Optimal Contingent Fiscal Policy in a Business Cycle Model*

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Abstract

The indeterminacy of the optimal fiscal policy that emerges in a stochastic setting has been characterized by Zhu (1992) and Chari, Christiano and Kehoe (1994). There are infinite paths of ex-post capital income tax rates and state-contingent debt return supporting the optimal allocations of consumption, investment and leisure. The main goal of this paper is to introduce identification constraints to determine the state-contingent fiscal policy and to characterize its cyclical properties. Two types of identification constraint will be considered: constraints on the stability of the debt path and constraints on the expectation mechanism. Results indicate that, independently of the identification constraint, the optimal ex-post capital income tax rate is zero and does not fluctuate.

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1. Introduction

Macroeconomics has long been interested in optimal taxation analysis in a dynamic framework, extending the work of Ramsey (1927). The public finance literature has been concerned with studying the optimality of distorting taxation, because ideal non-distorting lump sum taxes are not available for the fiscal authority. Then, we might call this analysis of optimal distorting taxation the second best option. The optimal fiscal policy problem deals with the optimal combination of taxes and debt return that support allocations that maximize household welfare and are consistent with the government spending path (generally exogenous). Although this question was initially treated in a deterministic setting, in recent years there has been a lot of work in a stochastic framework, Lucas and Stokey (1983), Bohn (1994), Zhu (1992), Chari, Christiano and Kehoe (1994, 1995), Marcet, Sargent and Seppälä (1996) and Scott (1997) are good examples.

One of the main results in the deterministic setting points out that the tax rate on capital income must be initially high, and tends to zero afterwards (see Chamley [1986] and, in an endogenous growth framework, Lucas [1990] and Jones, Manuelli and Rossi [1993,1997]).

Extending these models to the stochastic setting generates a significant change in the characteristics of the optimal fiscal policy. Introducing uncertainty yields the indeterminacy of the optimal fiscal policy: there are infinite state-contingent paths of debt return and capital income tax rates decentralizing the optimal allocations, so it is not possible to determine simultaneously these two policy variables contingent on the state of nature.

Chari, Christiano and Kehoe (1994) show that if the government has either state-contingent capital taxes or state-contingent debt return, it can support the optimal allocations. If the government restricts debt return to be state-uncontingent, we can obtain the path of the state-contingent capital income tax rate (called ex-post capital income tax rate by Chari, Christiano and Kehoe [1994]). Alternatively, if the capital income taxation is restricted to not depend on the current state of nature, we can obtain the state-contingent path of debt return that supports optimal allocations. However, we can always determine the so-called ex-ante capital income tax rate theoretically, which is defined as the ratio of the expected revenue of capital income taxation to the expected return of capital net of depreciation. This variable approximates the expected capital income tax rate.

Despite of the indeterminacy of the fiscal policy that arises when extending the analysis to the stochastic setting, there are other results in line with those obtained in the deterministic framework: Chari, Christiano and Kehoe (1994) show, under restrictive conditions for preferences, that the ex-ante capital income tax rate is zero, recalling Chamley's (1996) result.

The goal of this paper is to introduce an identification constraint in order to pin down one of the infinite state-contingent policies and to characterize its cyclical properties. Then, we will be able to study the differences with the *Ramsey policies* of Chari, Christiano and Kehoe (1994), who use the assumption of uncontigent debt return as an identification constraint. We think that assuming state-contingent return is more interesting because we know the nominal return of debt with certainty, but not the real return. In this paper we assume two kinds of constraints: restrictions on the stability of the debt path and restrictions on the expectation mechanism. Given the non-linearity and stochastic nature of the problem, we need to solve it numerically, and in order to do this we show, methodologically, how to apply the solution method of stochastic dynamic systems under rational expectations provided by Sims (1998).

The first identification assumption we impose is a stability condition that limits the growth of the debt path, which in turn cannot grow more than the other variables in the economy. In order to evaluate how this constraint determines the properties of the *Ramsey policies*, we compare the results with those obtained under another identification constraint: an exogenous stochastic process for the debt path, allowing stationary and non-stationary behavior.

The second kind of identification assumption is related to *sunspot equilibria*. There are many real economy calibrated models in which the theoretical structure (preferences, technology and endowments) is not enough to obtain a unique equilibrium. Farmer (1993) points out that this kind of model can be completed by specifying an expectation rule, being consistent with the hypothesis of rational expectations and market clearing, obtaining a unique equilibrium. In this paper, the theoretical structure of the model is enough to obtain the stochastic path of the optimal allocations and the optimal policies in the steady state, but not enough to obtain the stochastic path of the optimal fiscal policy. In order to do this, we add an expectations rule, assuming that one of the expectation errors associated with the Euler conditions of the household is exogenous. The main difference with the usual literature of *sunspot equilibria* is that we have determined the optimal allocations of private agents, whereas usually the real equilibrium is undetermined, as in Benhabib and Farmer (1994, 1996) and Benhabib and Perli (1994), or both the real and the nominal equilibrium - see Farmer (1997), and Chari, Christiano and Kehoe (1995).

We do not find significant differences in the properties of the capital income tax rates under the different identification restrictions implemented. The main results we obtain are: (i) *ex-post* capital income tax rates are zero, in contrast to the Chari, Christiano and Kehoe (1994) results, that obtain a very volatile tax rate on capital income, (ii) imposing stability conditions on the dynamic behavior of the debt path does not restrict the dynamic properties of the *ex-post* capital income tax rates, so the assumption does not seem to be very restrictive. Even when we impose a non-stationary stochastic path for the debt, the properties of the *ex-post* tax rate remain unaltered. (iii) When an exogenous stochastic path for one of the expectation errors is used as an identification constraint, the results about the capital income tax rate do not change. We also show that the expectation errors are independent when we assume one of them as an exogenous identification restriction. (iv) The labor income tax rates hardly fluctuate, taking on the persistence properties of the exogenous shocks.

The paper is organized as follows: section 2 describes the model. The Ramsey problem is presented in section 3. Section 4 reports the simulation results under different identification restrictions. Finally, section 5 concludes by summarizing the main findings.

2. The model

The economy consists of households, firms, and the government, represented by the neoclassical stochastic growth model. We assume a representative household and a representative firm that produces a single good.

2.1. Households

The household makes decisions by maximizing an expected flow of utility, subject to the budget constraint and taking wages and interest rates as given. Preferences at each period are represented by a utility function that includes consumption ($\tilde{c}_t$) and leisure ($1-n_t$), where the household is endowed with one unit of time:

$$\sum_{t=0}^{\infty} \beta^t U(\tilde{c}_t, 1-n_t) \ , \tag{1}$$

we assume a standard utility function:

$$U(\tilde{c}_t, 1-n_t) = \frac{\left(\tilde{c}_t^{1-\theta}(1-n_t)^{\theta}\right)^{1-\sigma}}{1-\sigma} \ , \tag{2}$$

where σ is the relative risk aversion, and θ is the preference for leisure. $\beta \in (0,1)$, is the discount rate.

Household income arises from renting capital and labor to the firm and from the bond returns. Labor and capital income are taxed. After-tax income is spent on consumption, investment and government bonds ($\tilde{b}_t$). The household budget constraint is:

$$\tilde{c}_t + \tilde{k}_t + \tilde{b}_t = (1-\tau_{w_t})\tilde{w}_t n_t + R_t \tilde{b}_{t-1} + \left[1+(r_t-\delta)(1-\tau_{k_t})\right]\tilde{k}_{t-1} \ , \tag{3}$$

where δ is the depreciation rate of the capital stock, τ_w y τ_k are tax rates on labor and capital income. R_t is the return on government bonds. In ecuation (3), the term in brackets on the right hand side represents the gross return after taxes and depreciation, where taxation on capital income has a depreciation tax credit.

2.2. Firms

The production function of the firm exhibits constant returns to scale, using labor and capital as inputs. This function incorporates a stochastic productivity shock (z_t):

$$\tilde{y}_t = F(n_t, \tilde{k}_{t-1}, z_t) \ , \tag{4}$$

where $F(\cdot)$ is a Cobb-Douglas production function with labor augmenting technological change:

$$\tilde{y}_t = \left(e^{\gamma \cdot t + z_t} n_t\right)^{\alpha} \tilde{k}_{t-1}^{1-\alpha} \ , \tag{5}$$

γ represents the exogenous growth rate. The productivity shock follows a stochastic process:

$$z_t = \rho_z z_{t-1} + \varepsilon_{z_t} \ , \quad \varepsilon_{z_t} \underset{iid}{\sim} N(0, \sigma^2_{\varepsilon_z}) \ , \quad |\rho_z| < 1 \ , \quad z_{-1} = 0 \ . \tag{6}$$

The competitive behavior of the firm ensures that input prices equal marginal productivities:

$$\tilde{w}_t = F_n(n_t, \tilde{k}_{t-1}, z_t) \ , \tag{7}$$

$$r_t = F_k(n_t, \tilde{k}_{t-1}, z_t) \ . \tag{8}$$

2.3. Government

The government finances the exogenous flow of government consumption by taxing labor and capital income and by issuing debt. The government budget constraint is:

$$\tilde{G}_t + R_t \tilde{b}_{t-1} = \tau_{w_t} \tilde{w}_t n_t + \tau_{k_t} [r_t - \delta] \tilde{k}_{t-1} + \tilde{b}_t \ . \tag{9}$$

Government consumption is given by:

$$\tilde{G}_t = G e^{\gamma t + g_t} \ , \tag{10}$$

where G is a constant and g_t is a shock that affects government consumption and follows a stochastic process:

$$g_t = \rho_g g_{t-1} + \varepsilon_{g_t} \ , \quad \varepsilon_{g_t} \underset{iid}{\sim} N(0,\ \sigma^2_{\varepsilon_g}) \ , \quad |\rho_g| < 1 \ , \quad g_{-1} = 0 \ . \tag{11}$$

2.4. Competitive equilibrium

In order to analyze the competitive equilibrium of the economy, the optimization problem of the household can be easily converted into stationary dividing variables by the gross rate of growth: $x_t = (\tilde{x}_t / e^{\gamma t})$, $x = c$, k , w , y , G , b , and modifying the discount rate appropriately[1]:

$$\underset{\{c_t,\, n_t,\, k_t,\, b_t\}_{t=0}^{\infty}}{Max} \qquad E_0 \sum_{t=0}^{\infty} \hat{\beta}^t\, U\left(c_t\, ,\ 1 - n_t\right) \tag{12}$$

subject to:

$$c_t + e^{\gamma} k_t + e^{\gamma} b_t = (1-\tau_{w_t}) w_t n_t + R_t b_{t-1} + \left[1 + (r_t - \delta)(1-\tau_{k_t})\right] k_{t-1}$$
$$k_{-1},\ b_{-1} \quad given \tag{13}$$
$$c_t,\ n_t,\ k_t \ \geq\ 0\ .$$

The competitive equilibrium of this economy is the set of paths $\{c_t\, ,\, n_t\, ,\, k_t\, ,\, b_t\}_{t=0}^{\infty}$ that maximizes the household expected flow of utility, subject to the budget constraint and given the input prices $\{r_t\, ,\, w_t\}$, the government policies $\{\tau_{w_t}\, ,\, \tau_{k_t}\, ,\, R_t\, ,\, G_t\}$, the stochastic processes of productivity and government consumption shocks, and the government budget constraint.

The aggregate resources constraint (feasibility constraint) emerges from the competitive equilibrium conditions:

$$c_t + e^{\gamma} k_t - (1-\delta) k_{t-1} + G_t = \ F(n_t\, ,\, k_{t-1}\, ,\, z_t) \ . \tag{14}$$

This expresion indicates that output in the economy is spent on private consumption, investment and public consumption.

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[1] Under CRRA preferences, as those specified in (2), the discount rate is: $\hat{\beta} = \beta e^{\gamma(1-\sigma)(1-\theta)}$

3. The Ramsey problem

The government solves the Ramsey problem in order to select optimally the fiscal policy tools. We adopt the primal approach, characterizing the optimal allocations that can be implemented as a competitive equilibrium with distorting taxation, subject to the feasibility constraint (14) and the so-called implementability constraint. This constraint represents the present value of the budget constraint of the household, eliminating prices and policy variables by using de Euler conditions of the competitive equilibrium. Then it is possible to calculate the optimal allocations separately from the fiscal policy variables.

Since Ramsey allocations are calculated, we obtain the set of policies (*Ramsey policies*) that support optimal allocations, from the conditions of competitive equilibrium. Throughout the analysis we assume that the government can commit itself to follow the fiscal policy plan.

3.1. Ramsey allocations

Allocations of consumption, hours and capital, initial tax rate on capital income and initial debt return emerge from:

$$\underset{\{c_t,\, n_t,\, k_t\}_{t=0}^{\infty}}{Max} \qquad E_0 \sum_{t=0}^{\infty} \hat{\beta}^t \, U\left(c_t,\ 1-n_t\right) \tag{15}$$

subject to:

$$c_t + e^{\rho} k_t - (1-\delta) k_{t-1} + G_t = F(n_t, k_{t-1}, z_t) \ , \tag{16}$$

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U_{c_t} c_t + U_{n_t} n_t\right] = U_{c_0}\left[R_0 b_{-1} + \left(1+(F_{k_0}-\delta)(1-\tau_{k_0})\right) k_{-1}\right] , \tag{17}$$

$$c_t,\ n_t,\ k_t \geq 0, \qquad b_{-1}, k_{-1} \quad given,$$

given the public consumption path, and where (16) represents the aggregate resources constraint (*feasibility constraint*) and (17) is the *implementability constraint*, $r_0 = F_{k_0}$ [2]. Following Lucas and Stokey (1983) and Lucas (1990), it is possible to demonstrate that (17) holds for any allocation that fulfills the competitive equilibrium conditions.

The implementability constraint is included, ir order to be easily solved, into the maximand, with λ representing the Lagrange multiplier that discounts the constraint. Objective function can be rewritten as:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U(c_t, 1-n_t) + \lambda\,(U_{c_t} c_t + U_{n_t} n_t)\right] - \lambda\, U_{c_0} \left[R_0 b_{-1} + \left(1+(r_0-\delta)(1-\tau_{k_0})\right) k_{-1}\right] . \qquad (18)$$

Ecuation (18) is an increasing function of τ_{k_0} and decreasing in R_0. Therefore the government has incentives to set an initial capital income tax rate as high as possible and an initial debt return as low as possible. The reason is that τ_{k_0} taxes capital returns and R_0 rewards the debt stock , both at $t=-1$, so the individual cannot react to the tax and debt return by varying investment and debt stock decisions. Following Chari, Christiano and Kehoe (1994) we assume that initial tax rate on capital income and debt return, τ_{k_0} y $R_0 b_{-1}$, are fixed.

For our later convenience we write:

$$W(c_t, 1-n_t, \lambda) = U(c_t, 1-n_t) + \lambda\,(U_{c_t} c_t + U_{n_t} n_t) \ , \qquad (19)$$

Therefore the government objective function is different at $t=0$ than at $t \geq 1$:

$$\begin{aligned} & W(c_0, 1-n_0, \lambda) - \lambda\, U_{c_0} \left[R_0 b_{-1} + \left(1+(F_{k_0}-\delta)(1-\tau_{k_0})\right) k_{-1}\right] \qquad t=0, \\ & W(c_t, 1-n_t, \lambda) \qquad t \geq 1. \end{aligned} \qquad (20)$$

Optimal allocations $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ satisfying optimal conditions of the problem given by (15)-17) depends on the multiplier λ, which discounts the implementability constraint. These paths are the *Ramsey allocations*, for such a λ, so that the paths of consumption, hours and capital stock satisfy the

[2] $F_{k_0} = \partial F(n_0, k_{-1}, z_0)/\partial k_{-1}$

optimal conditions of the Ramsey problem and the implementability constraint. Following Marcet, Sargent and Seppälä (1996), we iterate in value of λ (using the Gauss-Newton algorithm) until we find the implementability constraint is fulfilled. We evaluate numerically the constraint (C.I.) across 100 simulations as:

$$C.I. = \frac{1}{100}\sum_{i=1}^{100}\left[\sum_{t=0}^{T}\hat{\beta}^t\left(U_{c_{t,i}}c_{t,i}+U_{n_{t,i}}n_{t,i}\right)-U_{c_{0,i}}\left(R_0 b_{-1}+\left(1+(F_{k_{0,i}}-\delta)(1-\tau_{k_0})\right)k_{-1}\right)\right] \qquad (21)$$

where the conditional expectation has been aproximated at $t=0$ as the average of the 100 simulations for a large enough T ($T=1000$).

The convergence criterion used to find the multiplier λ, is reached when the numerical value of (21) is into the interval: $(-\,dt(C.I.)\,,\ d.t.(C.I.))$, where *dt(C.I.)* is the standard deviation of (21), obtained from the 100 simulations.

In order to solve Ramsey allocations numerically, the solution method proposed by Sims (1998) is implemented, extending it to the case of rational expectations non-linear systems (see Novales et al. [1999] for detailled applications of this method to standard models of real business cycles). Appendix A describes the application of this methodology to the Ramsey problem.

3.2. Ramsey policies

After optimal allocations of consumption, hours and capital have been computed, we use the optimal conditions of competitive equilibrium to obtain the debt path and the policies of labor and capital income tax rates and debt return that decentralize allocations.

Given the Ramsey allocations, the optimal labor income tax rate is pinned down from the competitive equilibrium condition that equals consumption-leisure marginal substitution rate to the inverse of labor marginal productivity after taxes:

$$\tau_{w_t} = 1 - \frac{U_{n_t}}{w_t U_{c_t}} \ . \qquad (22)$$

Nevertheless, in a stochastic framework an indeterminacy emerges that makes it impossible to obtain the tax rate on capital income and the debt return simultaneosly, both contingent to the state of nature. Competitive equilibrium first order conditions for capital and debt are:

$$e^{D} = E_t\left(\beta\frac{U_{c_{t+1}}}{U_{c_t}}\left[1+(r_{t+1}-\delta)(1-\tau_{k_{t+1}})\right]\right) = \beta\left(\frac{U_{c_{t+1}}}{U_{c_t}}\left[1+(r_{t+1}-\delta)(1-\tau_{k_{t+1}})\right]\right)-\varepsilon_{1_{t+1}}\ , \tag{23}$$

$$e^{D} = E_t\left(\beta\,\frac{U_{c_{t+1}}}{U_{c_t}}\,R_{t+1}\right) = \beta\,\frac{U_{c_{t+1}}}{U_{c_t}}\,R_{t+1}-\varepsilon_{2_{t+1}}\ , \tag{24}$$

The expectation operator in (23) and (24) imply that the after-tax returns on capital and bonds (weighted by marginal utility) must be equals “on average”. The government can implement many paths of capital income tax rates and debt return to decentralize Ramsey allocations satisfying this ex-ante arbitrage condition. Thus, conditions (23) y (24) cannot be used to compute the paths of capital income tax rate and debt return.

From a computational point of view, the indeterminacy of fiscal policy implies that the ecuations (9), (13), (23) y (24), are not enough to calculate the optimal paths of $\{\tau_{k_t}, b_t, R_t, \varepsilon_{1t}, \varepsilon_{2t}\}_{t=1}^{T}$, given the Ramsey allocations, and where ε_{1t} y ε_{2t} are the expectation errors associated to the Euler conditions (23) and (24).

The indeterminacy is the same showed by Zhu (1992) and Chari, Christiano and Kehoe (1994) in a similar model, though they use shocks with discrete support. As we mentioned previously, Chari, Christiano and Kehoe (1994) identify optimal fiscal policy by restricting the debt return to be uncontingent on the state of nature, then ecuation (24) can be transformed into:

$$R_{t+1} = \frac{e^{D}}{E_t\beta\frac{U_{c_{t+1}}}{U_{c_t}}}\ , \tag{25}$$

allowing us to compute debt return, since, given the *Ramsey allocations*, the right hand side of (25) is known $\forall t$. In other words, expectation $E_t \beta (U_{c_{t+1}}/U_{c_t})$ is known from computing the *Ramsey allocations* (see appendix A). Therefore, optimal fiscal policy is computed under the assumption of uncontigent debt return. In other words, private agents know the debt return for next period with certainty. This ad-hoc identification condition is imposed on the stochastic behavior of debt return, to allow us to determine the optimal fiscal policy.

In contrast to the work of Chari, Christiano and Kehoe (1994), our analysis is characterized because we assume an identification constraint that picks one of the many paths of optimal fiscal policy, in which both the capital income tax rate and the debt return are contingent on the state of nature.

The ex-ante tax rate on capital income can be always computed because this variable is defined by theory. Following Chari, Christiano and Kehoe (1994), the ex-ante tax rate is defined as the ratio of the expected value of revenues from capital income taxation to the expected value of the net return of capital, both terms weighted by marginal substitution rate between consumption today and tomorrow:

$$\tau_{k_t}(ex\text{-}ante) = \frac{E_t \beta \frac{U_{c_{t+1}}}{U_{c_t}} \tau_{k_{t+1}} (F_{k_{t+1}} - \delta)}{E_t \beta \frac{U_{c_{t+1}}}{U_{c_t}} (F_{k_{t+1}} - \delta)} \quad , \tag{26}$$

So long as the ex-ante tax rate is a ratio of expected values, if we have continual support shocks, having approximations of these expectation terms is not trivial. However the solution method proposed allows us to evaluate this tax rate, as we describe in appendix A. In this appendix, we also demonstrate, from the solution method, that the ex-ante tax rate on capital income is zero for all $t \geq 1$ under logarithmic preferences, and fluctuates around zero for a risk aversion coefficient different than one.

3.2.1. Restrictions on the dynamic and stochastic behavior of debt path.

Linear system consisting of (9), (13), (23) y (24) represents the dynamic evolution of the policy variables, given the *Ramsey allocations*. Because of the inherent indeterminacy of the fiscal policy, there is a continuum of paths for variables $\{\tau_{kt}, R_t, b_t, \nu_{1t}, \nu_{2t}\}$ that decentralizes the optimal allocations and solves such a system. $\{\tau_{kt}, R_t, \nu_{1t}, \nu_{2t}\}$ evolves to a stochastic steady state, while the debt stock can follow an explosive path, being optimal and compatible with a stable and stationary equilibrium for the remaining variables.

When we restrict the debt path to not growing higher than the other variables in the economy, we avoid an unstable behavior of the debt. In such a case we are selecting one of the infinite paths of optimal fiscal policy. There is a large set of restrictions that enforces the stability of the debt path. For example the condition $b_t = b_{ss}\ \forall t$ ensures that the debt is not exploited, because it is constant over time. However we focus on other kind of restrictions that also ensure that the debt path is stable.

The first kind of restrictions consists of the debt path evolving according to the policy variables $(\tau_{w_t}, \tau_{k_t}, R_t)$ and being compatible with the government budget constraint. The debt path will be stable, because all optimal paths of $(\tau_{w_t}, \tau_{k_t}, R_t)$ from the infinite solutions are stable. We need to know how the debt path depends on the policy variables. From the government budget constraint,

$$e^{\rho} b_t = R_t b_{t-1} + G_t - \tau_{k_t}(r_t - \delta)k_{t-1} - \tau_{w_t} w_t n_t \ , \tag{27}$$

we linearize around the deterministic steady state[3], taking into account that optimal allocations are known at any moment:

$$b_t = \frac{R_{ss}}{e^{\rho}}(b_{t-1} - b_{ss}) + \frac{b_{ss}}{e^{\rho}}(R_t - R_{ss}) + \frac{G_t}{e^{\rho}} - \frac{\tau_{k_t}}{e^{\rho}}(r_t - \delta)k_{t-1} - \frac{\tau_{w_t}}{e^{\rho}} w_t n_t \ , \tag{28}$$

[3] Note that the deterministic steady state is determined, since :

$$\nu_{1_{ss}} = \nu_{2_{ss}} = 0\,,\ \tau_{k_{ss}} = 0\,,\ R_{ss} = 1/\beta\,,\ b_{ss} = \frac{1}{e^{\rho} - R_{ss}}\left(G_{ss} - \tau_{ss} w_{ss} b_{ss}\right).$$

The linear aproximation given by (28), implies that the debt path is stable, because $(R_{ss} > e^{\rho})$. The debt path is not exploited if we cancel the unstable path. In order to find the unstable path we linearize the dynamic system consisting of (9), (13), (23) and (24), and we take the eigenvector of the transition matrix associated to the unstable eigenvalue, as the path to cancel. For the selected parameterization the stability condition selected is:

$$b_t - b_{ss} = .15(R_t - R_{ss}) + .26(\tau_{w_t} - \tau_{w_{ss}}) + .04(\tau_{k_t} - \tau_{k_{ss}}) \ , \qquad (29)$$

that guarantees that b_t is not exploited and is compatible with both the budget constraints of the household and the government, and satisfies the transversality condition of private assets[4].

It must be noticed that condition (29) enforces not only stability of the debt path, but also imposes stationarity. Since debt return and tax rates are stationary (do not grow in the steady state), then b_t fluctuates around b_{ss}.

The economic interpretation of this condition is that the debt return will be stable whenever deviations of debt between the value at moment *t* and the steady state are due to variations either in debt return or in the tax rates, according to the parameters in (29). Thus, if $R_t > R_{ss}$, the debt stock increases, because more debt needs to be issued to repay the outstanding debt. Moreover when $b_t > b_{ss}$, higher tax rates are needed in order to repay outstanding debt.

The other kind of identification condition that we implement is an exogenous stochastic process for the debt path. This condition also guarantees that the transversality condition holds. The dynamic

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[4] Given the transversality condition of the competitive equilibrium:

$$\lim_{j \to \infty} E_{t-1+j} \hat{\beta}^{t+j} U_{c_{t+j}}(k_{t+j} + b_{t+j}) = 0 \ ,$$

constraint (29) implies that $\lim_{j \to \infty} E_{t-1+j} \hat{\beta}^{t+j} U_{c_{t+j}} b_{t+j} = 0$, since the path b_{t+j} is stable, that is the growth rate of b_{t+j} is lower than that of $\hat{\beta}^{t+j} U_{c_{t+j}}$, and this last growth rate is lower than the gross debt return (*R*). The Ramsey allocations solve the optimal path of capital, so the transversality condition for capital in the Ramsey problem holds ($\lim_{j \to \infty} E_{t-1+j} \hat{\beta}^{t+j} U_{c_{t+j}} k_{t+j} = 0$, under logarithmic preferences). Thus, the transversality condition of the competitive equilibrium holds.

system for policies together with the stochastic path for the debt allow us to compute the optimal fiscal policy. The constraint is motivated in the next subsection.

3.2.2. *Exogenous debt path*

In order to asses to what extent imposing a stationary path for the debt can bound the properties of the *Ramsey policies*, we use an alternative identification condition: we evaluate exogenous stochastic processes for the debt path, whether stationary or not stationary, though such processes do not violate the transversality condition for the debt path. We assume first order autorregresive stochastic processes:

$$b_t = \bar{b}(1-\phi)+\phi b_{t-1}+\varepsilon_t^b \ , \quad \varepsilon_t^b \underset{iid}{\sim} N(0,\sigma_b) \ , \tag{30}$$

for several degrees of persistence (ϕ). Ludvigson (1996) settled these exogenous processes for debt paths in order to analyze how the degree of persistence affects the competitive decisions of private agents. This kind of constraint give us a large set of debt processes to study its implications on the properties of the *Ramsey policies*.

3.2.3. *Sunspot equilibria*

The last set of constraints we impose is an exogenous expectation mechanism for private agents, fulfilling the government budget constraint and computing the debt path residually from such a constraint. The indeterminacy of Ramsey policies emerges from the stochastic framework, and we cannot compute the Ramsey policies and the expectation errors of the Euler conditions of the competitive equilibrium. A way to have determined optimal policies is to take one of the expectation errors (η_{1t} o η_{2t}) exogenously, keeping the rational expectation assumption (the expectation error follows an i.i.d. process). Then it is possible that the prophecies of agents that do not depend on the "fundamentals" of the model, can be useful to explain economic behavior that the model structure cannot. In this way it is interesting to characterize the optimal policies compatible with the optimal allocations under this kind of assumption.

Therefore the main goal of the paper is to asses to what extent the optimal fiscal policy stochastic properties change under different assumptions about stability and persistence of the debt path or about the structure of an expectation error of the competitive equilibrium (*sunspot equilibria*). The analysis is carried out with parameter values calibrated by Chari, Christiano y Kehoe (1994) for the U.S. economy, and is presented in the next section along with simulation results.

4. Simulation results

Results from stochastic simulation of the different versions of the model are presented in this section. Each model has been simulated 100 times each with a length of 1200 periods. Calibrated parameters and initial conditions for capital stock, debt and capital income tax rates are the same as those discussed by Chari, Christiano and Kehoe (1994) when calibrating the model with U.S. economy data. Table 1 shows the calibrated parameters. The baseline model considers logarithmic preferences ($\mathrm{F} = 1$). Other versions of the model are also simulated: a high risk aversion model ($\mathrm{F} = 9$), a model with i.i.d. shock, and finally a model without shocks to government consumption is also simulated.

As was shown in section 3, the implementability constraint is included in the maximand to compute the *Ramsey allocations*. Thus, the objective function is different at $t = 0$ from $t\$1$. This question also affects to the *Ramsey policies*,which depend on the *Ramsey allocations*, thus the optimal policy rules at $t = 0$ ($\mathrm{J}_{w_0}, R_0, \mathrm{J}_{k_0}, b_0$), are different from those onwards ($\mathrm{J}_{w_t}, R_t, \mathrm{J}_{k_t}, b_t$ for $t\$1$). Under logarithmic preferences, the initial labor income tax rate is -50.9%, with -17.8% being the rate in the model with high risk aversion. With regard to the capital income tax rate at $t = 1$, goes from 306.3% with the logarithmic utility function, to 669.4% for the high risk aversion case. These values of tax rates are closed to those reported by Chari, Christiano and Kehoe (1994).

To compute the properties of the stochastic simulation, the first 200 periods are dropped. That ensures the stationarity of the statistics of both the optimal allocations and the optimal policies. Table 2 reports the statistics when the stability condition is used as an identification restriction, including both contingent capital income tax rate and debt return.

Analyzing taxation statistics, we can see that the average ex-post capital income tax rate is zero and constant over the business cyle (standard deviation of simulated tax rate is zero). Moreover, the optimal capital tax rate is uncorrelated with both productivity shock and government consumption, and it exhibits no persistence. The different stochastic procesess implemented for shocks and risk aversion do not change the properties described above. These results are quite different from those of Chari, Christiano and Kehoe (1994), which report a non-zero average and a very volatile ex-post capital income tax rate, correlated with technology shock and government consumption. The difference in results comes from the available fiscal tools for the government to absorb the economy shocks. Chari, Christiano and Kehoe (1994) restrict the debt return to be uncontingent, therefore it is known with certainty in the previous period. Thus, the optimal capital income tax rate becomes very volatile because the government cannot use debt return as a shock absorber, and capital income taxation contingent on the state of nature must be used. However, the identification condition we use allows the government to set both the debt return and the capital income tax rate as state-contingent , that is, government can use both policy variables as shock absorbers. Given the distorting nature of capital income taxation, the government finds it optimal to set debt return as a shock absorber, keeping a constant capital income tax rate over the business cycle.

With regard to the stochastic properties of labor income taxation, there are not differences with Chari, Christiano and Kehoe's (1994) results. The reason is that we also use the relationship of marginal substitution rate of consumption-leisure with after tax wages given in (22), to compute the labor income tax rate and that the differences with our paper do not affect computation of the *Ramsey allocations*.

Simulation results have been obtained by bounding the debt path to be stationary, and as we said above, this does not affect our computation of the labor income tax rate, but does affect our computation of optimal capital income taxation. The reason is that an identification constraint is needed to compute the capital income tax rate, so different kinds of identification conditions may change the optimal behavior of this tax rate. An interesting issue to adress is to what extent alternative identification conditions change the properties of contingent capital income taxation. In order to answer this question, we use an exogenous stochastic process for the debt path as an identification constraint, given in expression (30).

Parameter N defines the stationarity and stability of the debt path. Unconditional mean ($\bar{b}$) is selected to mimic the steady state value of debt, when it is decided endogenously.

Table 3 summarizes simulation results under several stochastic processes for the debt path. The results point out that, independently of persistence, ex-post capital income tax rate properties do not differ from those reported when the stability condition is used to identify optimal fiscal policy. This result indicates that imposing a stability condition does not bound the optimal properties shown by contingent capital income taxation.

Another condition that allow us to obtain the optimal contingent fiscal policy consists of assumptions about the expectation errors. In short, we would be replacing the initial identification constraint on debt path stabilitiy for another that affects the expectation mechanism. The question adressed is whether the alternative identification assumption changes characteristics of optimal capital income tax rate. In order to do this, *Ramsey policies* are simulated again, assuming that the expectation error $<_{2t}$ associated to the Euler condition of bonds (24), follows a white noise stochastic process, with different standard deviation sizes. Results reported in table 4 confirm the stochastic properties of optimal capital income taxation obtained under the previous identification conditions. An interesting simulation result is that the endogenous expectation error is uncorrelated with the exogenous error, and its size, measured by standard deviation, does not depend on the standard deviation assumed for the exogenous error.

5. Conclusions

Such as it has been established in previous studies, optimal combination of labor and capital income taxation and government debt returns exhibits indeterminacy when this question is analyzed in a stochastic setting. There are infinite combinations of capital income tax rates and debt return, both contingent on the state of nature, supporting the optimal allocations of consumption, hours and capital. The goal of this paper is to introduce identification constraints to select one among the infinite optimal fiscal policies and characterize its cyclical properties. Identification conditions are of two types: constraints on the stability of the debt path and constraints on the expectations rule.

When we impose a stability condition that bounds the growth of the debt path, we are limiting the space solutions, eliminating the unstable path of the dynamic system that describes the optimal fiscal policy. The results point out that the ex-post capital income tax rate is zero on average and constant, uncorrelated with government consumption and with the productivity shock and following an i.i.d. stochastic process.

In order to asses how these properties of optimal fiscal policy depend on the identification constraint, we replace this condition with an exogenous stochastic path for the debt. We allow for stationary and non-stationary processes varying the persistence parameter to analyze how this parameter influences the stochastic behavior of optimal fiscal policy. Our results on capital income taxation are robusts to this change in the identification condition.

With the second type of identification conditions, we analyze optimal fiscal policy by introducing an exogenous stochastic path for one of the expectation errors of the Euler conditions of the competitive equilibrium (following the *sunspot equilibria* literature). Once again, the properties of optimal capital taxation remain unaltered again.

Summing up, the main result of this paper indicates that the ex-post capital income taxation is zero and constant over the business cycle. This character remains unaltered with the other identification conditions implemented. This analysis suggest that in the optimum, the government uses debt return as a shock absorber, keeping the capital income tax rate constant. This result is quite different from that of Chari, Christiano and Kehoe (1994), who assuming uncontingent debt return obtain a very volatile capital income tax rate.

In accordance with the Chari, Christiano and Kehoe (1994) results, the optimal labor income tax rate hardly fluctuates and is highly persistent (first order autocorrelation is .86), with low standard deviation. Furthermore, the standard deviation of the statistics is very low, suggesting high precission when estimating these moments.

Extensions of this paper lead us in two directions. The first would be to extend the analysis to stochastic endogenous growth models. Jones, Manuelli and Rossi (1993) obtain, in a deterministic framework, different results from Chamley (1986) for long run capital income taxation. It would be interesting to ask whether the indeterminacy issue arise in an stochastic endogenous growth model and whether the properties of the optimal fiscal policies under different identification conditions are maintained. The second extension is to assess how optimal fiscal policy changes when we introduce money into the model.

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Appendix A. Solution method for Ramsey allocations.

For $t \geq 1$, the equations described below, (A.1)-(A.5), along with expresions (6), (10), (11) and (16) represent the set of conditions that allocations $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ must fulfill, with the exogenous $\{z_t, g_t\}_{t=0}^{\infty}$, given realizations for the innovations of the structural shocks (ε_{z_t}, ε_{g_t}):

$$-\frac{W_{c_t}}{W_{n_t}} = \frac{1}{F_{n_t}} \ , \tag{A.1}$$

$$e^{D}\left[1+\lambda\left(\frac{U_{cc_t}}{U_{c_t}}c_t+1+\frac{U_{cn_t}}{U_{c_t}}n_t\right)\right] = X_{1t}+X_{2t}+X_{3t} \ , \tag{A.2}$$

$$X_{1,t-1} = \hat{\beta}\frac{U_{c_t}}{U_{c_{t-1}}}(F_{k_t}-\delta)-\varepsilon_{1t} \ , \tag{A.3}$$

$$X_{2,t-1} = \hat{\beta}\frac{U_{c_t}}{U_{c_{t-1}}}-\varepsilon_{2t} \ , \tag{A.4}$$

$$X_{3,t-1} = \hat{\beta}\lambda\left(\frac{U_{cc_t}}{U_{c_{t-1}}}c_t+\frac{U_{c_t}}{U_{c_{t-1}}}+\frac{U_{cn_t}}{U_{c_{t-1}}}n_t\right)(1-\delta+F_{k_{t-1}})-\varepsilon_{3t} \ , \tag{A.5}$$

where X_{1t}, X_{2t}, X_{3t} in expresions (A.2) to (A.5), represents expectations that arise when the global conditional expectation in Euler condition of the problem defined by (15)-(17) is partitioned. The expectation term decomposition will be very useful when computing the ex-ante capital income tax rate later. $\varepsilon_{1t}, \varepsilon_{2t}, \varepsilon_{3t}$ represent the forecasting errors associated with the expectations. Summing up, we have twelve variables $\{c_t, n_t, k_t, G_t, g_t, z_t, X_{1t}, X_{2t}, X_{3t}, \varepsilon_{1t}, \varepsilon_{2t}, \varepsilon_{3t}\}$ to be solved and nine equations (A.1)-(A.5) , (6), (10), (11) and (16) each period. To compute all the system variables at each period, we need three aditional conditions, which will be conditions that cancel the non-converging subspace to the steady state.

An approximation to those conditions that cancel such subspaces emerges from the first order approximation of the previously mentioned system of nine equations around the steady state:

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t \ , \tag{A.6}$$

where matrices (9×9) Γ_0, Γ_1 contain the partial derivatives of each equation (A.1)-(A.5), (6), (10), (11) and (16) with respect to each variable $(c_t, n_t, k_t, z_t, G_t, g_t, X_{1t}, X_{2t}, X_{3t})$, evaluated in the deterministic steady state.

Vector y_t contains steady state deviations from the deterministic steady state $(c_t - c_{ss}, n_t - n_{ss}, k_t - k_{ss}, z_t, G_t - G, g_t, X_{1t} - X_{1,ss}, X_{2t} - X_{2,ss}, X_{3t} - X_{3,ss})$. Innovations and expectation errors are contained in vectors $\varepsilon_t = (\varepsilon_{zt}, \varepsilon_{gt})'$, $\eta_t = (\eta_{1t}, \eta_{2t}, \eta_{3t})'$. Matrices Ψ, Π are:

$$\Psi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A.7}$$

Equations that describe the dynamic stochastic path of allocations can simplified with logarithmic preferences, in particular it can be shown that if risk aversion is 1, then $X_{3_{t-1}} = 0 \ \forall t$, therefore $\eta_{3t} = 0 \ \forall t$.

The system consisting of (A.1)-(A.5), (6), (10), (11) and (16) can be simplified under this class of preferences, eliminating equation (A.5) and rewriting equation (A.2) as:

$$e^{D} \,'\, X_{1t} \% X_{2t}. \tag{A.8}$$

In such a case, we have two expectation errors, and to identify all variables in the dynamic system two stability conditions are required to cancel the non-converging subespace to the stochastic stationary equilibrium. Appendix B reports the numerical stability conditions for each type of preferences.

For a risk aversion strictly larger than one, the sytem linearization comes from (A.6). If $'_0$ were not singular, it would be:

$$y_t \,'\, '^{\&1}_0 \,'_1 y_{t\&1} \% '^{\&1}_0 Q g_t \% '^{\&1}_0 A 0_t \;, \tag{A.9}$$

that is,

$$y_t \,'\, \sum_{s'0}^{4} [M 7 M^{\&1}]^s \,'^{\&1}_0 [Q g_{t\&s} \% A 0_{t\&s}] \;, \tag{A.10}$$

where $M 7 M^{\&1} \,'\, '^{\&1}_0 \,'_1$ is Jordan decomposition.

Eigenvalues of $'^{\&1}_0 \,'_1$ determine the speed of convergence to the steady state and have some implications on the fulfilling of transversality conditions as well as for the identification of expectation errors. Thus, a set of sufficient conditions to have a stationary solution is for the economy to evolve into the subspace generated by the eingenvectors associated to the stable eigenvalues (inside the unit radius circle), and that the processes g_t , 0_t, to be stationary.

However, given the partition of the Euler condition in the Ramsey problem, it is obvious that $'_0$ is non invertible[5].

It can be shown that $'_0$, is of order $(9{\times}9)$, and it has a maximum range of 7 when $F...1$, while

[5] Notice that if the expectation were not partitioned, there would not be any problem of singularity for $'_0$ to compute the stability conditions. However, as we see later, this partition will be extremely useful to evaluate the ex-ante capital income tax rate.

with $\mathrm{F}\,'\,1$ the matrix is of order $(8{\times}8)$, with a range of 7 as maximum. Since matrix $'_{0}$ is singular, it is necessary to compute a QZ decomposition to obtain generalized eigenvalues and eigenvectors, see Sims (1998).

For any pair of square matrices $('_{0}, '_{1})$ there exist orthonormal matrices Q, Z, $(QQ' = ZZ' = I)$ and upper triangular matrices 7, S such that $'_{0} = Q'7Z'$ $'_{1} = Q'\,\mathrm{S}Z'$.

Premultiplying the system (A.6) by Q and replacing $Z^{)}y_{t}$ with u_{t}, we obtain:

$$7u_{t}\,'\,\mathrm{S}u_{t\&1}\%Q(\mathrm{Qg}_{t}\%\mathrm{A0}_{t}) \ . \tag{A.11}$$

We can rearrange matrices 7, S, in order to partition (A.11) in such a way that the below block corresponds to the equations associated to the unstable eigenvalues (those larger than $\hat{\$}^{\&1/2}$):

$$\begin{bmatrix} 7_{11} & 7_{12} \\ \mathbf{0} & 7_{22} \end{bmatrix} \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} \,'\, \begin{bmatrix} \mathrm{S}_{11} & \mathrm{S}_{12} \\ \mathbf{0} & \mathrm{S}_{22} \end{bmatrix} \begin{bmatrix} u_{1,t\&1} \\ u_{2,t\&1} \end{bmatrix} \% \begin{bmatrix} Q_{1\cdot} \\ Q_{2\cdot} \end{bmatrix} (\mathrm{Qg}_{t}\%\mathrm{A0}_{t}) \ . \tag{A.12}$$

A zero elemente in the diagonal of matrix 7 implies some identification lack in the system; we have two zero elements in the diagonal of 7, with $\mathrm{F}\ldots 1$. Since the elements in the diagonal of S in the same position are not zero, we have two infinite eigenvalues (so larger than $\hat{\$}^{\&1/2}$) that solve the system identification, that is, two stability condition emerge, along with the remaining finite eigenvalue larger than $\$^{\&1/2}$ (typical of saddle point solutions). This analysis allow us to identify the three expectation errors[6]. Therefore, bounding the space of solutions imply canceling the unstable paths associated with the unstable eigenvalues, that is:

$$u_{2t}\,'\,Z_{2\cdot}^{)}\,y_{t}\,'\,0 \ , \quad \text{œ}t \ , \tag{A.13}$$

that provides an approximated structure of relationships between the expectation errors and the

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[6] In the case of $\mathrm{F} = 1$, there are two eigenvalues larger than $\hat{\$}^{\&1/2}$: one is finite and the remaining is infinite; then the two expectation errors are identified.

innovations of the structural shocks:

$$Q_{2.}\,\mathrm{Q}\,\mathrm{g}_t \;'\; \& Q_{2.}\,\mathrm{A}\,0_t \;. \tag{A.14}$$

Appendix C reports the numerical expresions of (A.14) for the different versions of the model.

Since we have two infinite eigenvalues and another finite but unstable eigenvalue, expression (A.13) is a set of three equations that cancels the unstable subspace. As was pointed out, the optimal allocations follow different rules at $t\,'\,0$ than from then onwards, so the computational algorithm is different as we describe above.

At period $t\,'\,0$, given J_{k_0} and $R_0 b_{\&1}, z_{\&1}$, $g_{\&1}$ and the realization of the innovations of structural shocks (g_{z_0}, g_{g_0}), from (6), (10) and (11) particularized at moment 0, the three stability conditions and the Euler conditions of the Ramsey problem at $t\,'\,0$ are[7]:

$$\frac{W_{c_0}\&8U_{cc_0}[R_0 b_{\&1}\%\big(1\%(F_{k_0}\&*)(1\&\mathrm{J}_{k_0})\big)k_{\&1}]}{W_{n_0}\&8U_{cn_0}[R_0 b_{\&1}\%\big(1\%(F_{k_0}\&*)(1\&\mathrm{J}_{k_0})\big)k_{\&1}]\&8U_{c_0}F_{nk_0}(1\&\mathrm{J}_{k_0})k_0}\;'\;\&\frac{1}{F_{n_0}}\;, \tag{A.15}$$

$$\frac{e^{\mathrm{D}}}{U_{c_0}}\Big\{W_0\&8U_{cc_0}\Big[R_0 b_{\&1}\%\Big(1\%(r_0\&*)(1\&\mathrm{J}_{k_0})\Big)k_{\&1}\Big]\Big\}\;'\;X_{10}\%X_{20}\%X_{30}\;, \tag{A.16}$$

we can compute $6c_0,\ n_0,\ z_0,\ g_0,\ G_0,\ X_{10},\ X_{20},\ X_{30}>$. Equation (A.16) corresponds with (A.2) at moment $t\,'\,0$. Finally, the capital stock at $t\,'\,0$ is computed from the aggregate resources constraint (16).

At $t\,'\,1$, from $6\,c_0,\ n_0,\ k_0,\ z_0,\ g_0,\ G_0,\ X_{10},\ X_{20},\ X_{30}>$, the realization of structural innovations $\mathrm{g}_{z_1}, \mathrm{g}_{g_1}$, and using (A.1), (A.2), (6), (10), (11) and the three stability conditions,

[7] As was shown in last subsecction, we assume that the capital income tax rate and the debt outstanding, J_{k_0} and $R_0 b_{\&1}$, are fixed at the initial period, to the level calibrated by Chari, Christiano and Kehoe (1994).

$\{c_1, n_1, z_1, g_1, G_1, X_{11}, X_{21}, X_{31}\}$ are computed. Capital stock at period 1, emerges from (16). The solution is computed recursively for the periods from here on.

Once the allocation paths are computed, the expectation error paths associated to the expectational terms, are obtained from (A.3)-(A.5)[8].

From the described solution method, we obtain allocations of $\{c_t, n_t, k_t\}_{t=0}^{T}$ as a function of λ. As was pointed out, we iterate in λ until finding those allocations that fulfill the implementability constraint, given the convergence criterion showed in subsecction 3.1.

As a by-product of the solution method, we can demonstrate that the ex-ante capital income tax rate is zero for all $t \geq 1$ when risk aversion is one, and it fluctuates around zero when risk aversion is different than one.

Proof: In the definition of the ex-ante capital income tax rate, given by (26), we can see that the denominator corresponds with the expectation X_{1t}, defined by (A.3).

The numerator of (26) can be obtained from the competitive equilibrium conditions (that must be fulfilled by the previously computed Ramsey allocations). In particular, partitioning the expectation term of the Euler condition of capital of the problem defined by (12)-(13) and comparing with (A.3) and (A.4), we have:

$$\begin{aligned} e^{D} &= E_t\left[\frac{U_{c_{t+1}}}{U_{c_t}}\right] + E_t\left[\frac{U_{c_{t+1}}}{U_{c_t}}(F_{k_{t+1}}-\delta)\right] - E_t\left[\frac{U_{c_{t+1}}}{U_{c_t}}(F_{k_{t+1}}-\delta)\tau_{k_{t+1}}\right] \\ &= X_{2t} + X_{1t} - E_t\left[\frac{U_{c_{t+1}}}{U_{c_t}}(F_{k_{t+1}}-\delta)\tau_{k_{t+1}}\right]. \end{aligned} \tag{A.17}$$

Therefore:

$$E_t\left[\frac{U_{c_{t+1}}}{U_{c_t}}(F_{k_{t+1}}-\delta)\tau_{k_{t+1}}\right] = X_{1t} + X_{2t} - e^{D}, \tag{A.18}$$

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[8] Notice that(A.6) system equations are the only approximation in the solution method.

identifying terms with (A.2):

$$E_t \beta \frac{U_{c_{t+1}}}{U_{c_t}} (F_{k_{t+1}} - \delta) \tau_{k_{t+1}} = - X_{3t} + e^{\rho} \lambda \left(\frac{U_{cc_t}}{U_{c_t}} c_t + 1 + \frac{U_{cn_t}}{U_{c_t}} n_t \right) \quad . \tag{A.19}$$

With $\sigma = 1$ can be shown that $X_{3t} = 0 \ \forall t$. Moreover, it is clear that under logarithmic preferences, the term inside brackets is zero. Therefore, the ex-ante capital income tax rate zero for all $t \geq 1$.

When $\sigma \neq 1$, the ex-ante capital income tax rate is zero on average, because in the deterministic steady state we have from the Euler condition of the Ramsey problem:

$$\frac{e^{\rho}}{\beta} = 1 - \delta + F_{k_{ss}} \ , \tag{A.20}$$

and the Euler condition for capital stock in the competitive equilibrium:

$$\frac{e^{\rho}}{\beta} = 1 + (F_{k_{ss}} - \delta)(1 - \tau_{k_{ss}}) \ , \tag{A.21}$$

then, $\tau_{k_{ss}} = 0$. □

Appendix B. Numerical expresions of stability conditions.

This appendix summarizes the numerical expressions of (A.13), that is, the numerical stability conditions, given calibrated values of parameters.

The baseline model (logarithmic preferences), is simulated under alternative stochastic processes for productivity shocks and public consumption shocks. When both shocks follow a first order autorregresive process, the numerical stability conditions are:

$$\begin{pmatrix} -.8694 & 0 & .1275 & .0212 & 0 & -.0238 & .1782 & .4418 & 0 \\ -.4271 & 0 & -.0302 & .1044 & 0 & 0 & .1283 & -.8884 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} y_t = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \ , \tag{B.1}$$

with y_t defined in appendix A.

When both shocks follow i.i.d. processes, the numerical expresion of (A.13) is:

$$\begin{pmatrix} .8708 & 0 & -.1275 & 0 & 0 & 0 & -.1785 & -.4400 & 0 \\ .4294 & 0 & .0304 & 0 & 0 & 0 & -.1290 & .8933 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} y_t = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \tag{B.2}$$

If the public consumption shock is eliminated and the autorregresive structure is maintained for productivity shock, the stability conditions are:

$$\begin{pmatrix} .8708 & 0 & -.1275 & 0 & 0 & -.1785 & -.4400 & 0 \\ .4294 & 0 & .0304 & 0 & 0 & -.1290 & .8933 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} y_t^{*} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , \tag{B.3}$$

with y_t^{*} defined as in appendix A and eliminating the sixth element of such a vector.

Under high risk aversion ($\sigma = 9$), if both shock follow an autorregresive process, (A.13) is:

$$\begin{pmatrix} -.496 & .754 & .1120 & -.0174 & 0 & -.03 & .0228 & .347 & .2277 \\ .831 & .349 & -.0495 & -.1036 & 0 & 0 & -.0537 & .209 & .3574 \\ 0 & -.377 & .1097 & -.0693 & 0 & 0 & -.0728 & .898 & -.1727 \end{pmatrix} y_t = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \tag{B.4}$$

In (B.1), (B.2) y (B.3) we can see that last row of the matrix which premultiplies the steady state deviations vector is the same. So, under logarithmic preferences, this row is independent from the stochastic structure of the shocks. This numerical form also implies that expectation X_{3t} is constant over time, therefore the error associated with this expectation (η_{3t}) is zero $\forall t$, as it was shown in appendix A.

Appendix C. Numerical relations between structural innovations and expectation errors.

Numerical expresions of (A.14), that is the approximated relations between expectation errors and innovations of structural shocks, given the calibrated values of parameter, are detailed in this appendix.

The baseline model (logarithmic preferences) is simulated under alternative processes for both productivity and consumption shocks. As was pointed out before, with this kind of preferences, the expectation error $\eta_{3t} = 0 \ \forall t$. When both shocks follow an autorregresive stochastic process, numerical expresion of (A.14), is:

$$\begin{aligned} \eta_{1t} &= -.1105\varepsilon_{z_t} + .3731\varepsilon_{g_t} \ , \\ \eta_{2t} &= -.0159\varepsilon_{z_t} - .0912\varepsilon_{g_t} \ . \end{aligned} \tag{C.1}$$

(C.1.) exhibits a positive correlation of the expectation error η_{1t} with the innovation of productivity shock, and a positive correlation with the innovation of public consumption shocks as well. However, η_{2t} is negatively correlated with both shock innovations.

Simulating the model with i.i.d. processes for both stochastic shocks, the sign of correlations keep unaltered, although with different intensity, as we can see in:

$$\begin{aligned} \eta_{1t} &= -.0284\varepsilon_{z_t} + .2066\varepsilon_{g_t} \ , \\ \eta_{2t} &= -.0041\varepsilon_{z_t} - .1153\varepsilon_{g_t} \ . \end{aligned} \tag{C.2}$$

When the shock of public consumption is eliminated and the productivity shock maintains an autoregresive process, the expectation errors and the structural innovations appear related by:

$$\begin{aligned} \eta_{1t} &= -.4060\varepsilon_{z_t} \ , \\ \eta_{2t} &= -.0566\varepsilon_{z_t} \ , \end{aligned} \tag{C.3}$$

wich implies that the negative correlation between the expectation errors η_1 and η_2 with productivity shock innovation remains but wiht higher intensity.

When the model is simulated with high risk aversion, keeping the autorregresive structure for both shocks, the expectation error η_{3t} is not necessarily zero. The numerical expression of relations between expectation errors and structural innovations is given by:

$$\eta_{1t} = -.4319\,\varepsilon_{z_t} + 1.1867\,\varepsilon_{g_t}\ ,$$

$$\eta_{2t} = -.0427\,\varepsilon_{z_t} - .0302\,\varepsilon_{g_t}\ , \qquad \text{(C.4)}$$

$$\eta_{3t} = -.0399\,\varepsilon_{z_t} - .1618\,\varepsilon_{g_t}\ .$$

This expresion indicates that the expectation errors η_1 y η_2, keep the sign of correlation with structural innovation, while error η_3 is not more independent of structural innovations, being now negatively correlated with both.

Table 1. Baseline parameters.

Preferences:	
Discount rate ($)	.98
Risk aversion (F)	1
Preference for leisure (2)	.75
Technology:	
Output elasticity of labor (")	.66
Growht rate (D)	.016
Capital depreciation rate (*)	.07
Stochastic process of public consumption:	
Steady state public consumption (G)	.07
Autocorrelation of public consumption shock (D_g)	.89
Standard deviation of innovation of public consumption shock (F_g)	.07
Stochastic process of productivity shock:	
Autocorrelation of productivity shock (D_g)	.81
Standard deviation of innovation of productivity shock (F_g)	.04
Initial conditions:	
Outstanding debt ($R_0 b_{\&1}$)	.20
Capital stock ($k_{\&1}$)	1.05
Capital income tax rate (J_{k_0})	27.1%

Table 2. Stochastic simulation under stable behavior of the debt path. Properties of optimal tax rates. Statistics computed are means of 100 simulations of 1200 periods, where first 200 periods are dropped. Standard deviation of statistics is in parenthesis. NA indicates that the corresponding statistic is not well defined. Means and standard deviations are in percentage terms.

			Alternative stochastic processes for shocks	
	Baseline model	High risk aversion	Only technology shock	I.I.D.
	Labor income tax rate			
Mean	25.198 (.019)	22.588 (.012)	25.191 (.004)	25.198 (.002)
Standard deviation	.190 (.010)	.096 (.005)	.128 (.006)	.149 (.004)
Autocorrelation	.800 (.021)	.860 (.015)	.688 (.025)	-.069 (.025)
Correlation with public consumption	.731 (.033)	-.813 (.030)	NA	NA
Correlation with technology shock	.433 (.064)	-.468 (.067)	.541 (.038)	.929 (.007)
	Ex-post capital income tax rate			
Mean	.000 (.001)	.000 (.000)	.000 (.000)	.000 (.000)
Standard deviation	.002 (.022)	.000 (.001)	.000 (.000)	.000 (.001)
Autocorrelation	-.003 (.029)	-.002 (.015)	-.005 (.038)	-.002 (.014)
Correlation with public consumption	.000 (.047)	.008 (.037)	NA	-.005 (.035)
Correlation with technology shock	-.003 (.029)	.001 (.033)	.002 (.042)	-.001 (.021)

Table 3. Stochastic simulation under exogenous processes for the debt path. Properties of optimal capital income tax rate. Statistics computed are means of 100 simulations of 1200 periods, where first 200 periods are dropped. Standard deviation of statistics is in parenthesis. Means and standard deviations are in percentage terms. The exogenous stochastic path of the debt is: $b_t = \bar{b}(1-\rho)+\rho\, b_{t-1}+\varepsilon_t^b \qquad \varepsilon_t^b \sim N(0\ ,\sigma_b\ )$

	σ_b = .05					σ_b = .5				
	$\rho = 1.025$	$\rho = 1.01$	$\rho = 1$	$\rho = .95$	$\rho = 0$	$\rho = 1.025$	$\rho = 1.01$	$\rho = 1$	$\rho = .95$	$\rho = 0$
Mean	.000 (.000)	.000 (.000)	.000 (.000)	.000 (.002)	.000 (.000)	.000 (.000)	.000 (.000)	.000 (.000)	.000 (.001)	.000 (.002)
Standard deviation	.000 (.000)	.000 (.001)	.001 (.005)	.009 (.062)	.002 (.005)	.000 (.001)	.000 (.001)	.003 (.009)	.007 (.017)	.021 (.055)
Autocorrelation	-.008 (.036)	.001 (.077)	-.003 (.050)	.000 (.013)	.002 (.025)	-.006 (.079)	-.009 (.093)	.003 (.055)	-.004 (.059)	.014 (.067)
Correlation with public consumption	.002 (.052)	.009 (.060)	-.001 (.029)	.002 (.032)	-.001 (.032)	.000 (.059)	.009 (.057)	.001 (.033)	.003 (.032)	.000 (.034)
Correlation with technology shock	.004 (.057)	-.005 (.056)	.001 (.031)	-.002 (.035)	-.001 (.030)	.000 (.053)	.000 (.050)	-.005 (.032)	.006 (.027)	.001 (.029)

Table 4. Sunspot equilibria: exogenous expectation error in the Euler condition of debt. Stochastic simulation under different sizes of standard deviation of expectation error. Properties of optimal capital income tax rate and endogenous expectation error. Statistics computed are means of 100 simulations of 1200 periods, where first 200 periods are dropped. Standard deviation of statistics is in parenthesis. Means and standard deviations are in percentage terms.

	Ex-post capital income tax rate		
	F error = .5	F error = .005	F error = .00005
Mean	.000 (.000)	.000 (.000)	.000 (.000)
Standard deviation	.000 (.000)	.000 (.000)	.000 (.000)
Autocorrelation	-.003 (.013)	.002 (.025)	-.002 (.030)
Correlation with public consumption	.000 (.034)	.001 (.033)	-.002 (.029)
Correlation with technology shock	-.001 (.029)	.001 (.030)	-.003 (.030)
	Endogenous expectation error(optimality condition of capital)		
	F error = .5	F error = .005	F error = .000005
Standard deviation	.014 (.000)	.015 (.000)	.014 (.000)
Correlation with exogenous expectation error	.004 (.034)	.002 (.029)	-.002 (.033)