A NON-LINEAR TIME SERIES APROACH TO MODELLING ASYMMETRY IN STOCK MARKET INDEXES

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Abstract: In this paper we propose an approach to modelling non-linear conditionally heteroscedastic time series characterised by asymmetries in both the conditional mean and variance. This is achieved by combining a TAR model for the conditional mean with a Changing Parameters Volatility (CPV) model for the conditional variance. Empirical results are given for the daily returns of the S&P 500, NASDAQ composite and FTSE 100 stock market indexes.

Keywords: Changing Parameters Volatility model, TAR, Kalman Filter, EM algorithm.

1. Introduction

Due to the presence of asymmetric effects in the mean and in the time varying conditional variance, the complex behaviour of financial time series can be hardly captured by linear models. Hence, the last two decades have been characterised by a growing interest in the application of non linear time series modelling techniques to the analysis of financial data.

In order to simultaneously capture different aspects of non-linear time series behaviour, Tong (1990) first proposed to combine the use of a non-linear model for the conditional mean with a non linear model for the conditional variance. This idea was successively adopted in different frameworks by various authors (see Li and Li, 1996, Liu, Li and Li 1997, Lundbergh and Teräsvirta, 1998).

In this paper we propose an alternative modelling procedure in order to allow for asymmetry in both the conditional mean and variance. For modelling asymmetry we combine a threshold autoregressive strructure (TAR, Tong 1978) for the conditional mean with a Changing Parameters Volatility model for the conditional variance (CPV, Storti 1999). This class of models can be considered as a state-space generalisation of the CHARMA models proposed by Tsay (1987). With respect to conventional GARCH (Bollerslev, 1986) type models, the CPV model has two main advantages. First, interaction terms can be included in the conditional variance equation allowing to account for asymmetric effects. Second, it can incorporate time-varying parameters in the volatility model.

The performances of the proposed model in estimating the conditional variances of three different stock market indexes (S&P500, NASDAQ, FTSE) are assessed by means of a comparison with the results obtained estimating some different conditional heteroscedastic structures. In particular, we consider the classic GARCH model and two alternative specifications, the TARCH (Rabemananjara and Zakoian, 1993) and EGARCH (Nelson, 1991) models, which allow to capture asymmetric effects in the conditional variance.

The paper is organised as follows: section 2 illustrates the theoretical background underlying our proposal; the results of the empirical analysis are shown in section 3 where some final comments are also given.

2. Theoretical background and modelling procedure

2.1. Threshold models for the conditional mean

The Threshold AutoRegressive (TAR) models were first presented by Tong (1978) and further developed and applied in Tong and Lim (1980) and Tong (1983) (a more thorough discussion can be found in Tong,1990). A TAR model can be regarded as a piecewise linear autoregressive structure, which allows to obtain the decomposition of a complex stochastic system into smaller subsystems, based on the values assumed by a *threshold variable,* compared with a set of predetermined values, the *threshold values*.
Let $\{Y_t\}$ be a time series, a *TAR* model for Y_t is given by:

$$\begin{aligned} &Y_t = a_0^{(j)} + \sum_{i=1}^{p} a_i^{(j)} Y_{t-i} + h^{(j)} u_t \\ &X_{t-d} \in R_j \end{aligned} \quad (1)$$

where $\{u_t\}$ is i.i.d. with zero mean and finite variance, the threshold values, $\{w_0, w_1, ,..,w_l\}$, are such that $w_0<w_1<...<w_l$, $w_0= -\infty$ and $w_l= +\infty$ with $Rj = (w_{j-1},w_j]$, d is a positive integer.
If we choose as threshold variable a past realisation of Y_t, $\{Y_{t-d}\}$, the time series $\{Y_t\}$ follows a Self Exciting Threshold (SETAR) model.
To investigate the appropriateness of a threshold non-linear model instead of a simpler linear model, we perform the Tsay test for detecting threshold non-linearity. This testing procedure was presented in Tsay (1989) and successively refined and generalised by Tsay (1998). Assuming that the autoregressive order p and the delay d are known, we first perform an arranged regression based on the increasing order of the threshold variable and then use the predictive residuals calculated by recursive least squares to evaluate the test statistic. Under the null hypothesis of a linear model, the Tsay test is asymptotically distributed as a chi-squared random variable with ($p+1$) d.f..
In order to identify the model we follow the iterative procedure described in Tsay (1998). Given the identification of the threshold variable, often suggested by the nature of the specific problem, the results of the non-linearity test can be used to first select a set of possible delays $\{d_1,...,d_h\}$. For each combination of p, d and m, where m is the number of possible regimes (usually in the range of 2 or 3), we then use a grid search method and the AIC (Akaike Information Criterion) to select the threshold values and identify the final model.
Finally the model parameters in each regime can be estimated by the conditional least squares method.

2.2 A state space approach to the analysis of non-linear CH time series

Let u_t be an univariate series of prediction errors such that $(u_t|\boldsymbol{u}^{t-1})\sim N(0,h_t^2)$ and $Cov(u_t, u_{t-d}) = 0$, $\forall d \neq 0$. Also assume that u_t has finite moments up to the fourth order. A Changing Parameters Volatility (CPV) model (Storti, 1999) of order (r,s), with r and s integers, is defined as

$$u_t = \sum_{i=1}^{r} a_{i,t} u_{t\text{-}i} + \sum_{j=1}^{s} b_{j,t} h_{t\text{-}j} + e_t = \mathbf{C_t} \mathbf{x}_t + e_t \quad (2a)$$

$$\mathbf{x}_t = \mathbf{A}\ \mathbf{x}_{t-1} + \mathbf{q}_{\mathbf{t}} \quad (2b)$$

where e_t is a Gaussian white noise observation error $e_t \sim N(0,\sigma_e^2)$, $\mathbf{q}_{\mathbf{t}}$ $(n\times 1)$, with $n=r+s$, is a Gaussian serially uncorrelated system error, $\mathbf{q}_{\mathbf{t}} \sim N(\mathbf{0},\mathbf{Q})$, and $\mathbf{x}_t$ $(n\times 1)$ is an n-dimensional state vector with state variables given by the stochastically varying parameters $a_{i,t}$ $(i=1,...,r)$ and $b_{j,t}$ $(j=1,...,s)$. The observation matrix is

$$\mathbf{C}_t = [u_{t-1},...,u_{t-r} \mid h_{t-1},...,h_{t-s}]$$

while $\mathbf{A}$ is an $(n\times n)$ transition matrix of unknown coefficients. The specification of the model is completed by the usual assumptions

$$\mathbf{E}(\mathbf{q}_t e_z) = \mathbf{0} \ , \ \forall\{t,z\}$$

$$\mathbf{x}_0 \sim N(\mathbf{m}_0,\mathbf{P}) \ \ with \ \mathbf{E}\left|\mathbf{q}_t (\mathbf{x_0})'\right| = \mathbf{0} \ \ and \ \mathbf{E}(\mathbf{x}_0 e_t) = \mathbf{0} \ , \ \forall t$$

Under the above assumptions the model is *conditionally Gaussian* and the Kalman filter can be used to obtain a MMSE estimate of the state vector. The conditional variance is recursively estimated as

$$h_t^2 = \mathbf{C_t} \mathbf{P_{t|t\text{-}1}} (\mathbf{C}_t)' + \sigma_e^2 \quad (3)$$

where $\mathbf{P_{t|t\text{-}1}} = Var(\mathbf{x}_t \mid \mathbf{u}^{t-1})$. The conditional variance equation (3) can be also written as

$$\begin{aligned} h_t^2 = \sigma_e^2 &+ \sum_{i=1}^{r} p_{i,i(t)} u_{t-i}^2 + \sum_{j=r+1}^{r+s} p_{j,j(t)} h_{t-(j-r)}^2 + \sum_{i=1}^{r}\sum_{j=1}^{r} p_{i,j(t)} u_{t-i} u_{t-j} \\ &+ \sum_{i=r+1}^{s}\sum_{j=r+1}^{s} p_{i,j(t)} h_{t-(i-r)} h_{t-(j-r)} + \sum_{i=1}^{r}\sum_{j=r+1}^{s} p_{i,j(t)} u_{t-i} h_{t-(j-r)} \end{aligned} \quad (4)$$

with $p_{i,j(t)}$ being the element of place (i,j) in $\mathbf{P_{t|t\text{-}1}}$. Compared to conventional approaches, the CPV model has two main advantages. First, it allows for time varying parameters in the conditional variance specification (4). Second, interaction terms between past innovations and volatilities are easily included in the model. The choice $\mathbf{A} = \mathbf{0}$ yields a more parsimonious random coefficient version of the CPV model that we will call the *constrained CPV* model or, abbreviated, CPV-C. In a CPV-C model the conditional variance parameters are constant but the interaction terms are still present. It can be shown (Storti, 1999) that, if $\mathbf{A} = \mathbf{0}$ and the covariance matrix $\mathbf{Q}$ is diagonal, the resulting CPV-C model will have the same conditional variance as a Generalized Autoregressive Conditionally Heteroskedastic (GARCH) model (Bollerslev, 1986) of the same order. Similarly, for $s=0$ the model is a random coefficient autoregressive model of order *r*. A generalization of model (2) which allows for simultaneous modelling of conditional mean and variance is given by the *regression CPV* model

$$y_t = \mathbf{M}_t \boldsymbol{\beta} + u_t = \mathbf{M}_t \boldsymbol{\beta} + \mathbf{C}_{\mathbf{t}} \mathbf{x}_t + e_t \qquad (5a)$$

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{q_t} \qquad (5b)$$

where y_t is an observed time series, $\boldsymbol{M}_t$ is a vector $(1\times g)$ of endogenous or exogenous regressors and $\boldsymbol{\beta}$ a vector $(g\times 1)$ of unknown parameters. The model parameters $\mathbf{A}$, $\mathbf{Q}$, σ_e^2 and $\boldsymbol{\beta}$ can be estimated maximizing a Gaussian log-likelihood function expressed in the classical prediction error decomposition form. The formulation of model (5) is quite general and the regression term included into the observation equation can incorporate an ARMA type structure with exogenous explanatory variables. Also, replacing the constant parameter vector $\boldsymbol{\beta}$ by a time variable vector $\boldsymbol{\beta}_t$, we can easily accommodate for some common non-linear structures such as Threshold Autoregressive models.

Under the assumption of conditional normality, the log-likelihood function of a regression CPV model can be written as

$$\ell(y;\mathbf{A},\sigma_e^2,\mathbf{Q}) = -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log\left[(h_t)^2\right] - \frac{1}{2}\sum_{t=1}^{T}\frac{(e_{t|t-1})^2}{(h_t)^2} \qquad (6)$$

where

$$e_{t|t-1} = y_t - \mathbf{M}_t\boldsymbol{\beta} - \mathbf{C}_t\mathbf{x}_{t|t-1}$$

and

$$(h_t)^2 = \mathbf{C}_t\mathbf{P}_{t|t-1}\left(\mathbf{C}_t\right)'\sigma_e^2$$

with $\mathbf{x}_{t|t-1} = E(\mathbf{x_t} \mid \mathbf{y}^{t\text{-}1})$ and $\mathbf{P}_{t|t-1} = Var(\mathbf{x}_t \mid \mathbf{y}^{t\text{-}1})$. The unknown parameters in $\{\mathbf{A},\sigma_e^2,\mathbf{Q}\}$ can be estimated by maximising the Gaussian log-likelihood (6) using a version of the EM algorithm tailored for state space models by Wu *et al.* (1996). Alternatively *scoring* or *quasi-Newton* methods could also be used (see Watson and Engle, 1983, for a discussion on the application of the method of scoring in the context of state space models).

Finally, it is worth noting that, when forecasting from a CPV model, the conditional variance h_t^2 affects the estimate of the conditional mean $E(u_t \mid \boldsymbol{u}^{t-1})$ in two different ways. First, h_t^2 enters the state updating equation, second, the estimated $E(\mathbf{x}_t \mid \mathbf{u}^{t-1})$ does not necessarily have to be equal to zero.

3. Asymmetric effects in the CPV model

If the model order is such that s>0, asymmetric effects are introduced into the CPV-C conditional variance specification by means of the interaction terms between past shocks and volatilities. It follows that, in CPV-C type models, the effect of a past shock on the present volatility derives from the sum of two components. Of these, the first, as in GARCH models, is given by a linear functions of the past squared shock with no regard for its sign. Differently, the second adds a further contribution to the value of the conditional variance depending on the sign of the shock. This term is, in module, proportional to the magnitude of the shock rescaled by past conditional standard deviations. In order to clarify this point, we consider, as an example, the simple CPV-C(1,1) model with conditional variance equation given by

$$h_t^2 = \nu_0 + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1}^2 + \delta_1 u_{t-1} h_{t-1} \quad (7)$$

where $\nu_0 = \sigma_e^2$, $\alpha_1 = Q_{1,1}$, $\beta_1 = Q_{2,2}$ and $\delta_1 = 2Q_{1,2}$. As it is evident from equation (7), the asymmetry in the relation between the conditional variance and past residuals comes from the cross-term $\delta_1 u_{t-1} h_{t-1}$. In particular, for $\delta_1 < 0$, a positive penalty term will be added to h_t^2 if $u_{t-1} < 0$ while a positive quantity will be subtracted from h_t^2 if $u_{t-1} > 0$. A similar reasoning applies to the case in which $\delta_1 > 0$.

This suggests a simple testing procedure for verifying the presence of asymmetric effects in the relationship between conditional variance and past shocks by testing for δ_1 significantly different from 0. The presence of leverage effects can be tested by the hypothesis that $\delta_1 < 0$. Fig. 1 shows the simulated news impact curve for a particular CPV-C (1,1) model incorporating leverage effects.

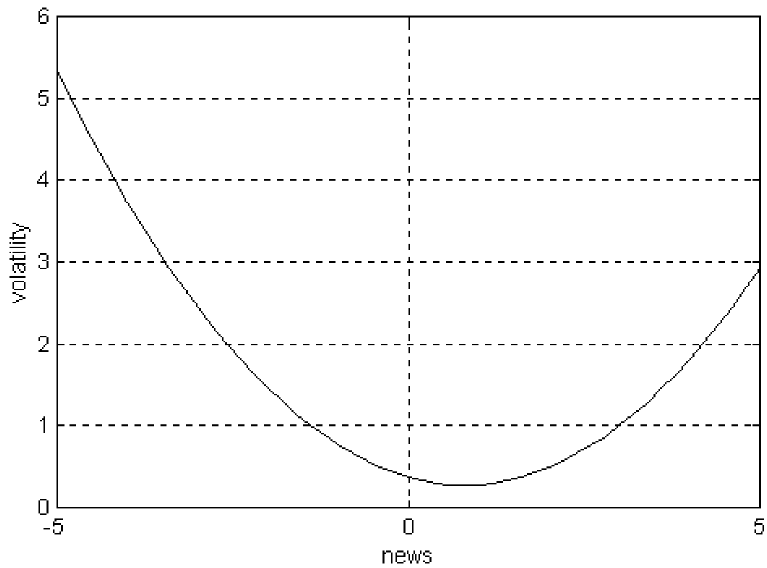

Fig. 1 Simulated news impact curve for a CPV-C(1,1) model with parameters ν_0=0.05 α_1=0.15, β_1=0.65 and δ_1=-0.34.

Also, an attractive feature of the model is that, when u_{t-1} and $\delta_1 < 0$, the magnitude of the penalty term added to the conditional variance will not depend only on the magnitude of the shock itself but it will be weighted, or, more properly, rescaled, by the conditional standard

deviation at time (t-1). So the greater will be the uncertainty associated with the negative shock u_{t-1}, and the greater will be the impact of u_{t-1} on h_t^2.
The same argument applies to higher order models even if, in high dimensional structures, care should be taken in the identification of the relevant lags of interaction in order to avoid to incur the so called *curse of dimensionality*.

3. Empirical results

In this section we present the results of an application of the proposed modelling approach to the analysis of some stock market indexes. In particular we consider the daily U.S. NASDAQ Composite, the U.S. Standard and Poor's 500 and the U.K. FTSE 100. The sample covers the period from 2 January 1996 to 16 November 1999. Returns are calculated as logarithmic first differences, $R_t = \nabla(\log X_t)$. The plots of the original data and returns are shown in Fig. 2.
Even if the algorithm for the maximisation of the likelihood function is able to naturally deal with the simultaneous estimation of the conditional mean and variance model parameters, in order to guarantee full comparability of the performances of the models considered in estimating the conditional variance of the series, we revert to a two stage modelling procedure.
First, in order to investigate the presence of a threshold type non-linear structure in the data we have performed Tsay's test for non-linearity. The results obtained for different values of the delay parameter are shown in Tab. 1. The test leads to reject the null hypothesis of linearity for all of the series considered. Theoretical and empirical evidence suggest that the behaviour of stock market returns is influenced by what has happened in the previous days. Therefore we choose as possible threshold variables lagged values of the returns. This leads to the identification of a SETAR model for the conditional mean. The test also indicates, as the best choice for the delay *d,* the one which corresponds to the highest value of the test statistic.
For all the series we have chosen the autoregressive order p to lie in the range [1, 5], and we have considered *m*=2,3 as the possible number of regimes. For each combination of d, p and s we have chosen the SETAR model that gives the minimum AIC.

Tab.1: *Tsay's threshold non-linearity test*

d		1	2	3	4	5
SP500	*Test*	13.740	16.087	14.981	40.573	14.679
	d.f.	5	5	5	5	5
FTSE	*Test*	5.197	8.122	8.300	26.560	11.349
	d.f.	4	4	4	4	4
Nasdaq	*Test*	5.625	3.939	4.333	17.341	7.239
	d.f.	3	3	3	3	3

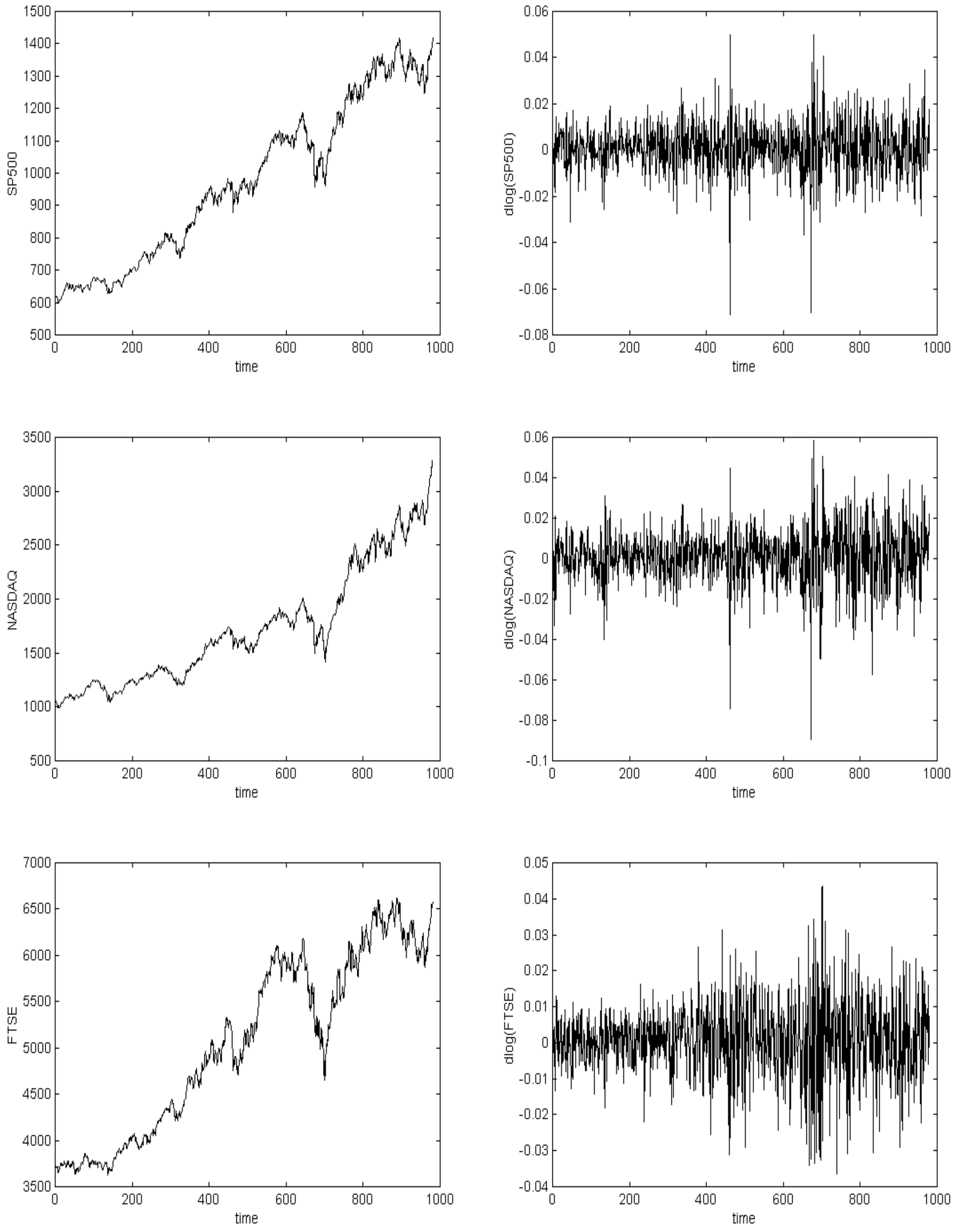

Fig. 2: From top to bottom: original data (left) and logarithmic first differences (right) of S & P 500, NASDAQ and FTSE 100.

The final models identified, after refining the thresholds and the orders, are a SETAR(3,5) with delay d=4 and w=0.001044 for the S&P 500, a SETAR(2,1) with d=4 and w=0.000315 for the FTSE 100, and a SETAR (2,2) with d=4 and w=0.00 for the NASDAQ series.
The least squares estimate of the models parameters and the corresponding standard errors (in parentheses) are shown in Tab. 2. In order to assess the fitting accuracy of the SETAR models we calculate some widely used loss functions (Tab.3).

Tab. 2 *Least squares estimates and standard error of SETAR models*

	Regime	a_0	a_1	a_2	a_3	a_4	a_5
SP500	**I**	0.00186 (0.00048)	-0.10279 (0.04038)	-0.10947 (0.03888)	-0.06195 (0.39965)		
	II	-0.00144 (0.00084)	0.10173 (0.05027)	0.15097 (0.05303)		0.18264 (0.07397)	-0.08585 (0.04483)
FTSE	**I**	0.00122 (0.00051)		-0.16896 (0.04565)			
	II	0.0008 (0.0004)	0.17209 (0.04342)				
Nasdaq	**I**	0.00190 (0.04392)		-0.09016 (0.00073)			
	II		0.07558 (0.04742)	0.12456 (0.04742)			

Tab.3: *Values of different loss functions for the estimation of the conditional mean*

	RMSE	**MAE**	**MAPE** ($\times10^{-6}$)	**THEIL** ($\times10^{-6}$)
S&P500	0.010778740	0.0080519	8.4626584	5.3523706
FTSE	0.010289465	0.0077406	1.5182596	0.99863828
Nasdaq	0.014082002	0.0103381	6.1528462	3.8766286

For each series we have then performed an ARCH-LM test (Engle, 1982) on the residuals of the threshold model estimated for the conditional mean. The results of the test and the analysis of the autocorrelation functions of the squared residuals suggest the presence of autoregressive conditional heteroskedasticity in the data. In order to detect any possible asymmetry in the conditional variance component, we look at the cross correlation between the squared standardized residuals and lagged standardized residuals (Fig. 3). These cross correlations should be zero if asymmetric effects are not present and be negative in presence of asymmetry. The estimated correlations show evidence in favour of the hypothesis of asymmetry for the S&P 500 and NASDAQ series while, for the FTSE 100, the values of the autocorrelation function lie inside the $\pm$ 2 s.e. confidence bands except for lag 5.
The next step is to identify and estimate a suitable CPV model for the conditional variance. In particular, we consider the parsimonious CPV-C specification. The order of the model to be fitted has been chosen to minimise the value of the Schwarz Criterion (SC). Tab. 4 reports the values of the SC for different model specifications together with the AIC and log-likelihood values. The search has been restricted within the intervals $1 \leq r \leq 2$ and $0 \leq s \leq 2$. For all the series a CPV-C (1,1) model is identified.

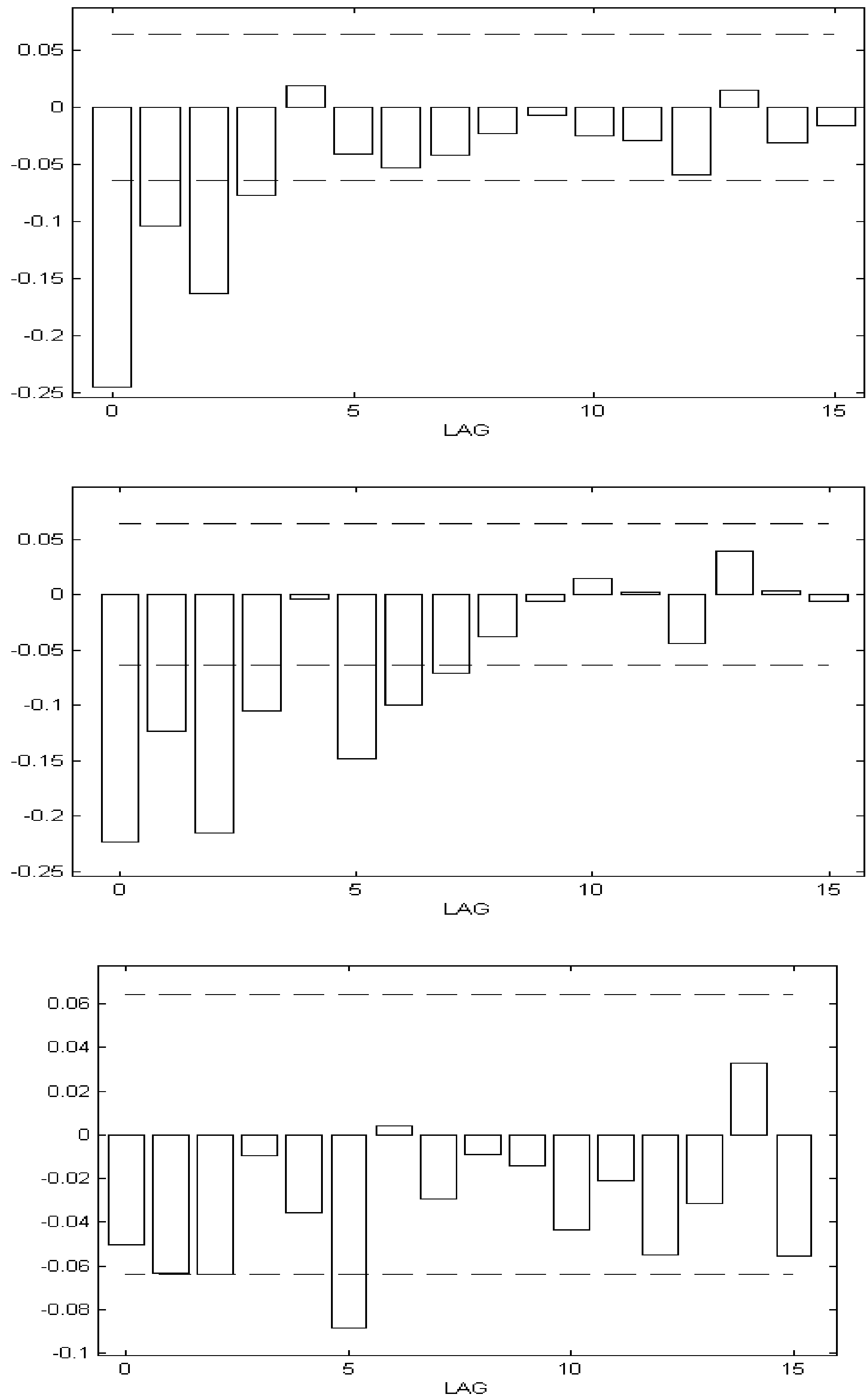

Fig 3: From top to bottom, cross-correlations between u_t^2 and u_{t-i} for S&P 500, NASDAQ and FTSE100.

Tab.4: *AIC, SC and log-likelihood values for different CPV models*

	Orders		AIC	SC	LL
SP500	1	0	-6.2352830	-6.2252595	3038.5828
	1	**1**	**-6.3338302**	**-6.3137833**	**3088.5753**
	1	2	-6.3250968	-6.2899861	3084.1596
	2	0	-6.2800987	-6.2600354	3059.2680
	2	1	-6.2980138	-6.2629031	3070.9837
	2	2	-6.3167281	-6.2615541	3084.0882
FTSE	1	0	-6.3331366	-6.3231214	3089.4041
	1	**1**	**-6.4353684**	**-6.4153379**	**3141.2421**
	1	2	-6.4465846	-6.4115026	3146.4867
	2	0	-6.3572922	-6.3372453	3100.0013
	2	1	-6.4224969	-6.3874149	3134.7560
	2	2	-6.4521051	-6.3969763	3153.1752
Nasdaq	1	0	-5.7456715	-5.735648	2800.1420
	1	**1**	**-5.9307591**	**-5.910712**	**2892.2797**
	1	2	-5.9172976	-5.882187	2885.7653
	2	0	-5.8292261	-5.809162	2839.9185
	2	1	-5.9275484	-5.892437	2890.7523
	2	2	-5.9244467	-5.869272	2893.2433

The maximum likelihood estimates and relative standard errors of the CPV-C model parameters are shown in Tab. 5.

Tab. 5: *ML parameter estimates and asymptotic s.e. (in parentheses)*

	σ_e^2	Q_{11}	Q_{12}	Q_{22}
SP500	0.0000163 (0.0000045)	0.0436 (0.0202)	-0.0898 (0.0178)	0.8161 (0.0513)
FTSE	0.00001433 (0.0000042)	0.1116 (0.0312)	-0.0739 (0.0196)	0.7466 (0.0642)
Nasdaq	0.0000149 (0.0000086)	0.1281 (0.0294)	-0.0852 (0.0168)	0.7878 (0.0411)

The negative sign of the Q_{12} parameter confirms the presence of a leverage effect as suggested by the cross-correlation analysis.
The results have been compared with those obtained using some different conditional variance specification. Namely we have considered the classical GARCH(p,q) model (Bollerslev, 1986), given by:

$$h_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-1}^2 + \sum_{j=1}^{p} \beta_j h_{t-1}^2;$$

the Exponential GARCH model (EGARCH, Nelson 1991), with conditional variance specification given by:

$$\ln(h_t^2) = \alpha_0 + \sum_{j=1}^{p} \beta_j \ln(h_{t-j}^2) + \gamma \left| \frac{u_{t-1}}{h_{t-1}} \right| + \lambda \frac{u_{t-1}}{h_{t-1}} ;$$

the TARCH model (Rabemananjara and Zakoian, 1993) which, as the EGARCH, includes asymmetric effects in the conditional variance:

$$h_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \gamma u_{t-1}^2 d_{t-1} + \sum_{j=1}^{p} \beta_j h_{t-j}^2$$

with d_t=1 if u_t<0 and d_t=0 otherwise.
The estimated parameters for the GARCH, E-GARCH and TARCH models fitted for the conditional variance of each of the three series are given in tables 6, 7 and 8, respectively.

Tab. 6 *Parameters estimates and standard error of GARCH models*

	α_0	α_1	α_2	β_1	β_2
SP500	0.00003	0.08702		0.88459	
	(0.00001)	(0.01303)		(0.02019)	
FTSE	0.00004	0.00892	0.03729	0.95061	
	(0.00002)	(0.025439)	(0.02654)	(0.01175)	
Nasdaq	0.00006	0.17887		0.78627	
	(0.00002)	(0.02410)		(0.02952)	

Tab. 7 *Parameters estimates and standard error of EGARCH models*

	α_0	γ_1	γ_2	λ_1	λ_2	β_1
SP500	-0.42264	-0.11529	0.24769	-0.21988	0.11488	0.96535
	(0.09501)	(0.05934)	(0.05503)	(0.042371)	(0.04484)	(0.00981)
FTSE	-0.13008	0.09291		-0.04770		0.99391
	(0.04337)	(0.02534)		(0.01579)		(0.00351)
Nasdaq	-0.81521	0.27703		-0.12445		0.93196
	(0.15429)	(0.04095)		(0.0831)		(0.01568)

Tab. 8 *Parameters estimates and standard error of TGARCH models*

	α_0	α_1	α_2	θ	β_1
SP500	-0.00005	0.08463	0.08399	0.16555	0.86356
	(0.00001)	(0.02673)	(0.02866)	(0.86356)	(0.01798)
FTSE	0.00004			0.06710	0.95807
	(0.00002)			(0.01579)	(0.01042)
Nasdaq	-0.00001	0.06712		0.20333	0.77876
	(0.000002)	(0.02720)		(0.03427)	(0.03343)

For each model, the performance in the estimation of the conditional variance has been assessed on the basis of five different loss functions, namely:

$$MSE_v = T^{-1}\sum_{t=1}^{T}[u_t^2 - \hat{h}_t^2]^2 \qquad LSE_v = T^{-1}\sum_{t=1}^{T}[ln(u_t^2) - ln(\hat{h}_t^2)]^2$$

$$MAE_v = T^{-1}\sum_{t=1}^{T}|u_t^2 - \hat{h}_t^2| \qquad LAE_v = T^{-1}\sum_{t=1}^{T}|ln(u_t^2) - ln(\hat{h}_t^2)|.$$

$$HMSE_v = T^{-1}\sum_{t=1}^{T}\left[\frac{u_t^2}{\hat{h}_t^2} - 1\right]^2$$

Bollerslev et al. (1994) point out that, in many cases, RMSE can be inappropriate, as a measure of goodness of fit, since it penalizes conditional variance estimates which are different from the realized squared residuals in a fully symmetrically fashion. Alternative loss functions which penalize conditional variance estimates asymmetrically are the logarithmic loss functions (LSE_v and LAE_v) and the Heteroscedasticity Adjusted MSE ($HMSE_v$).
The results obtained have been reported in Table 9. The last two column give the values of the maximised log-likelihood and the number of estimated parameters for each model[1].

Tab.9: *Values of different loss functions for the estimation of the conditional variance*

	RMSE$_v$	**MAE$_v$**	**LSE$_v$**	**LAE$_v$**	**HMSE$_v$**	**LL**	**NP**
S&P500							
CPV-C(1,1)	0.00025623	0.000119447	6.6631537	1.8223311	3.2258	3088.57	4
GARCH(1,1)	0.00026118	0.000123238	6.6482462	1.8229540	3.9526	3081.23	3
E-GARCH(2,1)	0.00025923	0.000119847	6.4474081	1.7788037	4.0150	3107.51	6
TARCH(2,1)	0.00026028	0.000121892	6.4796421	1.7875947	4.3201	3101.06	5
Nasdaq							
CPV-C(1,1)	0.00040197	0.000190241	6.9320506	1.8262975	2.6794	2892.27	4
GARCH(1,1)	0.00041594	0.000202138	7.0631666	1.8505669	3.2411	2880.56	3
E-GARCH(1,1)	0.00040443	0.000193965	6.9526359	1.8276174	3.1444	2893.50	4
TARCH(1,1)	0.00041071	0.000198711	6.9526060	1.8284924	3.0980	2892.42	4
FTSE							
CPV-C(1,1)	0.00017941	0.000105197	8.4181208	1.9044951	2.5588	3141.24	4
GARCH(2,1)	0.00017585	0.000104466	8.2114254	1.8518322	2.4131	3175.03	4
E-GARCH(1,1)	0.00017381	0.000103006	8.1523470	1.8429394	2.3494	3179.76	4
TARCH(1,1)	0.00017418	0.000102389	8.1419768	1.8405596	2.5915	3175.80	4

The performances of the CPV-C model results to be better for the series characterised by a substantial asymmetric component (S&P500, NASDAQ) than for the FTSE100, for which the cross-correlation analysis (Fig.2) does not show a so strong evidence in favour of the hypothesis of asymmetry. In particular for the NASDAQ series the CPV-C performs better than all the other models on the basis of all the loss functions used. For the S&P 500, again,

[1] Strictly, the maximised log-likelihood value of the CPV model is not comparable to those obtained for the other models (GARCH, EGARCH, TARCH) since, while the likelihood function for each of the latter is defined on the residuals of the model estimated for the conditional mean, the value reported for the CPV model refers to the classical prediction error decomposition form of the likelihood which is defined on the series of the one step-ahead prediction errors made in forecasting these residuals.

the minimum $RMSE_v$, MAE_v and $HMSE_v$ values are obtained for the CPV-C model while a worse performance is achieved on the basis of the logarithmic loss functions LSE_v and LAE_v. which exaggerate the interest in predicting when residuals are close to 0.

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