Small Dimension PDE for Discrete Asian Options*

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Abstract

This paper presents an efficient method for pricing discrete Asian options. Its contribution to the existing literature consists in targeting at smile and non proportional dividend effects. Using an homogeneity property, we show how to reduce an $n+1$ dimensional problem to a 2 or 3 dimensional one. We derive a PDE for the Asian option and solve it with the standard Crank Nicholson method. The dimension reduction impose us to interpolate and extrapolate our conditional price at each fixing date. Within a determistic volatility structure consistent with the smile, the homogeneity property is roughly conserved, thanks to a vega correction term. This allows us to stay in a two dimensional framework as in the Black Scholes case. We examine different numerical specifications of our finite difference (interpolation method, grid boundaries, time and space steps) as well as the extension to the case of non proportional discrete dividends, using a jump condition. We benchmark our results with Quasi Monte-Carlo simulation and a multi-dimensional PDE.

1 Introduction

Asian options are securities with a payoff depending on the average value of an underlying stock, index, interest rates over some time period. First introduced in Tokyo[1], Asian Options are among

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[1] hence the name of Asian options as opposed to American, European or Bermudean ones.

the most popular path-dependent options, since their characteristics capture, in a way, the whole trajectory of the underlying, with a reduced exposure to volatility in most cases. The common belief that these options should be cheaper than their corresponding string of vanilla options is not strictly accurate. However, it happens to often be the case in various practical cases (see Geman and Yor (1993)) for a discussion). In addition, Asian options are less sensitive to possible spot manipulations or extreme movements at settlement and offer much flexibility in the way the average is settled. From a trader's point of view, the delta of an Asian option naturally decreases since part of the average becomes known after an observation date. The hedging strategy is therefore eased, compared with regular options. Consequently, Asian options have become very attractive for investors since they provide a customized cheap way to hedge periodic cash-flows (see Longstaff (1995) for a discussion of the efficiency of Asian interest-rate options for corporations with reasonably predictable cash flows). Nonetheless, such options have turned out to be much more difficult to value than standard options.

Previous research was intensively focused on continuous time Asian options using Black-Scholes (1973) assumptions. However, traded Asian options are based on a discrete time sampling and the underlying security can exhibit a pronounced volatility smile as well as non-proportional dividends.

The existing very extensive literature has at least two major drawbacks. Previous works attempting at approximate closed forms solutions fail to adapt to more complex volatility models, like the Dupire (1993a), (1993b) and Deman and Kani (1994) ones, as well as to American type features. These works include the ones of Vorst (1996), (1992), Geman and Yor (1993), Turnbull and Wakeman (1991), Levy (1992), Jacques (1996), Zhang (1996) and Milevsky and Posner (1997). Vorst (1992) approximated the arithmetic Asian option with a modified geometric one. Geman and Yor (1993) found closed formula for the Laplace transform of the option, by means of Bessel processes. Turnbull and Wakeman (1991) used a lognormal density to approximate the sum of lognormal density. Levy (1992) and Jacques used an Edgeworth expansion to match higher moments. Zhang (1995) (1998) developed a Taylor expansion to derive an approximation based on the geometric Asian option. Recently, Milevsky and Posner (1997) suggested to use the asymptotic limit of the sum of lognormal density known as the reciprocal gamma density.

Works on numerical methods do not account for volatility smile and discrete Asian options: Kemma and Vorst (1990), Hull and White (1987), Carverhill and Clewlow (1992), Benhamou (2000), Rogers and Shi (1995), He and Takahashi (1996), Alziary et al. (1997) and Forsyth et al. (1998). Kemma and Vorst (1990) used Monte Carlo simulations, Hull and White (1987) binomial trees, Carverhill and Clewlow (1992) and Benhamou (2000) Fast Fourier Transform techniques, Roger and Shi (1995), He and Takahashi (1996), Alziary et al. (1997) finite differences and Forsyth et al. (1998) finite elements.

The motivation of this paper is to provide an efficient method for pricing discrete Asian options with a deterministic volatility as specified in Dupire (1993a), (1993b) and Deman and Kani (1994) as well as non-proportional discrete dividends. These two features are far more realistic than Black Scholes assumptions for equity derivatives pricing. Using an homogeneity property, we show how to reduce an $n+1$ dimensional problem to a 2 to 3 dimensional one. This is of considerable interest for the efficient computation of discrete Asian options. This generalizes to discrete Asian options the dimension reduction technique found for continuous Asian options by Rogers and Shi (1995). We show that the homogeneity property is coarsely conserved within a deterministic volatility structure, consistent with the smile as in the Dupire model. This is also true for the issue of non proportional discrete dividends, solved with a jump condition. We can still infer call prices from distinct ones, by means of the homogeneity property. We derive a PDE for the computation of the Asian option and solve it with the standard Crank Nicholson method. Because of the dimension reduction, we need to interpolate our conditional price at each fixing dates. The rest of the article tackles the issue of numerical specifications for the finite difference method (grid boundaries, time and space steps). We compare our result with a Quasi Monte Carlo simulation based on Sobol sequences.

The remainder of this paper is organized as follows. In section 2, we explain how to reduce the dimension of the problem using homogeneity property and a conditional expectation method. We introduce a modified strike variable. This leads to a 3 dimensional PDE which in the case of the Black Scholes diffusion is only 2 dimensional. In section 3, we explain how to account for non homogeneous situation either implied by the smile effect or by discrete non proportional dividends. Section 4 compares our method with a benchmark price given by a Sobol Quasi Monte Carlo simulation. We conclude briefly in section 6 giving some further developments.

2 How to reduce the Dimension?

2.1 Mathematical Framework

We consider a continuous time trading economy with an infinite horizon. The uncertainty is characterized by a complete probability space $(\Omega, \mathcal{F}, Q)$ where Ω is the state space, $\mathcal{F}$ is the σ-algebra representing the measurable events, and Q is the risk neutral probability measure, assumed to be unique in a complete market with no arbitrage opportunity. The information evolves according to the augmented right continuous complete filtration $\{\mathcal{F}_t, t \in \mathbb{R}^+\}$ generated by a standard one dimensional Brownian Motion $\{W_t, t \in \mathbb{R}^+\}$. We assume the evolution of the underlying price process $(S_t)_{t\in\mathbb{R}^+}$ is described by a Stochastic Differential Equation (2.1)

$$dS_t = r_t S_t dt + S_t \sigma(t, S_t) dW_t \tag{2.1}$$

with an initial condition $S_0 = x$, r_t is the deterministic risk free interest rate and $\sigma(t, S_t)$ is either constant (Black Scholes model) or deterministic (like in the Dupire and CEV models). We discuss the extension of this framework to the stochastic volatility case in section 4.1.4. The solution for the underlying process is given by

$$\begin{aligned} S_t &= x e^{\int_0^t \left(r_u - \sigma^2(u,S_u)\right)du + \int_0^t \sigma(u,S_u)dW_u} \\ &= X_v e^{\int_v^t \left(r_u - \sigma^2(u,S_u)\right)du + \int_v^t \sigma(u,S_u)dW_u} \end{aligned}$$

for $0 \leq v \leq t$. It is worth noticing that the underlying process is not perfectly homogeneous (of degree 1) with respect to x, as soon as the volatility structure $\sigma(t, S_t)$ depends on S_t. We denote by μ a density measure over the interval $[0, T]$ with a continuous density ρ_t and some atoms $\sum_{i=1}^n \alpha_i \delta_{t_i}$ at points $(t_i)_{i=1..n}$ representing some fixing dates with $t_n = T$, and $(\alpha_i)_{i=1..n}$ some weights, either positive or negative. The averaging measure μ is not necessary absolutely continuous with respect to the Lebesgue measure and is not necessary of total measure 1. This enables us to be very general, allowing for discrete or continuous-time averaging, fixed or floating strike as explained in exhibit 1. We focus at the following option payoff:

$$(A_T - K)^+$$

where K is a real number. The running average given by:

$$A_t = \int_0^t S_u \mu(du)$$

Asian option type	Discrete underlying	Continous underlying
Fixed strike	$\left(\frac{\sum_{i=1}^n S_t}{n} - K\right)^+$ $\mu(dt) = \frac{\sum_{i=1}^n \delta_{t_i}(t)}{n}$	$\left(\frac{\int_0^T S_t dt}{T} - K\right)^+$ $\mu(dt) = \frac{1_{[0,T]}(t)dt}{T}$
Floating strike	$\left(\frac{\sum_{i=1}^n S_t}{n} - S_T\right)^+$ $\mu(dt) = \frac{\sum_{i=1}^n \delta_{t_i}(t)}{n} - \delta_T(t)$	$\left(\frac{\int_0^T S_t dt}{T} - S_T\right)^+$ $\mu(dt) = \frac{1_{[0,T]}(t)dt}{T} - \delta_T(.)$

Exhibit 1: Payoff and measure for different type of Asian option the sign $\delta_u(.)$ denotes the Dirac function at the point $u \in \mathbb{R}$)

2.2 Determination of a small dimension PDE

A brute force PDE for a discrete Asian option with n fixing dates would consist in a $n+1$ dimensional PDE. The option depends on n variables $S_{t_1}, ..., S_{t_n}$ and the time t. This becomes soon intractable because of the high dimension of the problem. The complexity of a finite difference method increases exponentially with respect to the dimension. To reduce the dimension, we suggest two strategies: first, we extend the method of Rogers and Shi (1995), derived for the Black Scholes case, to non-constant volatility structures as imposed by the smile effect. Indeed, we see that this method is only appropriate for homogeneous diffusions. However, for a diffusion implied by the Dupire method, this is inappropriate. A second approach, which can offer a solution to this particular case, exploits the homogeneity property of the option underlying and price.

2.2.1 Traditional PDEs

Before embarking into some dimension reduction consideration, we show how to adapt traditional PDEs for the Asian option to non-constant volatility structures. The standard PDE derived for continuous-time Asian options (as explained in Ingersoll (1987) or Forsyth, Vetzal and Zvan (1998)) leads to the following expression in the case of non constant volatility structure:

$$C_t + \frac{1}{2}\sigma^2\left(t, S_t\right) S_t^2 C_{ss} + rSC_s + SC_I - rC = 0$$

where C_t respectively C_s, C_I, C_{ss} denotes the first order partial derivative function with respect to the time, respectively the underlying, the running sum or the second order partial derivative function with respect to the underlying. When deriving the PDE with respect to the running average denoted by A, we find the following PDE (see for instance Barraquand and Pudet (1996))

$$C_t + \frac{1}{2}\sigma^2\left(t, S_t\right) S_t^2 C_{ss} + rSC_s + \frac{1}{T}\left(S_t - A_t\right) C_A - rC = 0$$

The difference with the standard Black Scholes PDE comes from the dependence in the underlying of the volatility structure. For discrete Asian options, these equations transform to the same one dimensional PDE:

$$C_t + \frac{1}{2}\sigma^2\left(t, S_t\right) S_t^2 C_{ss} + rSC_s - rC = 0$$

with the condition at the observation date

$$C\left(t_i^-, S, A_{t_i^-}\right) = C\left(t_i^+, S, A_{t_i^-} + \alpha_{t_i} S_{t_i}\right)$$

where $C\left(t, S, A_t\right)$ denotes the call value at time t with underlying S and average A_t. This represents an infinite set of one-dimensional PDEs and is computationally time-consuming.

2.2.2 Change of variable

As suggested by Rogers and Shi (1995) in the case of the Black Scholes model, we can use a change of variable to reduce the dimension. Extending their works, we show that the PDE to be satisfied by the option price is a three dimensional one. The price of a call option is expressed as the expected value of the discounted pay-off under the risk neutral probability measure

$$C_t = \mathbb{E}_Q\left[e^{-\int_t^T r_s ds}\left(\int_0^T S_s \mu\left(ds\right) - K\right)^+ |\mathcal{F}_t\right] \tag{2.2}$$

We define

$$f\left(t, k, S_t\right) = \mathbb{E}_Q\left[\left(\int_t^T \frac{S_s}{S_t}\mu\left(ds\right) - k\right)^+ |\mathcal{F}_t\right] \tag{2.3}$$

The option price is given by

$$\begin{aligned}
C_t &= e^{-\int_t^T r_s ds}\mathbb{E}_Q\left[\left(\int_0^T S_s\mu(ds)-K\right)^+ |\mathcal{F}_t\right] && (2.4)\\
&= e^{-\int_t^T r_s ds}S_t\mathbb{E}_Q\left[\left(\int_t^T \frac{S_s}{S_t}\mu(ds)-\frac{K-\int_0^t S_s\mu(ds)}{S_t}\right)^+ |\mathcal{F}_t\right]\\
&= e^{-\int_t^T r_s ds}S_t f\left(t,\frac{K-\int_0^t S_s\mu(ds)}{S_t}\right)\\
&= e^{-\int_t^T r_s ds}S_t f(t,Y_t,S_t) && (2.5)
\end{aligned}$$

where

$$Y_t=\frac{K-\int_0^t S_s\mu(ds)}{S_t}$$

By Itô's formula,

$$dY_t=\left(-\mu(dt)-r_tY_t+\sigma^2(t,S_t)Y_t\right)dt-Y_t\sigma(t,S_t)dW_t$$

Since $e^{\int_t^T r_s ds}C_t$ is a martingale, its deterministic part should be equal to zero. We notice that the function $f : t,k,s \mapsto f(t,k,s)$ is jointly continuous in t,k and s, decreasing in t and decreasing convex in k. Assuming that the function f has enough smoothness to apply Itô's formula to the equation (2.5), we get

$$\begin{aligned}
d\left(e^{\int_t^T r_s ds}C_t\right) &= S_t\begin{pmatrix} f_t(t,Y_t,S_t)dt+f_k(t,Y_t,S_t)dY_t+\frac{1}{2}f_{kk}(t,Y_t,S_t)\langle dY_t\rangle\\ +f_s(t,Y_t,S_t)dS_t+\frac{1}{2}f_{ss}(t,Y_t,S_t)\langle dS_t\rangle\\ +f_{ks}(t,Y_t,S_t)\langle dY_t,dS_t\rangle\end{pmatrix}\\
&\quad +f(t,Y_t)dS_t+\langle dS_t,df(t,Y_t,S_t)\rangle
\end{aligned}$$

the deterministic term should be equal to zero, leading to

$$0=S_t\begin{pmatrix}\left(f_t+r_tf+\frac{1}{2}(Y_t\sigma(t,S_t))^2f_{kk}-r_tY_tf_k\right)dt-f_k\mu(dt)\\ +\left(r_tf_s+\frac{1}{2}\sigma^2(t,S_t)S_tf_{ss}-Y_t\sigma^2(t,S_t)f_{ks}\right)dt\end{pmatrix} \tag{2.6}$$

If the measure μ has a continuous density ρ_t and some atoms $\sum_{i=1}^n\alpha_i\delta_{t_i}$, and if we denote by $g(t,y,s)=e^{-\int_t^T r_s ds}f(t,y,s)$ the equation (2.6) can be rewritten as the following PDE

$$\frac{\partial}{\partial t}g+\mathcal{A}g=0 \tag{2.7}$$

with

$$\mathcal{A}=\begin{pmatrix}\frac{1}{2}y^2\sigma^2(t,s)\frac{\partial^2}{\partial y^2}-\left(\rho_t+\sum_{i=1}^n\alpha_i\delta_{t_i}+r_ty\right)\frac{\partial}{\partial y}\\ +\frac{1}{2}\sigma^2(t,s)s\frac{\partial^2}{\partial s^2}+r_t\frac{\partial}{\partial s}-y\sigma^2(t,s)\frac{\partial}{\partial s\partial y}\end{pmatrix}$$

at a point of an atom t_i, we get by an arbitrage argument (the value of g has to be continuous)

$$g\left(t_i^-,y\right)=g\left(t_i^+,y+\alpha_i\right)$$

The boundary condition is then equal to

$$g(T, y, s) = (\alpha_T - y)^+ \tag{2.8}$$

The call price is obtained by:

$$C_{t=0} = xg\left(0, \frac{K}{x}, x\right)$$

The variable Y_t can be interpreted as a conditional strike as shown in the next subsection.

2.2.3 Homogeneous case

For an homogeneous underlying (like the Black Scholes model), the function $f(t, k, S_t)$ does not depend on the underlying price S_t. It reduces to

$$f(t, k) = \mathbb{E}_Q\left[\left(\int_t^T S_s \mu(ds) - k\right)^+ |S_t = 1\right] \tag{2.9}$$

as shown in Rogers and Shi (1995) for instance. This property can also be applied to stochastic volatility models (like in Hull and White (1987), Wiggings (1987), Melino and Turnbull (1990), Stein and Stein (1991), Amin and Ng (1993) and Heston (1992)).

Furthermore, for the Black Scholes model, the volatility structure is a constant. The PDE (2.7) simplifies into a two dimensional one:

$$\frac{\partial}{\partial t} g + \widetilde{\mathcal{A}} g = 0 \tag{2.10}$$

with the diffusion operator given by:

$$\widetilde{\mathcal{A}} = \frac{1}{2} y^2 \sigma^2 \frac{\partial^2}{\partial y^2} - \left(\rho_t + \sum_{i=1}^{n} \alpha_i \delta_{t_i} + r_t y\right) \frac{\partial}{\partial y}$$

and still the same boundary conditions (2.8). The result is strong. For the two types of options, fixed and floating strike ones, the PDE is only two dimensional. However, this property is not easily adaptable to more complex volatility structure.

One of the important but often disregarded property of a geometric Brownian motion is its homogeneity property. This is an appropriate method for the Asian option when looked at as a conditional expectation calculation. The price of the discrete Asian option can be rewritten as the following conditional expectation:

$$C = \mathbb{E}_Q\left[\mathbb{E}_Q\left[\left(e^{-\int_0^T r_s ds}\left(\sum_{i=1}^{n} \alpha_i S_t - K\right)^+\right) \middle| S_{t_1}, ..., S_{t_{n-1}}\right]\right] \tag{2.11}$$

Such a conditional expectation can be interpreted as a call option with a strike equal to $\left(K - \sum_{i=1}^{n-1} \alpha_i S_t\right)$. The homogeneity (of degree one) of the call option price leads to the following remark. Denoting by $C(x, k)$ the call price with an initial underlying level of x and a strike of k, we have

$$C(x, k) = C\left(\frac{xk}{k_0}, k_0\right) \frac{k}{k_0} \tag{2.12}$$

The knowledge of call prices for one strike but different underlying levels is consequently equivalent to the one for all pair of strikes and underlying levels. The finite difference method provides call prices for different values of the underlying. Using an interpolation, we can infer a continuum of prices of the call option for different underlying value. This implies the knowledge of any call price.

The algorithm works as follows: it implies to calculate the call price between two fixing dates for a given level of strike. This is done by a Crank Nicholson method with a backward propagation. When the previous fixing date is reached, we infer call prices for different levels of strike by means of the homogeneity property. We continue the backward propagation.

3 PDE solving for the Homogeneous case

We concentrate on discrete Asian options, often of more interest. The equation in the case of the Black Scholes model is either the two dimensional one as explained on the preceding section or the simple Black Scholes equation, with at each fixing dates, the use of the homogeneity property.

3.1 Discretisation of the PDE: Crank Nicholson Method

We use a Crank Nicholson finite difference. The straightforward discretisation of the PDE derived for the Asian option provides a spurious solution. Indeed, it is more appropriate to use the logarithmic change of variable, as argued by Brennan and Schwartz (1978), Hull and White (1990). In the latter case, the diffusion operator is uniformly elliptic. We denote by $C_{i,j}$ the discretised function where the first variable i stands for the time, whereas the second one j for the space variable. We get the following discretised scheme:

$$\begin{aligned} & \frac{C_{i+1,j}-C_{i,j}}{\Delta T}+\left(r-\frac{\sigma^2}{2}\right)\left(\frac{C_{i+1,j+1}-C_{i+1,j-1}}{4\Delta S}+\frac{C_{i,j+1}-C_{i,j-1}}{4\Delta S}\right) \\ & +\frac{\sigma^2}{2}\left(\frac{C_{i+1,j+1}-2C_{i+1,j}+C_{i+1,j-1}}{\Delta S^2}+\frac{C_{i,j+1}-2C_{i,j}+C_{i,j-1}}{\Delta S^2}\right) \\ = \; & r\left(\frac{C_{i+1,j}+C_{i,j}}{2}\right) \end{aligned}$$

or after grouping the terms

$$\begin{aligned} & a_{i,j+1}C_{i,j+1}+a_{i,j}C_{i,j}+a_{i,j-1}C_{i,j-1} \\ = \; & a_{i+1,j+1}C_{i+1,j+1}+a_{i+1,j}C_{i+1,j}+a_{i+1,j-1}C_{i+1,j-1} \end{aligned}$$

with

$$\begin{aligned} a_{i+1,j+1} &= \left(r-\tfrac{\sigma^2}{2}\right)\tfrac{1}{4\Delta S}+\tfrac{\sigma^2}{2}\tfrac{1}{2\Delta S^2} & a_{i+1,j} &= \tfrac{1}{\Delta T}-\tfrac{\sigma^2}{2}\tfrac{2}{2\Delta S^2}-r\tfrac{1}{2} \\ a_{i+1,j-1} &= -\left(r-\tfrac{\sigma^2}{2}\right)\tfrac{1}{4\Delta S}+\tfrac{\sigma^2}{2}\tfrac{1}{2\Delta S^2} & a_{i,j+1} &= -\left(r-\tfrac{\sigma^2}{2}\right)\tfrac{1}{4\Delta S}-\tfrac{\sigma^2}{2}\tfrac{1}{2\Delta S^2} \\ a_{i,j} &= \tfrac{1}{\Delta T}+\tfrac{\sigma^2}{2}\tfrac{2}{2\Delta S^2}+\tfrac{r}{2} & a_{i,j-1} &= \left(r-\tfrac{\sigma^2}{2}\right)\tfrac{1}{4\Delta S}+\tfrac{\sigma^2}{2}\tfrac{2}{2\Delta S^2} \end{aligned}$$

This is solved by a standard LU method as explained in Press et al. (1992).

3.2 Interpolation and Extrapolation at observation dates

In order to illustrate this methodology, let's take the example of a very simple Asian option, that is a fixed strike, two fixings average call. We want to evaluate

$$C_0 = \mathbb{E}_Q \left[e^{-rt_2} \left(\frac{S_{t_1} + S_{t_2}}{2} - K \right)^+ | \mathcal{F}_0 \right]$$

We use the Crank-Nicholson algorithm with a final condition assuming that $S_{t_2} = K$ for instance. The grid will exhibit the values of

$$C_{i,j}(K) = C_i(S_j, K) = \mathbb{E}_Q \left[e^{-r(t_2 - t_i)} \left(\frac{K + S_{t_2}}{2} - K \right)^+ | S_{t_i} = S_j \right]$$

i.e. half the price of a call of strike K, for $t_i \in [t_1, t_2]$ and $S_j \in [S_{\min}, S_{\max}]$.

However, from date t_1, we rather need the following values for $S_j \in [S_{\min}, S_{\max}]$

$$C_{i,j} = \mathbb{E}_Q \left[e^{-r(t_2 - t_1)} \left(\frac{S_j + S_{t_2}}{2} - K \right)^+ | S_{t_1} = S_j \right]$$

that is the price of a call of strike $2K - S_j$. We therefore use the homogeneity property of the price of the call and we can easily replace the values we have on the grid by the values we need:

$$C_{i,j} = \frac{2K - S_j}{K} C_i \left(\frac{KS_j}{2K - S_j}, K \right)$$

Depending on j, there will be different ways to compute these values.

- if $S_j > 2K$, the call option will be exercised for any value of S_{t_2}. Therefore, we have

$$C_{i,j} = \frac{S_{t_2} e^{-r(t_2 - t_1)} + S_{t_2}}{2} - Ke^{-r(t_2 - t_1)}$$

- if $\frac{KS_j}{2K - S_j} \in [S_{\min}, S_{\max}]$, we will interpolate using the values we already have on the grid. We have implemented a simple linear interpolation.

- if $\frac{KS_j}{2K - S_j} \notin [S_{\min}, S_{\max}]$, we have to extrapolate outside the range of values already computed. For this kind of very in-the-money or very out-of-the-money option prices, we can assume that

$$C_{i,j} = \begin{cases} 0 & \text{for } \frac{KS_j}{2K - S_j} < S_{\min} \\ \frac{2K - S_j}{K} \left(\frac{KS_j}{2K - S_j} - Ke^{-r(t_2 - t_1)} \right) & \text{for } \frac{KS_j}{2K - S_j} > S_{\max} \end{cases}$$

3.3 Numerical Results

This methodology gives very satisfactory results. Exhibits 2 and 3 compare the price of the Asian call computed using Monte Carlo simulation with 10^6 paths and a PDE on a 100x100 grid with S ranging +/-3 standard deviations.

Strike	100%	110%
PDE	24.47	20.81
Monte Carlo	24.46	20.81

Exhibit 2: Price of the Asian call with $r = 0.05$, $t_1 = 1$, $t_2 = 2$ and $\sigma = 0.5$

Strike	100%	110%
PDE	12.40	7.87
Monte Carlo	12.40	7.87

Exhibit 3: Price of the Asian call with $r = 0.05$, $t_1 = 1$, $t_2 = 2$ and $\sigma = 0.2$

4 Extension to the non-homogeneous case

4.1 Taking account for the Smile

4.1.1 The different methods for the smile

The volatility smile is a key concept in option pricing. Research have concentrated over the last ten years extensively on this subject leading to a huge literature. Traditionally, it is divided into two different approaches: parametric and non parametric ones.

In the first type of methods, the equation of the evolution of the underlying process is specified. This description can consist either in a continuous diffusion process with a so called deterministic volatility (Rubinstein (1994), Dupire (1993b) and Derman and Kani (1994)) or a continuous diffusion with a stochastic volatility process (Hull and White (1987), Wiggings (1987), Melino and Turnbull (1990), Stein and Stein (1991), Amin and Ng (1993) and Heston (1992)) or a model with jumps (Aase (1993), Ahn and Thompson (1988), Amin (1993), Bates (1991), Jarrow (1984), Merton (1976)).

Other works, close in the spirit, assume constant elasticity of volatility distribution often called power-law (Rubinstein (1994), Cox Ross (1976)). This has also been reformulated by means of a mapping principle between normal and lognormal distributions (Hagan (1998), Pradier and Lewicki (1999)).

The second type of methods is the inference of the underlying distribution from market data, with no assumption on the evolution of the underlying process. This has been called the expansion method. One infers the different terms of the expansion and can rebuild the distribution (Jarrow and Rud (1982), Bouchaud et al. (1998), Abken et al. (1996)).

4.1.2 Case of deterministic volatility

The deterministic volatility model consistent with the smile has been introduced by Dupire (1993a), (1993b), Rubinstein (1994), Derman and Kani (1994). It assumes the volatility structure be a function of the time and the underlying process. The interest of this method lies in its little assumption about the underlying evolution. The "local volatility" (as opposed to the implied Black Scholes volatility) is proved to be only determined by market data as long as there are enough distinct call options quoted. This implies a liquid market with various call options. It is unfortunately not often the case; and an interpolation procedure is required. However, the main drawback is the instability of the local volatility surface over time and especially for long maturities, for which the inferred structure is frequently not very realistic.

With a known local volatility surface, we can derive the option price as being solution of the modified Black Scholes equation:

$$C_t + r_t S_t C_s + \frac{1}{2}\sigma^2\left(t, S_t\right) S_t^2 C_{ss} = rC$$

or using the change of variable $X = \log\left(S\right)$, we get to

$$C_t + \left(r_t - \sigma^2\left(t, x\right)\right) C_x + \frac{1}{2}\sigma^2\left(t, x\right) C_{xx} = rC$$

The Crank Nicholson method leads to the following discretisation scheme

$$\begin{aligned} & a_{i,j+1}C_{i,j+1} + a_{i,j}C_{i,j} + a_{i,j-1}C_{i,j-1} \\ = \; & a_{i+1,j+1}C_{i+1,j+1} + a_{i+1,j}C_{i+1,j} + a_{i+1,j-1}C_{i+1,j-1} \end{aligned}$$

with

$$\begin{aligned} a_{i+1,j+1} &= \left(r - \frac{\sigma\left(t_{i+1}, S_j\right)^2}{2}\right)\frac{1}{4\Delta S} + \frac{\sigma\left(t_{i+1}, S_j\right)^2}{2}\frac{1}{2\Delta S^2} & a_{i+1,j} &= \frac{1}{\Delta T} - \frac{\sigma(t_{i+1}, S_j)^2}{2}\frac{2}{2\Delta S^2} - r\frac{1}{2} \\ a_{i+1,j-1} &= -\left(r - \frac{\sigma\left(t_{i+1}, S_j\right)^2}{2}\right)\frac{1}{4\Delta S} + \frac{\sigma\left(t_{i+1}, S_j\right)^2}{2}\frac{1}{2\Delta S^2} & a_{i,j+1} &= -\left(r - \frac{\sigma(t_i, S_j)^2}{2}\right)\frac{1}{4\Delta S} - \frac{\sigma(t_i, S_j)^2}{2}\frac{1}{2\Delta S^2} \\ a_{i,j} &= \frac{1}{\Delta T} + \frac{\sigma(t_i, S_j)^2}{2}\frac{2}{2\Delta S^2} + \frac{r}{2} & a_{i,j-1} &= \left(r - \frac{\sigma(t_i, S_j)^2}{2}\right)\frac{1}{4\Delta S} + \frac{\sigma(t_i, S_j)^2}{2}\frac{2}{2\Delta S^2} \end{aligned}$$

4.1.3 Non-homogeneity with a parabolic parameterisation of volatility

A volatility parameterisation widely used on the market is a parabolic dependence with respect to the strike. We have used the following model for the Black Scholes volatility in our computations

$$\Sigma^{BS}\left(K, T\right) = \Sigma_0^{BS}\left(1 + Sm\frac{K - F(T)}{F(T)} + Cu\left(\frac{K - F(T)}{F(T)}\right)^2\right) \tag{4.1}$$

where $F(T)$ is the forward of maturity T, Sm and Cu are some parameters. Usually $Sm < 0$ and $Cu > 0$.

We have to recall that not all values of Sm and Cu are admissible over a given range for K. Indeed, with such a parameterisation, the price of the call may not be convex in K and this allows butterfly arbitrage opportunities (see Hull (1997) for a description of butterfly strategies). We suppose that Sm and Cu are such that the call prices are convex for the (K,T) values we use in the PDE algorithm.

Within this model, the volatility for a given absolute value of the strike depends on the value of S_0. The law of S_t depends not only on $\frac{S_t}{S_0}$, but also on S_0, and the call prices are no longer homogeneous of degree 1 with respect to (S_0, K).

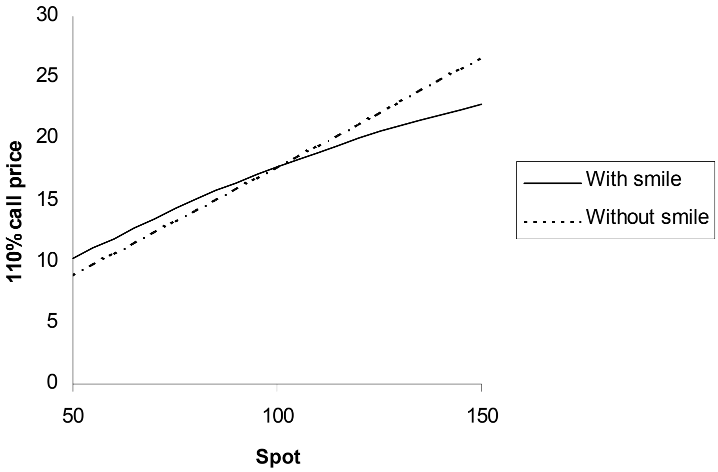

Figure 1: Prices of a call with a strike equal to 110% of spot. The parameters are $r = 0.05$, $T = 1$, $\Sigma_0^{BS} = 0.5$, $Sm = -0.25$, $Cu = 0.015$.

As shown on Figure 1, acting like in the homogeneous case would induce errors. We propose to correct these errors using the vega of the call price, with the following formula

$$C\left(\lambda x, \lambda K, \Sigma^{BS}(\lambda K)\right) \simeq \lambda\left[C\left(x, K, \Sigma^{BS}(K)\right) + Vega^{BS}\left(x, K, \Sigma^{BS}(K)\right)\left(\Sigma^{BS}(\lambda K) - \Sigma^{BS}(K)\right)\right] \tag{4.2}$$

This correction proves to be satisfactory for in-the-money and not very out-of-the-money calls. As shown on Figure 2, it is still acceptable for very out-of-the-money calls. We have to keep in mind that the price of very out-of-the-money calls is small, the error should stay small in absolute value.

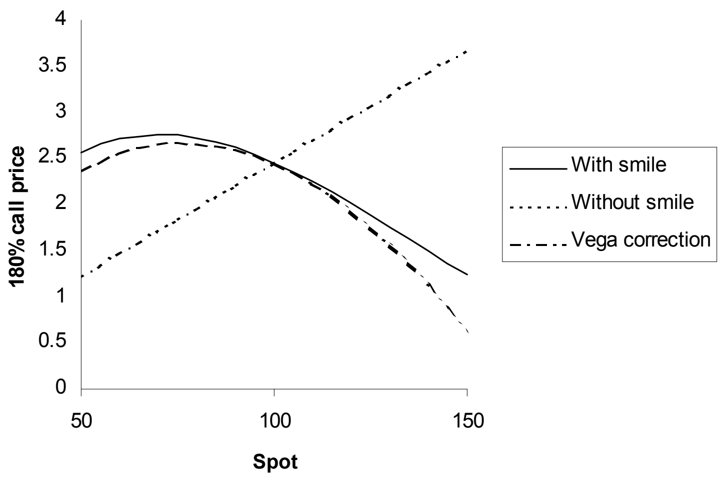

Figure 2: Prices of a call with a strike equal to 180% of spot. The parameters are $r = 0.05$, $T = 1$, $\Sigma_0^{BS} = 0.5$, $Sm = -0.25$, $Cu = 0.015$.

We can apply this technique to the interpolation described in Section 3.2. We have to be careful, however, since the Black-Scholes volatility we use in Equation 4.2 is a *forward start volatility*, starting on date t_1, while Equation 4.1 gives a volatility valid for options starting at $t = 0$. Computing the forward implied Black-Scholes volatility for all the strikes we need would add a dimension to our PDE algorithm. Instead, we use the implied *local* volatility at time t_1. This induces another error, but two facts reduce the impact of this error. First, the smile of market volatility tends to vanish with maturity. Second, if the fixings of the Asian option are close in time, as it is often the case, the local volatility is a good approximation of the forward Black-Scholes volatility between two fixings. Finally, this methodology seems to yield good results (see Section 5.)

4.1.4 Case of stochastic volatility

We briefly explain how to extend the results of this paper to stochastic volatility models. This type of models assumes the volatility structure to be stochastic (models of Hull and White (1987), Wiggings (1987), Melino and Turnbull (1990), Stein and Stein (1991), Amin and Ng (1993) and Heston (1992)). A stochastic volatility structure appears to be more realistic for long maturities. However, it is still more an art than a science to calibrate this kind of models. These models lead to add another dimension to the PDE so as to account for the stochastic volatility. In this paper, we did not implement this type of models.

4.2 Modelling dividends

Dividend modelling is a complicated issue for equity derivatives pricing. Dividends can be discrete or continuous, proportional or not. It is worthwhile examining for a given problem the implication(s) of assumptions on dividends in terms of realism, simplicity, and efficiency.

4.2.1 Advantage of continuous dividends

Continuous proportional dividends consists in a very tractable solution. Indeed, this assumption changes nothing but the risk free rate, which is diminished by the continuous yield of dividend stream. This hypothesis is often appropriate for an index. The different dividends have different issue dates and smoothens the dividend component. It is not the case for a single stock.

4.2.2 Why using discrete dividends?

For a single stock, it is more appropriate to introduce a non proportional discrete dividend. A more complicated assumption could be as well to have a stochastic dividend. However, we assume that the amount of the dividend is known.

We use the jump condition in our PDE algorithm

$$C\left(t_i^-, S\right) = C\left(t_i^+, S - D\right)$$

where $C\left(t, S\right)$ stands for the option price at time t with and underlying S. D is the discrete non-proportional dividend.

4.2.3 Effect on homogeneity

Even with a constant volatility, a non proportional dividend prevents us from using the homogeneity property of the Black-Scholes model. The presence of a non proportional dividend shifts the location of the distribution of $\frac{S_t}{S_0}$ by $\frac{D}{S_0}$. As in Section 4.1.3, the law of S_t depends not only on $\frac{S_t}{S_0}$, but also on S_0.

As long as the dividend is small compared to the underlying price, we can assume that the effect of the non-proportional dividend on the call price doubles if the spot value of the underlying is divided by two. We therefore propose the following approximation

$$C\left(\lambda x, \lambda K, D\right) \simeq \lambda C\left(x, K, D\right) + \left(1 - \lambda\right)\left(C\left(x, K, D\right) - C\left(x, K, 0\right)\right) \tag{4.3}$$

Figure 3 shows that this approximation works well. For important dividend values, however, we have to be careful of negative option prices. A coarse way around this problem is to floor prices to zero. We recall that call prices at these low spot levels are already very small. The absolute error should not be very important.

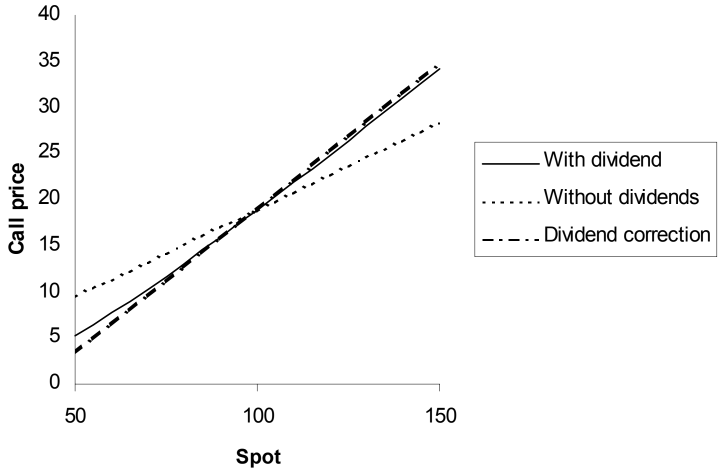

Figure 3: Prices of an at-the-money call. The parameters are $r = 0.05$, $T = 2$, $\sigma = 0.5$, and a dividend of 25 at $t = 1$.

In our implementation, we run twice the algorithm between two fixings: once with the dividends and once without. We then have all the data we need to use Equation 4.3.

5 Numerical Results

This section presents numerical results of our methodology in a simple framework. We price and Asian call with two fixings on dates t_1 and t_2. The continously compounding risk-free interest rate is supposed to be constant. The ex-dividend date is $\frac{t_1+t_2}{2}$. Finally, we have modelled the *local volatility* directly

$$\sigma(t, S) = \sigma_0 \left(1 + sm\frac{S - S_{ref}}{S_{ref}} + cu\left(\frac{S - S_{ref}}{S_{ref}}\right)^2\right) \tag{5.1}$$

with boundaries at 0 and 10. More realistic applications would use an *implied* local volatility instead, as described in Section 4.1.2.

Our results will be benchmarked with a 3-dimension PDE solver.

5.1 The choice of the finite-differences mesh

Exhibit 4 presents results for different numbers of time steps and space steps, as well as different ranges for the underlying level.

time and space steps	20	50	100	200
1 standard deviation	24.18	24.15	24.13	24.11
3 standard deviations	24.32	24.44	24.46	24.47
6 standard deviations	23.67	24.37	24.44	24.46

Exhibit 4: Prices of an at-the-money Asian call with $t_1 = 1$, $t_2 = 2$, $\sigma = 0.5$ and no smile or dividend.

In the remaining calculations, we take a 100x100 mesh and a range of 3 standard deviations for the underlying, which seems to be a good compromise between precision and computing time.

5.2 The "vega correction" in the case of a volatility smile

Figure 4 compares the value profiles we get on our grid at a date just before the fixing, i.e. just after we have used our interpolation procedure. Thanks to the vega correction of Section 4.1.3, the interpolation profile matches quite well the "real" profile obtained through a 3-dimensional PDE.

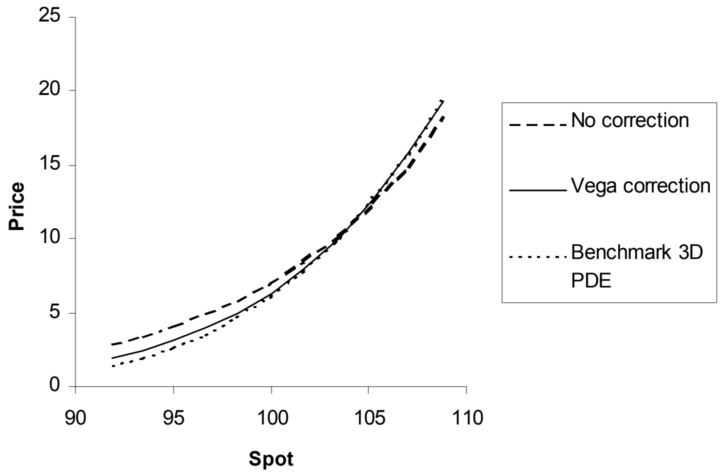

Figure 4: Price of a 110% call just before the fixing. The parameters are $\sigma = 0.5$, $sm = -1$, $cu = 0$ and no dividend.

Consequently, the accuracy of upfront prices is satisfactory, as shown on Exhibit 5

sm parameter	0	-0.2	-0.5	-1
3D benchmark	20.81	20.47	19.92	18.93
with vega correction	20.81	20.51	19.95	18.91
without vega correction	20.81	20.50	20.00	19.08

Exhibit 5: Prices of a 110% Asian call with $\sigma_0 = 0.5$, $cu = 0$ and no dividend.

5.3 The "dividend correction" in the case of non-proportional dividends

The dividend correction of Section 4.2.3 gives good results on the fixing date profile (see Figure 5.)

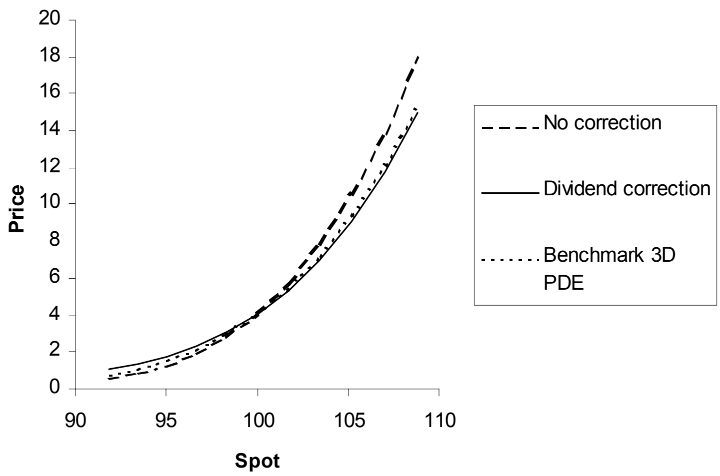

Figure 5: Price of an at-the-money call just before the fixing. The parameters are $\sigma = 0.5$, no smile and a dividend of 30.

And this transposes into very good results for the upfront price, as shown on Exhibit 6.

dividend value	0	5	10	30
3D benchmark	24.46	23.38	22.25	18.22
with dividend correction	24.46	23.36	22.26	18.25
without dividend correction	24.46	23.55	21.77	19.45

Exhibit 6: Prices of an at-the-money Asian call with, $\sigma = 0.5$ and no smile

6 Conclusion

In this paper, we have seen that we can price an Asian option efficiently with a 2-dimension PDE method. The contribution of this paper lies in two ways. We have examined the particular case of the discrete Asian option which is often ignored in the previous literature. We have used the homogeneity of the Black Scholes underlying to reduce the dimension. We have extended the results of Rogers and Shi (1995) to non-constant volatility structure. We have seen the importance of the homogeneity property. It is only in the case of the Black Scholes diffusion that the problem reduces to a two dimensional one. Indeed, with a deterministic volatility like in the Dupire (1993a), (1993b) and Derman and Kani (1994) models, an other variable needs to be added. This is because we have lost the homogeneity property. However, this homogeneity is coarsely satisfied and can be corrected. This enables us to keep on using the backward propagation in two dimensions as in Black Scholes. We have examined the impact of certain numerical specification for the finite difference method as well as the impact of discrete dividends.

There are many possible extensions to this paper. The first one would consist in finding additional features on the relationship between the different calls for non-constant volatility structure.

The homogeneity seems to handle this quite well. However, we have no boundary on the error term. A second enlargement of this work concerns other path dependent options, like ratchet options. The approach adopted here should be adaptable to this kind of options.

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