i, T_j) = \delta(L(T_j, T_j) - K_i)_+.$$

If the volatility $\sigma(L(t, T_j), t)$ is assumed to be a deterministic function of t and T_j, the forward LIBOR rate $L(t, T_j)$ is log-normally distributed under the forward measure $P_{T_{j+1}}$. With this assumption, the model yields the Black type formula for the caplet $c_t(K_i, T_j)$, which is consistent with market conventions. In the following section, we consider the spline functional approximation within the forward LIBOR rate model framework. Observing market caplet prices and pure discount bond prices, we can back out $\sigma(L(t, T_j), t)$ under $P_{T_{j+1}}$ which gives the best fit of market prices.

[2]In Miltersen, Sandmann and Sondermann's paper [26], the state variable of the PDE (6) is the ratio of two bond prices with different maturities.

3.2 The Numerical Procedure

Within the above forward LIBOR rate framework, our problem is to obtain a local volatility surface which minimizes the difference between the market observed forward caplet prices and theoretical forward caplet prices solved from the backward equation (6). Note that when finite difference methods are used to solve the theoretical forward caplet prices from equation (6), we can only solve caplets with a specific expiration date for one grid. The reason is that the underlying is the forward rates for different future time interval and the pricing of caplets is under different forward measures. However, it is difficult to interpolate local volatility function accurately given caplets with only one expiration date. Therefore, in addition to the prices observed at present time, we also used the historical data of caplets and bond prices with the same expiration date observed in the past.

At the current time t^*, we can observe the pure discount bond prices $B(t^*, T_{j+1})$ maturing at future time T_{j+1} and the market caplet prices $c_{t^*}^m(K_i, T_j)$ expiring at future time T_j with strike K_i, for $i = 1, \ldots, M$ and $j = 0, \ldots, N-1$. For the past time $t^* - k\delta$, for $k = 1, \ldots, \bar{k}$, we also observe the prices of $B(t^* - k\delta, T_{j+1})$ and $c_{t^* - k\delta}^m(K_i, T_j)$ for $i = 1, \ldots, M$ and $j = 0, \ldots, N-1$. For the simplicity, we will only consider the case of $\bar{k} = 1$ in the following discussion. Besides, the expiration date of caplets will be fixed to a specific T_j. However, the same procedure can be applied to caplets with expiration date T_j, for $j = 0, \ldots, N-1$. The local volatility of different forward rate $L(t, T_j)$, for $j = 0, \ldots, N-1$, will be estimated on different grids.

The forward caplets at time t^* is

$$\tilde{c}_{t^*}^m(K_i, T_j) = \frac{c_{t^*}^m(K_i, T_j)}{B(t^*, T_{j+1})}.$$

At time $t^* - \delta$, the forward caplet price was

$$\tilde{c}_{t^*-\delta}^m(K_i, T_j) = \frac{c_{t^*-\delta}^m(K_i, T_j)}{B(t^* - \delta, T_{j+1})}.$$

The theoretical forward caplet prices at time t^* and time $t^* - \delta$ with expiration date T_j solved from (6) are denoted as $\tilde{c}_{t^*}(K_i, T_j)$ and $\tilde{c}_{t^*-\delta}(K_i, T_j)$, for $i = 1, \ldots, M$. Under the forward measure $P_{T_{j+1}}$, they satisfy the backward PDE (6). The finite difference methods are used to solve PDE (6) for the theoretical forward caplet prices given an initial guess of local volatility surface $\sigma(L(t, T_j), t)$. Different from the setup of the grid in the equity market, where the current time is set at the lower boundary of time space, the past time $t^* - \delta$ is set at the lower boundary. This is illustrated in Figure (1).
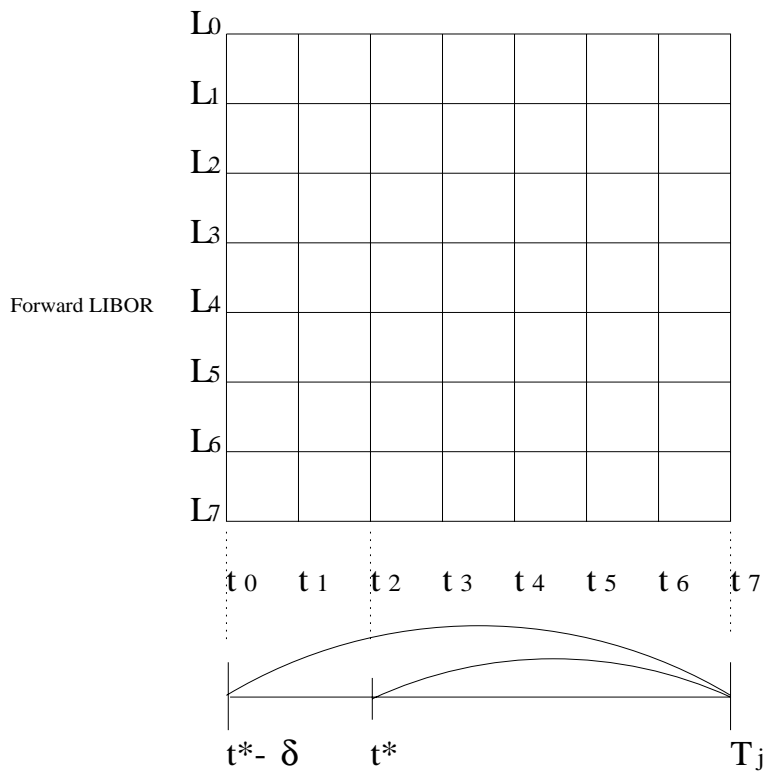
Figure 1: The set up of the grid

In the equity market, the call options with different expiration dates can be priced under one accumulator measure given a local volatility surface of stock prices. However, here we need to fix the expiration date to some specific T_j when the grid is set up, but allow the time to observe market prices to vary from $t^* - \delta$ to t^*. This requires the use of historical data of caplet prices. The recovered local volatilities from the past time $t^* - \delta$ to the current time t^* is the realized local volatilities and are not related to market expectation. The more useful and "meaningful" local volatilities are those from the current time t^* to future expiration date T_j. [3]

The local volatility function $\sigma(L(t, T_j), t)$ is represented by natural tensor product cubic splines with fixed spline knots in time space t and state space $L(t, T_j)$. To avoid the problem of underdetermination, the number of the spline knots cannot be over too many of the number of market observations. The spline knots are specified by the array $\theta = \{(L_p, t_q)\}$, for $p = 1, \ldots, \bar{p}$ and $q = 1, \ldots, \bar{q}$, where $\bar{p} * \bar{q}$ is equal to the market observations. The corresponding local volatility at knots is specified by the array $\bar{\sigma} = \{\sigma_{p,q}\}$, for $p = 1, \ldots, \bar{p}$ and $q = 1, \ldots, \bar{q}$. Given θ and $\bar{\sigma}$, the interpolating local volatility $\sigma(L(t, T_j), t)$ can be represented as $\zeta(L(t, T_j), t \mid \theta, \bar{\sigma})$.

The objective function is to minimize the difference between the theoretical forward caplet prices $\tilde{c}_{t^*-k\delta}(K_i, T_j, \zeta)$ and market forward caplet prices $\tilde{c}^m_{t^*-k\delta}(K_i, T_j)$, for $i = 1, \ldots, M$ and $k = 0, \ldots, \bar{k}$ by adjusting the local volatility at knots. It is

$$\min_{\bar{\sigma}} f(\bar{\sigma}) = \frac{1}{2} \sum_{k=0}^{k=\bar{k}} \sum_{i=1}^{i=M} \left(\tilde{c}_{t^*-k\delta}(K_i, T_j, \zeta) - \tilde{c}^m_{t^*-k\delta}(K_i, T_j) \right)^2. \qquad (7)$$

The computation procedure can be summarized as follows.

1. Set up the grid for the finite difference method. The grid covers the domain $[L_{min}, L_{max}] \times [t^* - \delta, T_j]$. L_{min} is the lower boundary of state space and L_{max} is the upper boundary of state space. Specify the spline knots array θ. for the allocation of the knots, we set $L_1 = L_{min}$, $L_{\bar{p}=L_{max}}$, $t_1 = 0$, and $t_{\bar{q}} = T_j$.

2. Initially guess the volatility vector $\bar{\sigma}$ on the spline knots.

3. Given θ and $\bar{\sigma}$, solve the theoretical forward caplets $\tilde{c}_{t^*}(K_i, T_j, \zeta)$ and $\tilde{c}_{t^*-\delta}(K_i, T_j, \zeta)$ with equation (6), for $i = 1, \ldots, M$ with the interpolated local volatility surface.

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[3]It is not necessary to assume the local volatility function is time invariance in this case. It needs to be re-estimated frequently, which means it changes with time.

4. Evaluate the objective function (7). If it doesn't meet the minimization criteria, update $\bar{\sigma}$ with optimization algorithm and go back to the previous step. Otherwise, terminate the optimization.

3.3 The Consistent Pricing of Bond Options

The aim to recover local volatility functions of forward LIBOR rates is to price other interest rate options consistently with market prices of caplets. In this section, we will show how to price European bond options and barrier bond options using the recovered local volatility functions.

Consider a bond option $c_t^B(K^B, T_j)$ with strike level K^B and maturity T_j on a pure discount bond $B(t, T_{j+1})$ maturing at time T_{j+1}. At time T_j, the payoff of the bond option is

$$c_{T_j}^B(K^B, T_j) = \left(B(T_j, T_{j+1}) - K^B\right)_+ = \left(\frac{B(T_j, T_{j+1})}{B(T_j, T_j)} - K^B\right)_+. \qquad (8)$$

Define the relative bond price $\tilde{B}_t(T_{j+1}, T_j)$ as

$$\tilde{B}_t(T_{j+1}, T_j) = \frac{B(t, T_{j+1})}{B(t, T_j)}.$$

Under the forward measure P^{T_j}, the relative bond price $\tilde{B}_t(T_{j+1}, T_j)$ follows the process

$$d\tilde{B}_t(T_{j+1}, T_j) = \tilde{B}_t(T_{j+1}, T_j) \gamma\left(\tilde{B}_t(T_{j+1}, T_j), t\right) dz^{P_{T_j}}. \qquad (9)$$

The function $\gamma\left(\tilde{B}_t(T_{j+1}, T_j), t\right)$ is the local volatility function of the relative bond price $\tilde{B}_t(T_{j+1}, T_j)$. The value of the bond option at the current time t^* is

$$c_{t^*}^B(K^B, T_j) = B(t^*, T_j) \mathbf{E}^{P_{T_j}}\left[(\tilde{B}_{T_j}(T_{j+1}, T_j) - K^B)_+ \mid \mathcal{F}_{t^*}\right]. \qquad (10)$$

Denote $\phi\left(\tilde{B}_t(T_{j+1}, T_j)\right)$ as the transition probability density of $\tilde{B}_t(T_{j+1}, T_j)$ conditional on $\tilde{B}_{t^*}(T_{j+1}, T_j)$ under measure P^{T_j}. [4] The expectation in equation (10) can be evaluated with respect to $\phi\left(\tilde{B}_{T_j}(T_{j+1}, T_j)\right)$, which is

$$
\begin{aligned}
c_{t^*}^B(K^B, T_j) &= B(t^*, T_j) \int_{K^B}^{\infty} \left(\tilde{B}_{T_j}(T_{j+1}, T_j) - K^B\right) \\
&\quad \phi\left(\tilde{B}_{T_j}(T_{j+1}, T_j)\right) d\tilde{B}_{T_j}(T_{j+1}, T_j).
\end{aligned}
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[4]It is more clear if we write the transition probability as $\phi\left(\tilde{B}_t(T_{j+1}, T_j) \mid \tilde{B}_{t^*}(T_{j+1}, T_j)\right)$. However, to simplify notations, we use $\phi\left(\tilde{B}_t(T_{j+1}, T_j)\right)$.

It is known $\phi\left(\tilde{B}_{T_j}(T_{j+1}, T_j)\right)$ satisfies Kolmogrov forward equation,

$$\frac{1}{2}\frac{\partial^2\left(\tilde{B}_t(T_{j+1}, T_j)^2\gamma\left(\tilde{B}\left(_t(T_{j+1}, t), t\right)^2\phi\right)}{\partial\tilde{B}_t(T_{j+1}, T_j)^2} = \frac{\partial\phi}{\partial t}. \tag{11}$$

$\gamma\left(\tilde{B}_t(T_{j+1}, T_j), t\right)$ can be obtained from local volatility functions $\sigma(L(t, T_j), t)$ of forward LIBOR rates $L(t, T_j)$ by the following equation.

$$\gamma\left(\tilde{B}_t(T_{j+1}, T_j), t\right) = -\frac{\delta L(t, T_j)\sigma\left(L(t, T_j), t\right)}{1 + \delta L(t, T_j)}. \tag{12}$$

Equation (12) can be derived by Itō's lemma. [5] Given $\sigma\left(L(t, T_j), t\right)$ backed out from market prices of caplets, we can recover $\gamma\left(\tilde{B}_t(T_{j+1}, T_j)\right)$ from equation (12) and solve the forward equation for ϕ with finite difference methods. The solved probability density ϕ is consistent with market prices of caplets and so is the bond option.

The same method can be used to price barrier options. Assume the barrier U is greater than the relative bond price $\tilde{B}_{t^*}(T_{j+1}, T_j)$ at current time t^*. Let the price at time t of a single barrier bond option on a pure discount bond $B(t, T_{j+1})$ be $c_t\left(U, 0; T_j, \left(B(T_j, T_{j+1}) - K^B\right)_+\right)$. The barrier option matures at time T_j and has strike level K^B. The option pays zero if the relative bond price $\tilde{B}_t(T_{j+1}, T_j)$ hits the barrier U before T_j. If the relative bond price never hits the barrier before T_j, the option pays $\left(B(T_j, T_{j+1}) - k^B\right)_+$ at the maturity. Under the forward measure P^{T_j}, the value of the barrier bond option at current time t^* is

$$\begin{aligned} c_{t^*}\left(U, 0; T_j, \left(B(T_j, T_{j+1}) - K^B\right)_+\right) &= B(t^*, T_j)\int_{K^B}^\infty \left(\tilde{B}_{T_j}(T_{j+1} - T_j) - K^B\right)_+ \\ &\quad \phi_U\left(\tilde{B}_{T_j}(T_{j+1}, T_j)\right)\mathrm{d}\tilde{B}_{T_j}(T_{j+1}, T_j). \end{aligned}$$

ϕ_U is the transition probability density of $\tilde{B}_t(T_{j+1}, T_j)$ not being absorbed at U. It satisfies the forward equation (11) with an absorbing boundary at U, which is

$$\phi_U(U) = 0.$$

Given recovered $\gamma\left(\tilde{B}_t(T_{j+1}, T_j), t\right)$, we can solve the forward equation with the absorbing boundary. The pricing of the single barrier option is simply

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[5] The derivation is given in the appendix.

evaluate the integral with respect to ϕ_U. For the double barrier options, the transition probability density $\phi_{U,L}$ for $\tilde{B}_t(T_{j+1}, T_j)$ not being absorbed can also be solved in the same way but with two abosrbing boundary conditions. They are

$$\phi_{U,L}(U) = 0,$$

and

$$\phi_{U,L}(L) = 0.$$

The issue to use recovered local volatility to price options such as European bond options or barrier bond options is that the local volatility functions are only recovered within the range of strikes of available market data. However, the options which need to be priced consistently may be sensitive to the local volatility outside the range of available data. In that case, The misspecification of local volatility outside the data range may cause pricing error.

In the following section, we will give computation examples of the numerical procedure to recover local volatility functions. We use the quasi-Newton method for the optimization algorithm and the Crank-Nicolson method for the finite difference scheme. In the first example, we simulate the market caplet prices with extended forward LIBOR rates model developed by Andersen and Andreasen [5] and back out the constant elasticity variance local volatility structure. The second example is to apply the method to the real market caplet data on three month GBP LIBOR.

4   The Computational Examples

4.1   The Example of Extended Forward LIBOR Rate Model

Andersen and Andreasen [5] investigate the extension of forward LIBOR rate model. They extend the forward rate process to be constant elasticity variance (CEV hereafter) process. In their extension, the local volatility function depends on forward LIBOR rate in power function. The closed form formula for caplets is obtained. We use the formula of this model to simulate the market caplet prices and use spline functional approximation to recover the CEV local volatility function.

In Andersen and Andreasen's paper [5], the forward LIBOR rate is expressed as

$$\mathrm{d}L(t, T_j) = L(t, T_j)^\alpha \lambda_{T_j}(t) \mathrm{d}z^{P_{T_{j+1}}},$$

where $\lambda_{T_j}(t)$ is a deterministic function. For $0 < \alpha < 1$ and an absorbing boundary at the level $L(t, T_j) = 0$, the forward caplet with strike level K,

expiration date T_j and tenor δ is

$$\tilde{c}_t(K, T_j) = \delta \left[L(t, T_j)(1 - \chi^2(a, b+2, c)) - K\chi^2(c, b, a) \right], \qquad (13)$$

where $a = \frac{K^2(1-\alpha)}{(1-\alpha)^2 v(t, T_j)}$, $b = \frac{1}{1-\alpha}$, $c = \frac{L(t, T_j)^{2(1-\alpha)}}{(1-\alpha)^2 v(t, T_j)}$, $v(t, T_j) = \int_t^{T_j} \| \lambda_{T_j}(u) \|^2 \mathrm{d}u$, and $\chi^2(.)$ is the non-central Chi-square distribution.

Given $\alpha = \frac{1}{2}$ and $\lambda_{T_j}(t)$ as a constant 0.06, in our previous notations, the local volatility function is

$$\sigma(L(t, T_j), t) = \frac{0.06}{L(t, T_j)^{\frac{1}{2}}}.$$

We use the numerical procedure described in the previous section to back out the local volatility function from the simulated market prices computed from equation (13). Sixteen market observations are simulated. The tenor δ is 0.25 year. The range of strikes are set from 0.085 to 0.12 with the interval of 0.05 and the time to expiry T_j is 1.00 year. The current time t^* set in the grid is 0.25 year and the past time is set at 0.00 year. Therefore the maturities of caplets are $T_j - t^* = 0.75$ year and $T_j - (t^* - \delta) = 1.00$ year. The forward LIBOR rate at current time t^* is 0.1 and 0.1075 at the past time $t^* - \delta$.

After 196 iterations, the value of objection function $f(\bar{\sigma})$ converges to $5.20983e - 12$. Figure (2) shows the comparison of simulated CEV volatility function and the recovered volatility function. It appears that the splines functional approach can accurately recover the local volatility function within the range of $[0.08, 0.13]$ in the forward LIBOR rate. It is the range of strikes that the simulated market caplets have. It appears that the fitting of local volatility curve at currentt time $t = 0.25$ year is better than the fitting at past time $t = 0.00$ year. The recovered local volatility surface in Figure (3) is very smooth, which is a good property for hedging and pricing other OTC products. The caplet prices calculated from equation (13) and those calculated from the recovered local volatility surface are compared in Figure (4). Table (1) and Table (2) show that the absolute fitting errors are below $0.11e - 5$ and the relative errors are below 0.0003% of the simulated caplet prices. If we compare the relative fitting errors across strikes, the fitting of in the money and at the money caplets is better than the fitting of out of the money caplets. For absolute errors, the fitting of at the money caplets is best. In general, the caplet prices calculated from recovered local volatility are very closed to those from analytical formula.
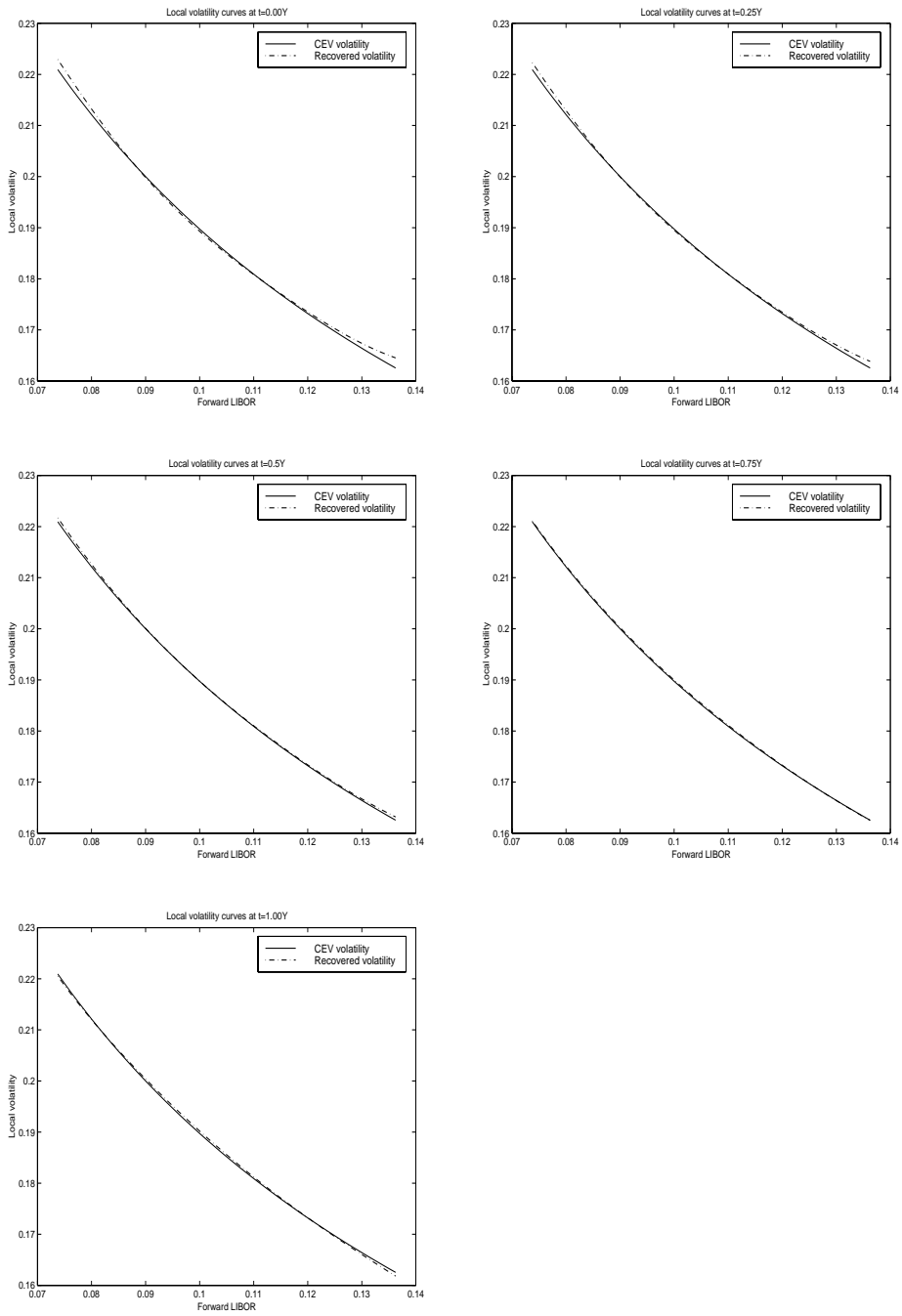
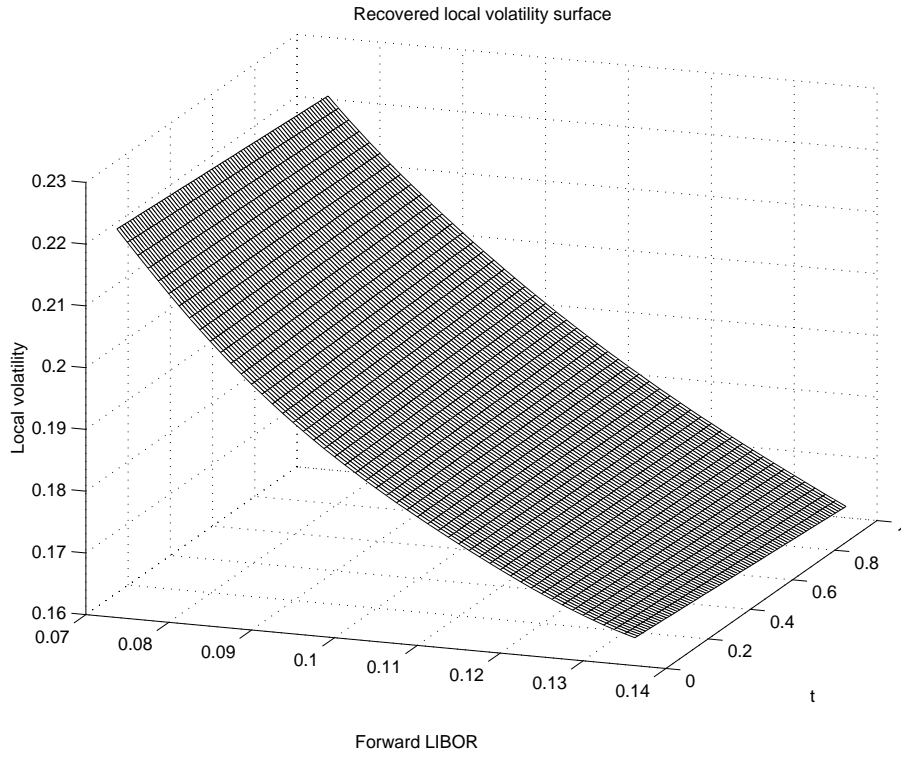Figure 2: The CEV volatility and recovered local volatility

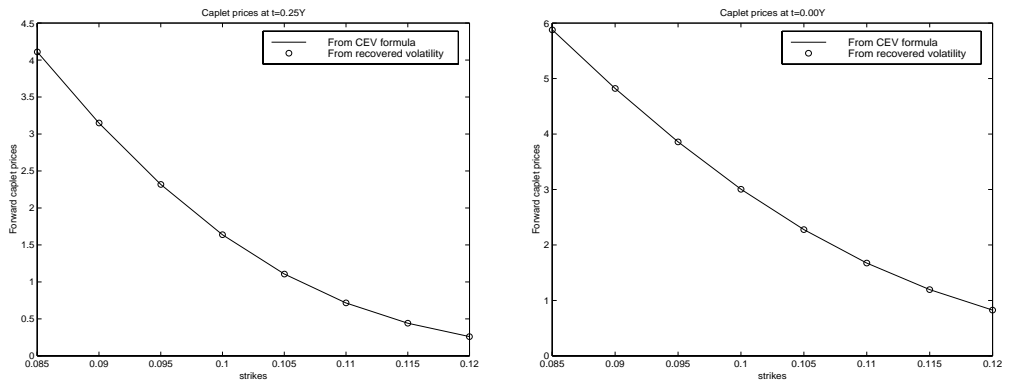Figure 3: The recovered local volatility surface from CEV caplet prices

Figure 4: The caplet prices from CEV formula and recovered local volatility

Table 1: The fitting of CEV caplet prices at t^*

strikes	CEV prices	fitted prices	absolute err.	relative err.(%)
0.085	4.110549	4.110548	0.011883e-5	0.002890e-3
0.090	3.147337	3.147336	0.033225e-5	0.010556e-3
0.095	2.317979	2.317980	0.011089e-5	0.047842e-3
0.100	1.637438	1.637437	0.037568e-5	0.022943e-3
0.105	1.107138	1.107136	0.109826e-5	0.099198e-3
0.110	0.715525	0.715525	0.054974e-5	0.076829e-3
0.115	0.441689	0.441689	0.090782e-5	0.205535e-3
0.120	0.260369	0.260368	0.070552e-5	0.270972e-3

Table 2: The fitting of CEV caplet prices at $t^* - \delta$

| strikes	CEV prices	fitted prices	absolute err.	relative err(%)
0.085	5.878858	5.878858	0.031553e-5	0.005367e-3
0.090	4.821871	4.821870	0.050986e-5	0.010573e-3
0.095	3.858341	3.858341	0.019956e-5	0.007172e-3
0.100	3.006119	3.006118	0.016912e-5	0.005625e-3
0.105	2.276918	2.276918	0.036679e-5	0.016109e-3
0.110	1.674574	1.674574	0.005828e-5	0.003480e-3
0.115	1.194856	1.194855	0.049868e-5	0.041735e-3
0.120	0.826732	0.826732	0.026178e-5	0.031665e-3
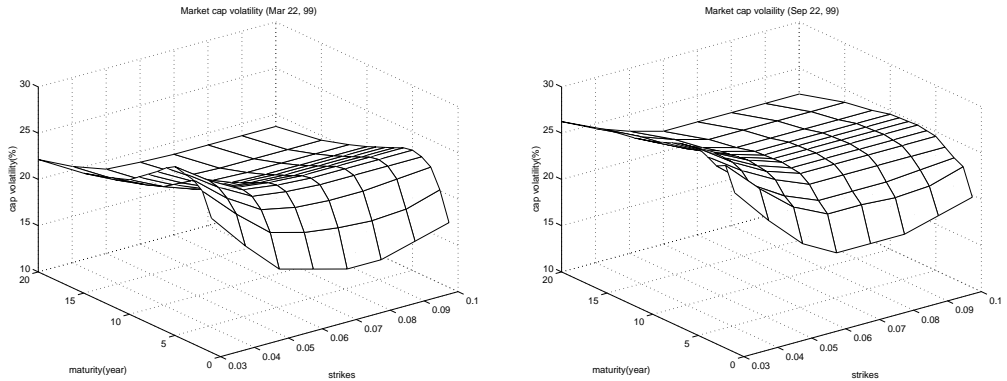
Figure 5: The market quoted cap volatility

4.2 The Implementation to Three Month GBP LIBOR Cap Prices

In this computation example, we tried to recover the local volatility function from real market cap prices on GBP three month LIBOR. We use the data on March 22, 1999 and September 22, 1999. Since the underlying is three month GBP LIBOR, the tenor δ is 0.25 year. The current time t^* is set at September 22, 1999 and the past time $t^* - \delta$ is at March 22, 1999. Figure (5) shows the market quoted cap volatility on March 22, 1999 and on September 22, 1999. There are eight strikes from 3% to 10% with 1% interval. The maturities of caps are up to 20 years. The cap volatilities on Setember 22, 1999 are higher and more noisy than those on March 22, 1999. This may be caused by Bank of England's announcement to raise short term interst rate on September 8, 1999.

The term structure data on both dates includes 1 month, 3 month, 6 month and 12 month GBP LIBOR rates and 2 year, 3 year 4 year, 5 year, 7 year, 10 year, 15 year and 20 year swap rates. The term structure of spot rates and forward rates are constructed nonparametrically. The forward rates are represented by Nelson and Siegel curves and their parameters are obtained by fitting the yield to maturity to market term structure data. [6] Given the yield to maturity $y(t, T)$, the pure discount bond $B(t, T)$ is

$$B(t, T) = \exp(-y(t, T)(T - t)),$$

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[6] If instantaneous forward rates $f(t, T)$ are represented by Nelson and Siegel curve,

$$f(t, T) = \beta_0 + \beta_1 \exp(-\eta(T - t)) + \beta_2(T - t) \exp(-\eta(T - t)),$$
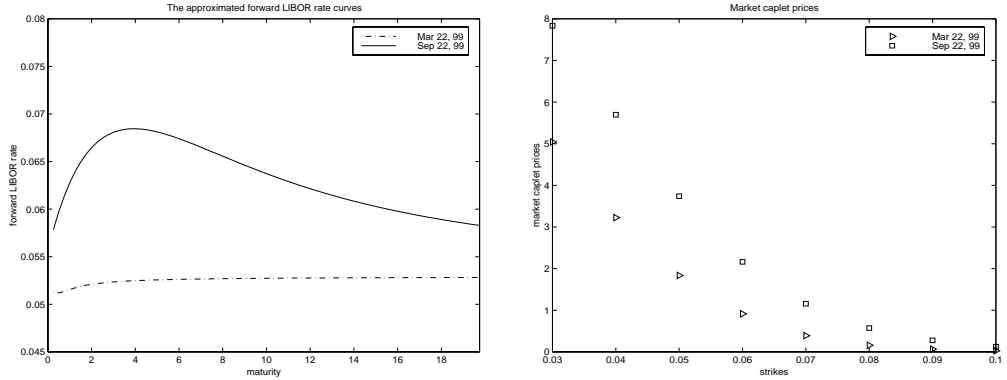
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Figure 6: The forward LIBOR rate curves and the market caplet prices

and the forward LIBOR rate is calculated from pure discount bonds prices by equation (3).

The cap prices are calculated from Black formula with flat cap volatility quoted in the market. We use Nelson and Siegel curves to approximate forward forward volatility and obtain caplet prices with forward forward volatility which can best fit the caplet prices. The caplets expiring on March 22, 2001 are used to recover the local volatility of three month forward LIBOR rates prevailing at time September 22, 1999 over the future time interval of March 22, 2001. Figure (6) shows the forward LIBOR rate curves and the caplet prices on March 22, 1999 and September 22, 1999. The underlying forward rate at time March 22, 1999 is 5.21% and 6.5% at time September 22, 1999. The pure discount bond which expires at March 22, 2001 is 0.8765 on March 22, 1999 and 0.8976 on September 22 1999.

For the set up of the grid, the number of time steps is 81 and the number of space steps is 162. The time interval is 0.025 and the state space interval is 0.0011. The same as the finite difference method used in our simulation example, we use Crank-Nicolson scheme to discritize the PDE (6). Since the market data is more noisy, we add upper and lower bounds in the optimization for $-3.00 < \bar{\sigma} < 3.00$. The spline knots are allocated at $[0.0005, 0.0435, 0.0865, 0.1295, 0.1736] \times [0.00, 0.50, 1.00, 1.50, 2.00]$. The

and the yield to maturity y(t,T) is

$$y(t,T) = \beta_0 + \left(\beta_1 + \frac{\beta_2}{\eta} \right) \frac{1 - \exp(-\eta(T-t))}{\eta(T-t)} - \frac{\beta_2}{\eta} \exp(-\eta(T-t)).$$

Table 3: The fitting of market forward caplet prices at Mar. 22.

strikes	Market prices	fitted prices	absolute error	relative err(%)
0.03	5.665982	5.660614	0.536727e-2	0.094727
0.04	3.622942	3.631808	0.886657e-2	0.244734
0.05	2.061762	2.057187	0.457453e-2	0.221874
0.06	1.026168	1.013474	1.269327e-2	1.236958
0.07	0.437058	0.451610	1.455230e-2	3.329604
0.08	0.179067	0.188121	0.905443e-2	5.056449
0.09	0.074724	0.071686	0.303704e-2	4.064352
0.10	0.030834	0.023641	0.719233e-2	23.326003

Table 4: The fitting of market forward caplet prices at Sep. 22.

strikes	Market prices	fitted prices	absolute error	relative err(%)
0.03	8.766952	8.767460	0.050870e-2	0.005802
0.04	6.378040	6.373046	0.499327e-2	0.078288
0.05	4.185346	4.187952	0.260760e-2	0.062303
0.06	2.421501	2.431652	1.015205e-2	0.419246
0.07	1.293898	1.282347	1.155027e-2	0.892672
0.08	0.639062	0.640564	0.150284e-2	0.235163
0.09	0.310231	0.307536	0.269450e-2	0.868401
0.10	0.139832	0.144828	0.499640e-2	3.573168

number of spline knots is the same as the number of observations.

It appears more difficult for the optimization to converge with the market data. The objective function is 0.000947 after 48 iteration. Figure (7) shows the recovered local volatility surface. It can be seen that the local volatility function is non-linear in both variables of time and forward LIBOR rate. The local volatility function appears to increase with time within the time horizon of 0.5 to 2 year, which may indicate that the market expects three month LIBOR rates to be volatile in the future 12 months. The fitting errors are presented in Table (3) and Table (4). The fitting is worse for out of the money caplets in terms of relative errors. Due to the fact that there are only 16 observations available, the splines may not be flexible enough to fit the noisy market prices. If there is more data available, we will have more freedom to choose the allocation of spline knots and improve the fitting.
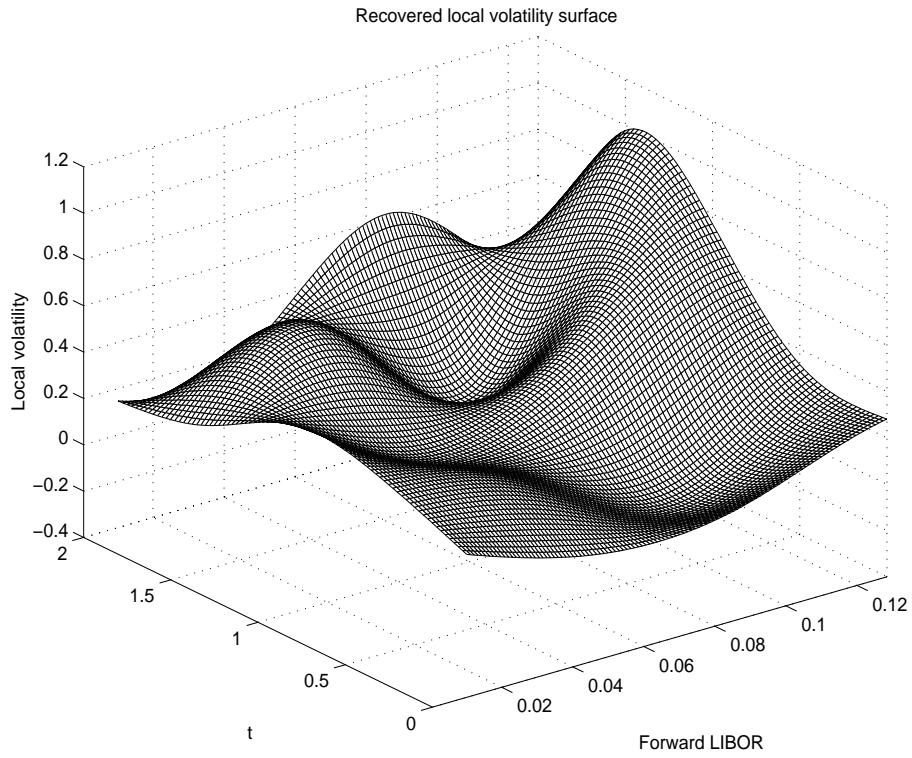
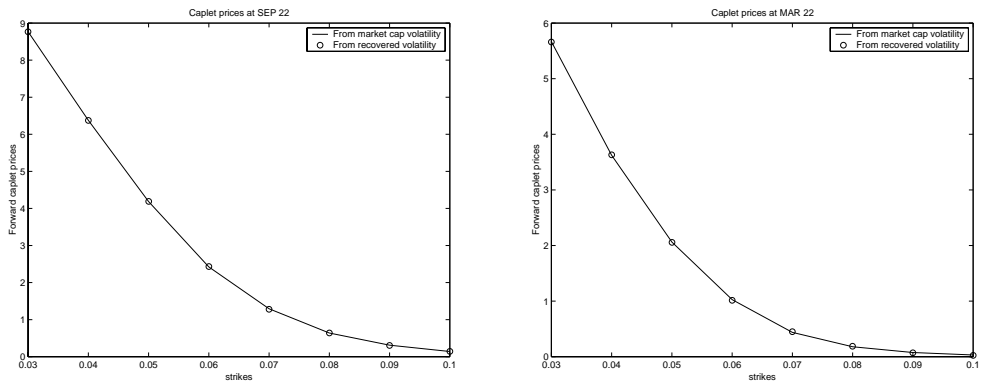Figure 7: The recovered local volatility surface from market prices

Figure 8: The market prices and prices from recovered local volatility

5   Conclusion

In this paper, we implement spline functional approach to approximate the local volatility surface of forward LIBOR rates. Since the forward LIBOR rate follows a martingale under the related forward measure $P_{T_{j+1}}$, the implied process of forward LIBOR rates can be recovered from caplet prices without the need to compute the local drifts. This reduces the computation and complexity of optimization. Given the local volatilities of forward rates, we show how to consistently price European bond options and barrier bond options.

We give two computation examples to demonstrate the numerical procedure. To accurately approximate the local volatility function, we need to use historical caplet prices which has the same underlying forward LIBOR rates. In the first example, the caplet prices are simulated by CEV volatility structure. It shows that the recovered local volatility surface is very closed to the simulated CEV volatility structure. In the second example, we use market data of three month GBP LIBOR cap volatility to recover the local volatility of forward LIBOR rate prevailing on September 22 1999 for March 22 2001. It is more difficult for the optimization to converge in the case of market data. Since we have the historical data for only two days, the number of spline knots is restricted. Therefore, the spline functions may not be flexible enough to fit the noisy market data very well. However, this can be improved if more data is available.

The approach to back out the local volatilities within the framework of forward LIBOR rate model can be extended to multifactors where interest rates are not perfectly correlated with each other. In that case, one needs to obtain the covariance structure of forward LIBOR rates. This can be calculated directly from historical forward LIBOR rates. The spline functional approach can also be implemented to forward swap rate model [19] to recover the local volatility functions of forward swap rates if the swaption prices across strikes and maturities are available. For further research, the application to other interest rate models and the empirical study to the evolution of the local volatility functions will be investigated.

Appendix

The relative bond price $\tilde{B}_t(T_j, T_{j+1})$ is

$$\tilde{B}_t(T_j, T_{j+1}) = \frac{B(t, T_j)}{B(t, T_{j+1})} = \frac{1}{\tilde{B}_t(T_j, T_{j+1})}.$$

Under the forward measure $P_{T_{j+1}}$, $\tilde{B}_t(T_j, T_{j+1})$ follows the process

$$\mathrm{d}\tilde{B}_t(T_j, T_{j+1}) = \tilde{B}_t(T_j, T_{j+1})\gamma\left(\tilde{B}_t(T_j, T_{j+1}, t)\right)\mathrm{d}z^{P_{T_{j+1}}},$$

where $z^{P_{T_{j+1}}}$ is a Brownian motion under the measure $P_{T_{j+1}}$. It is known that the relationship between $\gamma\left(\tilde{B}_t(T_j, T_{j+1}, t)\right)$ and the local volatility function $\sigma\left(L(t, T_j), t\right)$ of forward LIBOR rate $L(t, T_j)$ is [7]

$$\gamma\left(\tilde{B}_t(T_j, T_{j+1}), t\right) = \frac{\delta L(t, T_j)\sigma\left(L(t, T_j), t\right)}{1 + \delta L(t, T_j)}.$$

Let $Y = \tilde{B}_t(T_{j+1}, T_j) = \frac{1}{\tilde{B}_t(T_j, T_{j+1})}$ and apply Itō's lemma to Y.

$$\mathrm{d}Y = Y\gamma\left(\tilde{B}_t(T_j, T_{j+1}), t\right)^2 \mathrm{d}t - Y\gamma\left(\tilde{B}_t(T_j, T_{j+1}), t\right)\mathrm{d}z^{P_{T_{j+1}}}.$$

Change the measure from $P_{T_{j+1}}$ to P_{T_j}, where Y is a martingale. The processs of Y under P_{T_j} is

$$\mathrm{d}Y = -Y\gamma\left(\tilde{B}_t(T_j, T_{j+1}), t\right)\mathrm{d}z^{P_{T_j}},$$

where $z^{P_{T_j}}$ is a Brownian motion under P_{T_j}. So the process of $\tilde{B}_t(T_{j+1}, T_j)$ under P_{T_j} is

$$\mathrm{d}\tilde{B}_t(T_{j+1}, T_j) = \tilde{B}_t(T_{j+1}, T_j)\left(-\gamma\left(\tilde{B}_t(T_j, T_{j+1}), t\right)\right)\mathrm{d}z^{P_{T_j}}.$$

Compare above equation with equation (9), we have

$$\begin{aligned}
\gamma\left(\tilde{B}(T_{j+1}, T_j), t\right) &= -\gamma\left(\tilde{B}(T_j, T_{j+1}), t\right) \\
&= -\frac{\delta L(t, T_j)\sigma\left(L(t, T_j), t\right)}{1 + \delta L(t, T_j)}.
\end{aligned}$$

References

[1] Y. Ait-Sahalia. Nonparametric pricing of interest rate derivative securities. *Econometrica*, 64(3):527–561, 1996.

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[7] See Musiela and Rutkowski [28, page 344]. In their model, both γ and σ are assumed to be time dependent only. The relationship still holds if γ and σ are time and state variables dependent.

[2] Y. Ait-Sahalia and A. Lo. Non-parametric estimation of state-price densities implicit in financial asset prices. *Journal of Finance*, 53(2):499–548, 1998.

[3] K. Amin and A. Morton. Implied volatility functions in arbitrage-free term structure models. *Journal of Financial Economics*, 35(2):141–181, 1994.

[4] L. Andersen. *Five Essays on the Pricing of Contingent Claims*. PhD thesis, Århus Business School, 1996.

[5] L. Andersen and J. Andreasen. Volatility skews and extensions of the LIBOR market model. *Working Paper, General Re financial products, New York*, 1998.

[6] J. Andreasen. *Essays on Contingent Claim Pricing*. PhD thesis, Århus Business School, 1997.

[7] M. Avellaneda, C. Friedman, R. Holmes, and D. Samperi. Calibrating volaility surfaces via relative entropy minimization. *Applied Mathematical Finance*, 4, 1997.

[8] B. Bahra. Probability distributions of future asset prices implied by option prices. *Bank of England Quarterly Bulletin*, 36(3):299–311, 1996.

[9] A. Brace, D. Gatarek, and M. Musiela. The market model of interest rate dynamics. *Mathematical Finance*, 7:127–155, 1997.

[10] A. Brace and M Musiela. A multifactor Gauss Markov implementation of Heath, Jarrow and Morton. *Mathematical Finance*, 4(3):259–284, 1994.

[11] K.C. Chan, A. Karolyi, F. Longstaff, and A Sanders. An emperical comparison of alternative models of the short-term interest rate. *Journal of Finance*, 47(3):1209–1228, 1992.

[12] T. F. Coleman, Li Y., and Verma A. Recontructing the unknown local volatility function. *Journal of Computational Finance*, 2(3):77–102, 1999.

[13] G. M. Constantinides. A theory of the nominal term structure of interest rates. *Review of Financial Studies*, 5:531–552, 1992.

[14] J. Cox and Ross S. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3:382–402, 1976.

[15] B. Dupire. Pricing with a smile. *RISK*, 7(1):18–20, 1994.

[16] E. Durmen and I. Kani. Riding on a smile. *RISK*, 7(2):32–39, 1994.

[17] D. Heath, R. Jarrow, and A. Morton. Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica*, 60:77–105, 1992.

[18] J. Hull and A. White. The pricing of options on assets with stochastic volatility. *Journal of Finance*, 42:281–300, 1987.

[19] F Jamshidian. LIBOR and swap rate market models and measures. *Finance and Stochastic*, 1(4):293–330, 1997.

[20] R. Lagnado and S. Osher. A technique for calibrating derivative security pricing model: numerical solutions of an inverse problem. *Journal of Computational Finance*, 1:13–25, 1997.

[21] A. Levin. Recovering implied volatility and distribution from American future option prices using regularization method. *Working Paper, Bank of Canada*, 1998.

[22] A. Levin, D. Mcmanus, and D. Watt. The information content of Canadian Dollar future options. In *Proceedings of the Conference: Information in Financial Asset Prices*. Bank of Canada, May 1998.

[23] D. Madan and F. Milne. Contigent claims valued and hedged by pricing and investing in a basis. *Mathematical Finance*, 4(3):223–245, 1994.

[24] W. Melick and C. Thomas. Recovering an asset's implied PDF from option prices: An application to Crude Oil during the Gulf Crisis. *Journal of Financial Quantitative Analysis*, 32(1):91–115, 1997.

[25] R. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144, 1976.

[26] K. Miltersen, K. Sandmann, and D. Sondermann. Close form solutions for term structure derivatives with log-normal interest rates. *Journal of Finance*, 52(1):409–430, 1997.

[27] M. Musiela and M. Rutkowski. Continuous time term structure models: Forward measure approach. *Finance and Stochastics*, 1(4):261–291, 1997.

[28] M. Musiela and M. Rutkowski. *Martingale methods in financial modelling.* Springer, 1997.

[29] J. Rosenberg. Emperical tests of interest rate model pricing kernels. *Working Paper, Department of Finance, Stern School of Business, New York University*, 1999.

[30] J. Rosenberg and R. Engle. Emperical pricing kernal. *Working Paper, Department of Finance, Stern School of Business, Nen York University*, 1999.

[31] M. Rubinstein. Edgworth binomial tree. *Journal of Derivative*, 5(3):20–27, 1998.

[32] M. Rubintein. Implied binomial tree. *Journal of Finance*, 49(3):771–818, 1994.

[33] R. Stanton. A non-parametric model of term structure dynamics and the market price of interest rate risk. *Journal of Finance*, 52(5):1973–2003, 1997.