Patterns of consumption in discrete choice models with asymmetric interactions

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We study the role of asymmetric interactions in the consumption behaviour of a network of heterogeneous agents. We first analyze a model (Model I) where the interactions among agents are uniquely specified by their "social distance". In Models II and III we consider the opposite extreme, and assume that the strength of interactions among agents depend on a rather large number of underlying parameters. In this case, the strength of interactions can be modeled as a random variable, with positive and negative values, in analogy with spin glass models in statistical mechanics. The dynamical properties of all three models are explored, by numerical simulations, using three different evolution algoriths with: *parallel, sequencial* and *random-sequencial* updating rules. We focus on the long-time behaviour of the system wich, given the asymmetric nature of the interactions, can either converge into a fixed point or a periodic attractor. An interesting feauture of model II and III is the precence of very long transients before the system settles into an attractor and a very strong sensitivity to the initial conditions, as revealed by the damage spreading analysis.

I. INTRODUCTION

A great body of research has been devoted to the effects that direct interactions among consumers or firms have on macroeconomic variables (Aoki 1996, Brock 1995, Brock and Durlauf 1995, and references therein). Direct interactions among economic agents, usually refer to as social interactions (as opposed to market mediated interactions) are meant to capture how the decision of each individual is influenced by the choice of others in his reference group.

Statistical mechanics methods have been proven powerful to study the evolution and steady state properties of heterogeneous interacting populations: it can provide a framework for understanding how interdependencies can lead to the emergence of interesting and rich aggregate behaviour, where multiple equilibria may occur in the absence of any exhogenous coordination mechanism. In socioeconomic applications these models have been applied in the situation where individuals face a binary choice and where interactions are pairwise. Examples include the study of social networks, asset price behaviour, business cycles, and adoption of innovation technologies by firms (Blume 1993, 1995, Kirman 1997, Brock and Durlauf 1997).

Different alternatives have been considered in the social utility literature: global interactions (Brock and Durlauf 1995, Aoki 1995), where each individual tends to conform to the average behaviour of the entire population, as well local interactions, where each individual has an incentive to conform to a specified group of neighbours (Föllmer 1974, Blume 1995, Morris 2000) have been modeled. A stochastic interaction structure has also been considered. This has been implemented by considering, either fixed exogenously determined random communication links between any pair of agents or, by taking time-dependent links and letting the neighbouring composition evolve in a self-organized way (Benabou 1996, Durlauf 1996). In this case, agents would be able to form new alliances according to some fitness maximization scheme. Moreover, hetereogenous couplings can been introduced to account for the different strength in the interaction between different pairs. Also models have been introduced (Kirman 1997, Durlauf 1997, Cowan et al 1999), in which the interactions among agents depend on the agents' "distance" (such as, for example, differences in wealth).

In much of this social interaction literature though, the attention has been mainly focused on the case of positive, pairwise symmetric, spillover, i.e. the case where the payoff of a particular action increases when others behave similarly. In this contest it has been shown that the evolution is diffusive: even in the case of heterogenous agents, social interactions create conformity in behaviour or polarized group behaviour without relying on the presence of correlated characteristics amomg members of the same group. While models with positive, symmetric social interactions, which are analogous to ferromagnetic Ising models, have given numerous insights in a variety of contexts, by ruling out asymmetric interactions, interesting patterns of aggregate behaviour can be missed.

If all the couplings are positive, it is always possible to simultaneously satisfy the desire of each agent in a group to conform to the other agents in the same group. This is not always possible if some of the couplings can take negative values and this creates "frustration" in the system. As a consequence of disorder and frustrtation, these systems are characterized by a degeneracy in the equilibrium state and have therefore a potential role in the study of multiple equilibria in social and economic activities.

Asymmetry can be introduced in three different ways: (i) agents reciprocally want to distinguish from each other, in which case we have an analogy with antiferromagnetic models in physics, (ii) introducing antisymmetric interactions, in which case, agent A may want to imitate agent B but agent B wants to distinguish himself from A, or (iii) having unidirectional interactions, where agent A is influenced by agent B (positively or negatively) but where agent B is not affected by A's decision.

Non-symmetric pairwise interactions, although common in the study of neural networks and other biological systems (Kaufman 1969, see also Müller et al 1995 for a review), have recently been introduced in economics (Kirman 1997, Samuelson 1997). In a recent work, Cowan, et al (1998), introduced a model of consumption with asymmetric interactions where, the utility of an individual agent can be positively or negatively affected by the choices of other agents and consumption is driven by peering, imitiation and distinction effects. In Cowan et al (1998), consumers are ordered according to their social status and are affected by the behaviour of other agents depending on their relative location on the spectrum. Agents wish to distinguish themselves from those who are below and emulate their peers and those who are above in the social spectrum. The interplay between aspiration and distinction effects can generate consumption waves, which propragate through the system.

In this paper, we present three different models of consumption behaviour with social interdependencies, which include asymmetric and/or unidirectional interactions. We cast our formulation within the framework of statistical mechanics of disordered systems. In the first case (Model I), the microeconomic agents have pairwise interactions which are specified by a function of a single parameter, their "social distance" (such as, for example, differences in wealth), and hence are deterministic (even though the wealths can be random variables). This constitutes a reformulation of the Cowan et al. model expressed in a discrete time and space language. In the second and third cases (Models II and III, respectively), we consider the opposite extreme, and assume that the interactions among agents depend on a rather large number of underlying parameters. In this case, the interactions can be modeled as random variables, with positive and negative values (albeit constant in time), in analogy with spin glass models in statistical mechanics.

The dynamical properties of these models are explored, by numerical simulations, for the above choice of interactions among agents, using three different evolution algorithms with: *parallel, sequencial* and *random-sequencial* updating rules, depending on the order on which individual agents update their decision. Each one of these updating rules results in a different intresting dynamical behaviour. We study the attractors of each model that determine the steady state, long-time behaviour of the consumption behaviour. Depending on the evolution algorithm as well as the parameters of each model, we show that the attractors can be either fixed points or limit cycles with the possibility of quasi-periodic behaviour depending on the degree of the asymmetry. In the case of random asymmetric or unidirectional interactions the system might take extremely long time (that grows exponentially with the size of the system) to reach its steady-state behaviour, in which case the attractor itself might not be relevant for the dynamics. Moreover the transient exhibit chaotic behaviour, i.e. sensitive dependence on the initial conditions. The local stability of single trajectories, can be analyzed by using the technique of "damage spreading", introduced by Kaufmann (1969) in his study of cellular automata.

In section II we set the general framework by reviewing briefly the theory of discrete choice models in a stochastic environment. In section III we present our three models in terms of their interactions and the possible evolution mechanisms. Section IV contains our results from our simulations while we conclude in section V.

II. DISCRETE CHOICE MODELS

Discrete choice models start from the assumption that each agent faces a set of mutually exclusive alternatives, and chooses the one that yields greatest utility (see Anderson et al (1992) for a review). The problem can be formalized (Brock and Durlauf 1995, Durlauf 1997) by considering a population of N individuals, where each individual i chooses S_i with support $(-1, 1)$. The set of all possible sets of actions by the population, denoted by Ω, consists of all N-tuples $\tilde{S} = (S_1, \ldots, S_N)$.

Two families of models have been introduced to analyze the choice process in a probabilistic setting. The first family of models assumes that the decision rule is deterministic but the utility is stochastic. The idea behind this assumption is that even though individual behaviour might be deterministic, the modeller can only imperfectly observe the factors that influence individual choice and only has an imperfect knowledge of the utility function of another agent. Models in the second family assume that the utility is deterministic but the choice process is stochastic. These models capture the idea of bounded rationality of economic agents (Sargent 1995). Even if utility is deterministic, individuals might make an error in evaluating the importance of one or another characteristic associated with a certain alternative and do not necessarily select what is best for them. In this paper we shall adopt the later description, that of the choice as a stochastic process with the utility being deterministic (Aoki, 1995). This formulation will allow us to clearly demonstrate the effect of the asymmetric interactions on the macroeconomic dynamics of the system.

Following the observation that adjustments to the behaviour of economic agents are often made at discrete points in time and are of finite magnitude, Aoki proposes to use jump Markov processes to model evolutionary dynamics of a large collection of interacting microeconomic agents. Interactions of microeconomic units can then be specified in terms of transition probabilities of Markov chains (discrete Markov process with finite state-space). The initial condition and the state transition probability completely characterize the time evolution of a discrete-time Markov chain. The time evolution of the probabilities of states in terms of transition rates and the state's occupancy probability is given by the master equation:

$$\frac{\partial P(x',t)}{\partial t} = \sum_{x\neq x'} P(x,t)w(x'|x,t) - \sum_{x\neq x'} P(x',t)w(x|x',t) \tag{1}$$

where x, x' denote state space variables. From this equation one sees that for the stationary or equilibrium probability to exist the (full) balance condition should be satisfied:

$$\sum_{x\neq x'} P_e(x)w(x'|x) = \sum_{x\neq x'} P_e(x')w(x|x') \tag{2}$$

If moreover the probability flow balances for every pair of states (x, y), i.e. the detailed balance condition holds

$$P_e(x)w(y|x) = P_e(y)w(x|y) \tag{3}$$

it can be shown that the equilibrium distribution is path independent. More precisely, denoting by Ω the space state of a Markov chain, if we assume that the Markov chain is ergodic*, any state $x_i \in \Omega$ can be reached from an initial state x_0 through a sequence of intermediate states $x_1, x_2, ..., x_{i-1}$ so that

$$P_e(x_i) = P_e(x_0)\prod_{k=0}^{i-1} w(x_{k+1}|x_k)/w(x_k|x_{k+1}). \tag{4}$$

If the detailed balance condition holds it can be shown that the equilibrium distribution is given by the Gibbs distribution

$$P_e(x) = \frac{\exp(-V(x))}{\sum_{k\in\Omega}\exp(-V(k))} \tag{5}$$

where

$$V(x_i) - V(x_0) = -\log(w(x_i|x_0)/w(x_0|x_i)). \tag{6}$$

$V(x)$, being only dependent on the state x_i, is a potential. If follows that, although the transition rates for the states of microeconomic units is a crucial ingredient in the formulation of a discrete choice model, any dynamical process, as long as it satisfies the detailed balance condition with the same function P_e will reach the same asymptotic equilibrium distribution of states.

Before we procceed with the detailed description of the interactions, we should make some important remarks on the role of asymmetry. If there are asymmetric interactions the detailed balance condition, eq. (3) is no longer valid, and the long time evolution of the system does not necessarily converge to the Gibbs equilibrium distribution of eq. (5).

A. Interactions

We turn now to the description of the interactions. The (deterministic) utility $U(S_i)$ that agent i receives from action S_i, consists of two components (following the notation of Brock and Durlauf 1995)

$$U(S_i) = u(S_i) + s(S_i, \mu^e(\tilde{S}_{-i})) \tag{7}$$

where $u(S_i)$ represents agent's i private utility, and $s(S_i, \mu^e(\tilde{S}_{-i}))$ represents his/her social utility. Here $\tilde{S}_{-i}$ denotes the choices of all agents other than i, while the term $\mu^e(\tilde{S}_{-i})$ denotes the conditional probability measure agent i places on the choices of others at the time of making his own decision.

We shall write the social utility as

$$s(S_i, \mu^e(\tilde{S}_{-i})) = \sum_{j\in G_i} J_{ij} S_i E_i[S_j] \tag{8}$$

where G_i is the reference group of agent i and $E_i[.]$ represents the conditional expectation operator associated with agent i's beliefs. J_{ij} represents the interaction weight which relates i's choice to that of j's. Within this framework, choosing a particular realization of J_{ij}, the interactions among agents are completely defined. We shall further assume that agents make expectations about the choice of others according to

$$E_i[S_j(t+1)] = S_j(t) \tag{9}$$

i.e., they are deductive rather than inductive†. This amounts to assuming that agents are myopic or that they do not explore any intertemporal opportunities, *i.e.* there are no capital markets.

The private utility is assumed to depend linearly on S_i:

*More precisely, the chain should be irreducible, aperiodic and positive (see, for example, Hammersley and Handscomb 1975)

†This does not exhaust the analysis of learning mechanisms in the model and further generalizations might be considered (Brock and Durlauf 1995). A study where $E_i[S_j(t+1)] = f(S_j(t-1, t-2, \ldots)$, will appear in a separate study (Iori, Koulovassilopoulos)

$$u(S_i) = h_i S_i \qquad (10)$$

where the h_i can be chosen the same for all agents or can have different values for different i introducing heterogeneity.

Gathering all terms together, the utility of each individual i in a population of N agents, can be written as

$$U_i(t) = \frac{1}{N}\sum_{j=1}^{N} J_{ij} S_i(t) S_j(t) + h_i S_i(t) \qquad (11)$$

where we have rescaled the interactions J_{ij}, dividing by N, in order to keep U_i finite as $N \to \infty$. Given the utility at time t, each agent i takes his/her action $S_i(t+1)$ at the next time step, based on whether $U_i(t)$ is positive or negative. Note that the contribution of an asymmteric component of the interactions to the total utility $U(\tilde{S})$

$$U(\tilde{S}) = \sum_{i=1}^{N} U(S_i) \qquad (12)$$

is zero when calculating the double sum.

In order to complete the dynamics, we need to specify the order in which each individual updates his/her decision. We have considered three different such evolution mechanisms which are detailed in section III E below.

III. MODEL

A. General

According to our previous description, the state of our population of N agents, $\tilde{S}(t) = (S_1(t), \ldots, S_N(t))$, evolves according to a jump Markov process. Time evolves discretely in our model and at each step an agent i has a binary choice either to consume one indivisible unit of a good in which case $S_i = 1$, or not to consume, in which case $S_i = -1$. For some product that exists, or appears for the first time in the market, and given an initial state at time $t = 0$, each agent decides whether to consume or not at every subsequent time step, doing so if this action provides positive utility. For example, a new restaurant opens at time zero and in each subseqent time period agents decide whether to visit it or not.

The utility function $U_i(t)$ is specified in eq. (11). The local field h_i characterizes the intrinsic value of the good to agent i. This term contains all private factors that affect his consumption decision.

In this partial equilibrium set-up with only one good, the decision on whether to buy the good or not depends only on whether the "marginal" utility of buying one unit is positive. All agents are assumed to possess sufficient liquidity at all points of time and wealth constraints are never binding.

A product with $h_i = 0$ for all agents is called a "fashion" good, while a "status" good has a positive intrinsic value that might be well-suited to the characteristics or tastes of a particular class of consumers.

In the following subsections we shall consider three different cases of interactions among the N agents, by specifying the form of J_{ij}, and determine the time evolution of the system by specifying the transition probabilities $w(\tilde{S}|\tilde{S}', t)$.

B. Model I

We consider our population of N agents, ordered on a one-dimensional space and labeled by a variable w_i which represents their position in the social spectrum and in a broad sense their wealth. Agents' wealth is chosen randomly, from the uniform distribution in the interval $[0, W_0]$, and does not change with time. In this paper, wealth serves as an index of social status rather than the source of a budget constraint, as discussed below. A more realistic situation with consumers arranged over a multidimensional space (accounting, for example, for differences in age, education, etc.) should be considered, but for simplicity we will only discuss here the case $d = 1$.

Each agent interacts with all the others, and the coupling constants J_{ij} are functions of the agents' status according to:

$$J_{ij} = -J_A \arctan(w_i - w_j) + J_S\left[\frac{\pi}{2} - \arctan|w_i - w_j|\right] \qquad (13)$$

The coefficients J_A, J_S are taken positive. The asymmetric term, proportional to J_A, gives a negative contribution to the utility function U_i if $w_i > w_j$ and a positive contribution if $w_i < w_j$. This means that agent i wishes to distinguish herself from the poorer while imitating the richer. The second contribution, proportional to J_S, always generates positive utility and expresses peering effects among consumers of similar status. Both of these contributions saturate with distance to a constant value. In Fig. (1) J_{ij} is plotted as a function of $(w_i - w_j)$.

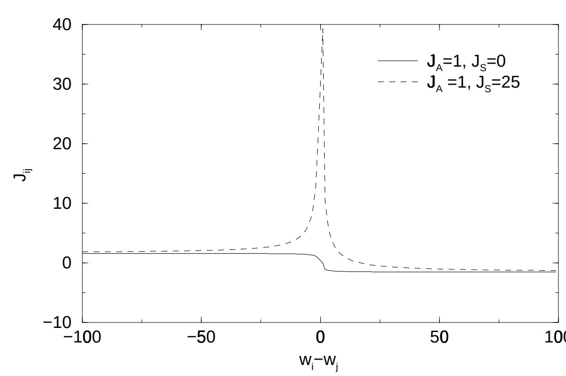

FIG. 1. J_{ij} for Model I as a function of $(w_i - w_j)$.

C. Model II: The asymmetric SK model

We shall consider, as the opposite extreme, the case where the strenght of interaction among agents depends on so many factors (example age, race, education, etc) that, for all practical purpose, it cannot be distinguished by a random variable. This model, in absence of the individual utility terms h_i, is known in statistical mechanics as the asymmetric Sherrington-Kirkpatrick model (henceforth called SK for short) (see Mezard et al 1987, for a review). In this case the utility U_i is still given by the expression in eq. (11), but now the couplings between spins are randomly chosen positive and negative variables, which do not change with time (quenched). In analogy to the SK model we define

$$J_{ij} = J_{ij}^S + kJ_{ij}^A \qquad k \geq 0 \tag{14}$$

where $J_{ii} = 0, J_{ij}^S = J_{ji}^S$ and $J_{ij}^A = -J_{ij}^A$ with the elements of J^S and J^A be independent Gaussian variables with zero mean, mean square equal to $J^2/N(1+k^2)$ and correlation

$$< J_{ij}J_{ji} >= \frac{J^2}{N}\frac{1-k^2}{1+k^2} \tag{15}$$

k defines the symmetry of the coupling. The case $k = 1$ correspond to the fully asymmetric case where J_{ij} and J_{ji} are completely uncorrelated. For $k = 0$ we recover the symmetric case while for $k = \infty$ we have the anti-symmetric case. The role of k in model I is played by J_A/J_S.

D. Model III

In the previous two models the agents are fully connected. We shall now consider a model where each agent interacts with only a fraction α of the entire population. The utility U_i is again given by the expression in eq. (11), where the pairwise interactions J_{ij} have values ± 1, randomly chosen, with however a fraction α of the $N \times N$ matrix J_{ij} randomly set to zero. This model contains asymmetry as interactions are *unidirectional*, that is, while agent i might be influenced by agent j, agent j is not influenced by i. We shall call α the "dilution" parameter. This type of model is similar to those considered in asymmetric neural networks where synaptic couplings are diluted. In the limit where $\alpha = 0$, this model is a spin glass model, where each agent interacts with all others, with random symmetric interactions $J_{ij} = \pm 1$, while for $\alpha \simeq 1$ the links are highly diluted and each agent interacts only with a small fraction of others.

E. Dynamics

Having specified the social interactions through J_{ij}, we now come to discuss the evolution mechanisms of the above system that determine the transition rates $w(x|x', t)$, as discussed in paragraph II. Our analysis will be restricted to the following three types of deterministic dynamics:

(a) *Parallel (or synchronous) dynamics:* At each time step t, all agents re-evaluate *simultaneously* their consumption decision, relative to that taken one step before, on the basis of the utility $U_i(t)$ they receive, according to:

$$S_i(t+1) = \mathrm{sgn}(h_i^t) = \mathrm{sgn}\left[\sum_{j=1}^{N} J_{ij}S_j(t) + h_i\right] \tag{16}$$

where we have implicitely absorbed the $1/N$ factor in the definition of J_{ij} and introduced for later convenience the notation:

$$h_i^t = h_i + \sum_{j\neq i} J_{ij}S_j. \tag{17}$$

(b) *Sequential dynamics:* This is the case of asynchronous updating in which the state of each spin is updated one by one in a serial manner, according to:

$$S_i(t+1) = \mathrm{sgn}\left[\sum_{j<i} J_{ij}S_j(t+1) + \sum_{j>i} J_{ij}S_j(t) + h_i\right] \tag{18}$$

(c) *Random sequential (Glauber) dynamics:* This is another case of asynchronous updating scheme, similar to the previous case, with the difference though that the order of updating is chosen randomly.

In cases (b) and (c) of asynchronous dynamics every agent coming up for a decision has full information about the state of the agent that have been updated before him. Note that the "field" h_i acts as a threshold above which the social interactions alter individual preferences.

A stochastic rule can be introduced by replacing the deterministic laws above by a probabilistic one

$$\mathrm{Prob}\left[S_i(t+1) = \pm 1\right] = \left[1 + \exp(\mp 2\beta h_i^t(t))\right]^{-1} \tag{19}$$

where h_i^t is defined in eq. (17) above, and $\beta = 1/T$ with T playing the role of a "social temperature" of the system.

For both synchronous and asynchronous dynamics, symmetric J_{ij} is a sufficient condition for the existence of detailed balance. Nonetheless the form of the asymptotic distribution differs in the two dynamics. In the following, we briefly review the properties of the model with $h_i = 0$. One can draw a number of general statements about the nature of fixed points, without relying on the detailed form of J_{ij}.

For *asynchronous* dynamics (sequential or random sequential) the distribution of configurations relaxes eventually to the Boltzmann distribution

$$P_e(\tilde{S}) \sim \exp(\beta U(\tilde{S})) \tag{20}$$

where U is the total utility, given in equation (12). In the noiseless ($T=0$) case, each contribution to the utility is increased (or remains constant) after each agent i updates her decision, as can be seen from

$$\begin{aligned} U_i(t+1) &= h_i^t(t) S_i(t+1) \\ &= h_i^t(t)\mathrm{sgn}(h_i^t(t)) = |h_i^t(t)| \\ &\geq h_i^t(t) S_i(t) = U_i(t). \end{aligned} \tag{21}$$

Since the total utility is bounded from above, the system will asymptotically be driven to a fixed point attractor which is either a local or a global minimum of the utility functional.

For sequential dynamics, there exist only fixed points for $k=0$ ($J_A=0$). Gutfreund, Reger and Young (1987) have proved that in the opposite extreme, for $k=\infty$, there exist only 2-cycles. Repeating their argument, one can easily verify that the updating rule of eq. (18) remains unchanged if we make the transformation $J_{ij} \to J'_{ij}$, given by

$$\begin{aligned} J'_{ij} &= J_{ij}, \qquad i > j \\ J'_{ij} &= -J_{ij}, \qquad i < j \end{aligned} \tag{22}$$

while simultaneously $S_i(t) \to S'_i(t)$, given by

$$\begin{aligned} S'_i(t) &= S_i(t), \qquad t \ \ even \\ S'_i(t) &= -S_i(t), \qquad t \ \ odd. \end{aligned} \tag{23}$$

Hence the dynamics with interactions J_{ij} is the same as that of interactions J'_{ij} if one inverts the decisions of agent i at every other time step. The above transformation is the "duality transformation" of Gutfreund et al. (1987), which maps the dynamics with asymmetry parameter k to that with asymmetry parameter $1/k$. Since for $k=0$ we have seen that the system has only fixed points, we can conclude that for $k=\infty$ there exist only 2-cycles. We should note here that this nice property is specific to sequential updating and is no more valid if the order of updating is different.

In the case of *synchronous dynamics* (as in the Little model of neural networks) the asymptotic distribution is still in the form of a Gibbs distribution but the potential is now different (Amit 1989)

$$P_e(S) \sim \exp\left[\sum_i \ln(2\cosh\left(\beta h_i^t\right))\right] \tag{24}$$

From this expression one can define an effective utility

$$\tilde{U} = 1/\beta \sum_i \ln(2\cosh(\beta h_i^t)) \tag{25}$$

which in the $T \to 0$ (i.e. $\beta \to \infty$) limit reduces to the deterministic updating rule of eq. (16), with

$$\tilde{U} = \sum_{j \neq i} J_{ij} S_i(t) S_j(t-1) + h_i S_i(t) \tag{26}$$

The dynamical process now maximizes $\tilde{U}$ (often called the stability function), since

$$\begin{aligned} \tilde{U}(t+1) &= \sum_i h_i^t(t) S_i(t+1) \\ &= \sum_i h_i^t(t)\mathrm{sgn}(h_i^t(t)) = \sum_i |h_i^t(t)| \\ &\geq \sum_j h_j^t(t) S_j(t-1) = \tilde{U}(t). \end{aligned} \tag{27}$$

For syncronous dynamics, if J_{ij} is symmetric ($J_{ji} = J_{ij}$), one sees that

$$\begin{aligned} \Delta\tilde{U} &= \tilde{U}(t+1) - \tilde{U}(t) \\ &= \sum_i [S_i(t+1) - S_i(t-1)] \sum_{j \neq i} J_{ij} S_j(t) \end{aligned} \tag{28}$$

implying that $\tilde{U}(t+1) \geq \tilde{U}(t)$. The $\tilde{U}(t)$ remains unchanged either when the consecutive states are identical, i.e. the system has reached a fixed point, or when the two states alternate $S_i(t+1) = S_i(t-1)$, i.e. the system has reached a 2-cycle. The existence of 2-cycles with symmetric interactions is a unique feature of synchronous dynamics.

If, on the other hand J_{ij} is antisymmetric ($J_{ji} = -J_{ij}$), then $\tilde{U}(t)$ changes by

$$\begin{aligned} \Delta\tilde{U} &= \tilde{U}(t+1) - \tilde{U}(t) \\ &= \sum_i [S_i(t+1) + S_i(t-1)] \sum_{j \neq i} J_{ij} S_j(t) \end{aligned} \tag{29}$$

and thus, again $\Delta\tilde{U}(t) \geq 0$, where the equality holds when $S_i(t+1) = -S_i(t-1)$. Therefore, for antisymmetric interactions the only attractors are 4-cycles, since the first and the third states are inverses of each other, as are the second and fourth state. It should be emphasized that these arguments apply to all three models above, as they do not rely on the detailed form of J_{ij}.

It is important to stress though the that number of fixed points is the same for sequential, parallel and for random sequential dynamics.

F. Damage Spreading

Another feature we shall address is the sensitivity of the system with respect a small perturbation in the initial conditions. In continuous dynamical systems a powerful indicator of chaoticity is the (maximum) Lyapunov exponent, which can be naively defined as the diverging rate of two initially close trajectories, in the limit of vanishing initial distance. This definition cannot be easily extended to discrete systems. First of all we need a notion of distance in discrete space. The natural definition in our system is to count the number of agents that make different choices (Hamming distance)

$$D(t) = \frac{1}{2N} \sum_{i=1}^{N} |S_i^{(1)}(t) - S_i^{(2)}(t)| \qquad (30)$$

$D(t)$ defines an equivalent of the Lyapunov exponent in continous systems and allows us to characterize the degree of chaoticity in the system. We can in fact check how an initial disturbance, called the damage, spreads through the system. Looking at two configurations S^1, S^2 which differ at $t = 0$ at only one lattice site, we can study the evolution of their Hamming distance and look whether the defect freezes in or grows as a function of time. Obviously for this analysis we need to evolve the two systems with the same realization of J_{ij} and h_i and the same rules (and stochastic noise for the $T \neq 0$ case).

IV. SIMULATIONS AND RESULTS

A. Model I

We have performed computer simulations to study the model in eq. (13) above for each of the evolution algorithms of section III E and different choices of the parameters and explored the various patterns observed. We studied both the time evolution of total consumption as well as the spacial distribution of consumption across the social spectrum. Our plots refer to relatively small lattice size $N = 800$, although our results do not qualitatively change with N. We have taken the agents' wealth to be uniformly distributed in the interval $(0, W_0)$ with $W_0 = 100$.

(a) Parallel dynamics

Initially we study the case of a fashion good where the intrinsic value of the good vanishes ($h_i = 0$) and explore the dependence on J_A, J_S. In this case the dynamics only depends on the ratio $k = J_A/J_S$. For $k \to \infty$ as we demonstrated in the last section, the system exhibits 4-cycles. For $k < \infty$ but large (J_S small) the system exhibits limit cycles, as depicted in fig. (2) for the total consumption. As J_S is increased the period of oscillations increases. Moreover consumption can start spontaneously without choosing it as an initial condition. At the beginning of each cycle consumption propagates rather slow but it spreads faster as it moves down to the poorer. S-shaped curves, similar to these have been observed in the case of diffusion of innovation (Rogers 1995).

In fig. (3) we show a typical pattern, for $J_A = 1, J_S = 25$, as a function of time (time increases from top to bottom). Notice that consumption propagates through the system as a travelling wave and that, it is the rich who start consuming first in order to distinguish themselves from the poor.

As k decreases (J_S increases) below a certain value $k = k_c$, waves disappear and the system is attracted towards a fixed point, developing a steady-state behaviour. This

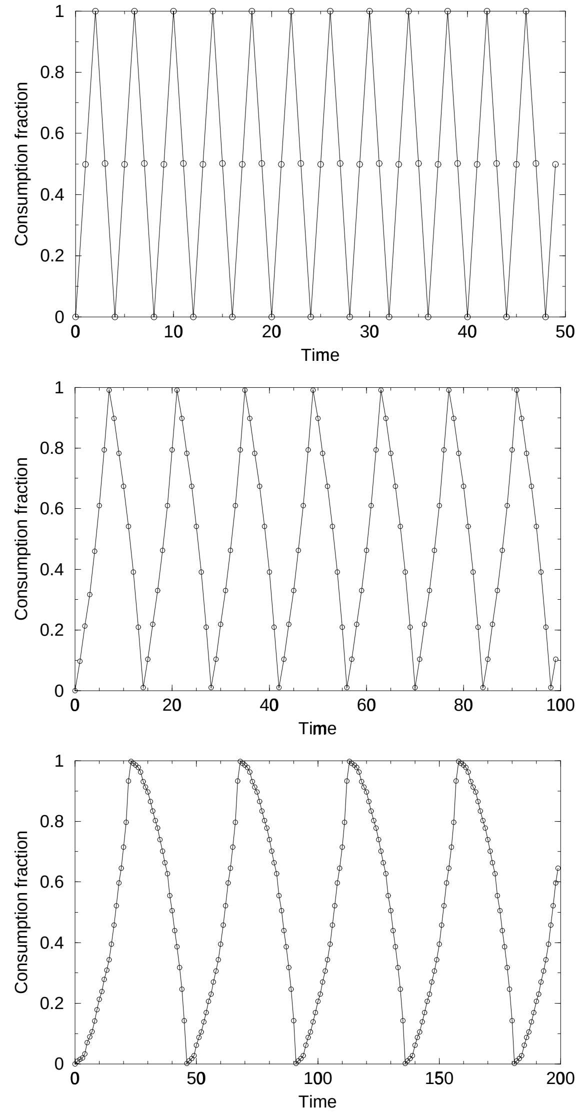

FIG. 2. Total consumption with parallel dynamics as a function of time in Model I. $J_A = 1$ and three different values of $J_S = 0, 15, 25$ are considered respectively from top to bottom. Here $N = 800, G = 0$ and for all agents $S_i(0) = -1$.

state is characterized in the $N \to \infty$ limit by the majority of agents either consuming or not consuming[‡] depending on the initial actions of the consumers at $t = 0$. As at the end of the social spectrum (richest or poorer) agents have nobody to imitate and/or distinguish from, these social classes do not always conform to the majority of the population.

Next we consider the case of a "status good" where $h_i \neq 0$. We have considered the case of a good for which its intrinsic value h_i for each agent i takes a different

[‡]These two configurations have the same utility (and $\tilde{U}$) as the transformation $S_i \to -S_i$ is a symmetry of the system.

FIG. 3. Wave dynamics in Model I, with parallel dynamics. The dark color corresponds to those agents consuming and light color to those not consuming. Wealth is increasing from left to right along the horizontal axis. The vertical axis represents time (here $T = 150$ steps) which increases from top to bottom. Here $h_i = 0, N = 800, J_A = 1, J_S = 25$.

random value from the interval $(-G, G)$ (the h_i do not change with time). For G relatively small, consumption exibits cycles, as shown on fig. (4). However as G increases, the amplitude of the cycles decreases and above a certain value G_c steady state behaviour emerges (it goes to a fixed point).

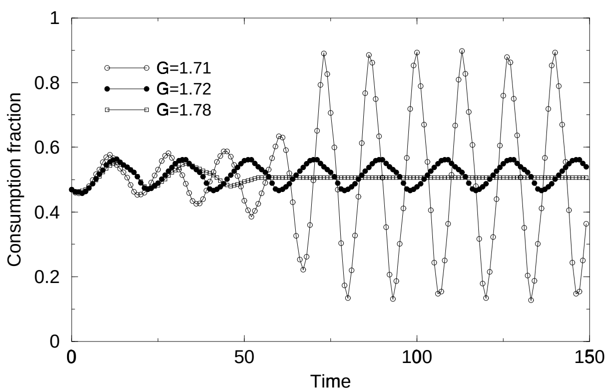

FIG. 4. Total consumption for Model I with parallel dynamics for $J_A = 1, J_S = 20$ and three values of G, for $N = 800$.

We have also considered the case where the good has an intrinsic value which is more suitable to the group of consumers whose wealth is distributed around a given value w_m choosing:

$$h = h(w_i) = \frac{G}{(w_i - w_m)^2} \quad (31)$$

and its consumption entails costs, which we choose random from the uniform distribution in the interval $(-C, 0)$. In this case, costs act as thresholds, and each agent buys the good if her marginal utility exceeds her marginal cost. As before, we assume that agents possess sufficient liquidity so that wealth is never a binding constraint. If $G = 0$ and costs exceed a certain value $C = C_c$, nobody is consuming. To demonstrate the interplay between the intrinsic value of the good (here it is G) and the costs, we fix the costs to be larger than C_c so that cycles do not emerge in the case of a fashion good ($G = 0$). Therefore it is only the intrinsic value of the good which can trigger consumption. The results which follow refer to the case $J_A = 1, J_S = 25, G = 1$ and $S_i(0) = -1$.

FIG. 5. Consumption behavior for Model I with parallel dynamics when $J_A = 1$, $J_S = 25$, $C = 0.5$, $G = 1$, and $w_m = 10, 50, 80, 85$ (from top to bottom). Dark (light) color is for those consuming (not consuming). Only when the good is suitable to the consumers located at the top of the social scale a consumption wave propagates. In all other case the good enters and finds a stable niche.

Different behaviours are found, depending on the position of the maximum (w_m) of $h(w_i)$. For $w_m < 85$, (we remind that the agents' wealth is distributed between zero and $W_0 = 100$), the good enters the social spectrum around w_m, possibly migrates through the closest social classes and then finds a stable niche (see top three cases in Fig. (5)). Eventually, when $w_m \sim W_0$, for costs not too high (but higher than C_c), waves emerge and spread throughout the whole social spectrum (bottom case in Fig. (5)).

We eventually simulated various copies of the same system that differ at $t = 0$ at only one point and studied the evolution of their Hamming distance. We found no sensitivity to the initial conditions with the different configurations always ending up to the same attractor.

(b) Sequential dynamics

We consider first the case of a fashion good ($h_i = 0$). For $k \to \infty$, as anticipated, 2-cycles emerge as depicted in fig. (6). Notice that after some transient period during which consumption is irregular, the system ends into a periodic attractor characterized by agents clustering into groups that syncronously alternate their decision. Starting with different initial conditions affects the number of clusters which form while the total consumption keeps oscillating around $N/2$. We also find that starting from two configurations which at $t = 0$ differ at one point, they end up into different attractors as indicated by the fact that the Hamming distance $D(t)$ reaches a non-zero plateau value as illustrated in the bottom of fig. (6).

As we reduce k waves emerges that propagate through the social spectrum, while for still lower k (i.e. larger J_S for fixed J_A) only fixed points exist. The dynamics of sequential updating thus look similar to that of parallel

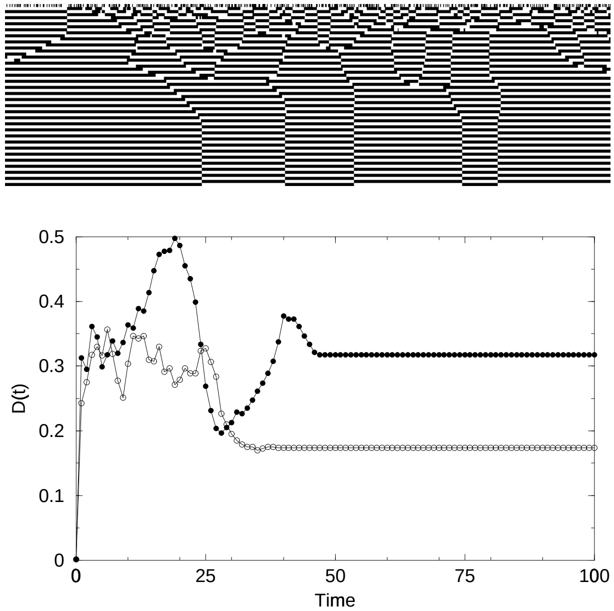

FIG. 6. Consumption behavior for model I with sequential dynamics when $J_A = 1$, $J_S = 0$, $G = 0$, in terms of the total consumption (top). Initially, 50% of the agents are randomly chosen to consume. On the bottom, the evolution of the Hamming distance is shown, in two cases, for two copies of the same system differing at $t = 0$ at only one point.

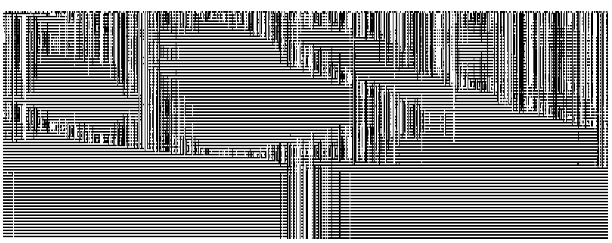

FIG. 7. Total consumption for model I with sequential dynamics when $J_A = 1$, $J_S = 0$, $G = 0.05, t = 150$. At $t = 0$ about 50% of the population is consuming.

dynamics at $k < \infty$, although for any given k the amplitude and period of the cycles are different (the period is shorter for sequential dynamics).

If we turn on a small personal utility, chosen randomly from the uniform distribution in $(-G, G)$, the behaviour of the system is distorted like in fig. (7) for $k \to \infty$, with an interesting transient behaviour before it settles to a 2-cycle state (the transient time depends on the initial configuration). Above a certain value of G the social interactions can not alter an individual's decision and steady-state behaviour emerges, where those with $h_i > 0$ are consuming while the others don't. We also studied the spreading of an initially small damage and found that $D(t)$ becomes asymptotically smaller as we increase G ($D \to 0$ for G large enough) indicating that the effect of non zero private utility is to reduce the sensitivity of the system to the initial conditions.

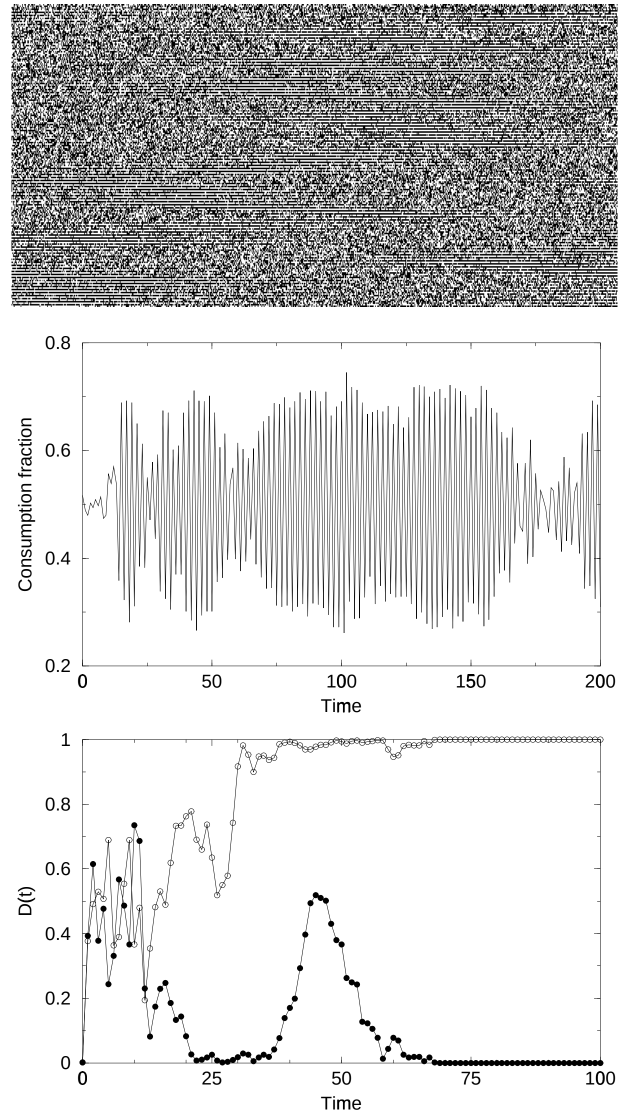

FIG. 8. Consumption behavior for Model I with random sequential dynamics for $J_A = 1$, $J_S = 0$, $h_i = 0$, for $t = 200$ (top). At $t = 0$ 50% of the agents (chosen randomly) are consuming. Displayed is also the total consumption (middle), and the hamming distance (bottom) between two different pairs of configurations which differ at $t = 0$ by one agent.

(c) Random sequential dynamics

Unlike the previous dynamical mechanisms, random sequential dynamics provides, from the economics viewpoint, a more natural picture of how individuals interact in a social network. For $k \neq 0$ no simple analytical statements about the dynamics can be proven, and the system appears to possess complex dynamics. Figure (8) shows the time evolution of a population of $N = 800$ agents for $k = \infty, G = 0$.

The bottom plot shows the evolution of a small damage at $t = 0$. The Hamming distance after some transient time goes to zero or one. For $G \neq 0$, we found that an initially small damage always healed with time.

As an interesting result we observe that the damage spreading technique in this model is non-universal with respect to the order of sites updating, as also found in a different context by Vojta et al. (1998).

For smaller k, waves emerge while above a certain value of J_S the population always ends up in a state where either all or nobody is consuming. The period of the attractor for the same value of k is shorter than with parallel dynamics.

(d) Remarks on Model I

Concluding this section we would like to remark that the existence of waves is not due to the specific choice of the potential in eq. (13) but occurs whenever the poor imitate the rich while the rich differentiate themselves from the poor. For example taking the J^S_{ij} as random positive symmetric variables and J^A_{ij} as random antisymmetric variables, positive for $i < j$ and negative eitherways, we can still find waves for parallel and sequential dynamics.

B. Model II: Asymmetric SK model

The asymmetric SK model, presented in section III C, has been studied in the statistical physics literature for $h_i = 0$ at increasing N, with both synchronous (Gutfreund et al. 1988, Nützel et al 1993, Crisanti et al 1993), and asynchronous dynamics (Crisanti et al 1987, Gutfreund et al. 1988). An interesting feature of this model is the presence of extremely long transients before the systems settles in the asymptotic regime. The system also exhibits chaotic behaviour. We shall review briefly the known results for $h_i = 0$ and then present our results for $h_i \neq 0$.

It has been shown that the average number of fixed points is the same for sequential, parallel and for random sequential updating, and it grows exponentially with the number of agents (N) for $0 < k < 1$,

$$< N_1(k) >= A(k)\exp(N\alpha_1(k))[1 + 0(1/N)], \qquad (32)$$

while $< N_1(k) >= 1$ at $k = 1$ and decreases exponentially with N at $k > 1$ (Gutfreund et al 1988).

As we have seen, with sequential dynamics there are only fixed points at $k = 0$, while in the opposite extreme, for $k = \infty$ there are only limit cycles of length two. Moreover, Gutfreund et al. have shown that a duality relationship holds for the average number of 2-cycles

$$< N^s_2(k) >= 1/2 < N_1(1/k) > \qquad (33)$$

where superscripts in this section will refers to sequential dynamics or parallel dynamics.

For parallel dynamics, as we have seen, for there are only fixed points and 2-cycles for $k = 0$ and only 4-cycles for $k = \infty$, while Bray and Young (1987) have shown that

$$< N^p_2(1/k) >= 2 < N_1(k) > \qquad (34)$$

while the average number of 4-cycles is

$$< N^p_2(1/k) >=< N_1(k) > \qquad (35)$$

A detailed study of the long time behaviour of the 2-cycles correlation function $C_2(t) =< \sum_i S_i(t-1)S_i(t+1) >$ has been performed by Nützel and Krey for parallel dynamics. We refer in the following to the symmetry parameter η as

$$\eta = \frac{1-k^2}{1+k^2} \qquad (36)$$

Note that $\eta = 1$ correspond to the symmetric case and $\eta = 0$ to the fully asymmetric case. Two relevant time scales can be introduced : the relaxation time τ and the cycle length l. These are defined as

$$l = \min_n[S(t+1) = S(t)] \qquad (37)$$

$$\tau = \min_n[S(l+n) = S(n)] \qquad (38)$$

The time τ is the time the system needs to reach the first cycle of length l. In general both τ and l depend on the initial condition and the realization of the J_{ij}. Averaging over different realizations of J_{ij} Nützel and Krey (1993) found that at $T = 0$ for $\eta > 0.5$ only fixed points or cycles with period two appear as traps, i. e. $< C_2(t) >\to 1$. For $\eta < 0.5$ also 2-cycles traps are found with high probability. Nonetheless while in the nearly symmetric case $< l >$ is size independent and the typical number of updates before the system relaxes onto the attractor grows as a power of N, the behaviour of both $< l >$ and $< \tau >$ is exponential in N for highly asymmetric couplings.

Similar results have been found in the case of random sequential updating by Crisanti and Sompolinski (1988) with the only difference that with random sequential updating the system always ends eventually in a metastable state, i.e a fixed point attractor for all values of η. The probability distribution of trapping time has been found to be log-normal with both parallel and random sequential dynamics.

We have studied the asymmetric SK model in the presence of a non-zero local field h_i, i.e. the case of a status good. We present here results in the case of parallel dynamics, while more systematic analysis will be presented elsewhere.

Given a population of $N = 100$ agents, and $k = 1$ (i.e. $\eta = 0$), the transient period before the system reaches an attractor, is of the order $t = 10^5 - 10^6$ time steps for $h_i = 0$. We have considered the case where the $h_i \neq 0$ are chosen for each agent randomly from the uniform

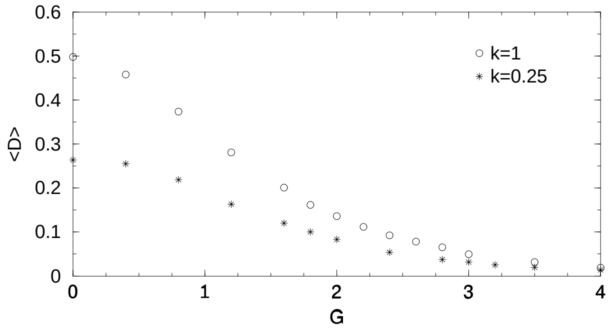

FIG. 9. The SK model with parallel dynamics with $N = 100$ agents, for $k = 1$ and $k = 1/4$, showing $< D >$ as a function of G. For $G > 2$ the system reaches the attractor after a few hundred steps.

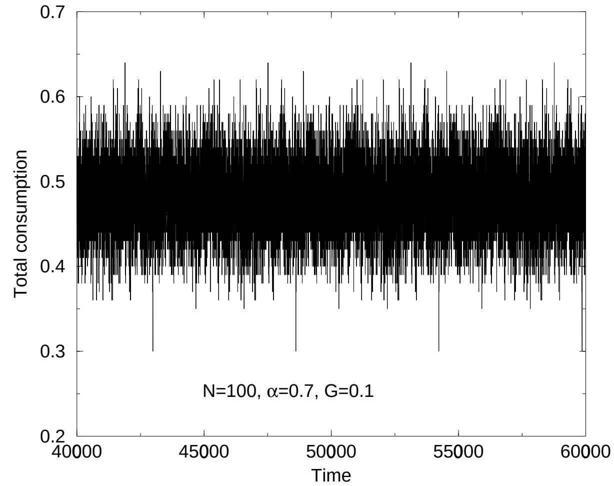

FIG. 10. Model III with parallel dynamics with $N = 100$ agents, for $\alpha = 0.7$ and $h_i = 0$ and at $t = 0$ 50% of the agents are randomly chosen tocd consume. The damage spreading distance $D(t)$ also plotted on the bottom.

distribution in the interval $(-G, G)$. We found that if the magnitude of the local field G is small, the system still exhibits chaotic behaviour (long l, τ). However as G increases the system reaches faster the attractor (the transient period is reduced as G increases), which can be either a fixed point or a cycle, while for values of G large enough the system is always trapped on a fixed point. Our results indicate that by increasing G the system passes from the chaotic to a "frozen phase". In fig. (9) we plot the average $D(t)$ as a function of G, for two values of k, where we have calculated the average over 1200 runs with different realizations of the couplings J_{ij} and initial conditions. Each run corresponded to about 10^5 time steps. The error bars are smaller than the size of the symbols. Above a certain value of $G > G_c^1$, which depends on the realization of the random couplings J_{ij}, the system falls on a cycle or fixed point rather quickly, while for still a higher value $G > G_c^2$ the social interactions can not alter the individual preferences and the system falls on a fixed point (steady-state behaviour) with no sensitivity on the initial conditions, as shown on fig. (9), $< D > \to 0$.

C. Model III: Unidirectional spin glass

In this section we study the dynamical properties of a sparsely connected network where αN^2 links (N is the size of the network) are randomly set to zero. For $\alpha = 0$ the system is a spin glass with symmetric interactions, while for $\alpha \to 1$ the number of unidirectional interacting pairs goes to zero. A model similar to this, but with a fixed coordination number K (i.e. with exactly K randomly chosen distinct agents connected to any other agent), has been studied in the literature (Derrida 1987, Derrida et al 1987, Kürten 1988, Bohr et al 1997). Both analytical arguments and numerical simulations seem to indicate that, above a critical value of α, the model undergoes a phase transition from a frozen phase to a chaotic phase where in the latter the average period of the limit attractor increases exponentially with the number of agents, N, while in the former it increases as a power of N. The behaviour of the Hamming in the two phases also differs in remaining confined and close to zero in the frozen phase, while evolving to a finite distance in the chaotic phase. Both the role of the dilution parameter α and of a constant field H have been investigated (Kürten 1988) showing that the chaotic regime disappears even at high α if H is high enough. In this paper we focus on the effects of random local fields h_i, on the dynamics of a network with random connectivity K_i. The fields, like in the previous models are uniformly distributed in the interval $(-G, G)$ and describe the individual preferences of our heterogenous agents. In fig (10) we show how, for a population of $N = 100$ with $\alpha = 0.7$ and $h_i = 0$, evolved according to parallel dynamics, the system exhibits a very long transient of about $\tau = 43,000$ steps after which it reaches a periodic attractor with period $l = 5610$. This means that diluting the interconnectivity to 30% a chaotic behaviour prevails. When increasing G we also find, like other studies (Kürten 1988), that the local fields h_i reduce the level of chaoticity of the system. A more systematic investigation of the system with other types of dynamics will be presented elsewhere.

V. CONCLUSIONS

In this paper we have focused on a potentially important mechanism that drives consumption decision: the interaction among heterogeneous consumers. Particular attention has been paid on the role of the asymmetry of interdependencies. In the sociology literature, interactions among individuals, belonging to similar or different social circles, are often seen as a major mechanism that determines new styles of behaviour. In model I we studied how peering, distinction and aspiration effects, in addition to the intrinsic values of a good, generate different consumption patterns, under the assumption that information on the consumption behaviour of agents is public (J_A and J_S are long range and saturate to a finite value at large distances). In model II and III the social interactions contain independent symmetric and antisymmetric influences that are random positive and negative, chosen from a Gaussian and uniform distribution respectively (it has in fact been shown (see e.g. Mezard et al 1987) that the qualitative results are independent of the probability distribution). This is a frustrated system, and for sufficient degree of asymmetry it exhibits chaotic behaviour, namely sensitivity to the initial condition. In both models the population is fully connected albeit of varying strength.

Collective behaviour may be affected by the structure of the communication channels. Models with imperfect information diffusion and social segregation abound in the literature (Ellison 2000, Chwe 2000, Morris 2000). We have examined, in model III, the case where individuals only communicate with a subset, chosen at random, of the entire population. The system also in this case is found to exhibit chaotic behaviour. The local nature of social interactions will be more systematically investigated in a future work.

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