A Computional General Equilibrium Model with Vintage Capital[1]

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Introduction

Computational general equilibrium models usually assume a putty putty technology: the capital intensity of the production process can be changed instantaneously and without cost. Thus, in a competitive framework, the factors of production fully and instantaneously adjust to current economic conditions. This means that "realistic" changes in real wages or in the cost of capital lead to very significant and quick moves in demand for labor and capital. Moreover, the quick adjustment of the capital stock should cause huge variations in the flows of investment.

However, actual employment and capital stock exhibit much weaker movements than those predicted above. Hence, the integration of this theoretical framework in a realistic model requires some improvements. One way to decrease the cost-sensitivity of production factors consists in assuming nonlinear adjustment costs (usually quadratic costs). This results in smoother dynamic adjustments of labor and capital. However, this specification rests upon an *ad hoc* assumption without strong empirical foundations. Moreover, it is a very unconvincing way to model irreversibility, and firing and hiring costs. Finally, the putty putty framework is unable to give simple and acceptable explanations for the medium term movements in the wage share in value added, which are observed in some European countries. Although adjustment costs smooth the dynamics of factor demands in the short run, they are far from sufficient to produce medium term changes in the income distribution between capital and labor.

A key feature of the putty putty specification, that is central to its empirical failure, is that all the vintages of capital have the same capital intensity. On the contrary, we would expect the current technology menu to be only available to the newly created units of production. This is precisely what the putty clay specification does. In this framework, current economic conditions affect the capital intensity of the new production units (their technological choice) and the number of these units created (the investment in the economy). The other production units keep the technology they were given at their creation. However, current economic conditions affect their profitability and lead to the scrapping of non-profitable units. Hence, the aggregate capital-labor ratio changes gradually with the flows of investment and the scrapping of old obsolete production units. Putty clay investment may thus provide medium term dynamics in the distribution of income.

This specification has some other advantages. The irreversibility of investment is embedded in the model and firing costs can easily be introduced. This gives a convincing foundation to the stickiness of employment.

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[1] The ideas developed here were inspired by a series of papers by Caballero and Hammour. We have benefited of very useful comments by Agnès Bénassy, Pierre-Yves Hénin, Paul Zagamé, Pierre Sicsic and Werner Roeger. Stéphanie Guichard made invaluable criticisms and suggestions on previous versions of the paper.

Despite all its advantages, the putty clay technology suffers from a serious drawback. Its implementation in a macroeconomic model is cumbersome for two reasons. First, the model has a long memory since it keeps track of all the vintages of capital created in the past, that are still in working order. Thus the model has “variables with long lags”. Second, the planning horizon of investors stretches far into the future. More precisely, the decision concerning the new production units involves forward variables that cover the expected lifetime of these units. The model has then “variables with long leads”.

However, these problems can be easily overcome nowadays. Models with variables presenting long leads and lags can be solved with powerful algorithms (for instance those implemented in Troll), and simulation time is decreasing with the improvement of personal computers.

The first section presents a model representing the production of goods and the demand of factors with a putty clay technology. In the second section we first close the model by completing its demand side, by introducing a “wage curve” and by assuming the equilibrium of the goods market. Then, we describe the determination of the equilibrium. The third section presents the results of the simulation of the model. First, the calibration is such that the steady state of the model is identical to the average situation of the French economy over 1991-1996. Then, we investigate whether the properties of existence and uniqueness of a solution to the model are satisfied. In particular, we illustrate the usefulness of the stability conditions highlighted in Laffargue (2000). Finally, the fourth section discusses the consequences of the introduction of monetary policy and nominal rigidities in the model.

1. Technology and factors demands

In this section, we introduce the specification of the technology of firms and we determine their decisions. At each date, firms build a number of new production units, which will start to produce one period later. They have to choose the capital intensity embodied in these units. Since this capital intensity cannot change in the future, the expected lifetime of the new production units is part of the decision-making. Both capital intensity and expected lifetime are set to maximize the expected discounted cash flows minus installation costs. Moreover, firms reassess the profit that each older unit would make during the current period if it were kept into activity. When this profit is negative the units are scrapped. Then, we can aggregate the decisions of firms and get the investment, the employment and the production of the current period.

a) Technology and investment cost

We consider a representative firm, on a perfectly competitive goods market, which must make choices at time t_0[2]. At this time, the firm decides to acquire k_{t_0} new units of capital. It also chooses the technology embodied in this capital. The technology menu is characterized by an *ex ante* first-order homogenous production function:

$$F(k_{t_0}, A_{t_0} n_{t_0}) = z[\alpha k_{t_0}^{1-1/\sigma} + (1-\alpha)(A_{t_0} n_{t_0})^{1-1/\sigma}]^{\sigma/(\sigma-1)}, \text{ with: } z, \sigma > 0, 0 < \alpha < 1.$$

This equation determines the amount of goods produced by combining k_{t_0} units of capital with n_{t_0} units of labor. A_{t_0} represents the efficiency of the technology available at time t_0. σ is the *ex ante* elasticity of factor substitution. The capital intensity κ_{t_0} chosen at time t_0 for the entire lifetime of the capital units is defined by:

$$\kappa_{t_0} = k_{t_0} / A_{t_0} n_{t_0}.$$

The production function can be rewritten as:

$$(1)\quad F(\kappa_{t_0}, 1) = z[\alpha \kappa_{t_0}^{1-1/\sigma} + (1-\alpha)]^{\sigma/(\sigma-1)}$$

In the future, the vintage of capital created at t_0 will either be used with capital intensity κ_{t_0} or be scrapped.

We define one unit of production created at time t_0 as the combination of 1 unit of labor and $\kappa_{t_0} A_{t_0}$ units of capital. This unit of production produces $A_{t_0} F(\kappa_{t_0}, 1)$ units of goods. At the beginning of period t_0, i.e. at time t_0, n_{t_0} new units of production are created. At the end of each period, a fraction δ of the firms goes bankrupt and stops producing. δ can be seen as the rate at which the production units disappear for all reasons but macroeconomic conditions

[2] As firms are identical, the choices made by any individual firm also hold at the aggregate level. Time t_0 represents the beginning of period t_0.

(bad management, mistakes or technical difficulties in the implementation of production). In that sense, δ is equivalent to an exogenous depreciation rate.

For every unit, production starts in the period following its installation. The units created at time t_0 are productive from the beginning of period t_0+1, i.e. at time t_0+1.

The aggregate investment made at time t_0 is defined by the following relation:

(2) $I_{t_0} = A_{t_0}\kappa_{t_0}n_{t_0}$

The cost of one unit of capital, including the installation costs, expressed in units of good produced, is exogenous and equal to $c_{i_{t_0}} > 0$.

Caballero and Hammour (1994, 1996) show that with this cost assumption, expected demand shocks are reflected in the current changes in the investment flows (creation), and have no effect on the scrapping age (destruction). Of course, the number of scrapped units changes with time since the different vintages of capital do not have the same size[3].

b) Value of a new production unit

We define r_t as the nominal interest rate of an asset with a maturity of one period, available at time t. We also define $w_t A_t$ as the nominal wage paid to each worker of the production unit for the work done during period t. By convention, we assume that this wage is paid at the end of period t. Let us consider a unit of production created at time t_0. When capital is scrapped, at time $t_0+T(t_0)$, firing the workers costs $p_{t_0+T(t_0)}x^f_{t_0+T(t_0)}A_{t_0+T(t_0)}$ where $p_{t_0+T(t_0)}$ is the price level at time $t_0+T(t_0)$ and $x^f_{t_0+T(t_0)}$ is the real firing cost in efficiency unit[4]. If the production unit goes bankrupt, however, no firing cost is incurred.

We define by V_{t,t_0} the present value of the cash flows of the production unit built at t_0, measured at the end of period t, i.e. the value of the production unit for the firms that have not gone bankrupt until that time. In case of bankruptcy, this value is zero.

We have the following arbitrage condition:

$$(1-\delta)V_{t+1,t_0} + (1-tcr_{t+1})\lfloor p_{t+1}A_{t_0}F(\kappa_{t_0},1) - w_{t+1}A_{t+1}\rfloor = (1+r_t)V_{t,t_0}$$

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[3] Caballero et Hammour (1994) also consider the case of a more general cost function: $c_{i_{t_0}} = c_0 + c_1\dfrac{I_{t_0}}{A_{t_0}}$ with $c_0 > 0$ and $c_1 > 0$. Assuming a positive value of c_1 leads to fluctuations in the scrapping age with expected changes in demand. Note that this assumption is different from that of models of Tobin's *q*: for Caballero and Hammour the investment appearing in the cost function is the aggregated value of investment in the economy, so the individual firm considers that it cannot change it.

[4] This means that the unit of production has been active from the beginning of period t_0+1 to time $T(t_0)$, that is for $T(t_0)-1$ periods.

where tcr is the corporate tax rate.

At the end of period *t*, the owner of a production unit can either sell it and invest the proceeds in the financial markets, or hold it. In the first case, he/she will get $(1+r_t)V_{t,t_0}$ at the end of period *t+1*. In the second case, the unit will yield an after tax profit at the end of period *t+1* equal to: $(1-tcr_{t+1})\lfloor p_{t+1}A_{t_0}F(\kappa_{t_0},1)-w_{t+1,t_0}A_{t+1}\rfloor$. Besides, at the end of period *t+1*, the unit will either go bankrupt, with probability δ, or still be in working order, with probability $(1-\delta)$.

It is now possible to define an initial condition and a terminal condition for the first difference equation above.

First, the assumption of free entry means that the financial value of a new production unit at the end of the period when it is built is equal to its building cost (taking into account the possibility of bankruptcy):

$$(1-\delta)V_{t_0,t_0} = p_{t_0}c_{i_{t_0}}A_{t_0}\kappa_{t_0}$$

The investor who decides to create a new production unit pays its building cost during the installation period t_0: $p_{t_0}c_{i_{t_0}}A_{t_0}\kappa_{t_0}$. He is aware that this unit has a probability δ to go bankrupt at the end of period t_0, and a probability $(1-\delta)$ to remain in working order. Its value at the end of period t_0, expected at the beginning of period t_0 is then: $(1-\delta)V_{t_0,t_0}$.

Second, when the production unit is scrapped, the owner has to pay for the firing costs. Note that the expected scrapping date of the new unit, $t_o+T(t_o)$, is not necessarily an integer number of years. We define $\overline{T}(t_o)$ as the integer part of the expected lifetime[5] of a new production unit, and $\Delta T(t_o)$ as its decimal part[6]. The value of the production unit, at the date it is scrapped, is equal to the firing costs. This defines the terminal condition.

In the model, the production units created at time t_0 will be productive for $\overline{T}(t_o)$ full periods of time, i.e. from the beginning of period t_0+1 until the end of period $t_0+\overline{T}(t_o)$. It will still be in working order at the beginning period $t_0+\overline{T}(t_o)+1$, but will be only productive during a fraction of period equal to $\Delta T(t_o)$. Hence, to write down the terminal condition, we must make additional assumptions on the value of nominal wage (in efficiency unit), price and firing costs at the date $t_o+\overline{T}(t_o)+1+\Delta T(t_0)$. We assume that these three variables at that

[5] The expected lifetime corresponds to the period of time during which a unit is in working order.

[6] Thus we have $T(t_0)=\overline{T}(t_o)+1+\Delta T(t_0)$. All the units of the same vintage are identical. We want to avoid scrapping choices to be discrete: at a given time, all the units of a given vintage would be either scrapped or kept in use. With a model using the year as basic time unit, this discrete choice would have introduced too strong discontinuities in the response of the model to shocks. A better assumption is that part of a given vintage may be scrapped and the rest may be kept in use. The assumption, made in the paper is that a vintage is scrapped at an intermediary time inside a period. This assumption is slightly different, but a little more practical.

date are equal to their geometrical interpolations between (the end of period) $t_o + \overline{T}(t_o)$ and (the end of period) $t_o + \overline{T}(t_o) + 1$:

$$w_{t_0+\overline{T}(t_0)+1+\Delta T(t_0)} = w_{t_0+\overline{T}(t_0)}^{\;1-\Delta T(t_0)} w_{t_0+\overline{T}(t_0)+1}^{\;\Delta T(t_0)}$$

$$p_{t_0+\overline{T}(t_0)+1+\Delta T(t_0)} = p_{t_0+\overline{T}(t_0)}^{\;1-\Delta T(t_0)} p_{t_0+\overline{T}(t_0)+1}^{\;\Delta T(t_0)}$$

$$x^f_{t_0+\overline{T}(t_0)+1+\Delta T(t_0)} = x^f_{t_0+\overline{T}(t_0)}{}^{1-\Delta T(t_0)} x^f_{t_0+\overline{T}(t_0)+1}{}^{\Delta T(t_0)}$$

We assume that the income related to the fraction of the period starting at the beginning of period $t_0 + \overline{T}(t_0) + 1$ and lasting $\Delta T(t_0)$ is received and taxed at the end of the period. We also assume that the probability to go bankrupt is independent of the length of the fraction $\Delta T(t_0)$. Finally, technical progress is assumed to be a continuous function of time given by: $A(t_0 + t) = A(t_0)(1+\gamma)^t$ for $t \geq 0$. We now have the terminal condition:

$$\begin{aligned} &V_{t_0+T(t_0),t_0} = \\ &+ A_{t_0}(1 - tcr_{t_0+\overline{T}(t_0)+1})\Delta T(t_0)\Big(p_{t_0+\overline{T}(t_0)}^{\;1-\Delta T(t_0)} p_{t_0+\overline{T}(t_0)+1}^{\;\Delta T(t_0)} F(\kappa_{t_0},1) - w_{t_0+\overline{T}(t_0)}^{\;1-\Delta T(t_0)} w_{t_0+\overline{T}(t_0)+1}^{\;\Delta T(t_0)}(1+\gamma)^{T(t_0)}\Big) \\ &- p_{t_0+\overline{T}(t_0)}^{\;1-\Delta T(t_0)} p_{t_0+\overline{T}(t_0)+1}^{\;\Delta T(t_0)}\Big[x^f_{t_0+\overline{T}(t_0)}{}^{1-\Delta T(t_0)} x^f_{t_0+\overline{T}(t_0)+1}{}^{\Delta T(t_0)}\Big] A_{t_0}(1+\gamma)^{T(t_0)} \end{aligned}$$

The value of the firm when it is created is the present value of future expected profits minus the firing costs at the expected scrapping age. Note that the creation of a new production unit at the beginning of period t_o enables firms to save a fraction of the firing costs incurred by scrapping units at the beginning of period $t_o + 1$ (when the unit created in t_o begins to produce). We define η the fraction of workers in a new unit coming from scrapped units. This saving ($\eta p_{t_0+1} x^f_{t_0+1} A_{t_0+1}$) is part of the value of the new unit.

The value of a new production unit is obtained by summing the arbitrage equation until its scrapping date:

$$\begin{aligned} V_{t_0,t_0} &= \eta p_{t_0+1} x^f_{t_0+1} A_{t_0}(1+\gamma) + A_{t_0} \sum_{s=t_0+1}^{t_0+\overline{T}(t_0)} (1 - tcr_s)\Big(p_s F(\kappa_{t_0},1) - w_s(1+\gamma)^{s-t_0}\Big)(1-\delta)^{s-t_0-1} \Big/ \left(\prod_{\tau=t_0}^{s-1}(1+r_\tau)\right) \\ &+ p_{t_0+\overline{T}(t_0)} A_{t_0}\left(\frac{p_{t_0+\overline{T}(t_0)+1}}{p_{t_0+\overline{T}(t_0)}}\right)^{\Delta T(t_0)} \\ &\Big\{\Delta T(t_0)(1 - tcr_{t_0+\overline{T}(t_0)+1})\Big(F(\kappa_{t_0},1) - \big(w_{t_0+\overline{T}(t_0)} / p_{t_0+\overline{T}(t_0)}\big)^{1-\Delta T(t_0)}\big(w_{t_0+\overline{T}(t_0)+1} / p_{t_0+\overline{T}(t_0)+1}\big)^{\Delta T(t_0)}(1+\gamma)^{T(t_0)}\Big) \\ &- x^f_{t_0+\overline{T}(t_0)}{}^{1-\Delta T(t_0)} x^f_{t_0+\overline{T}(t_0)+1}{}^{\Delta T(t_0)}(1+\gamma)^{T(t_0)}\Big\}(1-\delta)^{\overline{T}(t_0)} \Big/ \left(\prod_{\tau=t_0}^{t_0+\overline{T}(t_0)}(1+r_\tau)\right) \end{aligned}$$

c) Characteristics of the new production units

When a new unit is created, its owners must choose the technology embodied in it on the basis of its planned lifetime, and of the market conditions they expect during that period. The choices of the planned lifetime ($T(t_0)$) and of the capital intensity of this new unit (κ_{t_0}) result from the maximization of the value of this unit minus its installation cost:

$$\Psi_{t_0}\left(T(t_0),\kappa_{t_0}\right)=(1-\delta)V_{t_0,t_0}-p_{t_0}c_{i_{t_0}}A_{t_0}\kappa_{t_0}\,.$$

The first-order condition relative to the expected lifetime cannot be obtained directly since the objective function Ψ_{t_0} is not derivable in $T(t_o)$. However, Ψ_{t_0} depends on the two elements that define the expected lifetime: $\overline{T}(t_o)$ which is an integer, and $\Delta T(t_o)$ which is a real number between 0 and 1. The objective function is actually a three variable function: $\Psi_{t_0}\left(\overline{T}(t_0),\Delta T(t_0),\kappa_{t_0}\right)$. Firms determine the two components of the expected lifetime and the capital intensity of the new production units so that Ψ_{t_0} is maximized.

<u>*Expected life time*</u>

The integer part of the expected lifetime, $\overline{T}(t_o)$, is then defined by:

(3) $$\begin{cases}\Psi_{t_0}\left(\overline{T}(t_0),\Delta T(t_0),\kappa_{t_0}\right)-\Psi_{t_0}\left(\overline{T}(t_0)-1,\Delta T(t_0),\kappa_{t_0}\right)>0\\ \Psi_{t_0}\left(\overline{T}(t_0)+1,\Delta T(t_0),\kappa_{t_0}\right)-\Psi_{t_0}\left(\overline{T}(t_0),\Delta T(t_0),\kappa_{t_0}\right)<0\end{cases}$$

This system of inequations may have more than one solution. If it is the case, the whole model should be solved for each of these solutions. The retained solution would then be the one associated with the highest wealth of the owners of the unit.

However, we give in Appendix 1, reasons to believe that under reasonable economic assumptions, the system of inequations has a unique solution.

The decimal part of the expected scrapping date of the unit $\Delta T(t_o)$ is determined by the following first-order condition:

$$\frac{\partial\Psi_{t_0}(\overline{T}(t_0),\Delta T(t_0),\kappa_{t_0})}{\partial\Delta T(t_0)}=0$$

Thus:
(4)

$$\ln\left(\frac{p_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)}}\right)$$

$$\left\{\Delta T(t_0)(1-tcr_{t_0+\bar{T}(t_0)+1})\left((1+\gamma)^{-T(t_0)}F(\kappa_{t_0},1)-\left(\frac{w_{t_0+\bar{T}(t_0)}}{p_{t_0+\bar{T}(t_0)}}\right)^{1-\Delta T(t_0)}\left(\frac{w_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)+1}}\right)^{\Delta T(t_0)}\right)-x^{f\ 1-\Delta T(t_0)}_{t_0+\bar{T}(t_0)}x^{f\ \Delta T(t_0)}_{t_0+\bar{T}(t_0)+1}\right\}$$

$$+(1-tcr_{t_0+\bar{T}(t_0)+1})\left((1+\gamma)^{-T(t_0)}F(\kappa_{t_0},1)-\left(\frac{w_{t_0+\bar{T}(t_0)}}{p_{t_0+\bar{T}(t_0)}}\right)^{1-\Delta T(t_0)}\left(\frac{w_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)+1}}\right)^{\Delta T(t_0)}\right)$$

$$-\Delta T(t_0)(1-tcr_{t_0+\bar{T}(t_0)+1})\left(\frac{w_{t_0+\bar{T}(t_0)}}{p_{t_0+\bar{T}(t_0)}}\right)^{1-\Delta T(t_0)}\left(\frac{w_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)+1}}\right)^{\Delta T(t_0)}\ln\left(\frac{w_{t_0+\bar{T}(t_0)+1}/p_{t_0+\bar{T}(t_0)+1}}{w_{t_0+\bar{T}(t_0)}/p_{t_0+\bar{T}(t_0)}}\right)$$

$$-x^{f\ 1-\Delta T(t_0)}_{t_0+\bar{T}(t_0)}x^{f\ \Delta T(t_0)}_{t_0+\bar{T}(t_0)+1}\ln\left(\frac{x^f_{t_0+\bar{T}(t_0)+1}}{x^f_{t_0+\bar{T}(t_0)}}\right)$$

$$-\left(\Delta T(t_0)(1-tcr_{t_0+\bar{T}(t_0)+1})\left(\frac{w_{t_0+\bar{T}(t_0)}}{p_{t_0+\bar{T}(t_0)}}\right)^{1-\Delta T(t_0)}\left(\frac{w_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)+1}}\right)^{\Delta T(t_0)}+x^{f\ 1-\Delta T(t_0)}_{t_0+\bar{T}(t_0)}x^{f\ \Delta T(t_0)}_{t_0+\bar{T}(t_0)+1}\right)\ln(1+\gamma)=0$$

Capital intensity

The capital intensity chosen by investors for the new production units is then derived by the first-order condition: $\frac{\partial\Psi_{t_0}(\bar{T}(t_0),\Delta T(t_0),\kappa_{t_0})}{\partial\kappa_0}=0$.

(5)

$$F_1^{'}(\kappa_{t_0},1)\left\{\sum_{s=t_0+1}^{t_0+\bar{T}(t_0)}p_s(1-tcr_s)(1-\delta)^{s-t_0-1}/\prod_{\tau=t_0}^{s-1}(1+r_\tau)\right.$$
$$\left.+\Delta T(t_0)p_{t_0+\bar{T}(t_0)}(1-\delta)^{\bar{T}(t_0)}\left(\frac{p_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)}}\right)^{\Delta T(t_0)}(1-tcr_{t_0+\bar{T}(t_0)+1})/\prod_{\tau=t_0}^{t_0+\bar{T}(t_0)}(1+r_\tau)\right\}=p_{t_0}c_{i_{t_0}}/(1-\delta)$$

where $F_1^{'}(\kappa,1)$ is the marginal productivity of capital.

By combining this equation and the free-entry condition, we get (see Appendix 2):

(6)

$$\sum_{s=t_0+1}^{t_0+\bar{T}(t_0)}(1-tcr_s)\left(p_s F_2^{'}(\kappa_{t_0},1)-w_s(1+\gamma)^{s-t_0}\right)(1-\delta)^{s-t_0-1}/\left(\prod_{\tau=t_0}^{s-1}(1+r_\tau)\right)$$

$$+\,p_{t_0+\bar{T}(t_0)}A_{t_0}\left(\frac{p_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)}}\right)^{\Delta T(t_0)}$$

$$\left\{\Delta T(t_0)(1-tcr_{t_0+\bar{T}(t_0)+1})\left(F_2^{'}(\kappa_{t_0},1)-\left(w_{t_0+\bar{T}(t_0)}/p_{t_0+\bar{T}(t_0)}\right)^{1-\Delta T(t_0)}\left(w_{t_0+\bar{T}(t_0)+1}/p_{t_0+\bar{T}(t_0)+1}\right)^{\Delta T(t_0)}(1+\gamma)^{T(t_0)}\right)\right.$$

$$\left.-\,x^{f\ \ 1-\Delta T(t_0)}_{t_0+\bar{T}(t_0)}x^{f\ \ \Delta T(t_0)}_{t_0+\bar{T}(t_0)+1}(1+\gamma)^{T(t_0)}\right\}(1-\delta)^{\bar{T}(t_0)}/\left(\prod_{\tau=t_0}^{t_0+\bar{T}(t_0)}(1+r_\tau)\right)=-\eta p_{t_0+1}x^f_{t_0+1}(1+\gamma)$$

Capital intensity is determined by present and future interest rates. Future expected real wages are in turn defined by capital intensity. Thus, we have a factor cost frontier which involves the average of present and future costs whereas the traditional cost frontier derived from putty putty technology only depends on current costs.

d) Scrapping and aggregation

We now consider the decisions concerning the production units built before time t_0. For each vintage of capital, its capital intensity being already set, the investor checks whether it is still profitable. If not, it is discarded.

Under our assumptions, a production unit is built during a period (which is an integer number), but can be used for the whole or part of a period. Let us call $\bar{a}(t_0)$ the age of the oldest unit which is used at the beginning of period t_0[7], i.e. at time t_0, and which will be scrapped at time $t_0+\Delta a(t_0)<t_0+1$. This unit was built at period $t_0-\bar{a}(t_0)$. Its value at the end of this period is[8]:

$$V_{t_0-\bar{a}(t_0),t_0-\bar{a}(t_0)}=\eta p_{t_0-\bar{a}(t_0)+1}x^f_{t_0-\bar{a}(t_0)+1}A_{t_0-\bar{a}(t_0)}(1+\gamma)$$

$$+\,A_{t_0-\bar{a}(t_0)}\sum_{s=t_0-\bar{a}(t_0)+1}^{t_0-1}(1-tcr_s)\left(p_s F(\kappa_{t_0-\bar{a}(t_0)},1)-w_s(1+\gamma)^{s-t_0+\bar{a}(t_0)}\right)(1-\delta)^{s-t_0+\bar{a}(t_0)-1}/\left(\prod_{\tau=t_0-\bar{a}(t_0)}^{s-1}(1+r_\tau)\right)$$

$$+\,A_{t_0-\bar{a}(t_0)}\,p_{t_0-1}\left(\frac{p_{t_0}}{p_{t_0-1}}\right)^{\Delta a(t_0)}$$

$$\left\{\Delta a(t_0)(1-tcr_{t_0})\left(F(\kappa_{t_0-\bar{a}(t_0)},1)-\left(w_{t_0-1}/p_{t_0-1}\right)^{1-\Delta a(t_0)}\left(w_{t_0}/p_{t_0}\right)^{\Delta a(t_0)}(1+\gamma)^{\bar{a}(t_0)+\Delta a(t_0)-1}\right)\right.$$

$$\left.-\,x^{f\ \ 1-\Delta a(t_0)}_{t_0-1}x^{f\ \ \Delta a(t_0)}_{t_0}(1+\gamma)^{\bar{a}(t_0)+\Delta a(t_0)-1}\right\}(1-\delta)^{\bar{a}(t_0)-1}/\left(\prod_{\tau=t_0-\bar{a}(t_0)}^{t_0-1}(1+r_\tau)\right)$$

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[7] So, the unit has been active at this date for $\bar{a}(t_0)-1$ periods.

[8] Wage, price and firing costs within period t_0 are geometrical interpolations of their values in (the end of) periods t_0-1 and t_0.

$$\Psi_{t_0-\bar{a}(t_0)}\left(\bar{a}(t_0)-1,\Delta a(t_0),\kappa_{t_0-\bar{a}(t_0)}\right)=(1-\delta)V_{t_0-\bar{a}(t_0),t_0-\bar{a}(t_0)}-p_{t_0-\bar{a}(t_0)}c_{i_{t_0-\bar{a}(t_0)}}A_{t_0-\bar{a}(t_0)}\kappa_{t_0-\bar{a}(t_0)}$$

The age of the oldest production units used at the beginning of period t_0 is defined by:

$$\begin{cases}\Psi_{t_0-\bar{a}(t_0)}(\bar{a}(t_0)-1,\Delta a(t_0),\kappa_{t_0-\bar{a}(t_0)})-\Psi_{t_0-\bar{a}(t_0)}(\bar{a}(t_0)-2,\Delta a(t_0),\kappa_{t_0-\bar{a}(t_0)})>0\\ \Psi_{t_0-\bar{a}(t_0)}(\bar{a}(t_0),\Delta a(t_0),\kappa_{t_0-\bar{a}(t_0)})-\Psi_{t_0-\bar{a}(t_0)}(\bar{a}(t_0)-1,\Delta a(t_0),\kappa_{t_0-\bar{a}(t_0)})<0\end{cases}$$

Lets us define

$$U_{t_0-\bar{a}(t_0)}(\bar{a}(t_0)-1)=\Psi_{t_0-\bar{a}(t_0)}(\bar{a}(t_0)-1,\Delta a(t_0),\kappa_{t_0-\bar{a}(t_0)})-\Psi_{t_0-\bar{a}(t_0)}(\bar{a}(t_0)-2,\Delta a(t_0),\kappa_{t_0-\bar{a}(t_0)})$$

Then, we have:

(7)

$$U_{t_0-\bar{a}(t_0)}(\bar{a}(t_o)-1)=\frac{A_{t_0-1+\Delta a(t_0)}(1-\delta)^{\bar{a}(t_0)-2}}{\prod_{\tau=t_0-\bar{a}(t_0)}^{t_0-2}(1+r_\tau)}p_{t_0-1}$$

$$\left\{(1-tcr_{t_0-1})\left((1+\gamma)^{-\bar{a}(t_0)-\Delta a(t_0)+1}F(\kappa_{t_0-\bar{a}(t_0)},1)-\frac{w_{t_0-1}}{p_{t_0-1}}(1+\gamma)^{-\Delta a(t_0)}\right)\right.$$

$$+\left(\frac{p_{t_0}}{p_{t_0-1}}\right)^{\Delta a(t_0)}\left(\frac{1-\delta}{1+r_{t_0-1}}\right)$$

$$\left[\Delta a(t_0)(1-tcr_{t_0})\left((1+\gamma)^{-\bar{a}(t_0)-\Delta a(t_0)+1}F(\kappa_{t_0-\bar{a}(t_0)},1)-\left(\frac{w_{t_0-1}}{p_{t_0-1}}\right)^{1-\Delta a(t_0)}\left(\frac{w_{t_0}}{p_{t_0}}\right)^{\Delta a(t_0)}\right)-x^{f\ \ 1-\Delta a(t_0)}_{t_0-1}x^{f\ \Delta a(t_0)}_{t_0}\right]$$

$$-\left(\frac{p_{t_0-2}}{p_{t_0-1}}\right)^{1-\Delta a(t_0)}$$

$$\left[\Delta a(t_0)(1-tcr_{t_0-1})\left((1+\gamma)^{-\bar{a}(t_0)-\Delta a(t_0)+1}F(\kappa_{t_0-\bar{a}(t_0)},1)-\left(\frac{w_{t_0-2}}{p_{t_0-2}}\right)^{1-\Delta a(t_0)}\left(\frac{w_{t_0-1}}{p_{t_0-1}}\right)^{\Delta a(t_0)}\left(\frac{1}{1+\gamma}\right)\right)\right.$$

$$\left.\left.-\frac{x^{f\ \ 1-\Delta a(t_0)}_{t_0-2}x^{f\ \ \Delta a(t_0)}_{t_0-1}}{1+\gamma}\right]\right\}$$

The period of time during which the oldest units are in working order within period t_0 is defined by:

$$\frac{\partial\Psi_{t_0-\bar{a}(t_0)}(\bar{a}(t_0)-1,\Delta a(t_0),\kappa_{t_0-\bar{a}(t_0)})}{\partial\Delta a(t_0)}=0$$

(8)

$$\ln\left(\frac{p_{t_0}}{p_{t_0-1}}\right)$$

$$\left\{\Delta a(t_0)(1-tcr_{t_0})\left((1+\gamma)^{-\bar{a}(t_0)-\Delta a(t_0)+1}F(\kappa_{t_0-\bar{a}(t_0)},1)-\left(\frac{w_{t_0-1}}{p_{t_0-1}}\right)^{1-\Delta a(t_0)}\left(\frac{w_{t_0}}{p_{t_0}}\right)^{\Delta a(t_0)}\right)-x_{t_0-1}^{f\ \ 1-\Delta a(t_0)}x_{t_0}^{f\ \Delta a(t_0)}\right\}$$

$$+(1-tcr_{t_0})\left((1+\gamma)^{-\bar{a}(t_0)-\Delta a(t_0)+1}F(\kappa_{t_0-\bar{a}(t_0)},1)-\left(\frac{w_{t_0-1}}{p_{t_0-1}}\right)^{1-\Delta a(t_0)}\left(\frac{w_{t_0}}{p_{t_0}}\right)^{\Delta a(t_0)}\right)$$

$$-\Delta a(t_0)(1-tcr_{t_0})\left(\frac{w_{t_0-1}}{p_{t_0-1}}\right)^{1-\Delta a(t_0)}\left(\frac{w_{t_0}}{p_{t_0}}\right)^{\Delta a(t_0)}\ln\left(\frac{w_{t_0}/p_{t_0}}{w_{t_0-1}/p_{t_0-1}}\right)$$

$$-x_{t_0-1}^{f\ \ 1-\Delta a(t_0)}x_{t_0}^{f\ \Delta a(t_0)}\ln\left(\frac{x_{t_0}^{f}}{x_{t_0-1}^{f}}\right)$$

$$-\left(\Delta a(t_0)(1-tcr_{t_0})\left(\frac{w_{t_0-1}}{p_{t_0-1}}\right)^{1-\Delta a(t_0)}\left(\frac{w_{t_0}}{p_{t_0}}\right)^{\Delta a(t_0)}+x_{t_0-1}^{f\ \ 1-\Delta a(t_0)}x_{t_0}^{f\ \Delta a(t_0)}\right)\ln(1+\gamma)=0$$

It is now possible to define the aggregate level of employment and production. At date t, the production structure available is characterized by the series: $\{n_{t-a}(1-\delta)^a,\kappa_{t-a}\}$, where a is the age of the different production units in working order ($1\le a\le \bar{a}(t_0)$), and by $\Delta a(t_0)$. Aggregate employment and production capacity are obtained by summing these vintages[9]:

(9) $$N_{t_0}=\sum_{a=1}^{\bar{a}(t_0)-1}n_{t_0-a}(1-\delta)^a+\Delta a(t_0)n_{t_0-\bar{a}(t_0)}(1-\delta)^{\bar{a}(t_0)}$$

(10)

$$Y_{t_0}=A_{t_0}\left\{\sum_{a=1}^{\bar{a}(t_0)-1}(1+\gamma)^{-a}F(\kappa_{t_0-a},1)n_{t_0-a}(1-\delta)^a+\Delta a_{t_0}(1+\gamma)^{-\bar{a}(t_0)}F(\kappa_{t_0-\bar{a}(t_0)},1)n_{t_0-\bar{a}(t_0)}(1-\delta)^{\bar{a}(t_0)}\right\}$$

[9] In the above paragraph we have computed $\bar{a}(t_0)$ and $\Delta a(t_0)$ as the integer and the decimal parts of the age $a(t_0)$ of the oldest units which can generate a positive profit. However we could add the assumption that once a productive unit was scrapped, it cannot be put in use again. In this case, we should introduce the constraint $\bar{a}(t_0)\le\bar{a}(t_0-1)+1$ in our optimization problem. The maximum number of available units of production can be called the *physical productive capacity*. In general this capacity will not be saturated, the exception being a strong unanticipated increase in demand at time t_0.

2. Closure and equilibrium of the model

In this section we close the model in the simplest possible way. Then, we investigate how the current equilibrium of the economy is determined when its past and its expected future are known.

a) Aggregate demand and labor supply

The model is completed by adding three equations: the equilibrium of the goods market, the intertemporal arbitrage equation of a representative consumer and a wage curve.

Aggregate supply must be equal to total demand, which consists of real consumption and investment[10].

$$(11)\, Y_{t_0} = C_{t_0} + I_{t_0} c_{i_{t_0}}$$

Consumption verifies the following Euler's equation which assumes constant relative risk aversion (ρ) and time preference (β) of households. Current consumption depends on its expected level and on the real interest rate for the following period.

$$(12)\left(C_{t_0+1} / C_{t_0}\right)^{\rho} = \left(\frac{1+r_{t_0}}{1+\beta}\right)\frac{p_{t_0}}{p_{t_0+1}}$$

Finally, labor supply is defined as a wage curve linking the nominal wage in efficiency units to the price level, the wedge (reflecting the spread between the purchasing power of wages for the worker and the effective labor cost for firms) and the employment rate (where $\overline{N}_{t_0}$ is the available labor force), which accounts for the effect of labor market conditions on wage bargaining.

$$(13)\ln(w_{t_0}) = \varphi_0 + \ln(p_{t_0}) + \varphi_1 \ln(wedge_{t_0}) + \varphi_2 \ln(N_{t_0} / \overline{N}_{t_0})$$

The general equilibrium of the model at date t_0 is determined by equations (1) to (13). Monetary policy pegs the nominal interest rate (r_{t_0})[11] and fiscal policy fixes the wedge

[10] We could consider aggregate supply as the *economic productive capacity*, and introduce some stickiness in production price. To do that we can assume that demand is never rationed, so that effective production is equal to total demand but may differ from the *economic productive capacity*. The specification of the model would become somewhat more complicated under this assumption and should borrow heavily to the theory of disequilibrium economics. First, we should compute the equilibrium of the model as it is presented in this paper. We get the equilibrium prices and the *notional* demands and supplies. Then, we should compute the effective price as a combination of the price of the previous period and the current equilibrium price. Finally, we should compute the *effective* equilibrium of time t_0 consistent with this effective price (which includes effective number of working units, employment, nominal wage, etc.).

[11] In section 4 we will introduce a Taylor monetary rule.

($wedge_{t_0}$). Firing cost ($x^f_{t_0}$), the labor force ($\overline{N}_{t_0}$), and technical progress (A_{t_0}), are exogenous.

b) Neutrality and inflation in the model

Two remarks must be made at this point. First, if we multiply the wage rate and the price for all lags and leads by any positive number, the equations of the model are still verified. Thus, the model presents the property of neutrality. It can be rewritten in real terms.

Second, as new units start to produce at the following period, the real interest rate that firms take into account in their decision-making is defined by: $(1+r_{t_0})\dfrac{p_{t_0}}{p_{t_0+1}}$. Let us define the inflation rate expected next period by: $\pi a_{t_0} = \dfrac{p_{t_0+1}}{p_{t_0}}$. As nominal interest rate is exogenous, then the real interest rate is endogenous through πa_{t_0}.

Actually, our model has a structure which is very similar to the structure of the neoclassical growth model. Thus, we would expect our model to include the real interest rate as an endogenous variable. If monetary policy pegs the nominal interest rate, the expected inflation rate should be an endogenous variable. If the economy starts at time t_0, $\pi a_{t_0} = \dfrac{p_{t_0+1}}{p_{t_0}}$ should be determined by the model. However, the initial price level p_{t_0} (and so all the following prices) should be undetermined, which is unsurprising in a model without nominal anchor.

However, in our model, current and past inflation rates appear in the equations that determine the age of the oldest units in working order at the beginning of period t_0, and the fraction of time during which it will remain productive. If the current inflation rate π_{t_0} was determined by the model, the initial price level would be determined by $p_{t_0} = \pi_{t_0} p_{t_0-1}$ the nominal anchor would be given by the past price level.

At this stage, at the price of some inconsistency, we will consider π_{t_0} as an exogenous variable, and we will check that the local conditions of existence and uniqueness, given by Blanchard and Khan (1980), are satisfied by the model. They would not be satisfied if π_{t_0} was made endogenous (we would get an infinity of solutions parameterized by the initial value of the inflation rate $\pi_{t_{0-1}}$). This result would not be changed if the current inflation rate were introduced as an argument in the wage equation (13), as a result of staggered wage contracts. However, we will see that a Taylor monetary rule can solve these problems.

The complete model is given and commented in Appendix 3. The equations, given in this appendix, are those implemented in Troll programs.

c) Equilibrium of the model

With a putty clay specification, aggregate supply adjusts to aggregate demand in two ways. First, investment flows provide new additional production capacities. Second, the profitability of available vintages commands the amount of production units that have to be scrapped.
However, in the model presented here, the production units created at date t_0 starts being productive at the beginning of period $t_0 + 1$. So here investment flows accrue to today's demand but to tomorrow's supply. Thus, at each period, supply can only adjust through the scrapping of old production units. As the technology of these vintages is set, their profitability only depends on the current real wages (in efficiency units). Hence, the rate of scrapping of old production units is an increasing function of the current real wages.

Aggregate employment is defined by the amount of labor attached to each vintage of production still in working order. It decreases with the age of the oldest profitable production units. Thus aggregate employment decreases with real wages. As labor supply is represented by a wage curve according to which an increase in employment triggers an increase in wage claims, the equilibrium on the labor market determines real wages, aggregate employment and the age of the oldest units in working order.

Aggregate supply is obtained by summing all the vintages of production, profitable at current real wages. Current consumption is defined by the expected levels of next period consumption and the real interest rate. Then aggregate demand meets aggregate supply through the investment flows. For any given capital intensity of the new production units, investment flows adjust through the number of units created.

The technological choices for the new units depend on the current and expected values for the real interest rate and real wages.

3. Simulation of the model

a) The steady state and the calibration of the model

Appendix 4 gives the equations of the balanced growth path of the model. The basic time unit is one year. The calibration is such that the steady state equilibrium replicates OECD National Accounts average data (over the period 1991-1996) for aggregate production, employment and investment (Y, N, I), as well as for the wage rate, the price level and the nominal interest rate (w, p, r).

The parameters to be evaluated are the characteristics of the production function (z and α), the scale factor of the wage equation φ_0, and the risk premium μ, which is introduced in an *ad hoc* way as the difference between the return on investment and the interest rate. Values of the unobserved endogenous variables must also be computed: the expected lifetime of the new units of production, their capital intensity and their number: T, κ and n[12]. Actually, the parameters and the unobserved variables only depend on the wage share in value added and the investment rate (see Appendix 5). The calculations presented in Appendix 5 rest upon approximations that are necessary to go deeper in the analytical analysis. This enables us to discuss the impact of macroeconomic data on the value of parameters and unobserved variables, without much loss of accuracy. However, the values presented in Table 1 are properly computed, using the steady-state model presented in Appendix 4.

Table 1 shows the results for France, the United States and Japan. The wage shares and the investment rates correspond to their average values over the period 1991-1996. Note that in National Accounts investment flows include dwellings, infrastructure, and industrial buildings. Value-added also includes housing services.

It is important to note that the expected scrapping date of the new units of production corresponds to the date of scrapping of the production units that have not gone bankrupt before. This date corresponds to a decision related to macroeconomic conditions (technical progress and wages). Nevertheless, the probability to go bankrupt for any reasons but economic conditions is equal to δ at each period. Hence, each new production unit depreciates at rate δ until date T, when it is scrapped. The effective lifetime of a new unit can then be defined as:

$$D = \sum_{i=1}^{T}(1-\delta)^i = \left(\frac{1-\delta}{\delta}\right)\left(1-(1-\delta)^T\right)$$

[12] A the steady-state $a = T + 1$.

Table 1: Calibration of the model

1991-1996 average	France	United States	Japan
Share of wages in value added ($\frac{wN}{pY}$)	0.59	0.63	0.64
Investment rate ($\frac{I}{Y}$)	0.21	0.17	0.27
Lifetime of the new production units T (years)	36.91	32.19	34.60
Effective lifetime of a new production unit D (years):	25.75	23.43	24.64
Capital intensity (κ)	7.96	5.45	9.37
Production function parameters (for σ=0.99) α z	 0.42 0.89	 0.37 1.02	 0.35 0.92
Return on investment (%)	8.8	11.0	3.9

Firing costs are close to zero (they amount to 0.1% of real wage) to concentrate on the impact of the other variables; population and technical progress increase respectively at the rates 0.5% and 2% per annum. The probability to go bankrupt at the end of each period (δ) is equal to 2% and the installation costs are set to 1.
Source: OECD National Accounts.

Consumers are assumed to have a relative risk aversion and a time preference rate of respectively 0.5 and 5%. The wage curve is characterized by an elasticity of real wages to employment of 0.3, and to the fiscal wedge of 0.5. Finally, we assume that firms do not save on firing costs with the creation of new units ($\eta = 0$).

b) Blanchard and Kahn's conditions and simulation method[13]

The model calibrated on French data was rewritten in reduced variables, and its linear approximation was computed around the reference steady state. Transforming the original variables to reduced variables is equivalent to give a common trend of rate 0 to all variables. We can also define expanded variables such that their common trend is equal to the highest balanced growth rate present in the model. The eigenvalues of the linear approximation of the model written in expanded variables are equal to the eigenvalues of the linear approximation of the model written in reduced variables, multiplied by the highest balanced growth rate plus one. The requirement for the model to converge in the long run to its balanced growth path is severer when we consider expanded variables (stability in the expanded difference) than when we consider reduced variables (stability in the relative difference). The severity is intermediate

[13] The methodology used in this paragraph is developed in Laffargue (2000). All the computations were made under Troll, with the commands Lkroots and Stack.

when we consider the original variables of the model (which have different long run growth rates), but in this case, the coefficients of the linear approximation of the model depend on time, and the local conditions for the existence and the uniqueness of a solution, developed by Blanchard and Kahn (1980), do not apply.

The model has 73 non-redundant lead variables. When we consider the linear approximation with reduced variables, we have 73 eigenvalues with modulus larger than 1. So the Blanchard and Kahn's conditions are verified, and there is a unique solution path of the reduced form model. Moreover, the largest eigenvalue less than 1 is equal to 0.951. Since the highest growth rate in the model is the real GDP growth rate, which is assumed to be equal to 2.5%, the Blanchard and Kahn's are also verified for the expanded form model (0.951*1.025<1). Hence, stability in expanded difference and stability in relative difference are verified. These two conditions are sufficient to ensure the existence and the uniqueness of a solution path for the model in original variables.

We simulate the model with a relaxation second order method implemented in Troll (Stack algorithm). We assume that its original variables, which follow different trends, must converge in the long run to their balanced growth path. The simulation of the model over 150 periods takes about 10 minutes.

The model can be used to illustrate the usefulness of the concept of stability highlighted in Laffargue (2000). In the reduced form model, the largest eigenvalue with modulus less than 1 is related to the dynamics of consumption. For high relative risk aversion (ρ), this eigenvalue is closer to 1, but still stays below unity. Hence stability in relative difference is still satisfied. However, the Blanchard and Khan's conditions are not verified anymore for the expanded form model, because the previous eigenvalue time the long run real growth rate of the economy is larger than 1. For example, when $\rho = 5$, the highest eigenvalue with modulus less than 1 in the reduced form model is equal to 0.992. Thus, in the expanded form model, this eigenvalue becomes 0.992*1.025>1. This situation, when the Blanchard and Khan's conditions are verified for the linear approximation of the model written in reduced variables but not in expanded variables, is called pseudo-hysteresis by Laffargue. In this case we do not know if the model written in its original variables has a unique solution. In our case, the simulation algorithm diverges, which suggests, according to Boucekkine (1995), that there is no solution to the model.

c) Shock on the wage-setting relation

To illustrate the properties of the model, we consider a permanent change in the wage-setting relation in France. This shock may stem from institutional or political events that permanently shift the wage curve upward, such as an increase in the unions' power or more generous unemployment benefits which improve the outside opportunity of incumbent workers.

In the steady-sate, the real interest rate is determined by consumers' preferences. Hence, the rate of return and the capital intensity of the production units are unchanged. Considering the factor cost frontier, the wage rate that firms are able to pay is also unchanged. Thus, the upward shift in the wage curve leads to a decrease in total employment and an increase in unemployment. Since labor is the bounding production factor in the long run, output decreases in the same proportion as employment. There is no effect on the wage share in value-added in the long run.

However, the path to the new steady-state is characterized by changes in the distribution of income between production factors. Appendix 6 presents the results of this simulation (Charts A1 to A5). At the first period, real wages adjust upward (Chart A4). The profitability of existing production units deteriorates. The age of the oldest units in working order decreases and employment and aggregate supply fall (Chart A2). Then, aggregate demand (Chart A1) adjusts downward with investment flows (more precisely with the number of new units created). The characteristics of the new production units have changed since investors expect a transitory increase in real wages: the new units are more capital intensive and have a higher expected lifetime (Chart A3).

The following period, as fewer units were created, aggregate employment is lower. This smoothes the impact of the upward shift in the wage curve on current real wages. Then, the age of the oldest units in working order decreases less than in the previous period. Besides, the decrease in the creation of new units in the previous period negatively affects current supply, although these units are more productive. *Ex post*, the fall in aggregate supply is stronger than in the previous period, since the lower scrapping rate of old units does not compensate for the lower number of recently created units.

The model exhibits medium term dynamics. Real wages shift upward at the beginning and eventually returns slowly to their initial level, over 3 decades. Since the fall in GDP and employment is also very gradual, the increase in the wage share in value-added is long lasting (Chart A5). Hence, contrary to models assuming a putty putty technology, the putty clay framework provides medium term changes in the income distribution between labor and capital. Similarly, the fall in the investment rate, reflecting that the downward adjustment of investment flows precedes the fall in output, lasts also around 3 decades.

The echo effect[14], a familiar feature of the putty clay specification, is present in the model. The change in the production capacities of new units, following a shock, has an impact on aggregate supply at the time these vintages are scrapped. Here, the upward shift in the wage curve leads to a decrease in the productive capacity of all the units created after the shock. The production units created before the shock (i.e. more productive) are gradually scrapped and replaced by the new ones. This explains the even fall in aggregate supply. Yet after the scrapping of the last "pre-shock" vintage aggregate supply is affected by an upward shock, since the vintage scrapped at that period is the first (less productive) units created after the shock.

[14] Among the largest eigenvalues less than one, several are complex (see above). The largest one has a periodicity of 35 years which corresponds to the echo effect described here. As our model includes rational expectations, this echo effect is expected before the scrapping of the units created after the shock and creates sharp movements on variables before the 35 years. We can see such movements on the charts in Appendix 6. The two complex eigenvalues with a periodicity of 17 and 11 years can explain these observed changes.

	Real	Imaginary	Magnitude	Period
74:	0.93563499	0.16994744	0.95094425	34.96881647
76:	0.94334400	0.00000000	0.94334400	
77:	0.88160129	0.33087799	0.94164806	17.49963710
79:	0.80466035	0.47414700	0.93396662	11.79988748

4. Monetary policy, price determinacy and *hysteresis*

The assumption of a monetary rule that maintains constant the nominal interest rate is not very realistic. The concern of the central bank for the level of the inflation rate in the economy can conveniently be rendered by a Taylor-type monetary rule, according to which the nominal interest rate increases with the inflation rate. Moreover, we have assumed until now that the expected inflation rate was endogenous, but that the effective inflation rate was exogenous. This is logically inconsistent and we will now introduce a relation between both rates, which will be endogenous. In this case, if the central bank pegs the nominal interest rate, the Blanchard and Khan's conditions are not verified (one eigenvalue larger than one is missing). However, with a Taylor monetary rule this problem will disappear.

a) Monetary policy rule and inflation determinacy

With an exogenous nominal interest rate (r_{t_0}), the model determines the inflation rate expected next period ($\pi a_{t_0} = \frac{p_{t_0+1}}{p_{t_0}}$) and the current inflation rate ($\pi_{t_0} = \frac{p_{t_0}}{p_{t_0-1}}$) is exogenous.

Now, we will suppose that the nominal interest rate is endogenous and follows a backward Taylor rule:
$r_{t_0} = rr^* + \pi^* + \alpha(\pi_{t_0} - \pi^*)$, where rr^* and π^* are the steady-state values of the real interest rate and inflation rate.

We will introduce the consistency equation: $\pi a_{t_0} = \pi_{t_0+1}$

Under the condition $\alpha > 1$[15], the integration of the Taylor rule increases by one the number of eigenvalues with modulus larger than 1. As one more non-redundant lead variable has also be included in the model, since $\pi a_{t_0} = \pi_{t_0+1}$, the models, both in reduced and expanded form, satisfy the Blanchard and Khan's conditions[16].

If we start to simulate the model at time t_0, π_{t_0} and π_{t_0-1} appear in the equation of the model. π_{t_0-1} is assumed to be given by history and π_{t_0} is computed by the model.
Now, if p_{t_0} represents the price level, we have :

$$\pi_{t_0} = \frac{p_{t_0}}{p_{t_0-1}}.$$

So, the current price level is determined by current inflation and the price level of the previous period. The price level permanently depends on the price level observed at time $t_0 - 1$: p_{t_0-1}.

---

[15] Actually, α must slightly exceed unity since the dynamics of the inflation rate is not exclusively determined by the Taylor rule. The inflation rate dynamics also affects the choice of the firms with regard the scrapping of old units and the expected lifetime of new ones. However, here $\alpha = 1.1$ is sufficient to increase by one the number of eigenvalues with modulus larger than 1.ensure the stability of the model.

[16] Now there are 109 eigenvalues with modulus larger than 1.

This hypothesis is related to an eigenvalue equal to 1 when the model is written in reduced variables (see Laffargue, 2000).

b) Simulating a monetary policy shock in a model with hysteresis

The nominal interest rate, the current inflation rate and the price level are added to the list of endogenous variables of the model of the previous section, together with three equations: a Taylor rule (with $\alpha = 1.5$), $\pi a_{t_0} = \pi_{t_0+1}$ and $p_{t_0} = \pi_{t_0} p_{t_0-1}$.
We can take into account nominal rigidities, coming from multi-period wage contracts by adding an inflation term in equation (13), which becomes:

$$\ln(w_{t_0} / p_{t_0}) = \varphi_0 + \varphi_1 \ln(Wedge_{t_0}) + \varphi_2 \ln(N_{t_0} / \overline{N}_{t_0}) - \varphi_3 \ln(p_{t_0} / p_{t_0-1})$$

We assume $\varphi_3 = 0.5$.

This model still satisfies the Blanchard and Khan's conditions. There are 109 non-redundant lead variables and 109 eigenvalues with modulus larger than 1 in the reduced form model as well as in the expanded form model. One eigenvalue is equal to unity. The largest eigenvalue less than one is equal to 0.952. Hence, as previously, the model satisfies the two stability conditions (in reduced and expanded forms), and has a unique solution path.

To illustrate the dynamics of the model with an endogenous monetary policy, we simulate an expected transitory shock on the nominal interest rate (5 years). For instance, this transitory shift in the usual monetary rule can stems from the central bank concern for the situation of financial institutions. The results of the simulation are presented in Appendix 6 (Charts A6 to A10).

The transitory downward shift in the monetary rule triggers an increase in the inflation rate expected next period so that the real interest rate, as defined in this paper, is almost unchanged (Chart A9). The equilibrium of the first period is barely affected by the shock. At the second period, however, the increase in the rate of inflation triggers a fall in real wages since nominal wages are sticky. The lower labor cost then leads to an increase in the age of the oldest units in working order as well as in aggregate supply and aggregate employment (Chart A7). Firms create more production units and investment flows increase. The positive impact on GDP is stronger during the 5 years of the monetary policy shift. However, GDP keeps track of the increase in the size of the vintages for around 15 more years.

Conclusion

This paper presents a macro-economic model assuming putty clay investment and perfect foresight, in the line of the works by Caballero and Hammour. When calibrated on French data, the model is shown to have a unique solution path for reasonable values of the parameters. More precisely, the methodology presented in Laffargue (2000) sates that the stability of a model with variables with different trends is ensured when the linear approximations of the model in both reduced and expanded form satisfy the Blanchard and Khan's conditions. These two conditions are verified for the parameters retained here. We show however that for high risk-aversion of households, the stability of the model in the original variables is no more satisfied, although Blanchard and Khan's conditions are still verified for the reduced form model. This illustrates the relevance of this methodology.

The model has variables with long leads and long lags, but it can be simulated easily using the Stack algorithm implemented in Troll. We first simulate a permanent shift in the wage curve. The putty clay framework provides medium term dynamics in the distribution of income between production factors that putty putty models lack. As highlighted by Caballero and Hammour (1998) in the case of France, putty clay investment may thus be the appropriate way to model factor demands in European countries.

Appendix 1: The uniqueness of solution for the system of inequations (3)

The multiplicity of solutions for the system of inequations (3) depends on the monotony of the expression $U_{t_0}(\bar{T}(t_o)) = \Psi_{t_0}\left(\bar{T}(t_0), \Delta T(t_0), \kappa_{t_0}\right) - \Psi_{t_0}\left(\bar{T}(t_0)-1, \Delta T(t_0), \kappa_{t_0}\right)$ relative to $\bar{T}(t_o)$. More precisely, the change in the investor's wealth resulting from the lengthening of the unit lifetime from $t_0 + \bar{T}(t_o) + \Delta T(t_o) - 1$ to $t_0 + \bar{T}(t_o) + \Delta T(t_o)$ has a unique solution if the sign of $U_{t_0}(\bar{T}(t_o))$ changes only once. We have:

$$
\begin{aligned}
U_{t_0}(\bar{T}(t_o)) = {} & \frac{A_{t_0}(1+\gamma)^{\bar{T}(t_0)+\Delta T(t_0)}(1-\delta)^{\bar{T}(t_0)-1}}{\prod_{\tau=t_0}^{t_0+\bar{T}(t_0)-1}(1+r_\tau)} p_{t_0+\bar{T}(t_0)} \\
& \Bigg\{ (1-tcr_{t_0+\bar{T}(t_0)}) \left((1+\gamma)^{-\bar{T}(t_0)-\Delta T(t_0)} F(\kappa_{t_0},1) - \frac{w_{t_0+\bar{T}(t_0)}}{p_{t_0+\bar{T}(t_0)}} (1+\gamma)^{-\Delta T(t_0)} \right) \\
& + \left(\frac{p_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)}} \right)^{\Delta T(t_0)} \left(\frac{1-\delta}{1+r_{t_0+\bar{T}(t_0)}} \right) \\
& \left[\Delta T(t_0)(1-tcr_{t_0+\bar{T}(t_0)+1}) \left((1+\gamma)^{-\bar{T}(t_0)-\Delta T(t_0)} F(\kappa_{t_0},1) - \left(\frac{w_{t_0+\bar{T}(t_0)}}{p_{t_0+\bar{T}(t_0)}} \right)^{1-\Delta T(t_0)} \left(\frac{w_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)+1}} \right)^{\Delta T(t_0)} \right) - x^{f\;\;1-\Delta T(t_0)}_{t_0+\bar{T}(t_0)} x^{f\;\;\Delta T(t_0)}_{t_0+\bar{T}(t_0)+1} \right] \\
& - \left(\frac{p_{t_0+\bar{T}(t_0)-1}}{p_{t_0+\bar{T}(t_0)}} \right)^{1-\Delta T(t_0)} \\
& \left[\Delta T(t_0)(1-tcr_{t_0+\bar{T}(t_0)}) \left((1+\gamma)^{-\bar{T}(t_0)-\Delta T(t_0)} F(\kappa_{t_0},1) - \left(\frac{w_{t_0+\bar{T}(t_0)-1}}{p_{t_0+\bar{T}(t_0)-1}} \right)^{1-\Delta T(t_0)} \left(\frac{w_{t_0+\bar{T}(t_0)}}{p_{t_0+\bar{T}(t_0)}} \right)^{\Delta T(t_0)} \left(\frac{1}{1+\gamma} \right) \right) \right. \\
& \left. - \frac{x^{f\;\;1-\Delta T(t_0)}_{t_0+\bar{T}(t_0)-1} x^{f\;\;\Delta T(t_0)}_{t_0+\bar{T}(t_0)}}{1+\gamma} \right] \Bigg\}
\end{aligned}
$$

Consider a date far enough in the future so that firms' expectations on the tax rate tcr, the firing cost (in efficiency unit) x^f, and the interest rate r are constant, as well as the inflation rate $\frac{p}{p_{-1}} = 1+\pi$. At this horizon the real wage rate in efficiency unit $\frac{w}{p}$ is also considered to be constant. Then, the change in wealth coming from the increase of the unit's lifetime by one period becomes:

$$U_{t_0}(\overline{T}(t_o)) = \frac{A_{t_0} p_{t_0} (1+\gamma)^{\overline{T}(t_0)+\Delta T(t_0)}}{(1-\delta)} \left(\frac{(1+\pi)(1-\delta)}{1+r} \right)^{\overline{T}(t_0)} (1-tcr)$$

$$\left\{ (1+\gamma)^{-\overline{T}(t_0)-\Delta T(t_o)} F(\kappa_{t_0},1) \left(1 + \Delta T(t_o) \left(\frac{1-\delta}{1+r} \right) (1+\pi)^{\Delta T(t_o)} - \frac{\Delta T(t_o)}{(1+\pi)^{1-\Delta T(t_o)}} \right) \right.$$

$$- \frac{w}{p} \left((1+\gamma)^{-\Delta T(t_0)} + \Delta T(t_o) \left(\frac{1-\delta}{1+r} \right) (1+\pi)^{\Delta T(t_o)} - \frac{\Delta T(t_o)}{(1+\gamma)(1+\pi)^{1-\Delta T(t_o)}} \right)$$

$$\left. - \frac{x^f}{1-tcr} \left(\left(\frac{1-\delta}{1+r} \right) (1+\pi)^{\Delta T(t_o)} - \frac{1}{(1+\gamma)(1+\pi)^{1-\Delta T(t_o)}} \right) \right\}$$

This expression is decreasing in $\overline{T}(t_o)$. For high values of $\overline{T}(t_o)$, it has the same sign as:

$$-(1-tcr)\frac{w}{p} \left(1 + \Delta\overline{T}(t_o) \left(\frac{1-\delta}{1+r} \right) (1+\pi)^{\Delta T(t_o)} - \frac{\Delta T(t_o)}{(1+\gamma)(1+\pi)^{1-\Delta\overline{T}(t_o)}} \right)$$

$$- \frac{x^f}{1-tcr} \left(\left(\frac{1-\delta}{1+r} \right) (1+\pi)^{\Delta T(t_o)} - \frac{1}{(1+\gamma)(1+\pi)^{1-\Delta T(t_o)}} \right)$$

which is negative for economically plausible values of the variables. If for $\overline{T}(t_o) = 1$, this expression takes a positive value, then its sign changes only once and there is a unique solution for system of inequations. Note that if for $\overline{T}(t_o) = 1$, this expression is negative, we get the solution $\overline{T}(t_o) = 0$. Since the previous reasoning assumes values of $\overline{T}(t_o)$ far enough in the future, the existence of another solution, less far away, cannot be ruled out. It could be the case if the investor expects very important changes in the firing costs, the tax rate and the real interest rate. We will neglect this possibility in what follows.

Appendix 2: The computation of the factor costs frontier (equation (6))

As F is homogenous of order 1, we have:
$F(\kappa,1)=\kappa F_1^{'}(\kappa,1)+F_2^{'}(\kappa,1)$,
where $F_2^{'}(\kappa,1)$ is the marginal productivity of labor, measured in efficiency unity[17].

Hence, the optimization condition (5) is equivalent to :

$$
\begin{aligned}
&[F_2^{'}(\kappa_{t_0},1)/\kappa_{t_0}]\left\{\sum_{s=t_0+1}^{t_0+\bar{T}(t_0)} p_s(1-tcr_s)(1-\delta)^{s-t_0-1}/\prod_{\tau=t_0}^{s-1}(1+r_\tau)\right.\\
&\left.+\Delta T(t_0)\,p_{t_0+\bar{T}(t_0)}(1-\delta)^{\bar{T}(t_0)}\left(\frac{p_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)}}\right)^{\Delta T(t_0)}(1-tcr_{t_0+\bar{T}(t_0)+1})/\prod_{\tau=t_0}^{t_0+\bar{T}(t_0)}(1+r_\tau)\right\}=\\
&[F(\kappa_{t_0},1)/\kappa_{t_0}]\left\{\sum_{s=t_0+1}^{t_0+\bar{T}(t_0)} p_s(1-tcr_s)(1-\delta)^{s-t_0-1}/\prod_{\tau=t_0}^{s-1}(1+r_\tau)\right.\\
&\left.+\Delta T(t_0)\,p_{t_0+\bar{T}(t_0)}(1-\delta)^{\bar{T}(t_0)}\left(\frac{p_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)}}\right)^{\Delta T(t_0)}(1-tcr_{t_0+\bar{T}(t_0)+1})/\prod_{\tau=t_0}^{t_0+\bar{T}(t_0)}(1+r_\tau)\right\}\\
&-p_{t_0}c_{i_{t_0}}/(1-\delta)
\end{aligned}
$$

The expression of the production unit value is used to eliminate F in this equation:

$$
\begin{aligned}
&[F_2^{'}(\kappa_{t_0},1)/\kappa_{t_0}]\left\{\sum_{s=t_0+1}^{t_0+\bar{T}(t_0)} p_s(1-tcr_s)(1-\delta)^{s-t_0-1}/\prod_{\tau=t_0}^{s-1}(1+r_\tau)\right.\\
&\left.+\Delta T(t_0)\,p_{t_0+\bar{T}(t_0)}\left(\frac{p_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)}}\right)^{\Delta T(t_0)}(1-tcr_{t_0+\bar{T}(t_0)+1})(1-\delta)^{\bar{T}(t_0)}/\prod_{\tau=t_0}^{t_0+\bar{T}(t_0)}(1+r_\tau)\right\}=\\
&V_{t_0,t_0}/(A_{t_0}\kappa_{t_0})+\sum_{s=t_0+1}^{t_0+\bar{T}(t_0)}(1-tcr_s)[w_s(1+\gamma)^{s-t_0}/\kappa_{t_0}](1-\delta)^{s-t_0-1}/\prod_{\tau=t_0}^{s-1}(1+r_\tau)\\
&+p_{t_0+\bar{T}(t_0)}\left(\frac{p_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)}}\right)^{\Delta T(t_0)}\left\{\Delta T(t_0)(1-tcr_{t_0+\bar{T}(t_0)+1})\left(\frac{w_{t_0+\bar{T}(t_0)}}{p_{t_0+\bar{T}(t_0)}}\right)^{1-\Delta T(t_0)}\left(\frac{w_{t_0+\bar{T}(t_0)+1}}{p_{t_0+\bar{T}(t_0)+1}}\right)^{\Delta T(t_0)}(1+\gamma)^{T(t_0)}\right.\\
&\left.+x^f_{t_0+\bar{T}(t_0)}{}^{1-\Delta T(t_0)}x^f_{t_0+\bar{T}(t_0)+1}{}^{-\Delta T(t_0)}(1+\gamma)^{T(t_0)}\right\}(1-\delta)^{\bar{T}(t_0)}/\prod_{\tau=t_0}^{t_0+\bar{T}(t_0)}(1+r_\tau)/\kappa_{t_0}\\
&-\eta p_{t_0+1}x^f_{t_0+1}(1+\gamma)/\kappa_{t_0}-p_{t_0}c_{i_{t_0}}
\end{aligned}
$$

If we combine this equation with the free-entry condition, we get equation (6).

---

17 $F_2^{'}(\kappa_{t_0},1)=\dfrac{\partial F(k_{t_0},A_{t_0}n_{t_0})}{\partial(A_{t_0}n_{t_0})}$.

Appendix 3: The computation of the equilibrium of the model at period t_0 (13 equations)

The following equations determine the equilibrium of the economy at time t_0, when the past and the expected future are given. The terminal conditions (for instance at time $t_0 + 200$) are computed by the steady state model of Appendix 3.

We define $\omega_{t_0} = \dfrac{w_{t_0}}{p_{t_0}}$ as the real wage in efficiency unit, $\pi a_{t_0} = \dfrac{p_{t_0+1}}{p_{t_0}}$ as the inflation rate expected next period, and $\pi_{t_0} = \dfrac{p_{t_0}}{p_{t_0-1}}$ as the inflation rate of the current period.

Firms' choices for new units, as well as aggregate supply and aggregate employment, refer to variables over a period of time or at a date that is endogenous. In Troll, variables are real numbers. So it is not possible to index a lag or a lead by a variable of the model (even though this variable happens to take only integer values). It is also impossible to sum over a period of time that is endogenous. Lastly, the command that yields the integer part of a variable is not recognized by the modeling procedures of Troll (actually it cannot be derived). The solution we have implemented rests upon the operator *sign(x)*, which is equal to 1 if *x* is positive, -1 if *x* is negative and 0 if it is equal to zero. The trick consists in summing, over a period that covers the date or the time period we are interested in, the expressions of the equations times a combination of *sign* operators. This combination is equal to 0 for the dates we want to avoid and to 1 for the dates we want to retain. Note that we can explicitly refer to the variables $\overline{T}(t_0)$ and $\overline{a}(t_0)$ in the algebraic part of the formula.

1st block

(1) $F(\kappa_{t_0},1) = z[\alpha\kappa_{t_0}^{1-1/\sigma} + (1-\alpha)]^{\sigma/(\sigma-1)}$

(3)

$$\overline{T}(t_0) = \frac{1}{2}\sum_{i=1}^{N}\Big\{1 + sign\Big[(1-tcr_{t_0+i})\Big((1+\gamma)^{-i-\Delta T(t_0)}F(\kappa_{t_0},1) - \omega_{t_0+i}(1+\gamma)^{-\Delta T(t_0)}\Big)$$
$$+ (1+\pi a_{t_0+i})^{\Delta T(t_0)}\left(\frac{1-\delta}{1+r_{t_0+i}}\right)$$
$$\Big[\Delta T(t_0)(1-tcr_{t_0+i+1})\Big((1+\gamma)^{-i-\Delta T(t_0)}F(\kappa_{t_0},1) - {\omega_{t_0+i}}^{1-\Delta T(t_0)}{\omega_{t_0+i+1}}^{\Delta T(t_0)}\Big) - {x^f_{t_0+i}}^{1-\Delta T(t_0)}{x^f_{t_0+i+1}}^{\Delta T(t_0)}\Big]$$
$$- (1+\pi a_{t_0+i-1})^{\Delta T(t_0)-1}$$
$$\Big[\Delta T(t_0)(1-tcr_{t_0+i})\Big((1+\gamma)^{-i-\Delta T(t_0)}F(\kappa_{t_0},1) - {\omega_{t_0+i-1}}^{1-\Delta T(t_0)}{\omega_{t_0+i}}^{\Delta T(t_0)}/(1+\gamma)\Big) - {x^f_{t_0+i-1}}^{1-\Delta T(t_0)}{x^f_{t_0+i}}^{\Delta T(t_0)}/(1+\gamma)\Big]\Big]\Big\}$$

(4)

$$\frac{1}{2}\sum_{i=1}^{N}\left(1-sign(\overline{T}(t_0)-i-0{,}5)sign(\overline{T}(t_0)-i+0{,}5)\right)$$

$$\Big\{\ln\left(1+\pi a_{t_0+i}\right)\left[\Delta T(t_0)(1-tcr_{t_0+i+1})\left((1+\gamma)^{-\overline{T}(t_0)-\Delta T(t_0)}F(\kappa_{t_0},1)-\omega_{t_0+i}{}^{1-\Delta T(t_0)}\omega_{t_0+i+1}{}^{\Delta T(t_0)}\right)-x^f_{t_0+i}{}^{1-\Delta T(t_0)}x^f_{t_0+i+1}{}^{\Delta T(t_0)}\right]$$

$$+(1-tcr_{t_0+i+1})\left((1+\gamma)^{-\overline{T}(t_0)-\Delta T(t_0)}F(\kappa_{t_0},1)-\omega_{t_0+i}{}^{1-\Delta T(t_0)}\omega_{t_0+i+1}{}^{\Delta T(t_0)}\right)$$

$$-\Delta T(t_0)(1-tcr_{t_0+i+1})\omega_{t_0+i}{}^{1-\Delta T(t_0)}\omega_{t_0+i+1}{}^{\Delta T(t_0)}\ln\left(\frac{\omega_{t_0+i+1}}{\omega_{t_0+i}}\right)$$

$$-x^f_{t_0+i}{}^{1-\Delta T(t_0)}x^f_{t_0+i+1}{}^{\Delta T(t_0)}\ln\left(\frac{x^f_{t_0+i+1}}{x^f_{t_0+i}}\right)$$

$$-\left(\Delta T(t_0)(1-tcr_{t_0+i+1})\omega_{t_0+i}{}^{1-\Delta T(t_0)}\omega_{t_0+i+1}{}^{\Delta T(t_0)}+x^f_{t_0+i}{}^{1-\Delta T(t_0)}x^f_{t_0+i+1}{}^{\Delta T(t_0)}\right)\ln(1+\gamma)\Big\}=0$$

(5)

$$F_1^{'}(\kappa_{t_0},1)\Big\{\frac{1}{2}\sum_{i=1}^{N}\left(1+sign(\overline{T}(t_0)-i+0{,}5)\right)\left((1-tcr_{t_0+i})(1-\delta)^{i-1}\prod_{\tau=1}^{i}(1+\pi a_{t_0+\tau-1})/(1+r_{t_0+\tau-1})\right)$$

$$+\frac{1}{2}\sum_{i=0}^{N}\Big[\left(1-sign(\overline{T}(t_0)-i-0{,}5)sign(\overline{T}(t_0)-i+0{,}5)\right)$$

$$\Delta\overline{T}(t_0)(1-tcr_{t_0+i+1})\left(1+\pi a_{t_0+i}\right)^{\Delta\overline{T}(t_0)-1}(1-\delta)^{i}\prod_{\tau=1}^{i+1}(1+\pi a_{t_0+\tau-1})/(1+r_{t_0+\tau-1})\Big]\Big\}=c_{i_{t_0}}\;/(1-\delta)$$

(6)

$$\frac{1}{2}\sum_{i=1}^{N}\left(1+sign(\overline{T}(t_0)-i+0{,}5)\right)(1-tcr_{t_0+i})(1-\delta)^{i-1}\left(F_2^{'}(\kappa_{t_0},1)-(1+\gamma)^{1}\omega_{t_0+i}\right)\prod_{\tau=1}^{i}(1+\pi a_{t_0+\tau-1})/(1+r_{t_0+\tau-1})$$

$$+\frac{1}{2}\sum_{i=0}^{N}\Big\{\left(1-sign(\overline{T}(t_0)-i-0{,}5)sign(\overline{T}(t_0)-i+0{,}5)\right)(1+\pi a_{t_0+i})^{\Delta\overline{T}(t_0)-1}(1-\delta)^{\overline{T}(t_0)}\prod_{\tau=1}^{i+1}(1+\pi a_{t_0+\tau-1})/(1+r_{t_0+\tau-1})$$

$$\left(\Delta\overline{T}(t_0)(1-tcr_{t_0+i+1})\left(F_2^{'}(\kappa_{t_0},1)-\omega_{t_0+i}{}^{1-\Delta\overline{T}(t_0)}\omega_{t_0+i+1}{}^{\Delta\overline{T}(t_0)}(1+\gamma)^{\overline{T}(t_0)+\Delta T(t_0)}\right)-x^f_{t_0+i}{}^{1-\Delta\overline{T}(t_0)}x^f_{t_0+i+1}{}^{\Delta\overline{T}(t_0)}\right)\Big\}$$

$$=-\eta(1+\pi a_{t_0})(1+\gamma)x^f_{t_0+1}$$

This system of 5 equations determines $\overline{T}(t_0)$, $\Delta T(t_0)$, κ_{t_0}, $F(\kappa_{t_0},1)$ and πa_{t_0}.

2nd block

(7)

$$\bar{a}(t_o)=\frac{1}{2}\sum_{i=1}^{N}\Big\{1+sign\Big[(1-tcr_{t_0-1})\Big((1+\gamma)^{-i-\Delta a(t_0)+1}F(\kappa_{t_0-i},1)-\omega_{t_0-1}(1+\gamma)^{-\Delta a(t_0)}\Big)$$

$$+\left(1+\pi_{t_0}\right)^{\Delta a(t_0)}\left(\frac{1-\delta}{1+r_{t_0-1}}\right)$$

$$\Big[\Delta a(t_0)(1-tcr_{t_0})\Big((1+\gamma)^{-i-\Delta a(t_0)+1}F(\kappa_{t_0-i},1)-\omega_{t_0-1}{}^{1-\Delta a(t_0)}\omega_{t_0}{}^{\Delta a(t_0)}\Big)-x^f_{t_0-1}{}^{1-\Delta a(t_0)}x^f_{t_0}{}^{\Delta a(t_0)}\Big]$$

$$-\left(\frac{1}{1+\pi_{t_0-1}}\right)^{1-\Delta a(t_0)}$$

$$\left[\Delta a(t_0)(1-tcr_{t_0-1})\left((1+\gamma)^{-i-\Delta a(t_0)+1}F(\kappa_{t_0-i},1)-\omega_{t_0-2}{}^{1-\Delta a(t_0)}\omega_{t_0-1}{}^{\Delta a(t_0)}\left(\frac{1}{1+\gamma}\right)\right)\right.$$

$$\left.-x^f_{t_0-2}{}^{1-\Delta a(t_0)}x^f_{t_0-1}{}^{\Delta a(t_0)}/(1+\gamma)\Big]\Big]\right\}$$

(8)

$$\frac{1}{2}\sum_{i=1}^{N}\left(1-sign(\bar{a}(t_0)-i-0.5)sign(\bar{a}(t_0)-i+0.5)\right)$$

$$\Big\{\ln(1+\pi_{t_0})$$

$$\Big[\Delta a(t_0)(1-tcr_{t_0})\Big((1+\gamma)^{-\bar{a}(t_0)-\Delta a(t_0)+1}F(\kappa_{t_0-i},1)-\omega_{t_0-1}{}^{1-\Delta a(t_0)}\omega_{t_0}{}^{\Delta a(t_0)}\Big)-x^f_{t_0-1}{}^{1-\Delta a(t_0)}x^f_{t_0}{}^{\Delta a(t_0)}\Big]$$

$$+(1-tcr_{t_0})\Big((1+\gamma)^{-\bar{a}(t_0)-\Delta a(t_0)+1}F(\kappa_{t_0-i},1)-\omega_{t_0-1}{}^{1-\Delta a(t_0)}\omega_{t_0}{}^{\Delta a(t_0)}\Big)\Big\}$$

$$-\Delta a(t_0)(1-tcr_{t_0})\omega_{t_0-1}{}^{1-\Delta a(t_0)}\omega_{t_0}{}^{\Delta a(t_0)}\ln\left(\omega_{t_0}/\omega_{t_0-1}\right)$$

$$-x^f_{t_0-1}{}^{1-\Delta a(t_0)}x^f_{t_0+1}{}^{\Delta a(t_0)}\ln\left(x^f_{t_0}/x^f_{t_0-1}\right)$$

$$-\left(\Delta a(t_0)(1-tcr_{t_0})\omega_{t_0-1}{}^{1-\Delta a(t_0)}\omega_{t_0}{}^{\Delta a(t_0)}+x^f_{t_0-1}{}^{1-\Delta a(t_0)}x^f_{t_0}{}^{\Delta a(t_0)}\right)\ln(1+\gamma)=0$$

For ω_{t_0} given (loop variable), these equations determine $\bar{a}(t_0)$ and $\Delta a(t_0)$.

(10)

$$Y_{t_0}=\frac{A_{t_0}}{2}\sum_{i=1}^{N}\Big\{\left(1+sign(\bar{a}(t_0)-i-0.5)\right)(1+\gamma)^{-i}(1-\delta)^{i}F(\kappa_{t_0-i},1)n_{t_0-i}$$

$$+\left(1-sign(\bar{a}(t_0)-i-0.5)sign(\bar{a}(t_0)-i+0.5)\right)\Delta a(t_0)(1+\gamma)^{-\bar{a}(t_0)}(1-\delta)^{\bar{a}(t_0)}F(\kappa_{t_0-i},1)n_{t_0-i}\Big\}$$

This equation determines Y_{t_0}.

(12) $\left(C_{t_0+1}/C_{t_0}\right)^{\rho} = \left(\dfrac{1+r_{t_0}}{1+\beta}\right)/(1+\pi a_{t_0})$

This equation determines C_{t_0}.

(11) $Y_{t_0} = C_{t_0} + I_{t_0} c_{i_{t_0}}$

This equation determines I_{t_0}.

(2) $I_{t_0} = A_{t_0} \kappa_{t_0} n_{t_0}$

This equation determines n_{t_0}.

(9)

$$N_{t_0} = \frac{1}{2}\sum_{i=1}^{N}\Big\{\big(1+sign(\bar{a}(t_0)-i-0.5)\big)(1-\delta)^{i}\, n_{t_0-i}$$
$$+\big(1-sign(\bar{a}(t_0)-i-0.5)sign(\bar{a}(t_0)-i+0.5)\big)\Delta a(t_0)(1-\delta)^{\bar{a}(t_0)}\, n_{t_0-i}\Big\}$$

This equation determines N_{t_0}.

(13) $\ln(\omega_{t_0}) = \varphi_0 + \varphi_1 \ln(wedge_{t_0}) + \varphi_2 \ln(N_{t_0}/\overline{N}_{t_0})$

This equation determines ω_{t_0}.

Endogenous variables (13)

$\bar{a}(t_0)$: age of the oldest production units in working order at the beginning of the current period $\Delta a(t_0)$: fraction of the current period during which the oldest production units at the beginning of the current period are effectively in working order.

C_{t_0}: consumption at the current period

$F(\kappa_{t_0},1)$: production capacity of a production unit implemented in the current period

I_{t_0}: investment of the period, productive from the next period

κ_{t_0}: reduced capital intensity of the new production unit (capital divided by efficiency and employment) implemented during the next period

n_{t_0}: number of production units built during the current period and employment required by these units at the following periods.

N_{t_0} : employment in the current period

πa_{t_0} : expected inflation rate for the next period

$\overline{T}(t_0)$: integer part of the lifetime of the unit implemented during the current period

$\Delta T(t_0)$: decimal part of the lifetime of the unit implemented during the current period

ω_{t_0} : real wage rate divided by efficiency A_{t_0}

Y_{t_0} : current real GDP

Exogenous variables

A_{t_0} : efficiency for the period; $A_{t_0} = A_{t_0 -1}(1+\gamma)$

$\overline{N}_{t_0}$: available labor force

r_{t_0} : interest rate of the period

$wedge_{t_0}$: fiscal wedge

tcr_{t_0} : tax rate on profit

$x^f_{t_0}$: reduced firing cost (i.e. deflated by $p_{t_0} A_{t_0}$)

$c_{i_{t_0}}$: real cost of the investment implemented during the current period

π_{t_0} : inflation rate for the current period

Appendix 4: The steady-state of the model

The growth rates
$\dot{n}$: population growth rate
π : inflation rate
γ : technical progress growth rate
$1+g=(1+\gamma)(1+\dot{n})$: production growth rate

Endogenous variables (13)
κ $F(\kappa,1)$ $\overline{T}$ ΔT πa $\overline{a}$ Δa and ω, growing at rate 0
C I and Y, growing at rate g
n and N, growing at rate $\dot{n}$

The expressions using the operator *sign* in Appendix 3 simplify here, since there is no more reference to lead or lag variable in the steady-state model.

The levels (13 equations)

1st block

(1) $F(\kappa,1)=z[\alpha\kappa^{1-1/\sigma}+(1-\alpha)]^{\sigma/(\sigma-1)}$

(3)
$$\overline{T}=\frac{1}{2}\sum_{i=1}^{N}\Big\{1+sign\Big[(1-tcr)\big((1+\gamma)^{-\Delta T-i}F(\kappa,1)-\omega(1+\gamma)^{-\Delta T}\big)$$
$$+\left(1+\pi a\right)^{\Delta T}\left(\frac{1-\delta}{1+r}\right)\left[\Delta T(1-tcr)\big((1+\gamma)^{-\Delta T-i}F(\kappa,1)-\omega\big)-x^{f}\right]$$
$$-\left(\frac{1}{1+\pi a}\right)^{1-\Delta T}\left[\Delta T(1-tcr)\big((1+\gamma)^{-\Delta T-i}F(\kappa,1)-\omega/(1+\gamma)\big)-x^{f}/(1+\gamma)\right]\Big]\Big\}$$

(4)
$$\ln(1+\pi a)\left[\Delta T(1-tcr)\big((1+\gamma)^{-\overline{T}-\Delta T}F(\kappa,1)-\omega\big)-x^{f}\right]$$
$$+(1-tcr)\big((1+\gamma)^{-\overline{T}-\Delta T}F(\kappa,1)-\omega\big)$$
$$-\ln(1+\gamma)\big(\Delta T(1-tcr)\omega+x^{f}\big)=0$$

(5)
$$F_{1}^{'}(\kappa_{t_0},1)\left\{\frac{1}{2}\sum_{i=1}^{N}\big(1+sign(\overline{T}-i+0{,}5)\big)\left(\frac{(1-\delta)(1+\pi a)}{1+r}\right)^{i}\right.$$
$$\left.+\frac{1}{2}\sum_{i=0}^{N}\big(1-sign(\overline{T}-i-0{,}5)sign(\overline{T}-i+0{,}5)\big)\Delta T\left(1+\pi a\right)^{i+\Delta T}\left(\frac{1-\delta}{1+r}\right)^{i+1}\right\}=c_{i}/(1-tcr)$$

(6)

$$\frac{1}{2}\sum_{i=1}^{N}\left(1+sign(\overline{T}-i+0,5)\right)\left(F_2^{'}(\kappa,1)-(1+\gamma)^{i}\,\omega\right)\left[(1-\delta)(1+\pi a)/(1+r)\right]^{i}$$
$$+\frac{1}{2}\sum_{i=0}^{N}\left\{\left(1-sign(\overline{T}-i-0,5)sign(\overline{T}-i+0,5)\right)(1+\pi a)^{i+\Delta T}\left[(1-\delta)/(1+r)\right]^{i+1}\right.$$
$$\left.\left(\Delta T(F_2^{'}(\kappa,1)-(1+\gamma)^{i+\Delta T}\,\omega)-(1+\gamma)^{i+\Delta T}\,x^{f}\;/(1-tcr)\right)\right\}=-\eta(1-\delta)(1+\pi a)(1+\gamma)x^{f}\;/(1-tcr)$$

2nd block

(2) $I = A\kappa n$

(12) $1+r=(1+\beta)(1+\pi a)(1+g)^{\rho}$

(11) $Y = C + I.c_i$

(10)

$$Y = AF(\kappa,1)n\left\{\frac{1}{2}\sum_{i=1}^{N}\left(1+sign(\overline{a}-i-0.5)\right)\left[\frac{(1-\delta)}{(1+\gamma)(1+\dot{r})}\right]^{i}+\Delta a(1+\gamma)^{-\overline{a}}\left(\frac{1-\delta}{1+\dot{r}}\right)^{\overline{a}}\right\}$$

(9)

$$N = n\left\{\frac{1}{2}\sum_{i=1}^{N}\left(1+sign(\overline{a}-i-0.5)\right)\left(\frac{1-\delta}{1+\dot{r}}\right)^{i}+\Delta a\left(\frac{1-\delta}{1+\dot{r}}\right)^{\overline{a}}\right\}$$

(7)

$$\overline{a}(t_o)=\frac{1}{2}\sum_{i=1}^{N}\left\{1+sign\left[(1-tcr)\left((1+\gamma)^{-i-\Delta a+1}F(\kappa,1)-\omega(1+\gamma)^{-\Delta a}\right)\right.\right.$$
$$+(1+\pi)^{\Delta a}\left(\frac{1-\delta}{1+r}\right)\left[\Delta a(1-tcr)\left((1+\gamma)^{-i-\Delta a+1}F(\kappa,1)-\omega\right)-x^{f}\right]$$
$$\left.\left.-\left(\frac{1}{1+\pi}\right)^{1-\Delta a}\left[\Delta a(1-tcr)\left((1+\gamma)^{-i-\Delta a+1}F(\kappa,1)-\omega/(1+\gamma)\right)-x^{f}\;/(1+\gamma)\right]\right]\right\}$$

(8)

$$\ln(1+\pi)\left[\Delta a(1-tcr)\left((1+\gamma)^{-\overline{a}-\Delta a+1}F(\kappa,1)-\omega\right)-x^{f}\right]$$
$$+(1-tcr)\left((1+\gamma)^{-\overline{a}-\Delta a+1}F(\kappa,1)-\omega\right)$$
$$-\left(\Delta a(1-tcr)\omega+x^{f}\right)\ln(1+\gamma)=0$$

(12) $\ln(\omega)=\varphi_0+\varphi_1\ln(wedge)+\varphi_2\ln(N/\overline{N})$

Note that from equations (3), (4), (7) and (8), we get: $\overline{T} = \overline{a} - 1$[18].

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[18] More precisely, according to equations (4) and (8), we have: $\overline{a}(t_o) + \Delta a(t_o) = \overline{T}(t_o) + \Delta T(t_o) + 1$.
Let us consider that $\Delta a(t_o) = \Delta T(t_o)$. Equation (7) can be rewritten using equation (3) as:

$\overline{a}(t_o) = \overline{T}(t_o) + \frac{1}{2}(1 + sign(X)) - \frac{1}{2}(1 + sign(X(-N))$ where

$$X(-i) = (1 - tcr)\left((1+\gamma)^{-i-\Delta a+1} F(\kappa,1) - \omega(1+\gamma)^{-\Delta a}\right)$$
$$+ (1+\pi)^{\Delta a}\left(\frac{1-\delta}{1+r}\right)\left[\Delta a(1-tcr)\left((1+\gamma)^{-i-\Delta a+1} F(\kappa,1) - \omega\right) - x^f\right]$$
$$- \left(\frac{1}{1+\pi}\right)^{1-\Delta a}\left[\Delta a(1-tcr)\left((1+\gamma)^{-i-\Delta a+1} F(\kappa,1) - \omega/(1+\gamma)\right) - x^f /(1+\gamma)\right]$$

By definition, N is high enough for $X(-N))$ to be negative. Moreover, when all the production units produce for at least a fraction of time $\Delta a(t_o)$, X is positive. Thus, we have: $\overline{a}(t_o) = \overline{T}(t_o) + 1$.

Appendix 5: Calibration of the model

Let us introduce the new auxiliary parameter ξ, the ratio of firing costs to yearly wages, given by: $x^f = \xi \frac{w}{p}$.

We define $R = r + \mu - \pi a$ as the real return for investors, where r is the nominal interest rate, πa is the inflation rate (expected next period) and μ is a risk premium.

To discuss the impact of National Account data on the values of the parameters and the unobserved data, we need an analytical solution of the steady-state model. This cannot be easily done as long as the expected lifetime is divided between an integer part and a decimal part. We choose here to compute the first-order condition relative to the expected lifetime as if the model were written in continuous time. Then, we use the discrete time equivalent to this condition for replacing equations (3) and (4) of Appendix 4. The lifetime time variable is assumed to be an integer. Thus we neglect the decimal part in the aggregation equations. Although these assumptions are somehow inconsistent, we believe they lead to relations between variables that can be qualitatively interpreted.

The value of a new unit, in a continuous time framework, can be written as:

$$V_{t_0,t_0}(T(t_0)) = \int_{\tau=0}^{T(t_0)} (1 - tcr_{t_0+\tau})(A_{t_0} F(\kappa_{t_0},1) - A_{t_0+\tau} \frac{w_{t_0+\tau}}{p_{t_0+\tau}}) e^{\int_{\theta=0}^{\tau} e^{-r_{t_0+\theta}} d\theta - \delta\tau + \pi\tau} d\tau - A_{t_0+T(t_0)} x^f_{t_0+T(t_0)} e^{-\int_{\theta=0}^{T(t_0)} e^{-r_{t_0+\theta}} d\theta - (\delta - \pi)\tau} + \eta A_{t_0} x^f_{t_0}$$

with $A_{t_0+\tau} = A_{t_0} e^{\gamma\tau}$.

The first-order condition relative to the expected lifetime of the new unit is the following at the steady-state is as follows:

$$(1 - tcr)\left(F(\kappa,1)e^{-\gamma T} - \omega\right) + (r + \delta - \pi - \gamma)x^f = 0$$

whose equivalent in a discrete time framework:

$$(1 - tcr)\left(F(\kappa,1)(1+\gamma)^{-T} - \omega\right) - (r + \delta - \pi - \gamma)x^f = 0 \qquad \text{(3bis)}$$

This equation (3bis) replaces equations (3) and (4) of Appendix 4. Equations (7) and (8) of this appendix are also replaced by a similar formula. The other equations of the steady-state model are those of Appendix 4 with $\Delta T = \Delta a = 0$, $a = T + 1$ and $\overline{a} = a$. We refer to number of these equations in Appendix 4 in what follows. By definition we have $a = T + 1$, so we concentrate on the expected lifetime T.

The equation (3bis) above combined with aggregate production and aggregate employment (equations (10) and (9)) gives the expected lifetime of the new unit:

$$(1+\gamma)^{-T} = \frac{h_T(1+g)}{h_T(1+n\&)}\left(\frac{wN}{pY}\right)\left(1 - \xi(R + \delta - \gamma)\right)$$

where $h_T(x) = \left(1 - \left(\frac{x}{1-\delta}\right)^{-T}\right) / \left(\frac{x}{1-\delta} - 1\right)$

Without firing costs ($\xi = 0$), the previous equation determines the expected lifetime of the production units in terms of the share of wages in value-added and the growth rates of population and technical progress.

Note that firing costs affect positively the lifetime since the later these costs are charged the lower is their present value. Then, the expected lifetime also depends positively on the real return when firing cost are taken into account.

The capital intensity and the production capacity of a new unit are derived from the expected lifetime. It is also the case for the number of new units created at each period, i.e. the employment attached to the new units:

$$\kappa = \frac{I}{AN} h_T(1+r) \qquad F(\kappa,1) = \frac{Y}{AN}\frac{h_T(1+r)}{h_T(1+g)} \qquad \text{and} \qquad n = \frac{N}{h_T(1+r)}$$

The production function (1) ties the scale factor z to coefficient α:

$$\alpha z^{1-1/\sigma} = \frac{F(\kappa,1)^{1-1/\sigma} - z^{1-1/\sigma}}{\kappa^{1-1/\sigma} - 1}$$

Equation (5) can then be rewritten using the above relation:

$$z^{1-1/\sigma} F(\kappa,1)^{1/\sigma} = F(\kappa,1) - \frac{c_{i_0}}{(1-tcr)h_T(1+R)}\left(\kappa - \kappa^{1/\sigma}\right)$$

Combining the previous equation and equation (6) leads to the following relation:

$$h_T(1+R)F(\kappa,1) - \frac{c_{i_0}}{(1-tcr)}\kappa = \omega\left(h_T\left(\frac{1+R}{1+\gamma}\right) + \xi\left(\frac{(1+\gamma)^T}{(1+r)^{\bar{T}+1}} - \eta\right)\left(\frac{1-\delta}{1-tcr}\right)\right)$$

which can be rewritten as:

$$\frac{Y}{h_T(1+g)} = \left(\frac{1}{h_T(1+R)}\right)\frac{c_{i_0} I.}{(1-tcr)} + \left(\frac{h_T(R-\gamma)}{h_T(1+R)} + \xi\left(\frac{(1+\gamma)^T}{(1+r)^{\bar{T}+1}} - \eta\right)\left(\frac{1-\delta}{1-tcr}\right) / h_T(R)\right)\frac{A\omega N}{h_T(1+r)}$$

This equation expresses the distribution of total income between capital and labor. It can be rearranged to depend only on the wage share in value added ($\frac{A\omega N}{Y}$) and the investment rate $\frac{I}{Y}$.

The scale factor is then defined by:

$$z = F(\kappa,1)\left(1 - \frac{c_0 p_I}{(1-tcr)\,p.h_T(R)}\left(\frac{\kappa - \kappa^{1/\sigma}}{F(\kappa,1)}\right)\right)^{\sigma/(\sigma-1)}$$

and parameter α by:

$$\alpha = \frac{\left(\frac{F(\kappa,1)}{z}\right)^{1-1/\sigma} - 1}{\kappa^{1-1/\sigma} - 1}$$

Appendix 6: Simulation results

1- Upward shift in the wage curve

Chart A1 : Aggregate demand adjustment

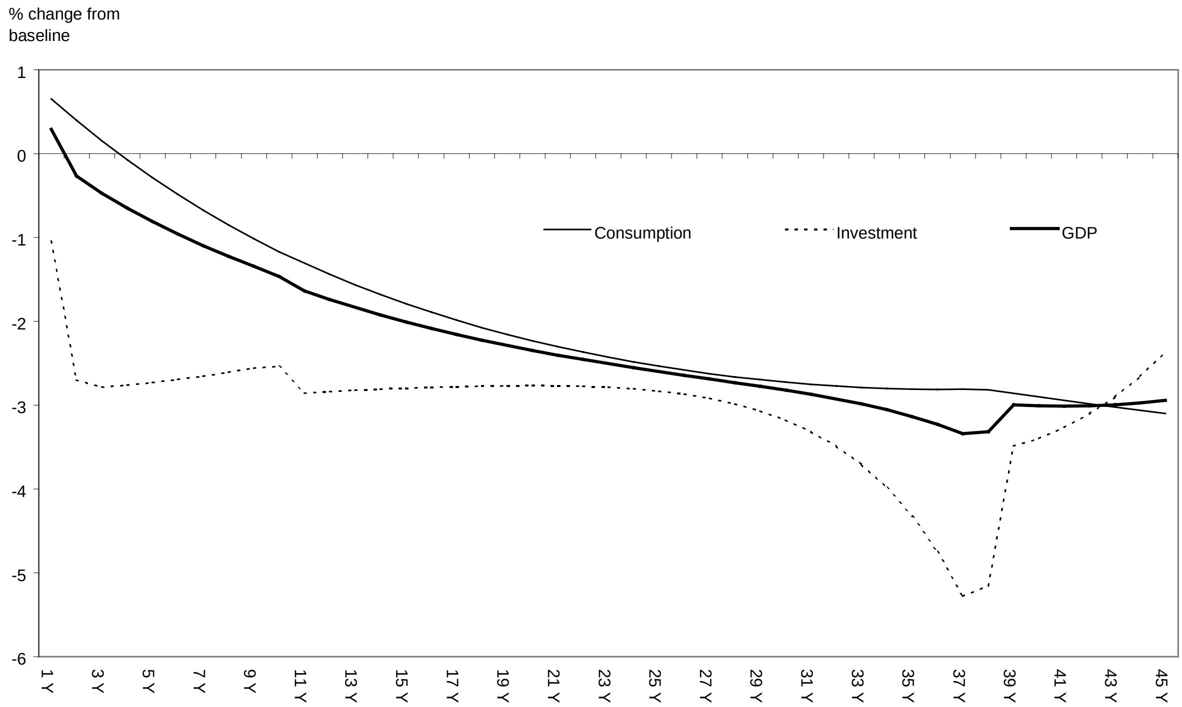


Chart A2 : Aggregate supply

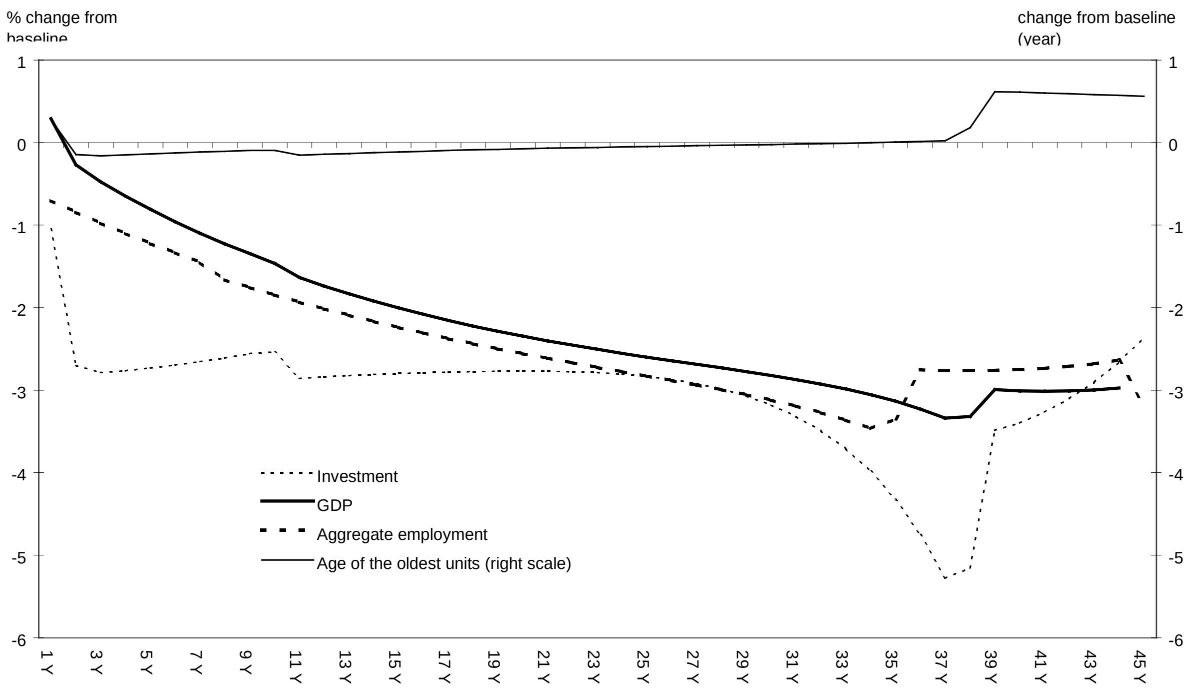


Chart A3 : Technology choices for the new units

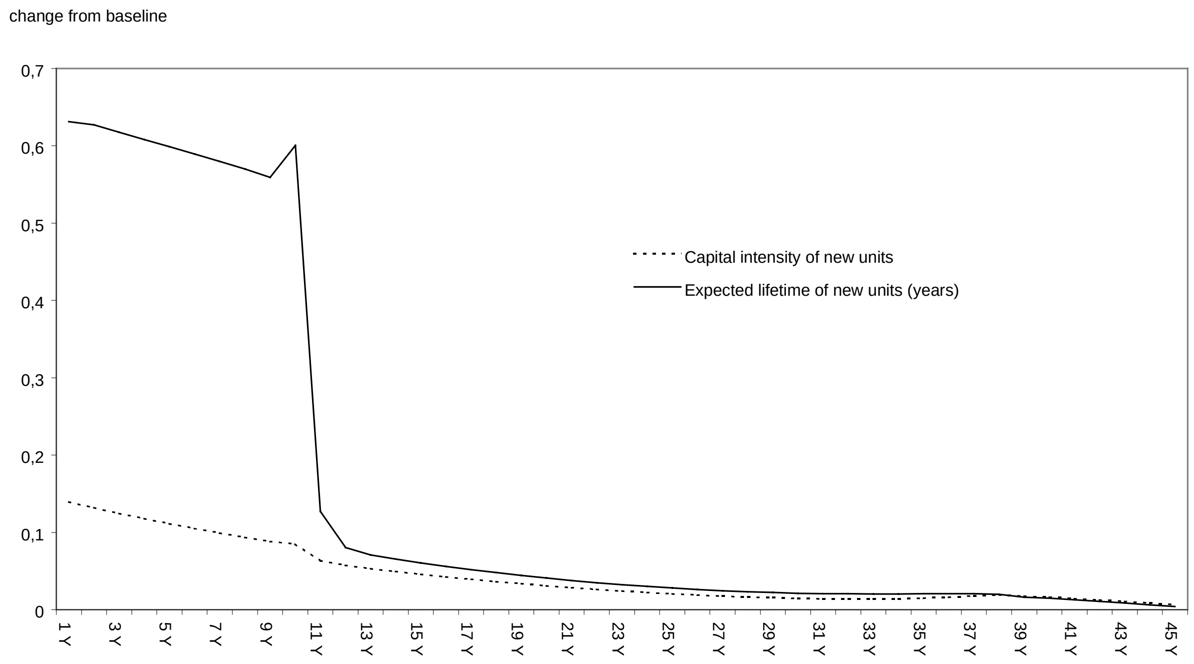


Chart A4 : Real interest rate and wage rate

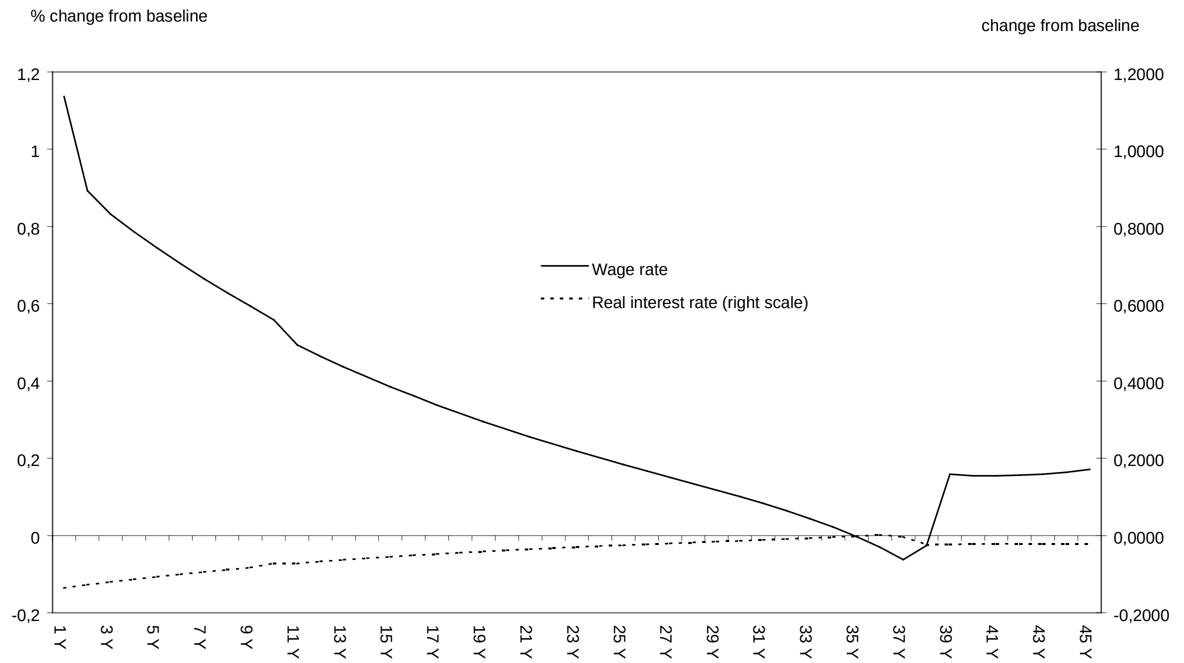
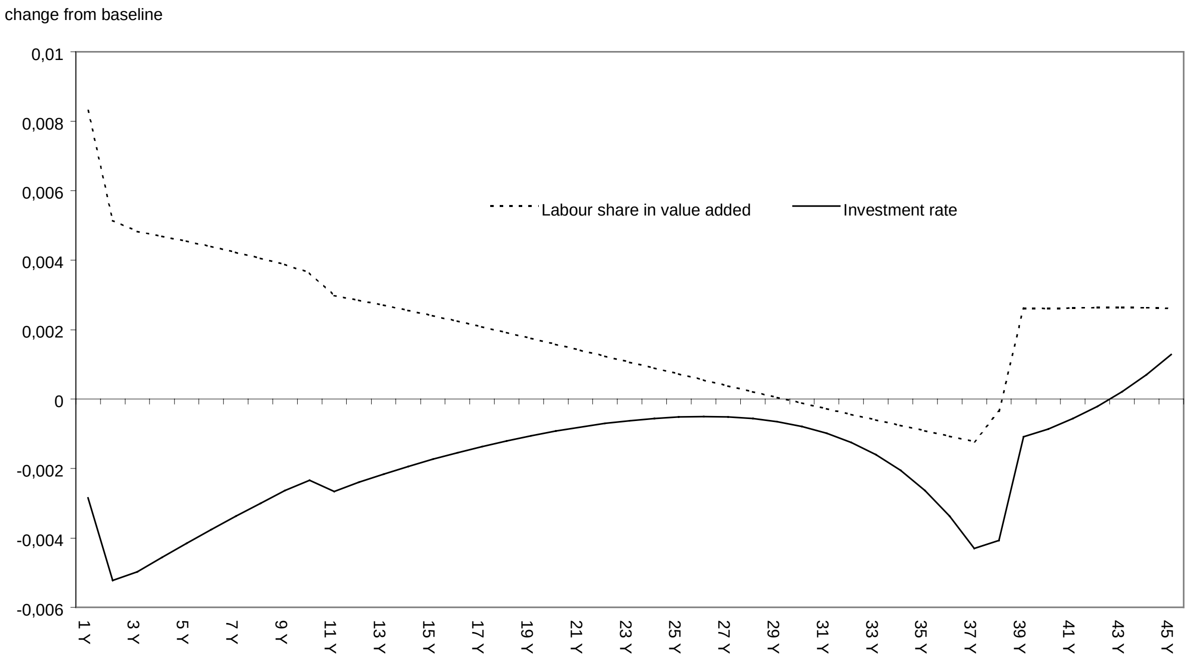


Chart A5 : Labor share in value added and investment rate



2- Transitory shock on nominal interest rate

Chart A6 : Aggregate demand adjustment

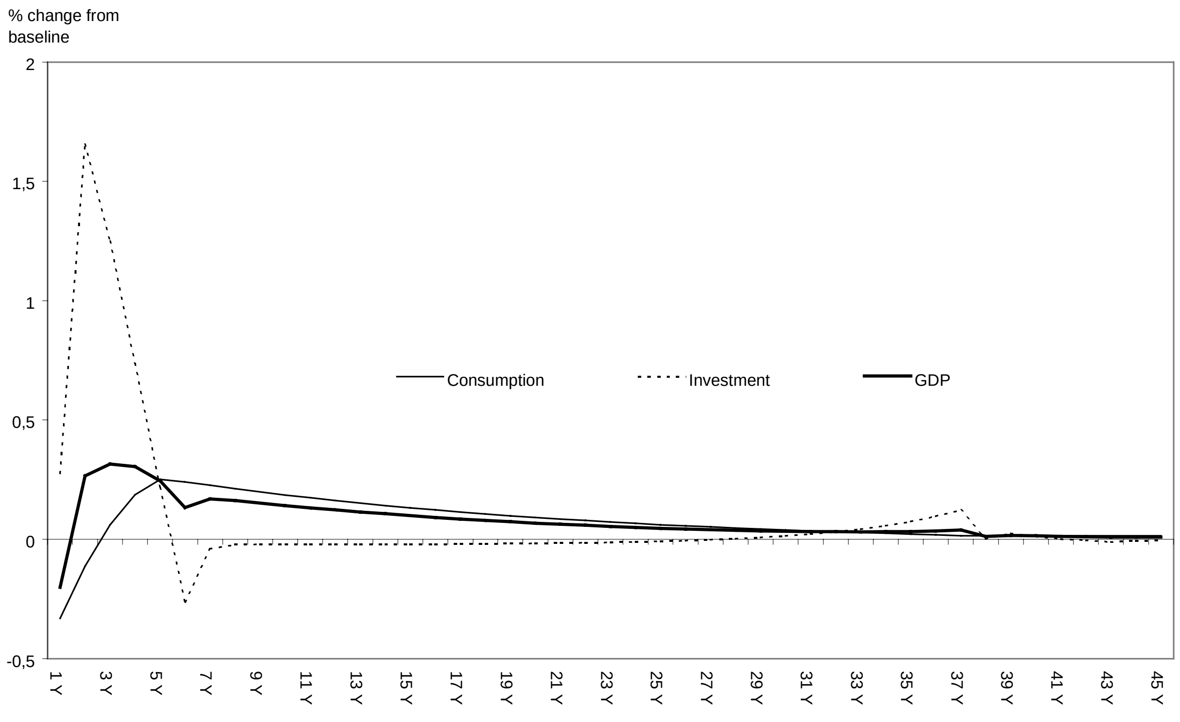


Chart A7 : Aggregate supply

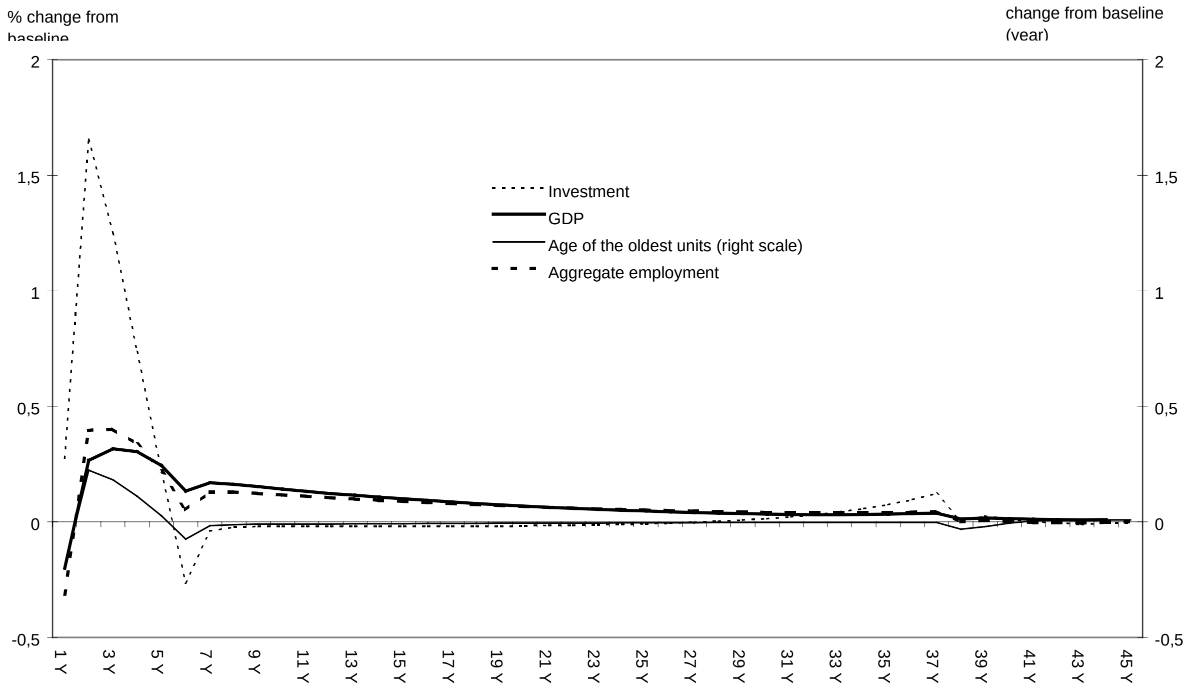


Chart A8 : Technology choices for the new units

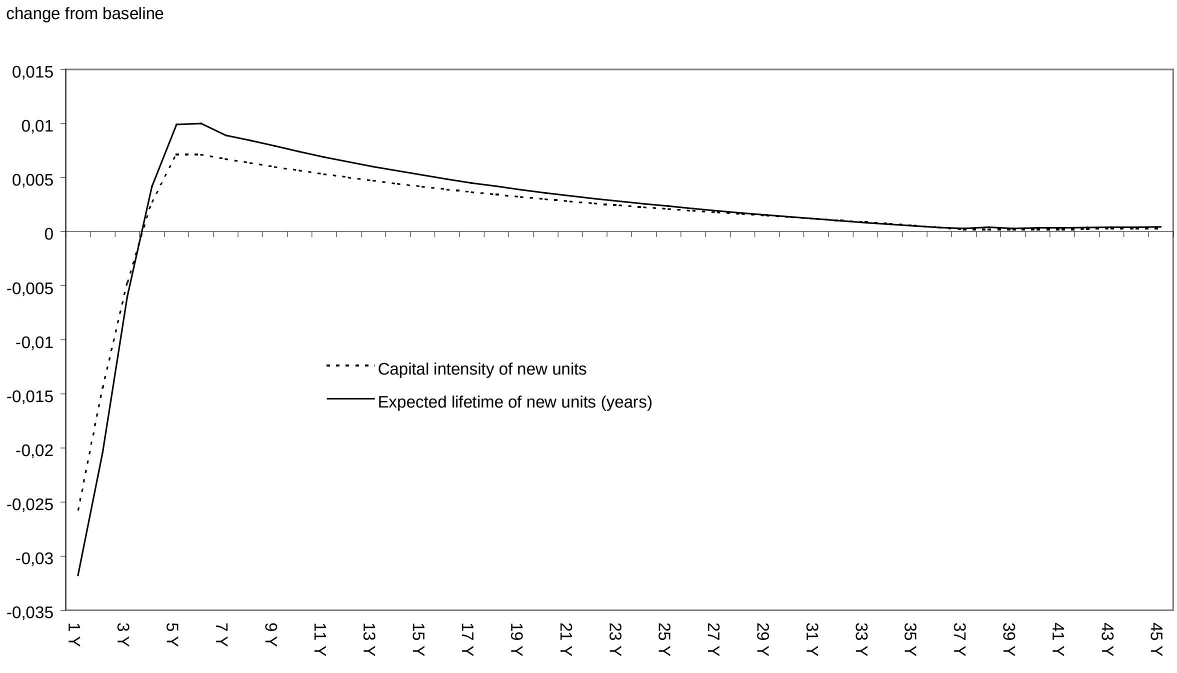


Chart A9 : Real interest rate and wage rate

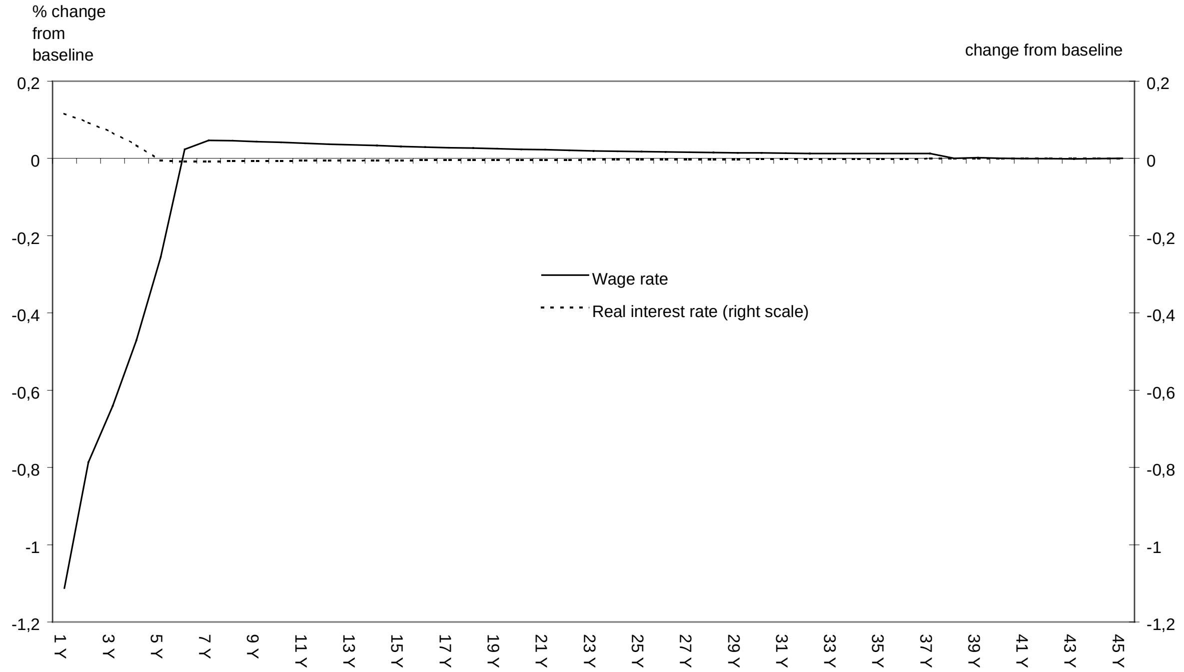
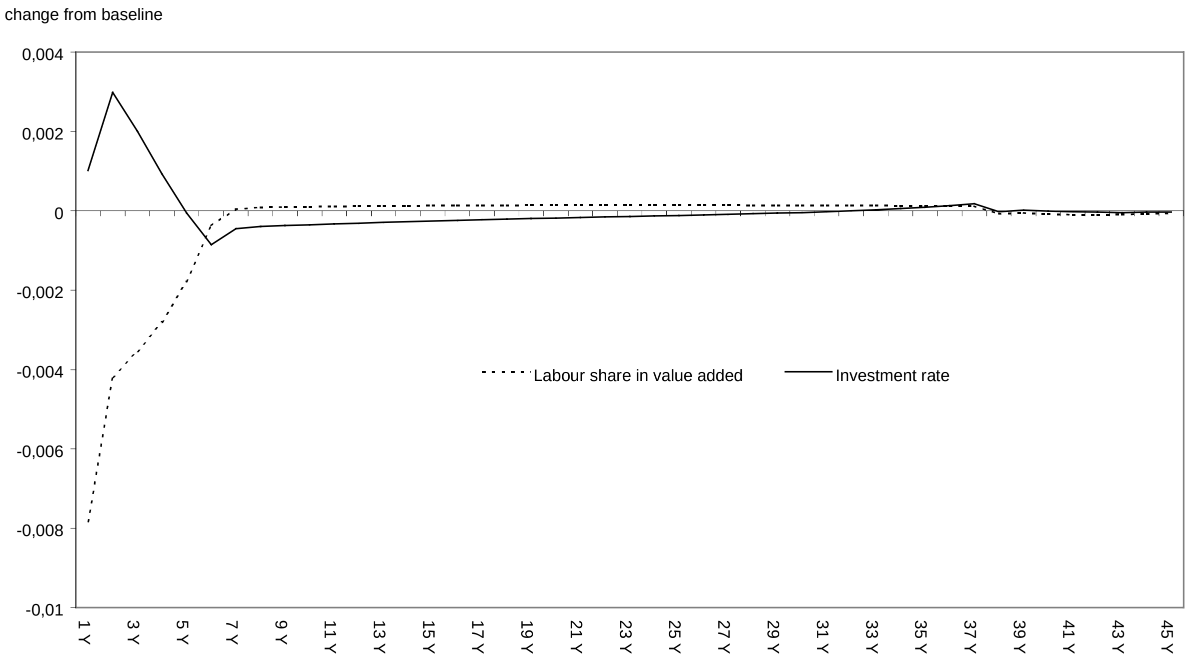


Chart A10 : Labor share in value added and investment rate



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