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Computation of Non-Linear Continuous Optimal Growth Models:

Experiments with Optimal Control Algorithms and Computer Programs

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ABSTRACT: In this paper we discuss the use of optimal control methods for computing non-linear continuous optimal growth models. We have discussed various recently developed algorithms for computing optimal control, involving step-function approximations, Runge-Kutta solutions of differential equations, and we suggest that the discretisation approach is preferable to methods which solve first-order optimality conditions. We have surveyed some powerful computer programs MATLAB-RIOTS, MISER, and OCIM for computing such models numerically. These programs have no substantial optimal growth modelling applications yet, although they have numerous engineering and scientific applications. A computer program named SCOM by MATLAB-CONSTR is developed in this study. Results are reported for computing the Kendrick-Taylor optimal growth model using RIOTS and SCOM programs based on the discretisation approach. References are made to the computational experiments with OCIM and MISER. The results are used to compare and evaluate mathematical and economic properties, and computing criteria. While several computer packages are available for optimal control problems, they are not always suitable for particular classes of control problems, including some economic growth models. The MATLAB based RIOTS and SCOM, however, offer good opportunities for computing continuous optimal growth models. It is argued in this paper, that optimal growth modellers may find that these recently developed algorithms and computer programs are relatively preferable for a large variety of optimal growth modelling studies.

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1. Introduction

The methods and models of computational optimal growth economics are often formulated as optimal control problems. Such models may describe a national economy, with state functions being vector valued, since various commodities are involved. The consumption may be varied, within some limits, to optimise some utility functional, which may involve both state and control functions over a planning period. In these models, there are in general explicit bounds (such as floor to consumption, or a ceiling to capital or pollution), as well as discontinuous control functions.

Analytic solutions are available for some of the simpler models. Often some sort of *steady state* has been studied, but not the rate of approach to the steady state from given initial conditions, or the transition to give terminal conditions (such as, for example, a stated minimum for terminal capital, so that a process may continue). It is noted that the available data are necessarily in discrete time (i.e. annual or quarterly data). While discrete-time optimal control models can of course be computed, a continuous-time model may give a more intelligible picture of what is happening. Also, the Pontryagin theory for optimal control applies to optimal control in continuous time, but only under serious restrictions to optimal control in discrete time. This paper will hence consider continuous-time models.

A computer package allows exploration, both of the sensitivity to parameter changes, and also of the domains of the various parameters for which solutions of an optimal control exist. Some recently developed computer packages, coding different algorithms for continuous optimal control such as MATLAB, RIOTS, MISER and OCIM have current applications to other areas of control, but have not, until now, been used for computing optimal growth models. The objective of this paper is to demonstrate the applications of these recently developed algorithms and computer programs to the computation of optimal growth models. In this study a general MATLAB optimal control program called Step Function Optimal Control (SCOM) is also developed on MATLAB based on one of the recently developed algorithms (step function discretisation approach) which can be used to solve optimal control models. These recently developed algorithms and programs have several advantages (discussed below) in economic modelling. This paper also provides a survey of different algorithms and computer programs for implementing continuous computational optimal growth models to determine the conclusions on the relative suitability of the algorithms and software discussed and developed in the present study. Detailed results of two computer programs, RIOTS and SCOM are reported and references to computing experiences with OCIM, and MISER for similar problems are made for comparison.

The experiments in this study using the Kendrick and Taylor model (Chakravarty 1969; Kendrick and Taylor 1971) have produced economic results and computing experience which are consistent with the results generated in the previous computational experiments (stated below) with the model. The global optimality of the model results is also proved in this paper.

The paper is organised as follows. Section 2 presents an optimal growth model developed by Kendrick and Taylor (1971) and its optimal control specification. Section 3 discusses different algorithms and computer programs for solving optimal control models, while Section 4 discusses the Kendrick-Taylor model numerical implementation by some computer programs. Mathematical properties of the model results and the sensitivity studies are reported in Section 5. Section 6 refers to the experiences for computing optimal growth models by computer programs, such as OCIM, and MISER. Section 7 contains the conclusion.

2. An Optimal Growth Model

The Kendrick-Taylor model for economic growth (Kendrick and Taylor 1971) has been used as a test problem for computational approaches. With capital stock k(t) at time t as the state function, and rate of consumption c(t) at time t as the control function, this model has the form:

$$\text{MAX} \quad \int_0^T e^{-\rho t} c(t)^{\tau}\, dt \qquad \text{subject to}$$

$$k(0) = k_0, \quad (d/dt)k(t) = \zeta e^{\delta t} k(t)^{\beta} - \sigma k(t) - c(t), \quad k(T) = k_T$$

This model does not state any explicit bounds for k(t) and c(t). However, both the formulas and their interpretation requires that both k(t) and c(t) remain positive. However, with some values of u(t), the differential equation for k(t) can bring k(t) down to zero.

Consistent with the mainstream practice in optimal growth modelling, the numerical optimal growth programs in this paper are specified within the framework of the elements of an optimal growth program of the following form:

1) an optimality criterion contained in an objective function which consists of the discounted integral of the utilities provided by consumption at every period;
2) the finite planning horizon;
3) a positive discount rate; and
4) the boundary conditions given by the initial values of the variables and parameters and by the terminal conditions.

This model can be expressed by a standard kind of optimal control model, which may be written as follows:

$$\mathrm{MIN}_{x(t),u(t)} \quad F^0(x,u) = \int_0^T f(x(t),u(t),t)dt + \Phi(x(T)) \quad \text{subject to:}$$

$$x(0) = x_0, \quad (d/dt)x(t) = m(x(t),u(t),t),$$

$$u_L(t) \le u(t) \le u_u(t) \qquad (0 \le t \le T).$$

Here the state function is x(t) = k(t), the control function u(t) = c(t),

$$(f(t),u(t),t) = -e^{\rho t}u(t)^{\tau}; \qquad \Phi(x(T)) = \mu(x(T) - x^*)^2$$

(with a minus sign to convert to a minimisation problem), and

$$m(x(t),u(t),t) = \zeta e^{\delta t}x(t)^{\beta} - \sigma x(t) - u(t).$$

The terminal constraint for x(T) has been replaced by a *penalty cost* term $\Phi(x(T))$, with a parameter x* (approximately x_T, but may need adjustment). In general, x(t) and u(t) are vector valued.

The relevant mathematical issues that are investigated in the context of an optimal growth model are the following: existence, uniqueness and globality of the optimal policy solution, and stability of the dynamic system in the equilibrium or steady state position (Intriligator 1971).

The control issues of analytical and policy importance in control models are estimability, controllability, reachability, and observability (Sengupta and Fanchon 1997). As these characteristics of control models can be studied only for linear control models, we will not investigate these issues in this paper.

The Kendrick-Taylor model was numerically implemented by Kendrick and Taylor (1971) by several algorithms such as search and quasi-linearisation methods based on the discrete Pontryagin maximum principle. In another experiment, Keller and Sengupta (1974) solved the model by conjugate search and Davidson algorithm based on the continuous Pontryagin maximum principle. While these algorithms were implemented by some special purpose computer programs, a GAMS version (a general purpose commercially available program, Brooke et al. 1997) of the Kendrick-Taylor model is also available.

3. Algorithms and Computer Programs for Optimal Control

3.1 A Classification of Algorithms

Algorithms for optimal control (Li 1993; Mufti 1970; Teo et al. 1991) have been variously based on: (i) dynamic programming; (ii) solving first-order necessary conditions of the Pontryagin theory; (iii) applying some approximation methods (steady state solution, numerical methods based on approximation and perturbation, and method of simulation); (iv) approximating the control by step functions; and (v) applying mathematical programming algorithms to a discretized version of the control problem.

Most of these methods (especially methods (i) to (iv)) are based on the first order maximisation conditions of a control problem, which Schwartz (1996) has called ‘indirect methods’. About the suitability of these methods for computing optimal control, Schwartz (1996, p. 1) has made the following statement:

> 'The main drawback to indirect methods is their extreme lack of robustness: the iterations of an indirect method must start close, sometimes very close, to a local solution in order to solve the two-point boundary value subproblems. Additionally, since first order optimality conditions are satisfied by maximisers and saddle points as well as minimizers there is no reason, in general, to expect solutions obtained by indirect methods to be minimizers.'

Against these indirect methods, he has advocated the superiority of the direct methods, the principles of which, according to him are as follows (p. 1):

> 'Direct methods obtain solutions through the direct minimization of the objective function (subject to constraints) of the optimal control problem. In this way the optimal control problem is treated as an infinite dimensional mathematical programming problem.'

The superiority of this direct method or the control discretisation method has also been advocated and implemented increasingly by others in recent years (Craven 1995; Teo et al. 1991). The direct method considers an optimal control problem as a mathematical programming problem in infinite dimensions (see e.g. Tabak and Kuo 1971), and then discretizes to permit computation.

The discretisation method has also several advantages: a wide variety of relatively large-scale and complicated optimal control problems can be solved by this approach and the method is relatively accurate and efficient compared to the indirect methods.

For the above reasons, we have adopted the discretisation (step function or spline) algorithmic approach in this paper. The two computer programs used in this paper for numerical computation are based on this approach.

3.2 The Discretisation Method for Optimal Control

The above optimal control problem can be implemented by the gradient search method based on step functions – discretisation of the control variable. It is shown in Teo et al. (1991) that a substantial class of optimal control problems can readily be put in the standard form given below; and that in many cases, the control function can be sufficiently approximated by a step-function. For simplicity, the time interval is now scaled from [0,T] to [0,1]. A standard version of such an optimal control model may be written as follows:

$$\mathrm{MIN}_{x(.),u(.)} \quad F^0(x,u) = \int_0^1 f(x(t), u(t), t)dt + \Phi(x(1)) \quad \text{subject to} \tag{1}$$

$$x(0) = a, \quad \dot{x}(t) = m(x(t), u(t), t) \quad (0 \le t \le 1), \tag{2}$$

$$x(1) = b, \tag{3}$$

$u_1(t) + u_2(t) \le 1$. $u_1(t) + u_2(t) \le 1$,

$$g(u(t)) \le 0 \quad (0 \le t \le 1). \tag{4}$$

The constraint (4) will often take the particular form of simple bounds :

$$q(t) \le u(t) \le r(t), \tag{5}$$

(A more general constraint such as $u_1(t) + u_2(t) \le 1$ could also be considered.)

Here the state function x(.) (assumed piecewise smooth) and the control function u(.) (assumed piecewise continuous) are, in general, vector-valued; the inequalities on u(t) are taken pointwise. It will be useful to handle the terminal constraint x(1) = b by adding a penalty term to the objective function $F^0(x,u)$. Thus, the objective becomes:

$$F(x,u) := F^0(x,u) + \tfrac{1}{2}\mu|x(1) - b^*|^2, \tag{6}$$

where $b^* \approx b$ (see later discussion), μ is a positive parameter, and the terminal constraint (3) is now omitted.

The differential equation (2), with initial condition, determines x(.) from u(.); denote thus x(t) = Q(u)(t). Then the objective function becomes:

$$J(u) := F^0(Q(u), u) + \tfrac{1}{2}\mu|Q(u)(1) - b^*|^2 \tag{7}$$

The interval [0, 1] is now divided into N equal subintervals, and u (.) is approximated by a step-function taking values $u_1, u_2, \ldots, u_N$ on the successive subintervals. Thus, the discretisation is achieved by restricting u(.) to such a step-function, and the optimisation is over the values $u_1, u_2, \ldots, u_N$. Then, from (2), x(.)

is a polygonal function, determined by its values x_0, x_1, …, x_N at the gridpoints t = 0, 1/N, 2/N, …, 1. Since, because of the jumps at gridpoints, the right hand side of the differential equation (2) is now not a smooth function of t, the differential equation solver chosen must be suitable for such functions. From the standard theory, the gradient J'(u) is given by:

$$J'(u)z = \int_0^1 (f + \lambda(t)m)_u (x(t), u(t), t)) z(t) dt, \tag{8}$$

where the costate function λ(.) satisfies the adjoint differential equation:

$$-\dot{\lambda}(t) = (f + \lambda(t)m)_x (x(t), u(t), t), \qquad \lambda(1) = \mu(x(1) - b^*) \tag{9}$$

(Here suffixes x and u denote gradients with respect to x(.) and u(.).)

As mentioned, the differential equation solver must handle discontinuities at gridpoints. Many standard solvers do not. For example, MATLAB 5.2 includes six ODE solvers, of which only one – designated for stiff differential equations – is useful for solving $\dot{x}(t) = u(t)$ when u(.) is a step-function. A better approach is to modify slightly the well-known *fourth order Runge–Kutta* method. If t is in the subinterval [j/N, (j+1)/N], then u(t) must take the appropriate value u_j, and not (for example) u_{j+1} when t = (j+1)/N. This is easily achieved by recording j as well as t. With this approach, there is no need to further divide the subintervals, in order to integrate. If more precision is required, the number N of subintervals is increased. Note that (2) is solved forwards (starting at t = 0) and (9) is solved backwards (starting at t = 1). Once the differential equation (2) is solved, the objective J(u) becomes a function $\hat{J}(u_1, u_2, ..., u_N)$ of N variables. Two approaches are available:

a. Compute objective values from (2), (1) and (6), but not to compute gradients from (9) and (8). This assumes an optimising package that can estimate gradients by finite differences from (2), (1) and (6).

b. Compute objective values from (2), (1) and (6), and also compute gradients from (9) and (8).

In either case, the optimising package is required to optimise the objective, subject to simple bounds (4). (It is also possible to deal with bounds by using penalty terms, see Craven 1995.)

When as in some economic models (see e.g. Kendrick and Taylor 1971), fractional powers of the functions occur, e.g. with $x(t)^{\beta}$ included in m(x(t), u(t), t) with $0 < \beta < 1$, then the differential equation (2) will bring $x(t) < 0$ for some choices of u(.), causing the solver to crash. Of course, only positive values of x(t) have meaning in such models. Even without explicit constraints on u(.), the requirement $x(.) > 0$ forms an *implicit constraint*, which does not have a simple form. In such a case, it may be better to use, as in (a), finite-difference approximations to gradients, since they may be valid, as useful approximations, over a wider domain.

In the *augmented Lagrangian* algorithm (see e.g. Craven 1995), constraints are replaced by penalty terms, as in (2), added to the objective, with the *target value* b* in (4) depending on a Lagrange multiplier, and the penalty parameter μ is finite, and typically need not be very large. If there are few such constraints, the problem may be considered as one of parametric optimisation, varying the parameter b*, even without computing the Lagrange multipliers.

In solving the differential equation (9), interpolation is required, to estimate values of x(t) at times t between gridpoints. (In SCOM, linear interpolation was used.) Similarly, using (8) to obtain:

$$\partial J(u) \big/ \partial u_j = \int_{j/N}^{(j+1)/N} (f + \lambda(t) m)_u (x(t), u(t), t)\, dt \qquad (10)$$

requires either interpolation of x(.) and $\lambda(.)$, or an Euler approximation of the integral. (The latter has been used, but seems too approximate for some problems). Of course, the number N of subintervals x can be increased, to get better precision.

The problem must be put into the above standard form, if necessary by scaling the time to an interval [0, 1]. As described in Craven, de Haas and

Wettenhall (1998), a nonlinear scaling of time is sometimes appropriate, to get good accuracy with a moderate number of subintervals.

With this background for discretisation and solution to the differential equation, we can summarise the algorithmic blocks and steps in SCOM and RIOTS as follows:

> SCOM: approximation based on a step function, a differential equation solution based on Runge-Kutta method and optimisation based on a conjugate gradient search method.
>
> RIOTS: various spline approximations, and the differential equation solution by Runge Kutta methods and the optimisation problem by projected descent or sequential quadratic programming methods.

3.3 Computer Programs

Computer programs which can be used for a wide variety of continuous control problems have become available only recently (Amman, Kendrick, and Rust 1996; Cesar 1994; Tapiero 1998; Teo et al. 1991). There are some packages which are well known to economists such as DUAL, MATHEMATICA etc.; see references in Amman, Kendrick and Rust (1996). Craven (1995) covers the theory and algorithm for optimal control with a list of possible computer programs including the OCIM program. Cesar (1994) also has references to some other computer programs. A recent survey of optimal control packages is also given in Tapiero (1998). RIOTS is developed by Schwartz, Polak and Chen (1997).

The various optimisation algorithms that some of the optimal control programs encode are as follows:

1. sequential quadratic programming (MISER, Jennings et al. 1991), MATLAB's optimiser "constr" (MATLAB 1997), and SCOM;
2. conjugate gradient methods (OCIM, Craven et al. 1998); and
3. projected descent methods and sequential quadratic programming (RIOTS_95, Schwartz 1996).

They can behave differently, in particular if the function is only defined on a restricted domain, since an optimisation method may want to search outside that domain. For example, the RIOTS_95 package uses various spline approximations to do this, and solves the differential equations by projected descent methods.

SCOM adopts a simpler approach, followed by the MISER package for optimal control, which approximates the control function by a step-function, then solves the differential equations by sequential quadratic programming. Conjugate gradient methods may also be used (as in OCIM). Different implementations may behave differently, in particular if the function is only defined over a restricted domain, since an optimisation method may want to search outside that domain. While a step-function is apparently a crude approximation, it has been shown in various instances (e.g. Craven 1995; Teo et al. 1991) to produce accurate results. The reason is that integrating the dynamic equation (d/dtx(t) = …) to obtain x(t) is a smoothing operation, which attenuates high-frequency oscillations. It is pointed out in Craven (1995) that if this attenuation is sufficiently rapid, the result of step-function approximations converges to the exact optimum as N $\to$ ∞. Some assumption of this qualitative kind is in any case necessary, in order to ensure that the chosen finite dimensional approximations allow a good approximation to the exact optimum.

From the alternatives discussed above we have adopted RIOTS and SCOM for the following reasons: (1) these two programs can solve a large class of optimal control problems with various types of objective functionals, differential equations, constraints on the state and control functions, and terminal conditions (such as bang bang, minmax, and optimal time control problems, see for theory Sengupta and Fanchon (1997) and Teo et. al., (1991)) very easily and conveniently, (2) they are based on a widely used mathematical package MATLAB, (3) these two programs implement the recently developed discretisation algorithmic approach, and (4) RIOTS is commercially available for research in this area.

4. The Kendrick-Taylor Model Implementation

To allow comparison with numerical results in Kendrick and Taylor (1971), the following numerical values were used:

$$T = 10, t = 0.1, d = 0.02, z = 0.842, b = 0.6, s = 0.05, k_T = 24.7.$$

The Kendrick-Taylor model, with the parameter values listed above, has been computed using the RIOTS-95 package on a Pentium, and also with some of the parameters varied, as listed below. For comparison, the model has also been computed using the SCOM package on an iMac computer. The latter package also uses MATLAB (version 5.2) and the constrained optimisation solver *constr* from MATLAB's *Optimisation Toolbox,* but none of the differential equation software used by RIOTS_95. Thus both computations share MATLAB's basic arithmetic and display software, but the implementations are otherwise independent.

4.1 Results: The Numerical Structure of Optimum Growth of the Economy

The results of the computed optimum growth paths of aggregate variables that show the empirical process of growth of the economy are reported in Figures 1a and 1b. The Kendrick-Taylor model was solved for 20 periods, for a total time of 10 years, using RIOTS_95.

Figure 1a. Alternative Model Runs with RIOTS: Consumption

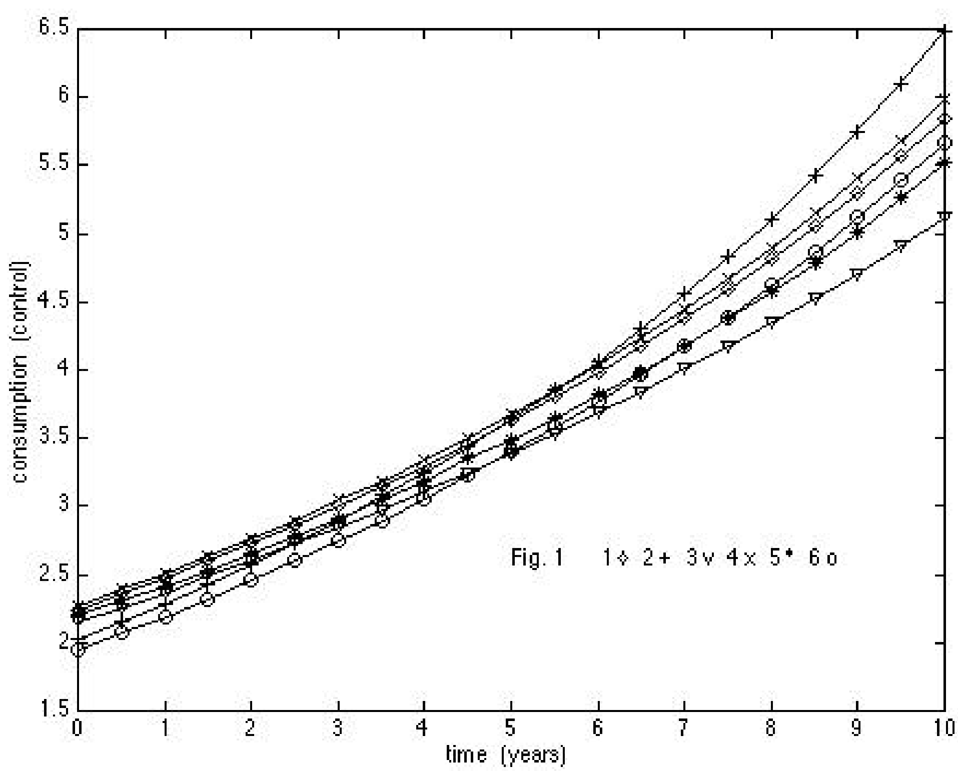
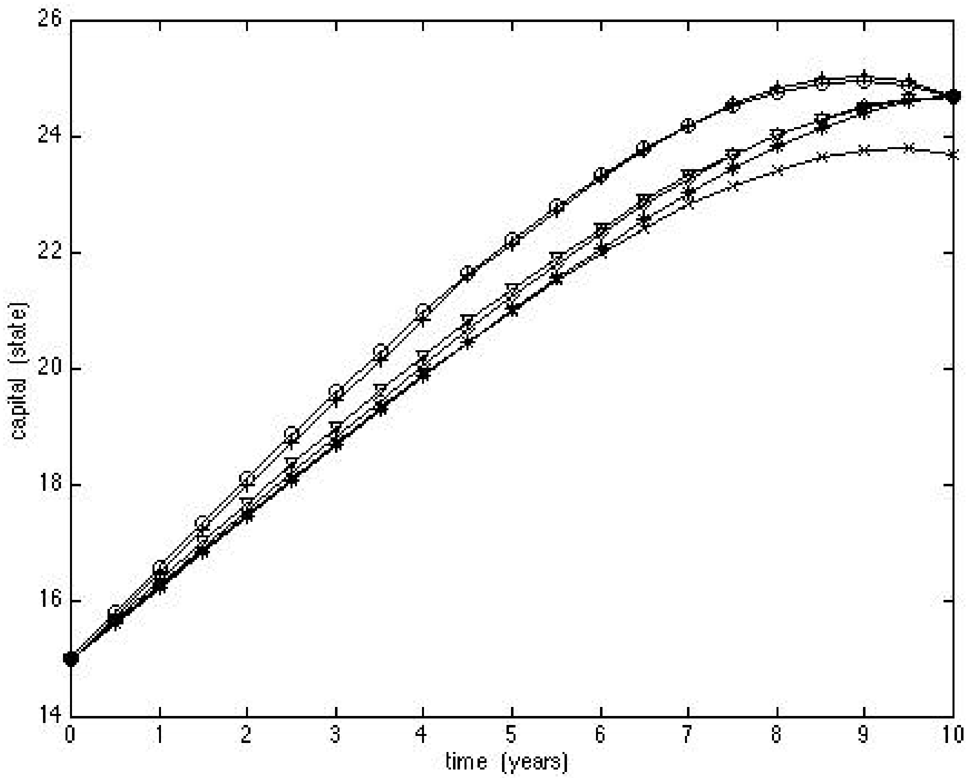


Figure 1b. Alternative Model Runs with RIOTS: Capital



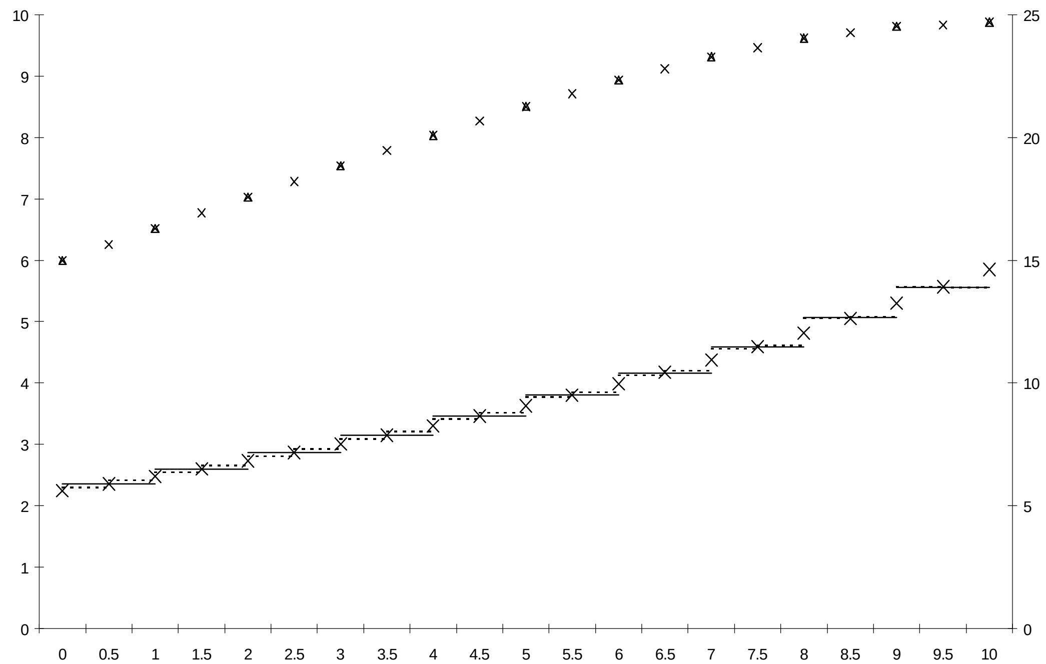
Run 1 shows the base case solution of the model. For sensitivity analysis, the following set of alternative values of the parameters of the model was adopted in six model runs, shown in Table 1.

Table 1. Alternative Values of the Parameters of the Model

Runs	ρ	δ	k_T	ξ
Run 1	0.03	0.02	24.7	0.842
Run 2	0.01	0.02	24.7	0.842
Run 3	0.03	0.01	24.7	0.842
Run 4	0.03	0.02	23.7	0.842
Run 5	0.08	0.02	24.7	0.822
Run 6	0.01	0.01	24.7	0.842

Figure 2 shows state (above) and control (below) for N = 10, compared with N = 20, as computed by SCOM. For comparison, the control points shown x were obtained with RIOTS_95. The state is the same, to graphical accuracy, as the three computations.

Figure 2. Effect of Number of Subdivisions



5 Mathematical and Economic Properties of the Results

The significant properties of the optimally growing economy characterised by the trajectories of the Kendrick-Taylor model computed here numerically are that the economy grows along a unique stable equilibrium path and along this path the society's saving (which is equal to investment) and consumption are optimal and the work force is employed at the full employment level. The values of the costate variable in the Kendrick-Taylor model at different time periods are the shadow prices along the dynamic equilibrium path of the economy and provide the dynamic valuation or pricing system of the economy.

Mathematical properties such as existence, uniqueness, stability and other properties of a steady state solution of the Kendrick-Taylor model are important in deriving and understanding the above stated economic implications of the growth model.

The first issue is the existence of an optimal growth trajectory of the model. Consider first a discrete-time version of the Kendrick-Taylor model, in which the control u is an n-dimensional vector. Suppose that lower and upper bounds are imposed on u(.). Then the set U(n) of feasible controls is bounded, as well as closed; hence the objective function, being continuous, reaches its extremum at a point of U(n); thus, the optimum is attained. Now computing an optimum for the continuous-time version of the model involves the tacit assumption that its optimum is a limit of the optima for suitable finite-dimensional approximations. Assuming this, the attainment of the continuous-time optimum would follow as a limiting case. However, the attainment, and uniqueness, of the optimum can be deduced in another way. The Pontryagin theory gives first-order necessary conditions for an optimum, and these conditions have a solution. If the model were a convex problem, then the first order necessary conditions would also be sufficient for an optimum, thus proving its existence. But convexity is lacking, in particular because the equality constraint (the differential equation) is not linear. However, it is shown below that there is a transformation of the model into a convex problem; hence the solution is a global optimum.

If there is a transformation of the variables k(t), c(t) in the Kendrick-Taylor model to new variables x(t), u(t), so that the transformed problem is a convex problem then the original problem has a property called *invex* (Craven 1995); and it follows then that a local optimum is also a global optimum. The following calculation shows that the Kendrick-Taylor model, with the parameters stated above (from Kendrick and Taylor 1971), is transformable to a convex problem. This will also hold with various other values of the parameters. For this purpose, the Kendrick-Taylor problem may be considered as a minimisation problem, to minimise the negative of the given objective function.

First define $K(t) := k(t)e^{\sigma t}$. Then:

$$\dot{K}(t) = \zeta e^{rt} K(t)^{\beta} - e^{\sigma t} c(t),$$

where r = q + (1 - β)σ. This removes the σk(t) term in the differential equation.

Next, define a new state function $x(t) = K(t)^{\gamma}$, where γ = 1 - β will be assumed later. Denote also θ = (1 - γ)/γ. Then:

$$\dot{x}(t) = \gamma\zeta e^{rt} x(t)^{(\beta+\gamma-1)/\gamma} - \gamma e^{\sigma t} x(t)^{-\theta} c(t)$$

The further substitution $c(t) = x(t)^{\theta} u(t)$ reduces the differential equation to the linear form:

$$\dot{x}(t) = \gamma\zeta e^{rt} - \gamma e^{\sigma t} u(t)$$

What becomes of the integrand $-e^{\rho t} c(t)^{\lambda}$ in the objective function? It becomes $-e^{\rho t} x(t)^{\kappa} u(t)^{\lambda}$, where κ = θλ. Note that with the numerical values cited for λ and θ, κ = 0.15, and κ + λ < 1. Since c(t) and k(t) must be positive, the domain of (x(t), u(t)) is a subset of R^2_+, depending however on t. The Hessian matrix of $-x^{\kappa}u^{\lambda}$ is then:

$$\begin{pmatrix} -\kappa(\kappa-1)x^{\kappa-2}u^{\lambda} & -\kappa\lambda x^{\kappa-1}u^{\lambda-1} \\ -\kappa\lambda x^{\kappa-1}u^{\lambda-1} & -\lambda(\lambda-1)x^{\kappa}u^{\lambda-2} \end{pmatrix}$$

Since 0 < κ < 1 and 0 < λ < 1, the diagonal elements of the Hessian are positive. The determinant is calculated as:

$$x^{2\kappa-2}u^{2\lambda-2}\kappa\lambda(1-\kappa-\lambda)$$

which is positive if κ + λ < 1.

Values λ = 0.1 and β = 0.6 have been used in the present model (e.g. Kendrick). Invexity, and consequent global optimisation, is thus shown for these values, and for other values near them.

For the dynamic systems, stability of the system can be determined by the characteristic roots of the Jacobian matrix formed by linearizing the differential equations of the variables and evaluated at the steady state point. Sensitivity analysis of numerical models can shed light on the stability of the model.

The results in Figures 1a and 1b show the RIOTS results of how changes in ρ, δ, k_T, and ξ affect the optimum growth trajectories of consumption and capital. Figure 1b shows that changes in ρ and k_T (terminal capital) have relatively higher impact on the dynamic path of capital accumulation compared to the changes in the rate of technical progress and the elasticity of marginal utility with respect to changes income. In regard to the value of optimal control (the optimal level of consumption), the impacts are similar. The optimal control variable is relatively more sensitive (see Figure 1a) to changes in the discount rate and the terminal level of capital. These sensitivity experiments suggest that the social time preference and the terminal conditions are significant determinants of the structure of optimal growth paths of an economy.

RIOTS was also used to examine the sensitivity of the Kendrick-Taylor model to some other changes in the data. Figure 3 shows the trajectory of capital when the initial capital is varied from 15.0 to 14.0 or 16.0, though not changing the terminal constraint on the capital. Figure 4 shows the trajectory of the capital with the initial value fixed at 15.0, but the terminal value varied from 24.7 to 27.0 and 22.4. The time scale here is 0 to 20; the turnpike effect noted by Kendrick and Taylor is only apparent on a longer time scale, say 0 to 50. Figure 5 shows the effect of varying the exponent β in the dynamic equation from its given value of 0.6. The upper curve has $\beta = 0.5$, compared with the middle curve for $\beta = 0.6$. The lower curve is what RIOTS gives when $\beta = 0.7$, but with the warning that no feasible point was found. (A computation with SCOM confirms that the model is not feasible for this case.) This graph is included, to point out that when the constraints cannot be satisfied, a computer output need not be any sort of approximation to a solution. In this instance, an implicit constraint is violated, since both state and control run negative.

RIOTS computed an optimal solution to the Kendrick-Taylor model in 20 iterations. The speed is due in part to the use by RIOTS of C code – although the functions defining the problem were entered as M-files (requiring however some computer jargon for switching). SCOM was slower - fairly rapid convergence to the optimum objective value, but much slower convergence to the optimal control function. Some more development is needed here with terminating tolerances for gradient and constraints.

The computing experience with the Kendrick-Taylor model in this paper is largely similar to its previous computing experiences.

Fig 3 Change initial condition

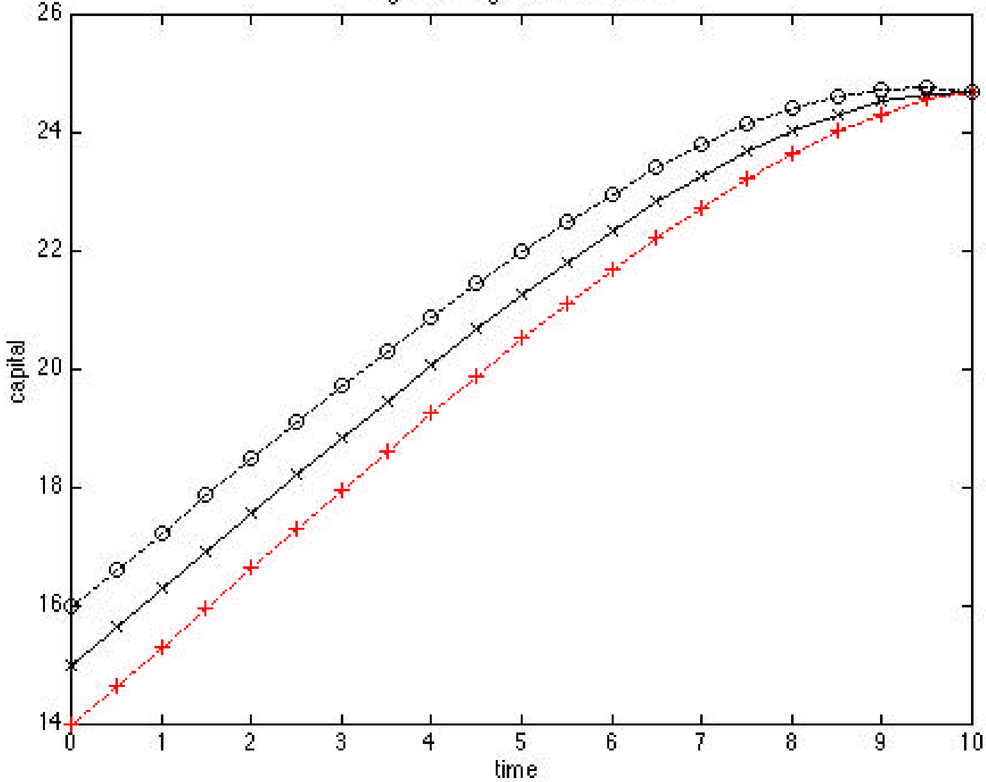
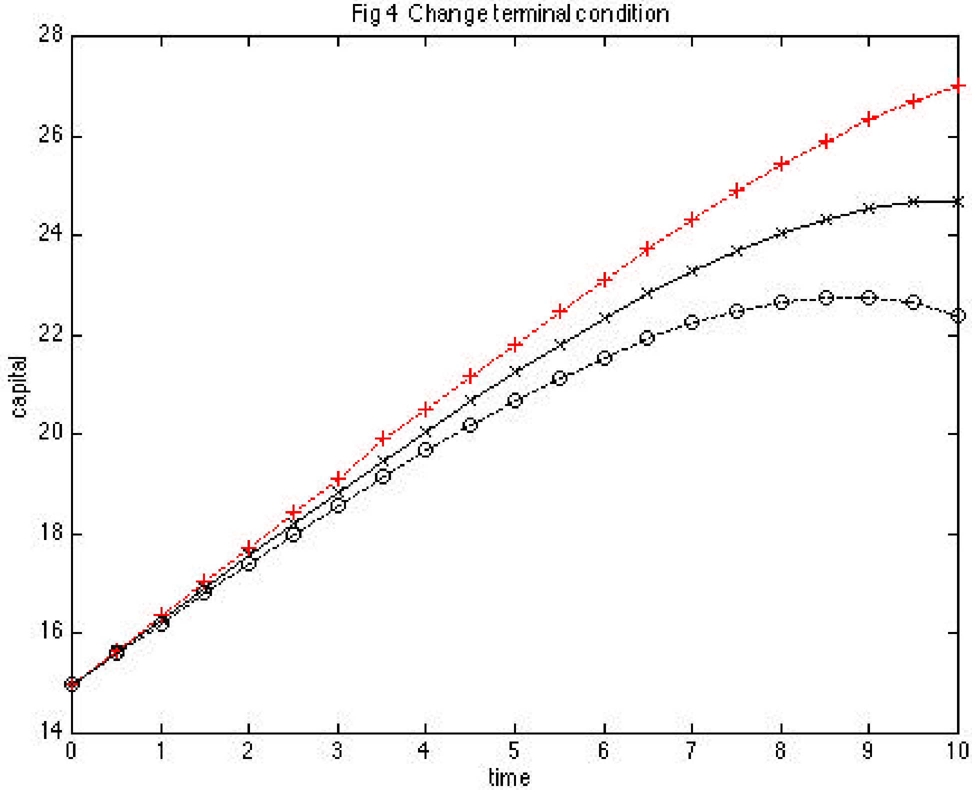
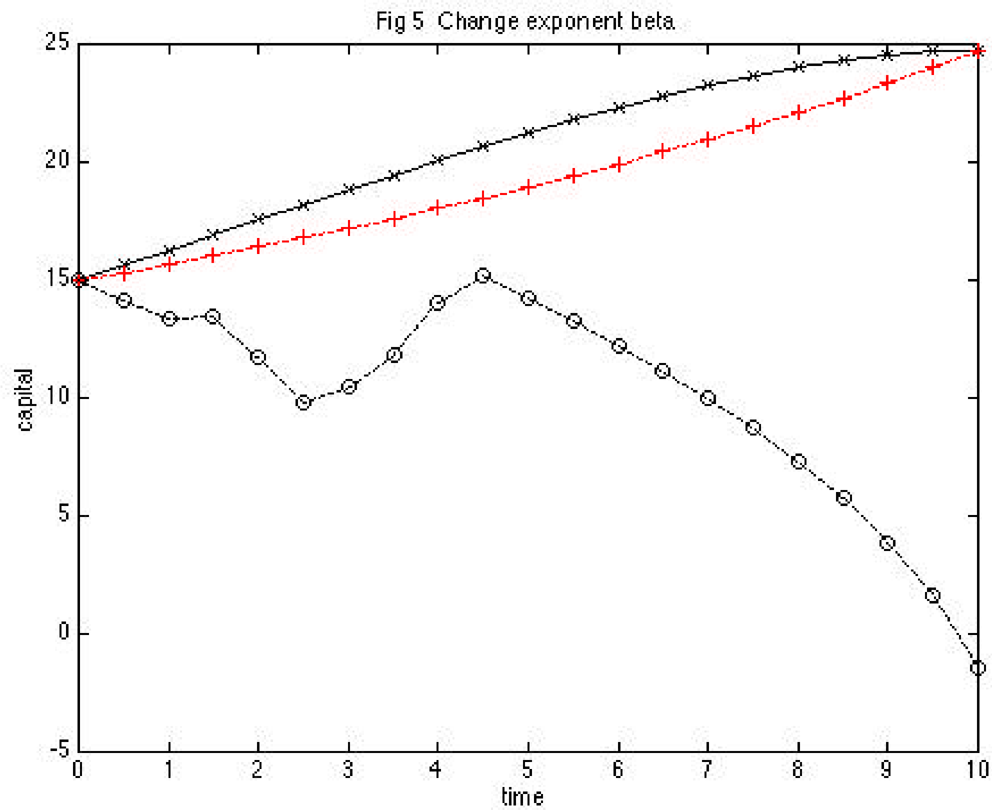


Fig 4 Change terminal condition





6. Computation by Other Computer Programs

While the Kendrick-Taylor model was not computed with MISER, some related economic models were attempted, without success because sufficient accuracy was not obtained for the gradients of the functions. The experience with SCOM suggests that the fractional-power terms in the model may lead to computational difficulty, since some values for the control function may lead to negative values for the state function, for which the fractional powers are undefined.

7. Conclusion

Algorithms and computer packages for solving a class of optimal control problems in continuous time, using the MATLAB system, but in a different way from the RIOTS_95 package, which also uses MATLAB and the RIOTS system, have produced plausible economic results. In the SCOM approach (as in the MISER and OCIM packages), the control is parameterised as a step-function; and SCOM uses MATLAB's "constr" package for constrained optimization as a subroutine. End-point conditions are simply handled using penalty terms. Much programming is made unnecessary by the matrix features built into MATLAB. Economic models may present computational difficulties because of implicit constraints, and there is some advantage using finite difference approximations for gradients. The RIOTS system can produce a set of results close to these produced by the SCOM approach. While several computer packages are available for optimal control problems, they are not always suitable for a wide class of control problems. The MATLAB based RIOTS and SCOM are user-friendly, easily transferable and efficient computer programs that offer good opportunities for computing continuous optimal growth models. Economic modellers will find these algorithm and computer programs suitable to their work.

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