Preference relations in ranking multivalued alternatives using stochastic dominance: case of the Warsaw Stock Exchange

by

Grażyna Trzpiot

Department of Statistics
Academy of Economics,Katowice
ul. 1- Maja 50, 40-287 Katowice, Poland

Abstract: This study used stochastic dominance tests for ranking alternatives under ambiguity, to build an efficient set of assets for a different class of investors. We propose a two step procedure: first test for multivalued stochastic dominance and next calculate the value of preference relations. The empirical part of paper was set by results from the Warsaw Stock Exchange.

Key words: ambiguity, stochastic dominance, efficiency criteria, preference relations

1. Introduction

While Stochastic Dominance has been employed in various forms as early as 1932, it has been since 1969-1970 developed and extensively employed in the area of economics, finance and operation research. In this study the first, second and third order stochastic dominance rules are discussed for ranking alternatives under ambiguity with an emphasis on the development in the area of financial issues. The first part of paper reviews the Stochastic Dominance properties. While the second part of the paper deals with the effectiveness of the various Stochastic Dominance rules in financial application.

2. Stochastic Dominance

In decision situations we have to compare many alternatives. When alternatives take uncertain character we can evaluate the performance of alternatives only in a probabilistic way. In finance, for example, problems arise with stock selection when we need to compare return distributions. The construction of a local preference relation already requires the comparison of two probability distributions. Stochastic dominance is based on a model of risk averse preferences, which was done by Fishburn (1964) and was extended by Levy and Sarnat (1984,1992).

DEFINITION 1. Let F(x) and G(x) be the cumulative distributions of two distinct uncertain alternatives X and Y, with support bounded by $[a, b] \subset R$ and $F(x) \neq G(x)$ for some $x \in [a, b] \subset R$. X dominates Y by first, second and third stochastic dominance (FSD, SSD, TSD) if and only if

$$H_1(x) = F(x) - G(x) \leq 0 \text{ for all } x \in [a,b] \quad (F \text{ FSD } G) \tag{1}$$

$$H_2(x) = \int_a^x H_1(y)dy \le 0 \text{ for all } x \in [a,b] \quad (F\ SSD\ G) \tag{2}$$

$$H_3(x) = \int_a^x H_2(y)dy \le 0 \text{ for all } x \in [a,b] \quad (F\ TSD\ G) \tag{3}$$

For definition of FSD and SSD see Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970). Whitmore (1970) suggested the criterion for TSD. The relationship between the three stochastic dominance rules can be summarised by the following diagram: FSD $\Rightarrow$ SSD $\Rightarrow$ TSD, which means that dominance by FSD implies dominance by SSD and dominance by SSD in turn implies dominance by TSD.

When, in decision situations, we have an ambiguity on value of ranking uncertain alternatives, then we map a point probability to an ambiguous outcome. Probability distribution maps probabilities to outcomes described by intervals. Probability mass, summing to one, is distributed over the subintervals of the outcome space. The outcome space is continuous, X is an interval in R and $p(A_j)$ denote the probability mass attributed to the subinterval of the outcomes space, with no future basis for establishing the likelihood of a specific value in that subinterval. Ambiguities in outcomes can be represented by a set of probability distributions. Each family has two extreme probability distributions on outcome space X. Lower probability distribution is identified by probability mass concentrated onto minimum element or value in the subset or interval A_j. Upper probability distribution is identified by probability mass concentrated onto maximum element or value in the subset or interval A_j.

DEFINITION 2. Lower probability distribution for all values $x_i \in X$, we say

$$p_*(x_i) = \sum_{j:\, x_i = \min\{y \,:\, y \in A_j\}} p(A_j) \tag{4}$$

According to this definition we have: $\sum_i p_*(x_i) = 1$.

DEFINITION 3. Upper probability distribution for all values $x_i \in X$, we say

$$p^*(x_i) = \sum_{j:\, x_i = \max\{y \,:\, y \in A_j\}} p(A_j) \quad , \tag{5}$$

Now we also have: $\sum_i p^*(x_i) = 1$.

In case of the point values of random variable both distributions (lower and upper probability distributions) are exactly the same: $p_*(x_i) = p^*(x_i) = p(x_i)$ and we have a probability distribution in the classical sense.

EXAMPLE 1 We determine lower and upper probability distributions for random variable **X**, which outcomes are multivalued, include in some intervals A_j:

A_j	[2, 4]	[3, 4]	[4, 5]	[5, 6]
$p(A_j)$	0,5	0,2	0,2	0,1

Table 1. Probability distribution for random variable **X**

According to the definitions 2 and 3 we have lower and upper probability distributions for random variable **X**:

x_j	2	3	4	5	6
$p_*(x_j)$	0,5	0,2	0,2	0,1	-
$p^*(x_j)$	-	-	0,7	0,2	0,1

Table 2. Lower and upper probability distributions for random variable **X**

Our approach now is to use stochastic dominance for ranking multivalued alternatives by using lower and upper probability distributions of each alternative (Langewisch and Choobineh (1996)).

DEFINITION 4. Let two distinct uncertain multivalued alternatives **X** and **Y** have lower probability distributions respectively $F_*(x)$ and $G_*(x)$, upper probability distributions respectively $F^*(x)$ and $G^*(x)$, with support bounded by $[a, b] \subset R$ and $F_*(x) \neq G^*(x)$ for some $x \in [a, b] \subset R$. We have multivalued first, second and third stochastic dominance if and only if

$$H_1(x) = F_*(x) - G^*(x) \le 0 \text{, for all } x \in [a,b] \quad \text{, } (\mathbf{X} \text{ FSD } \mathbf{Y}) \tag{6}$$

$$H_2(x) = \int_a^x H_1(y)dy \le 0, \text{ for all } x \in [a,b] \quad \text{, } (\mathbf{X} \text{ SSD } \mathbf{Y}) \tag{7}$$

$$H_3(x) = \int_a^x H_2(y)dy \le 0, \text{for all } x \in [a,b] \text{, } (\mathbf{X} \text{ TSD } \mathbf{Y}) \tag{8}$$

EXAMPLE 2 **(**Trzpiot (1998a)) Let take the random variables **C** and **D** whose outcomes are multivalued, include in some intervals A_j as follows:

A_j	[0,1]	[1, 2]	[2, 3]	[3, 4]
p(**C**)		0,2	0,4	0,4
p(**D**)	0,3	0,15	0,55	-

Table 3. Probability distributions for random variables **C** and **D**

We can determine lower and upper probability distributions for random variables **C** and **D** and next we can check that **C** TSD **D** (third degree multivalued stochastic dominance)**.**

3. Stochastic Dominance rules in portfolio selection

We have an appropriate investment criteria for the three alternative risk-choice situations. Stochastic dominance theorems assume that a given class of utility function can describe a decision-maker's preference structure. We initially assume that no information is available on the shape of the utility function, apart from the fact that it is non-decreasing. An efficiency criterion is a decision rule for dividing all potential investment alternatives into two mutually exclusive sets: an efficient set and an inefficient set. Firstly, using stochastic dominance tests we reduce the number of investment alternatives by constructing an efficient set of alternatives appropriate for a given class of investors. At the second step, we can make the final choice of the alternatives in accordance to particular preferences of the investor.

The FSD rule places no restrictions on the form of the utility function beyond the usual requirement that it be nondecreasing. Thus this criterion is appropriate for risk averters and risk lovers alike since the utility function may contain concave as well as convex segments. Owing to its generality, the FSD permits a preliminary screaming of investment alternatives eliminating those alternatives which no rational investor (independent of his attitude toward risk) will ever choose.

The SSD is the appropriate efficiency criterion for all risk averters. Here we assume the utility function to be concave. This criterion is based on stronger assumptions and therefore, it permits a more sensitive selection of investments. On the other hand, the SSD is applicable to a smaller group of investors. The SSD efficient set must be a subset of the FSD efficient set; this means that all the alternatives included in the FSD efficient set, but not necessarily vice versa.

The TSD rule is appropriate for a still smaller group of investors. In addition to the risk aversion assumption of SSD, the TSD also assumes decreasing absolute risk aversion. The population of risk averters with decreasing absolute risk aversion is clearly a subset for all risk averters, and the TSD efficient set is correspondingly a subset of the SSD efficient set: all TSD efficient portfolios are SSD efficient, but not vice versa.

The three stochastic dominance criteria, FSD, SSD and TSD, are optimal in the sense that given the assumptions regarding the investors preferences (describing as a class of utility functions), the application of the corresponding stochastic dominance criterion ensures a minimal efficient set of investment alternatives. For a more detailed description of utility functions belong to the three classes of the utility function divided all investors to groups by stochastic dominance test see Quirk and Saposnik (1962), Levy and Kroll (1970), Levy (1992), Langewisch and Choobineh (1996).

4. Preference relations in ranking multivalued alternatives using stochastic dominance

When we verified some of the stochastic dominance we also observed additionally that the dominance is not equivalent. Comparing results of ranking alternatives we can observe, that in one type of stochastic dominance the overlapping area of the two comparing distributions are changing but the type of stochastic dominance is still the same. For the investor, when we compare the return distributions, it can be a different situation, so we need the method for ranking preference inside of one type of stochastic dominance. We present preference relations that could help globally ranking alternatives. When one of the type of stochastic dominance is verified, we can calculate the degree of the decision maker preference by using the preference relation.

DEFINITION 5 For two distinct uncertain alternatives X and Y, f(x) and g(x) are the density functions, for $x \in [a, b] \subset R$, F(x) and G(x) are the cumulative distributions, μ_f and μg are the means of the alternatives X and Y, we define the index

$$\Phi(f,g) = \frac{|\mu_f - \mu_g|}{\int_a^b |H_1(x)| dx} \qquad (9)$$

According to the type of dominance this index may take different values in [0, 1]. These values should reflect a certain degree of the decision-maker’s preference relatively to the considered attribute. The clarification of the level of the decision maker’s preference impose us to introduce two other functions with values in [0, 1]:

DEFINITION 6 For two distinct uncertain alternatives X and Y, f(x) and g(x) are the density functions ($p_f(x)$ and $p_g(x)$ are probability distributions for the discrete case, respectively for X and Y), for $x \in [a, b] \subset R$, F(x) and G(x) are the cumulative distributions, SV_f and SV_g are semi –variances of the alternatives X and Y then we define:

$$\Psi(f,g) = \begin{cases} 1 - \int_a^b \min(f(x), g(x)) dx, \text{ in the continous case} \\ 1 - \sum_x \min(p_f(x), p_g(x)), \text{ in the discrete case} \end{cases} \qquad (10)$$

$$\theta(f,g)=\frac{\left|SV_f - SV_g\right|}{\int_a^b \left|H_2(x)\right|dx}\;, \qquad (11)$$

From these three functions it is possible to define a degree of credibility of the preference relation of the alternative X to the alternative Y.

DEFINITION 7 For two distinct uncertain alternatives X and Y, with respect to definition 5 and 6, we define the preference relation of the alternative X to the alternative Y as:

$$\delta(f,g)=\begin{cases}\Psi(f,g), & \text{if FSD}\\ \Psi(f,g)\cdot\Phi(f,g), & \text{if SSD and not FSD}\\ \Psi(f,g)\cdot\Phi(f,g)\cdot\theta(f,g), & \text{if TSD and not SSD}\\ 0, & \text{otherwise}\end{cases} \qquad (12)$$

The degree of preference decreases progressively as we go from the dominance FSD to the dominance TSD. This degree of credibility of the preference relation will allow us to know the nature of the preference relation between two alternatives X and Y basis of the characteristic obtained for three functions by type of dominance, in the case of each dominance. The important properties of δ are: antireflexivity, asymmetry and transivity (Martel, Azondekon, Zaras(1994)). It is easy to apply this relation for rank multivalued outcomes, which we firstly rank by multivalued stochastic dominance.

5. Empirical application of multivalued stochastic approximations: evidence from the Warsaw Stock Exchange

Continuous observations of the price of assets from the Warsaw Stock Exchange are the empirical example of multivalued random variables. Values of the price of the asset are from an interval: from minimal price to maximal price, each day. Daily we have empirical realisation of multivalued random variables. As an example of application of the theory from the previous points we made an analysis of the daily rate of return assets from the Warsaw Stock Exchange in June 1997. We determined multivalued rates of return for the set of assets from the Warsaw Stock Exchange, and then we applied the multivalued stochastic dominance for ranking alternatives. We can compare alternatives used stochastic dominance tests for ranking alternatives under ambiguity, to establish an efficient set of asset. The next step of the procedure is to apply to an efficient set of asset a preference relation δ to make the final ranking of the set of assets.

We started by taking the price of a group of 14 asset: ANIMEX, BPH, BRE, BSK, BUDIMEX, DEBICA, ELEKTRIM, MOSTOSTALEXP, OKOCIM, OPTIMUS, ROLIMPEX, STALEXPORT, UNIVERSAL, WBK, which were observed at Warsaw Stock Exchange in June 1997. From the set of information about price we count the multivalued rate of return. In financial application we have each value from time series, in our analysis - the rate of return, in the same probability 1/n, according to the time of observations (see Levy and Sarnat (1984)). So we are able to build lower and upper probability distributions for the set of assets and next we can apply the multivalued stochastic dominance for ranking alternatives.

We determined multivalued rates of return for the set of assets from the Warsaw Stock Exchange in June 1997, and then we applied the multivalued stochastic dominance for ranking alternatives. For whole analysis of all 14 assets, we should match each of two assets. We present the results of analysis in table 4, we read this table from left to the top, for example 2 SSD 3 (Trzpiot (1998b)).

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	-											SSD		
2		-	SSD			SSD						SSD		
3			-									SSD		
4				-								SSD		
5	SSD				-							SSD		
6						-						SSD		
7				SSD			-		FSD			SSD		
8								-				SSD		
9									-			SSD		
10										-		SSD		
11											-	SSD		
12												-		
13													-	
14				SSD								SSD		-

Table 4. Results the analysis of the set of assets from the Warsaw Stock Exchange in June 1997 by stochastic dominance

Where: 1) ANIMEX, 2) BPH, 3) BRE, 4) BSK, 5) BUDIMEX, 6) DEBICA, 7) ELEKTRIM, 8) MOSTOSTALEXP, 9) OKOCIM, 10) OPTIMUS, 11) ROLIMPEX, 12) STALEXPORT, 13) UNIVERSAL, 14) WBK.

From these results we have the implications that STALEXPORT was dominated by all assets. According to stochastic dominance rule in portfolio selection the investors can choose different assets to their efficient set. The investor neutral to the risk can add to efficient set: ELEKTRIM (because of FSD). The investor with aversion to the risk can add to efficient set: BPH, BUDIMEX, WBK (because of SSD). We can notice that in our research period of time was not TSD that means that it was difficult time for invest for investors with decreasing aversion to the risk.

Most of the observed stochastic dominance is SSD, so we need to compare the quality of these relations. We can calculate value of the preference relations δ for lower and upper distributions, which were important for multivalued stochastic dominance tests. The degree of preference decreases progressively as we go from the dominance FSD to the dominance SSD. This degree of credibility of the preference relation will allow us to know in the case of each dominance, the nature of the preference relation between two comparing assets based on the type of dominance. We present the results of analysis in table 5, read this table from left to the top, for example $\delta(2, 3) = 0{,}5378$.

δ	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	-											0,4229		
2		-	0,5378			0,4320						0,4371		
3			-									0,4203		
4				-								0,4405		
5	0,3295				-							0,4229		
6						-						0,4225		
7				0,4138			-		0,4560			0,4148		
8								-				0,4242		
9									-			0,4545		
10										-		0,4540		
11											-	0,4736		
12												-		
13												0,4238	-	
14				0,5000								0,5504		-

Table 5. Results of analysis of the set of assets from the Warsaw Stock Exchange in June 1997 by the preference relations δ
where: 1) ANIMEX, 2) BPH, 3) BRE, 4) BSK, 5) BUDIMEX, 6) DEBICA, 7) ELEKTRIM, 8) MOSTOSTALEXP, 9) OKOCIM, 10) OPTIMUS, 11) ROLIMPEX, 12) STALEXPORT, 13) UNIVERSAL, 14) WBK.

Now we have additional information by value of preference relations δ. As an example we can notice that all assets in different degree dominate STALEXPORT. We can propose for the investor with aversion to the risk efficient set (it was choosing by SSD) with the higher value of δ: BPH, WBK, and ROLIMPEX (the number of assets depends on how many assets we want to take to the portfolio).

After these two steps of analysis: test for multivalued stochastic dominance and calculating value of preference relations δ, the investor can choose an efficient set of assets, according to individual preferences. Next he can choose a method for creating an individual portfolio.

6.Conclusion

Multivalued stochastic approximations have an application in this class of problems when the classical point of view from random variables is not enough, when we have a set as an outcomes of random variables. The area of applications is very wide. When we determine multivalued stochastic variables, we can do some empirical applications. We can define multivalued stochastic dominance, and then we can do some analysis on the stock exchange. We can use the same method as in classical stochastic dominance and calculate the value of preference relations δ, which help in ranking the set of assets. The empirical examples are the illustration of the fact, that we have a number of nondominated alternatives. In the situation, where dominance cannot be shown, the investors may be satisfied by information about any of nondominated alternatives, or they may look for some additional information and repeat analysis.

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