

The *El Farol* Problem and the Internet: Congestion and Coordination Failure

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Abstract

The *El Farol* or bar attendance problem provides a simple paradigm for analyzing public goods like the Internet which may simultaneously suffer from congestion and coordination problems. This paper reviews the *El Farol* problem and surveys previous solutions, which typically involve complex learning algorithms. A simple adaptive strategy is proposed, and the strategy is investigated via simulation. The algorithm is analyzed in a few simple cases.

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1 Introduction

El Farol is a bar in Santa Fe. The bar is popular, but becomes overcrowded when more than sixty people attend on any given evening. Everyone enjoys themselves when fewer than sixty people go, but no one has a good time when the bar is overcrowded. How can, or how do, people choose whether to go to *El Farol* on any given evening?

Brian Arthur's [1] original formulation of the problem utilized a deterministic model of "inductive learning" at the level of individual agents. He observed that even when agents use a set of very simplistic or unrealistic rules of thumb to predict bar attendance, the number of people at the bar tends to hover near sixty. How is it possible for so many independent decision makers to arrive at an aggregate solution that is so near to optimal? This is the *El Farol* "problem".

1.1 *El Farol* and the Internet

This paper proposes *El Farol* as a simplified model of a class of congestion and coordination problems that arise in modern engineering and economic systems. The *El Farol* problem emphasizes the difficulty of coordinating the actions of independent agents without a centralized mechanism. Furthermore, because the level of congestion at *El Farol* depends on the actions of many individual agents, the problem adds the complexity of an endogenously changing environment. These features make it an especially useful tool for analyzing information technology systems which are characterized by decentralized decision making and rapid endogenous changes in the operating environment. For example, despite rapid technological advances and constantly expanding bandwidth, the Internet can become congested when a large number of people independently decide to visit the same web site or to download files at the same time. The level of congestion is endogenously determined by the actions of hundreds or thousands of users [8].

Standard models of congested public (freely available) resources (beginning with the classic papers by Hardin [6] and Vickrey [15]) focus on the marginal costs that an individual user imposes on other potential users. For example, each person who visits a popular web site increases the download time of other users. Explicitly charging users for these unobserved

costs can help eliminate the socially inefficient congestion of a scarce resource [11]. However, these models typically analyze equilibrium solutions in which all agents are fully informed about the structure of the problem and the behavior of other agents. Consequently, the relationship between agents' behavior and the congestion they experience is easily discerned. This reliance on information-intensive equilibrium solutions limits the applicability of these models to modern information systems such as the Internet.

In contrast, the *El Farol* problem focuses on the interaction between individual learning strategies and the environment that agents' face. Congestion arises in the deterministic version of the *El Farol* model because agents constantly attempt to predict the aggregate behavior of other agents, which simultaneously depends on all agents' predictions. If agents' could perfectly predict the behavior of all other potential bar goers, *El Farol* would never experience congestion. Using *El Farol* to model the Internet environment focuses attention on congestion that arises from coordination failure across agents as well as from absolute constraints on bandwidth.

1.2 How do agents decide to attend *El Farol*?

Arthur [1] first proposed *El Farol* as an example of a problem that requires boundedly rational decision making [10]. He aimed to mimic the decision making processes that people use when deductive logic fails, for instance, when the problem exceeds fundamental constraints on the ability to process information, as with the game of chess, or when the solution requires agents to form subjective beliefs about the behavior of other agents, as with poker or the board game Diplomacy.

In Arthur's solution, agents predict how many others will attend *El Farol* each time using a simple kind of inductive reasoning and decide whether or not to attend themselves accordingly. Each agent maintains a number of "rules of thumb" such as simple averages, moving averages, linear or nonlinear filters to formulate predictions, and then acts on the rule with the best performance in the recent past. If they predict attendance will be less than sixty then they go to the bar, if they predict attendance will be greater than sixty then they stay at home³.

³If agents' behavior has a random element the *El Farol* problem achieving mean attendance of 60 is

Figure 1, taken from [1], illustrates the dynamic behavior of Arthur’s model. The mean attendance is very close to 60. However, attendance varies greatly, often exceeding 70 or dropping below 50. About half of the time more than 60 people visit the bar, all of whom have a bad time. Looking closely at the set of rules actively used by individual agents, Arthur says, “... the predictors self-organize into an equilibrium pattern or ‘ecology’ in which the active predictors... average 40% above 60, 60% below 60. This emergent ecology is almost organic in nature.” Edmonds [4] expands Arthur’s approach by endowing the agents with an expanded set of predictors, and by allowing communication between agents (including the ability for agents to ‘lie’ to each other). His simulations confirm Arthur’s results: agents continually change their prediction strategies and an average of about 60 people attend each evening.

John Casti refers to the *El Farol* problem as the “most important problem in complex systems theory” and uses it to frame his definition of a complex adaptive system as one with “a medium-sized number of intelligent, adaptive agents interacting on the basis of local information.” (p. 10, [3]) Although the dynamics of Arthur’s system are entirely deterministic (only the initial values of agents parameters are chosen randomly) the resulting pattern of attendance appears random. Casti conjectures that attendance is a deterministic random process, i.e., that it exhibits chaotic behavior. The uncertainty or apparent randomness in the system is entirely endogenous, created by the interaction between the number of agents attending the bar and the set of prediction rules active at any given time.

According to Arthur, the dynamic behavior observed in simulations arises because successful predictions shared by many agents will be self-defeating. Whenever attendance exceeds or falls below 60, a large number of agents must be wrong in their predictions, no matter how the predictions are made. Suppose that 61 agents choose to attend the bar on a given evening; these 61 incorrectly predicted that fewer than 60 would attend. On the other hand, suppose that 59 attended. Then the other 41 incorrectly predicted that

simple. For example, if each agent’s probability of attending on a given night is 60% then approximately 60% of the agents go at each time. In the game theory literature a random choice made by an individual agent is referred to as a “mixed strategy”. An explicit definition of mixed strategy equilibria for the *El Farol* problem requires an explicit reward structure which identifies the costs and benefits of staying at home and attending a crowded or uncrowded bar.

more than 60 would attend. Consequently, many agents are wrong every time even though mean attendance equals 60. The number of agents who make wrong predictions depends on the variance, not the mean, of attendance. These repeated incorrect predictions drive the continual churning of rules that Arthur observes in his simulations. Furthermore, Arthur notes that “any commonalty of expectations gets broken up: If all believe *few* will go, *all* will go. But this would invalidate that belief. Similarly, if all believe *most* will go, *nobody* will go, invalidating that belief. Expectations will be forced to differ.” Thus Arthur argues that agents must utilize a heterogeneous set of predictive rules, undermining the “rational expectations” approach that prevails in many economic models.

1.3 Overview of Results

In this paper, we show that agents need not utilize different rules, nor must they constantly switch between rules, when deciding whether to attend *El Farol*. We propose a simple adaptive scheme (based on adaptive signal processing techniques similar in spirit to [16]) which, if followed by all agents, leads to an outcome close the socially optimal attendance of 60 agents per night. The next section motivates this adaptive strategy and details the algorithm. Section 3 examines the generic behavior of the adaptive strategy using simulations. The analysis of section 4 concretizes the simulation results by examining the behavior of the algorithm in certain simple cases. The final section places the adaptive solution in the broader context of congestion problems on the Internet and suggests several areas for further investigation.

2 An Adaptive Agent-based Solution

Arthur’s solution to the *El Farol* problem is cast in terms of an individual agent’s strategies for predicting the aggregate behavior of the system, which then determine the agent’s decision to attend. Beneath this simple formulation lies an infinite regress on other agents’ beliefs: agents who believe that other agents base their decisions on predictions of total attendance should base their decision on predictions of other agents’ predictions of total attendance, in turn those agents should base their predictions on other agents’ predictions

of their predictions of total attendance, and so on *ad infinitum*⁴. After peering into this predictive house of mirrors Casti claims “The upshot of the *El Farol* problem is that there exists no deductive chain of reasoning that will enable the individual to best decide whether to go to the bar or not.”

The approach taken here also proposes a solution in terms of the behavioral algorithm used by individual agents. The agent’s decision process, however, is less complex than Arthur’s predictive strategies: agents do not predict the aggregate behavior of the system, rather they base their decision to go to the bar or to stay home on their recent experiences at *El Farol*. Agents’ attendance decisions are parameterized by a rate or period that determines the frequency of attendance. Initially each agent attends *El Farol* once every c time periods. When $c = 2$, for example, the agents attends the bar every other time. When $c = 3$ the agent attends every third time, and so on.

It is tautological that people prefer to experience good times rather than bad, to repeat the enjoyable and to minimize the unpleasant. In response to a pleasant experience at an uncrowded bar the agent goes more often (decreases c). Similarly, in response to an unpleasant experience at a crowded bar the agent goes less often (increases c). Over time, the agent’s collected experiences at *El Farol* are encoded in the parameter c : an agent who has had many pleasant experiences at *El Farol* goes frequently (has a low c) while one who has had a series of bad experiences goes rarely (has a high c). The adaptive scheme proposed here is analogous to habit formation (or reinforced learning) on the part of agents. For instance, consider logging onto an internet service provider such as America On Line. If only a few people are also online, then emails and downloads occur rapidly and the user has a “good time”. But if too many people are online, then service is slow, graphics files are tedious to download, and the user has a bad experience. Note that agents’ only observe the

⁴John Maynard Keynes recounts a similar regress in a beauty contest sponsored by a local newspaper. Readers are asked to vote for the most beautiful contestant pictured in the paper, but are only eligible for the prize drawing if they voted for the winning contestant. The winner, of course, is determined by the largest number of votes sent in by other readers. A forward looking reader would cast their vote not for the contestant they found the most attractive, but for the contestant they believe other readers will find most attractive. Knowing this, even more perceptive readers would choose to vote for the contestant they believe other readers will believe most other readers will find most attractive, and so on.

congestion level online or attendance at *El Farol* when they themselves log on or attend.

The adaptive strategy for the *El Farol* problem resembles the algorithm for the “number guessing game” described in the first chapter of Johnson [9]. When new information reveals that the current guess is too low (or too high), then the guess is increased (or decreased). Although in the *El Farol* scenario the unknown quantity varies over time and depends directly on other agents’ guesses, the simplicity of the adaptive scheme and its structural similarity to other algorithms suggest strategies for analyzing the resulting behavior.

When exactly 60 agents attend *El Farol* the bar is neither overcrowded nor undercrowded: all bar-going agents enjoy themselves but no agent who chose to stay home would have been better off at the bar. Indeed, all bar-going agents would have been worse off if one more agent had chosen to attend. How might agents respond to this knife-edge scenario? Interpreting the adaptive strategy as a type of habit-formation suggests that agents would continue to increase their frequency of attendance because all attending agents experienced a good time. Interpreting the strategy as a type of reinforcement learning suggests that agents would neither increase nor decrease their frequency of attendance because an increase in attendance beyond 60 would result in a worse outcome for all attendees. In order to allow for both scenarios the specification of the adaptive strategy includes a ‘dead zone’ (indicated by a parameter d) below the point at which the bar becomes crowded. Agents neither increase nor decrease their frequency of attendance c when attendance at *El Farol* falls in the ‘dead zone.’ Results presented below indicate that the behavior of the system can depend critically on the treatment of such borderline cases⁵ and on the assumption that agents only learn attendance at the bar on nights that they themselves attend.

2.1 Algorithm Statement

To write this strategy in symbolic form, suppose there are M agents and N spaces at *El Farol* before it becomes crowded. Let d represent the size of the dead zone. Note that d can be 0 or 1 as discussed above, or even larger. Let c_i be the frequency (period) with which the i th agent wishes to attend and p_i , the ‘phase’, be the number of timesteps until the i th

⁵Arthur does not specify how agents deal with borderline cases in his original paper.

agent attends next (thus p_i is always less than or equal to c_i). Let μ_i be a stepsize parameter that defines how much the i th agent changes c_i in response to new information. Note that the stepsize is small and varies across agents. The time (iteration) counter is denoted by k . Since the c_i and p_i change as time evolves, $c_i(k)$ and $p_i(k)$ designate the instantaneous values of c_i and p_i at the time k . Thus the pair $[c_i(k), p_i(k)]$ represents the state of the i th agent at time k , and the concatenation of all M pairs gives the state vector for the entire system.

Let

$$N(k) = \sum_{i=1}^M 1_{[p_i(k) \leq 1]} \quad (1)$$

count the number of agents attending at time k , where 1_A is the indicator function that is one if the event A is true and zero otherwise. The evolution of the $c_i(k)$ and $p_i(k)$ is defined by

$$\begin{aligned} c_i(k+1) &= \begin{cases} c_i(k) & \text{if } p_i(k) > 1 \\ \text{Max}(1, c_i(k) + \mu_i(k) \text{Dsgn}(N(k) - N)) & \text{if } p_i(k) \leq 1 \end{cases} \\ p_i(k+1) &= \begin{cases} p_i(k) - 1 & \text{if } p_i(k) > 1 \\ c_i(k+1) & \text{if } p_i(k) \leq 1 \end{cases} \end{aligned}$$

where $\text{Dsgn}(x)$ represents the signum function with dead zone that is positive when $x > 0$, negative when $x - d \leq 0$, and zero otherwise. The $\mu_i(k)$ term is a stepsize that scales the agents response.

For each agent, the initial values of $c_i(0)$ and $p_i(0)$ are chosen randomly. Each agent's 'phase' term p_i counts down until it drops below one. Meanwhile, the counter c_i remains unchanged. Once p_i reaches one, the agent attends the bar. At this point, c_i is increased by μ_i if bar attendance exceeds N (the bar is crowded), decreased by μ_i if the bar attendance is lower than $N - d$ (the bar is uncrowded), and remains unchanged if attendance falls in the dead zone just below the cutoff point N . The phase p_i is then reset to the current (updated) value of c_i . The counter c_i always remains positive because of the $\text{Max}(\cdot)$ function. An alternative statement of the algorithm is:

$$\begin{aligned} c_i(k+1) &= \text{Max}(1, c_i(k) + \mu_i(k) 1_{[p_i(k) \leq 1]} \text{Dsgn}(N(k) - N)) \\ p_i(k+1) &= (p_i(k) - 1) 1_{[p_i(k) > 1]} + c_i(k+1) 1_{[p_i(k) \leq 1]} \end{aligned} \quad (2)$$

2.2 Derivation of the Algorithm

If there were a social planner who could coordinate and enforce attendance at *El Farol*, how many agents would be chosen to attend each night? When 61 agents simultaneously decide to attend they could all be made better off by a social planner who denies entry to one agent: the 60 agents at the bar would be free to enjoy themselves in an uncrowded environment, and the one agent that was denied entry would prefer staying home to attending a bar crowded with 61 agents. Similarly, if 59 agents decide to attend then the social planner would induce one more agent to attend and improve the outcome for that agent, who enjoys his or her time at a uncrowded bar, without diminishing the enjoyment of the other 59 patrons. A centralized social planner with the power to dictate bar attendance would choose attendance of exactly 60 people each night. The bar would never be under crowded (insuring that as many people as possible enjoyed themselves) and it would never be overcrowded (so no one would ever have a bad time). When attendance exactly equals 60 the outcome is Pareto efficient: there is no way to make any one agent better off without making some other agent worse off. Conversely, any deviation away from 60 people attending represents an inefficient outcome.

The exact cost of deviations away from the optimum depends on a richer specification of individual agents preferences and on the functional form of a social welfare function that aggregates individual utility. Social welfare typically depends not only on mean attendance, but also on the variance around the mean. As the simulations presented in the next section demonstrate the variance of attendance can be relatively low.

The algorithm for individual agents detailed above is now derived as an approximation to an instantaneous gradient descent for minimization of the global cost or social welfare function

$$J(k) = |\text{avg}\{N(k)\} - N| \tag{3}$$

where

$$\text{avg}\{N(k)\} = \frac{1}{w} \sum_{j=k-w+1}^k N(j) \tag{4}$$

is the average of the number of attendees over a window of length w . In other words, when each individual agent uses the proposed adaptive strategy, the system as a whole tends

to minimize deviations away from the moving average of attendance over time. Note that this global cost function weights deviations above and below the optimum of 60 equally. Substituting (1) into (4) gives

$$\text{avg}\{N(k)\} = \frac{1}{w} \sum_{j=k-w+1}^k \sum_{i=1}^M 1_{[p_i(j) \leq 1]}.$$

The two sums can be reordered

$$\text{avg}\{N(k)\} = \frac{1}{w} \sum_{i=1}^M \sum_{j=k-w+1}^k 1_{[p_i(j) \leq 1]}.$$

But the event $p_i(j) \leq 1$ occurs once every c_i timesteps, so for a large window length w and a small stepsize $\mu_i(k)$, the event $p_i(j) \leq 1$ occurs approximately $[\frac{w}{c_i}]$ times where $[x]$ is the largest integer contained in x . Hence

$$\text{avg}\{N(k)\} \approx \frac{1}{w} \sum_{i=1}^M [\frac{w}{c_i}].$$

For large windows w , this gives the approximation

$$\text{avg}\{N(k)\} \approx \frac{1}{w} \sum_{i=1}^M \frac{w}{c_i} \approx \sum_{i=1}^M \frac{1}{c_i}.$$

Hence

$$\frac{d\text{avg}\{N(k)\}}{dc_i} \approx -\frac{1}{c_i^2}. \quad (5)$$

The typical gradient strategy updates the state by

$$c_i(k+1) = c_i(k) - \mu_i(k) \frac{dJ(k)}{dc_i(k)}, \quad (6)$$

though in the *El Farol* problem this update occurs only when the agent attends the bar.

With $J(k)$ as in (3), and ignoring the singularity at $N(k) = N$,

$$\frac{dJ(k)}{dc_i(k)} = \text{sgn}(\text{avg}\{N(k)\} - N) \frac{d\text{avg}\{N(k)\}}{dc_i(k)}.$$

Combining with (5), this becomes

$$\frac{dJ(k)}{dc_i(k)} \approx -\text{sgn}(\text{avg}\{N(k)\} - N) \frac{1}{c_i^2(k)}.$$

Replacing $\text{avg}\{N(k)\}$ by its instantaneous value gives

$$\frac{dJ(k)}{dc_i(k)} \approx -\text{sgn}(N(k) - N) \frac{1}{c_i^2(k)}$$

which is an instantaneous approximation to the gradient of $J(k)$. Substituting this into (6) gives

$$c_i(k+1) = c_i(k) + \frac{\mu_i}{c_i^2(k)} \operatorname{sgn}(N(k) - N) \quad (7)$$

which holds whenever $1_{[p_i(k) \leq 1]}$. Combining (7) with the phase terms $p_i(k)$ (so that the update to $c_i(k)$ occurs only when the i^{th} agent is part of the sum $N(k)$) gives the complete algorithm (2) where $\mu_i(k)$ is identified with $\frac{\mu_i}{c_i^2(k)}$. The ‘sgn’ function can be readily generalized to incorporate the dead zone by suitable modification of the cost function (3), a standard procedure in the adaptive filtering literature, see for instance [12]. In this case, the global cost function treats deviations above the optimum and below the dead zone equivalently.

Alternatively, one could consider the squared cost function

$$J(k) = (\operatorname{avg}\{N(k)\} - N)^2. \quad (8)$$

Repeating the logic of (5) to (7), which leads to a kind of least mean square update [7]

$$c_i(k+1) = c_i(k) + \frac{\mu_i}{c_i^2(k)} (N(k) - N). \quad (9)$$

Here the cost functions weights large deviations away from the optimum more heavily but again treats overcrowding and undercrowding symmetrically. The full algorithm incorporating the phase is identical to (2) with the $\operatorname{Dsgn}(\cdot)$ function removed.

2.3 Comparison with Standard Adaptive Algorithms

Iterative minimization of a cost function using approximate gradient descent methods is a standard signal processing strategy [7]. Perhaps the most common technique is the Least Mean Square (LMS) algorithm [16] that has been modified in many ways to improve convergence or tracking (such as normalized LMS [9]), to decrease numerical complexity (such as the signed regressor LMS [14]), or to decrease its sensitivity to disturbances [5], [17]). This section details the relationship between standard signal processing algorithms and the similarly structured algorithms derived above for the *El Farol* problem.

All the algorithms have the basic gradient form that the new estimate is the old estimate plus a correction term defined as a scaled version of the (negative) gradient. The $\frac{\mu_i}{c_i^2(k)}$ term

is reminiscent of the stepsize update in the normalized LMS since it decreases the update with larger $c_i(k)$. The $N(k) - N$ term is analogous to the ‘error’ term in LMS, while the $\text{sgn}(N(k) - N)$ is the equivalent error term from the “signed-error” LMS. Although the $\frac{1}{c_i^2(k)}$ may appear to be problematic, it is actually well behaved since $c_i(k)$ is never less than one. The presence of the ‘knife-edge’ is paralleled by the sgn function, and sign functions with dead zones are commonplace in the adaptive signal processing literature [12].

Despite these similarities, the origin of the problem renders the *El Farol* algorithms quite different. First, the cost function $|\text{avg}\{N(k)\} - N|$ is not the same as $\text{avg}\{|\hat{y}(k) - y(k)|\}$ (as used to derive the signed-error LMS algorithm in [13]) since the direct analog would be $\text{avg}\{|N(k) - N|\}$. Similarly, the cost function for LMS has the average (or the expectation in a stochastic setting) outside the square, rather than inside as in (8).

Second, $N(k)$ is not a fixed (linear or nonlinear) target function with parameters to be identified. Since all agents are interchangeable, there are many possible equilibria which will minimize the cost. Hence the error surface is not quadratic nor even unimodal in $c_i(k)$, and proving convergence is a more subtle affair. Finally, the presence of the phase terms $p_i(k)$ means that convergence is only possible to a periodic state, and not to a fixed equilibrium. These issues are explored further in section 4.

3 Generic Behavior of the Adaptive Solution

A few example simulations illustrate the dynamics of a bar-going society in which each agent utilizes the strategy (2) defined above. Although details of the various simulations differ depending on the size of the population, the capacity of the bar, the size of the dead zone and the initial conditions, Figure 2 represents the typical behavior of attendance over time. In this case, $M = 100$, $N = 60$ and $d = 5$. The initial randomly chosen values of the counters $c_i(0)$ and phases $p_i(0)$ ’s result in too few people attending the bar⁶.

Perhaps the most striking aspect of these simulations is that the number of attendees converges within a few hundred iterations to the dead zone below the optimal value of

⁶Different sets of initial conditions which result in too many agents attending at the start exhibit similar periods of transience and similar long run behavior.

attendance. After the initial transience resolves itself, spikes of attendance above 60 occur infrequently. All agents base their attendance decisions on local information, that is, on their own experiences, but nonetheless attendance largely remains in the dead zone just below the optimal choice of a central planner. In comparison, both Arthur’s inductive learning and Edmond’s genetic programming approaches generate far greater excursions away from the optimal value.

The simulations also reveal an interesting pattern in the behavior of individual agents. Figure 3 shows values of the counters $c_i(K)$ at the final iteration $K = 2000$ of a typical simulation run. Fifty-four agents have $c_i(K)$ equal to 1, indicating that they go to *El Farol* every time. The remaining agents with much larger counters vie with each other for the remaining open slots. The occasional spikes in attendance in Figure 2 occur when several of the remaining agents happen to go simultaneously. While initial conditions such as the stepsize μ_i and the phase $p_i(0)$ help determine which agents attend every period and which agents attend infrequently, the same pattern of attendance develops.

Thus the population has segregated itself into a group of ‘regulars’ who go to the bar every time and a group of ‘casuals’ who only attend occasionally. Asked about their experiences at *El Farol* the regulars say “It’s great, I go there all the time and it’s almost never crowded.” Asked the same question the casuals say “I hardly ever go, and when I do, it always seems crowded.” Both groups are right: their different perspectives reflect different experiences and different data about the environment. This division of the population does not appear in the algorithm statement or the corresponding global cost function, rather, it is an emergent property of the adaptive solution to the *El Farol* problem. When agents follow this adaptive strategy, *El Farol* looks more like *Cheers*. As the following section demonstrates, the system defined by the algorithm (2) has a large number of equilibria; this type of habit formation or reinforcement learning tends to select those equilibria in which one group of agents attends regularly and another group attends sporadically.

Recall that in Arthur’s and Edmonds’ simulations agents know the history of attendance at the bar regardless of whether they attend or not. The final simulation, shown in Figure 4, investigates the effect of allowing the agents to update their counters at every iteration, whether they have personally attended the bar or not. Observe that the variance about

$N = 60$ is much larger than in Figure 2, and there is a cloud of values around 30 indicating that seats in the bar often remain unfilled. When all agents have access to the same information about attendance they all respond simultaneously and the individual decisions of the agents collectively overshoot or undershoot the optimum, creating inefficient variations in attendance. Somewhat paradoxically, the adaptive solution functions better when agents' have heterogeneous and limited information sets.

This insight suggests that universal access to information may reduce the global efficiency of other systems with endogenously fluctuating conditions. For example, in financial markets in which current prices depend critically on individual expectations of future prices, the rapid simultaneous dissemination of information may lead to bubbles or overshooting relative to fundamental or underlying economic conditions. This issue may also arise in decentralized algorithms for managing traffic flows on the Internet: globally available information about the level of congestion may lead to an instantaneous over-response that merely shunts the congestion from point to point rather than distributing it evenly across possible routes. In contrast, more selective or slower distribution of information may induce a smaller response, giving the entire system time to adapt smoothly to changing conditions.

4 Analysis of the Adaptive Solution

The analysis of the system focuses on defining and characterizing the equilibria of the system. In addition, under certain circumstances equilibria fail to exist: some agents attend less and less frequently, their counters diverging towards infinity, albeit at an ever slower rate.

In a strict formal sense, the system defined by (2) with $M > N$ has no equilibria because the phase terms p_i continually decrease from c_i to zero, only to be reinitialized at c_i . However, if agents' counters (the vector of c_i 's) do not change then neither does the pattern of attendance at the bar: the only externally observable quantity, $N(k)$, is completely predictable. Consequently an equilibrium in terms of attendance can be defined as follows:

Proposition 4.1 *Consider (2) with $\mu_i(k) = \mu_i$ fixed for all k . Suppose that $c_i(k) = c_i^*$ for all $k \geq K$. Then $N(k)$ is periodic for $k \geq K$.*

Proof: Since the $c_i(k)$ are fixed, $1_{[p_i(k) \leq 0]}$ is periodic with period c_i^* . Hence the sum of such indicator functions is periodic with period (at most) $c^* = \prod_{i=1}^M c_i^*$, and $N(k)$ is c^* periodic.

In the following, an equilibrium of the system (2) refers to the c_i portion of the state and the corresponding periodic pattern of attendance, in accordance with the proposition. Note that attendance at the bar need not be constant in equilibrium.

The *El Farol* problem contains a knife-edge response to increased attendance: the transition from an uncrowded to a crowded bar depends on the behavior of any one agent⁷. Consequently, when agents use predictive behavioral rules like those utilized by Arthur, the definition and analysis of equilibria depends crucially on how the agent accounts for his or her own behavior. For example, suppose that attendance at *El Farol* for the last ten periods has been:

60, 60, 60, 60, 60, 60, 60, 60, 60, 60

and that all agents predict attendance of 60 next period. How should an individual agent decide whether or not to attend in this case? Common sense suggests that agents who have attended the bar every period should continue to attend every period. On the other hand, agents who have not attended at all in the last ten periods should remain at home because the addition of another agent will result in attendance of 61. Agents who have attended irregularly should replicate their previous pattern of attendance.

A formal treatment of the knife-edge case when attendance exactly equals 60 could alter the predictive rule to account for the agent's own behavior: agents should attend if they predict 59 or fewer agents *other than themselves* will attend and stay home if they predict 60 or more agents *other than themselves* will attend. In this scenario, the *El Farol* problem has well-defined equilibria and all agents can utilize the same successful predictive rule. The heterogeneity in agents' actions results from the heterogeneity in information: each agent's information set is unique because only the agent knows whether or not they were among the bar attendees at any point in time. The key issue is that an equilibrium outcome in the *El*

⁷Although this is an extreme assumption, similar nonlinearities arise in congestion models. No delays occur at a bottleneck when traffic is below capacity but once the capacity long delays can appear rapidly. The performance of a computer network can degrade rapidly for all users with the addition of one more user [8]. Note that the introduction of a 'dead zone' makes this transition less extreme.

Farol model requires heterogeneity of behavior across agents. When agents do not account for their own behavior this implies that agents must draw different conclusions from the same data set necessitating heterogeneous prediction rules.

While the adaptive algorithm proposed here does not rely directly on agents' prediction of attendance, similar issues arise in defining equilibria. When $d = 0$ the knife edge eliminates the possibility of equilibria. Unless the same N agents, all with $c_i = 1$, attend each period at least one agents' counter changes. This implies that the remaining $M - N$ agents never attend, which in turn implies that their counters have diverged. The following result demonstrates a simple case in which the c_i of certain agents must diverge towards infinity.

Proposition 4.2 *Suppose M is given, $N = M - 1$, $d = 0$, and that $\mu_i \ll 1$ for all i . Suppose further that the system (2) has evolved (or is initialized) so that there are $M - 1$ 'regulars' with $c_i < 1 - \mu_i$. The remaining agent, designated agent m , has $c_m \gg 1$. Then $c_m(k) \rightarrow \infty$ as $k \rightarrow \infty$.*

Proof: For simplicity, suppose that $\mu_i(k) = \mu_i \forall k$. Suppose the m th agent first attends the bar at time k_1 . The regulars attend by assumption and so it is overcrowded. Hence

$$c_i(k_1 + 1) = c_i(k_1) + \mu_i \text{ for all } i.$$

Because $c_i(k_1) < 1 - \mu_i$, $c_i(k_1 + 1) < 1$ for all the regulars but $c_m(k_1 + 1) > 1$. Hence only the $M - 1$ regulars attend at timestep $k_1 + 1$. Thus the bar is uncrowded at timestep $k_1 + 1$ and $c_i(k_1 + 2) = c_i(k_1 + 1) - \mu_i = c_i(k_1)$ for all $i \neq m$, returning the regulars to their starting place, but leaving the m th agent with larger c_m . This exact sequence of events happens each time the m th agent attends, and hence c_m increases by μ_m each time. Since c_m directly specifies how often the m th agent attends, c_m diverges at an ever decreasing rate.

Such agents can increase their counters forever. Presumably, real humans in such a situation would eventually withdraw and go to some other bar. The result depends on a peculiar initial condition and on the assumption that $d = 0$. If $d \geq 1$, then this mechanism for divergence fails because the casual agents can eventually synchronize their periods into simple integer multiples in which the pattern of attendance remains within the moderate region.

The existence of a ‘dead zone’ allows for a multitude of equilibria. To get an idea of the number of possible equilibria, consider a single ‘slot’ of the M possible. This could be occupied by a customer who attends every time. It could also be occupied by two customers who alternate, i.e., who have different phases. In general, any number could alternate on successive evenings, and still only occupy a single space at the bar.

For example, consider the $M = 5, N = 3, d = 1$ case. (These three parameters determine the number of possible equilibria.) Equilibria of (2) are related to the number of different ways that M can be partitioned into sums of integers. For instance, $M = 5$ can be partitioned in six ways

$$\begin{array}{llll} (a) & 1 + 1 + 1 + 1 + 1 & (b) & 2 + 1 + 1 + 1 \\ (d) & 3 + 1 + 1 & (e) & 3 + 2 \\ & & (f) & 4 + 1 \end{array}$$

Let each of the summands represent (the integer part of) the c_i^* value for the agents in a candidate equilibrium configuration. For instance, a “3” means that three agents alternate filling one bar seat, while a “1” means that a single agent attends every time. If there are more than N terms in a given partition (such as in (a) and (b) above) then more than N agents attend the bar. When (2) is iterated, the $\text{Dsgn}(\cdot)$ term is positive, and the c_i will change. Hence this cannot represent an equilibrium configuration. Similarly, if there are fewer than N terms (such as in (e) and (f)), the $\text{Dsgn}(\cdot)$ term will be negative and again this is not an equilibrium. If however, the number of terms in the partition equals N (as in (c) and (d)) then all slots are exactly filled each time, $\text{Dsgn}(\cdot)$ is identically zero, and it is an equilibrium. The same argument holds in general.

Counting the number of possible equilibria requires two functions. For $M > N$, let $\phi(M, N)$ represent the number of ways that the integer M can be partitioned into a sum of exactly N integers. $\phi(5, 3)$ is illustrated above. Let $\psi(n)$ represent the number of ways that 1 can be written as a sum of exactly n fractions of the form $\frac{1}{i_1} + \frac{1}{i_2} + \dots + \frac{1}{i_n}$ for integers i_k . $\psi(n)$ counts the number of ways that n agents can alternate to occupy a single bar stool. For instance, the “3” in (d) above can represent three agents each of whom comes every third time $\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$, or it can represent a single agent who comes every second time in collaboration with two agents who come every fourth time $\frac{1}{2} + \frac{1}{4} + \frac{1}{4}$. The total number of equilibria is thus a combination of ϕ and ψ .

Proposition 4.3 *The algorithm (2) with $d = 1$ has at most $\sum_{j=1}^{\phi(M,N)} \prod_{k=1}^N \psi(i_k^j)$ equilibria.*

Proof: As shown above, there are $\phi(M, N)$ different configurations in which the equilibria may occur. For $j = 1, 2, \dots, \phi(M, N)$, each of these has the form $i_1^j + i_2^j + \dots + i_N^j$. Since $\psi(n)$ counts the number of ways that n agents can alternate to occupy a single bar stool, $\prod_{k=1}^N \psi(i_k^j)$ counts the number of possible equilibria within each configuration. Summing over all the configurations gives the desired result.

When $d > 1$, then there are more equilibria than $\phi(M, N)$. For example, consider the case $M = 5$, $N = 4$, with $d = 2$. Then (b)-(f) are equilibria by the same argument as above. But there are also others. The pair of agents representing the “2” in case (b) might actually consist of two agents attending every n th time (though out of phase with each other). The population of the bar would then fluctuate between three (when neither attends) and four when one attends. In neither case is there a change in the corresponding c_i values because of the nontrivial dead zone. Thus the kinds of arguments used in the previous proposition can only provide a lower bound on the number of kinds of equilibria. For general d , the number of equilibria of (2) is a nontrivial counting problem.

5 Conclusion

While Arthur [1] first introduced *El Farol* as an example of a problem requiring boundedly rational agents, we have shown that this model is also interesting as a simple model of congestion and coordination behaviors such as occur in an overcrowded Internet.

Arthur believed that any solution to the *El Farol* problem would require heterogeneous agents, that is, agents who pursue different strategies. In contrast, we have presented a simple adaptive solution which can be followed by all agents, and which can readily solve the problem. Each agent in the adaptive solution is characterized by a parameter that determines how often the agent attends, and a stepsize that determines how much to change the parameter in response to each visit to the bar. Our agents, like Arthurs, are boundedly rational, and are completely deterministic.

The adaptive solution to the *El Farol* problem differs from Arthur’s strategy in several ways. We do not require agents to make explicit predictions of the state of the bar, and

we allow them to use only the information that they have readily available, i.e., their own experiences. This makes our model more realistic (it is not clear how the agents in Arthur’s model learn what happened at the bar when they are absent). Finally, we allow the ‘don’t care’ zones. Arthur is not explicit about how his agents handle the transition from uncrowded to crowded.

Arthur’s solution, in which each agent maintains a bank of strategies leads to patterns of attendance that fluctuate considerably above and below the optimal. When crowded, none of the agents enjoys themselves. When undercrowded, there is a wasted resource represented by the empty seats at the bar. The adaptive solution, in contrast, leads to patterns of attendance with much smaller variance, and hence much less waste. Generically, we believe the adaptive method will lead to convergence to an optimal solution, one at which the bar is neither under nor over crowded.

The adaptive solution thus provides a simple mechanism whereby a large collection of decentralized decision makers, each acting in their own best interests and with only limited knowledge, can solve a complex congestion and social coordination problem. Moreover, convergence to the solution is relatively rapid (depending on the initial conditions) and robust. We have also demonstrated that under certain singular initial conditions it is possible for certain of the agents to ‘withdraw’ from the society. Nonetheless, the population of the bar remains near its optimal value.

Do we believe that customers of *El Farol* tick off the time till they can go again, increasing or decreasing a counter with each new visit? Of course not. But the incentives are in agreement with the commonsense idea that people tend to minimize bad experiences and maximize good ones. Moreover, the global behavior of the population is consistent with certain kinds of coordination phenomena. For instance, users of an Internet provider can spread demand over much of the day even though everyone might prefer (all else being equal) to log on in the middle of the afternoon. By developing certain habits (for instance, always logging on at the same time) users send signals to others to avoid these times. In this way, demand is smoothed.

There are many ways to generalize the adaptive solution to the *El Farol* problem. For instance, different people have different tolerances for what constitutes a crowd or an unac-

ceptable delay. Each agent could also have a parameter that represents their tolerance for congestion. Additionally, to more closely model the Internet situation, one might incorporate time-of-day or day-of-week as a parameter in the process of logging on. It would also be instructive to create a hybrid situation in which a number of Arthur-like agents and a number of adaptive agents compete for spaces at the bar. We believe that the more goal directed adaptive agents will again become regulars at the bar, relegating Arthur's more haphazard agents to squabbling over the few remaining seats.

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Figure 1: In Arthur's simulations, agents use a collection of predictors to decide whether to attend *El Farol*. Observe the large variance about the optimal 60 attendees.

Figure 2: When all agents use the adaptive solution, the number of attendees only rarely exceeds the critical $N = 60$.

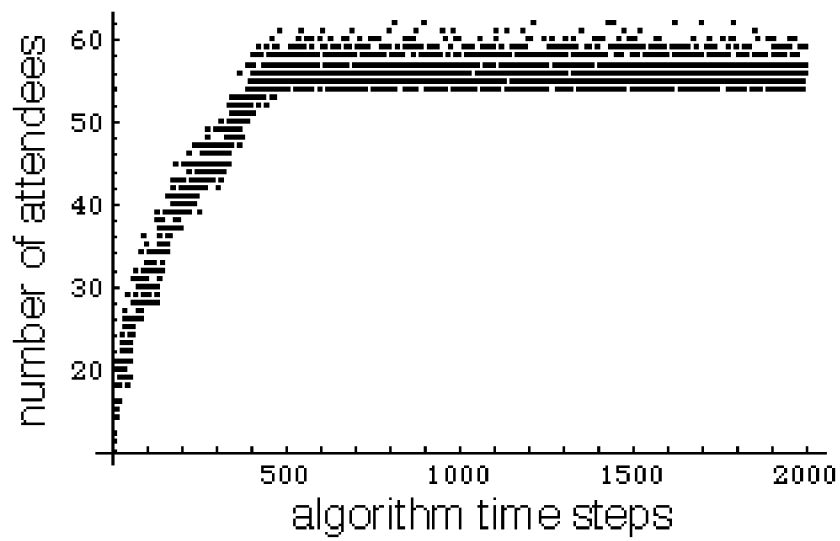


Figure 3: An emergent property of the adaptive solution is that the population divides itself into ‘regulars’ and ‘casuals’.

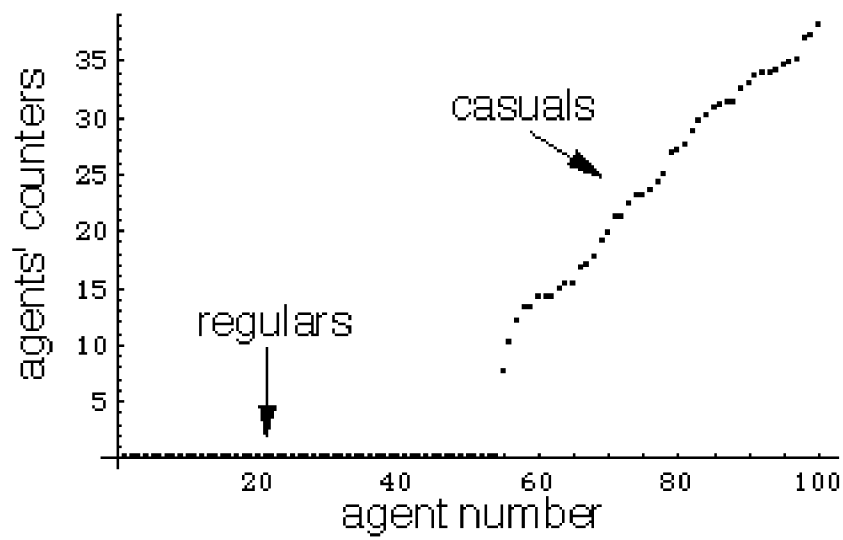


Figure 4: In this simulation, each agent updates its counter at every time step. Using more information does not necessarily lead to better results.

