ARCH Models and Option Pricing: the Continuous Time Connection

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Abstract

To implement continuous time option pricing models in which ARCH models can be used as direct or indirect approximators of stochastic volatility, we construct continuous time economies exhibiting equilibrium dynamics to which most asymmetric ARCH models converge in distribution as the sample frequency gets infinite. In the candidate economies, volatility is a diffusion that allows for the "leverage effect", and has a variance that is proportional to the square of the volatility itself. Such characteristics introduce nonlinearities in the resulting pricing models; models of the term structure of interest rates, for instance, are not affine, but are analytically treated here, using a method of iterated approximations. The convergence foundations of the pricing models considered here are based on a class of ARCH models that is large enough to make these pricing models incorporate realistic patterns of volatility of the Markovian type.

KEYWORDS: Stochastic volatility, ARCH, incomplete markets, term structure.

1 Introduction

The increased importance played by conditional volatility in financial economics has led researchers (e.g., Hull and White (1987), Wiggins (1987), Longstaff and Schwartz (1992), Heston (1993)) to extend early asset pricing theories (e.g., Black and Scholes (1973), Merton (1973), Vasicek (1977)) to the case in which volatility evolves in a stochastic manner. On an empirical standpoint, time-varying volatility is well captured by the ARCH-type models introduced by Engle (1982) and Bollerslev (1986). Furthermore, the contribution of Nelson (1990)[1] made clear that some basic ARCH models can reasonably be considered as approximations of a specific class of diffusion processes, which in turn are so frequently used to set up theoretical models. Fornari and Mele (1997*a*) extended the early approximation results of Nelson to an encompassing class of ARCH models, namely the Asymmetric Power ARCH (A-PARCH) model of Ding, Granger, and Engle (1993). Analogous results are presented in Fornari and Mele (1997*b*) for their sign- and volatility- switching ARCH.

The purpose of this article is two-fold.

First, it deepens the previous results by deriving a closed-form solution for the instantaneous correlation between (continuous time versions of) a financial asset price process and its instantaneous volatility, as implied by the A-PARCH model. The negative, instantaneous correlation naturally emerges in correspondence with the "Black-Nelson effect" (Black (1976) and Nelson (1991)) − negative shocks introduce more volatility than positive ones of the same size −.[2] In an independent paper, Duan (1997) elegantly accomplishes similar tasks for models encompassing subclasses of the A-PARCH, but here we account for all the encompassed models as well as for more general distributional assumptions of the discrete model, much in the spirit of Fornari and Mele (1997*a*). (Specifically, we suppose errors to be general error distributed.) We find that the correlation is constant, and that its modulus never reaches unity for a reasonably wide set of parameters' values; this implies that under our approximation scheme markets are likely to be incomplete in models including high frequency asymmetric-type ARCH as generating processes.

As originally suggested by Nelson (1990), the kind of results presented here can give rise to the possibility of using ARCH as *direct* approximators of continuous time stochastic volatility models. A second possibility is to use ARCH as *auxiliary* models in simulation-based schemes; see Gouriéroux and Monfort (1996) for a full account of simulation-based inference, and Pastorello, Renault, and Touzi (1995) for a methodology

[1]See, also, Nelson (1992) and Nelson and Foster (1994).

[2]This kind of property was shown to hold by Nelson (1990) for the exponential ARCH of Nelson (1991).

in which volatilities are filtered using option data. It is beyond the scope of this paper to provide a detailed, empirical analysis of such issues. We shall only provide an illustration of the first possibility in the Monte Carlo application to be discussed below, and a succinct outline of the second possibility in the term structure model that will also be described below.

The second purpose of the paper is to make a step forward in the option pricing domain. Specifically, the paper develops evaluation theory of contingent claims in economies in which *equilibrium* observables (namely, stock prices and interest rates) are diffusion processes with the same structural form (and correlation structure) as the one that is approximated by most asymmetric ARCH models. We consider two natural applications: European-type stock option pricing, and the determination of the term structure of interest rates. The main analytical difficulty that is encountered here is that, in continuous time, ARCH models predict that the variance of volatility is proportional to the *square* of volatility. In the case of the term structure models, for instance, the property in question is a disturbing property if one wishes to get tractable models − notably, affine models − permitting closed-form solutions. See Duffie and Kan (1996); and Fornari and Mele (1998*a*) for a thorough discussion.

The paper's main contributions to option pricing can be summarized as follows.

In the stock-option pricing application, we are only able to approach the problem by implementation of Monte Carlo experiments built-up upon the previous convergence results; Lamoureux and Lastrapes (1993) were the first proponents of such a line of investigation (see also, Engle and Mustafa (1992), for a related approach). The main aim here is to get a first assessment of the empirical relevance of our convergence results. We consider data generating processes with possibly "non standard volatility concepts", such as σ^{δ}, where σ is the instantaneous standard deviation of the primitive asset price process, and $\delta \in \mathbb{R}_{+}$. By using ARCH as direct approximators of continuous time stochastic volatility models applied to US data, we find that the price of a European option estimated via Nelson's (1990) approximation results is higher than the price of the same option estimated via the approximation results presented here; we show that the difference is monotonously increasing from in- to out-of-the-money options, reaching the order of more than one hundred per cent.

In the term structure application, we are able to present an analytical approach based on a method of iterated approximations.[3] It is inspired from relatively new work by Chen (1996) in a different context, and is based on a functional iteration of a certain benchmark affine pricing rule under the action of the associated Arrow-Debreu state price; here the

[3] In the analytical approach that was undertaken in earlier work (Fornari and Mele (1994) and (1995)), analytical obstacles were overcome by specifying risk premia in such a way that our final models were, in fact, observationally equivalent to models with deterministic volatility.

state price has the simple mathematical interpretation of the Green's function used to represent the fundamental solution in standard partial differential equations theory. In this application, further, we outline a procedure in which the approximation results can be used to obtain indirect estimators (Gouriéroux, Monfort and Renault (1993)) of the model's parameters.

The present article has to be considered in complement to other existing attempts to nest ARCH-type models in fully articulated option pricing schemes, namely Amin and Ng (1993), Duan (1995), Kallsen and Taqqu (1998) and Hobson and Rogers (1998), among others. Both Amin-Ng and Duan models, for instance, are in discrete-time: this implies non-linearities which impede the obtention of closed-form solutions. Both Kallsen-Taqqu and Hobson-Rogers, to cite a second example, work in continuous time, but they propose that agents have access to an information set so large that it allows to get around market incompleteness. None of the above cited articles provides theory of the term structure of interest rates. Abstracting from ARCH issues, our model is also the first analytical model in which the term structure of interest rates with stochastic volatility is treated within a rigorous, equilibrium-sounded framework.[4]

The paper is organized as follows. The next section presents the economy; section 3 contains the approximation result; section 4 presents the option pricing applications; section 5 concludes, and the appendices contain technical details omitted in the main text.

2 The models

The program in this section is to construct continuous time economies displaying stochastic equilibrium dynamics to which a well chosen (discrete-time) sequence of ARCH models converges weakly as the sampling frequency becomes higher and higher. In subsection 2.1, we present the first primitives of such candidate economies. In subsections 2.1 and 2.2, we impose restrictions that allow for the construction of two *distinct* models.

The first model is designed for the evaluation of stock options with stochastic volatility, a subject on which there is a vast theoretical literature and to which we shall have very little to add. Our main concern lies in introducing a generalized version of the Hull and White (1987) model that is to be useful in applications such as the one presented in subsection 4.1.

The second model is designed for the evaluation of default-free bonds with stochastic

[4]Longstaff and Schwartz (1992) present an equilibrium model in which the volatility of the instantaneous interest rate is stochastic. In their model, however, volatility is driven by the same Brownian motions driving the instantaneous interest rate. Hence, volatility does not add any further information than that contained in the interest rate dynamics.

volatility, a subject on which relatively little attention has been devoted in the theoretical literature and to which we positively contribute.

In all models, we shall keep close to the approximation results to be presented in section 3. It is emphasized that the restrictions we impose in subsections 2.2 and 2.3 possibly yield two distinct, mutually incompatible models; see remark 2.1 below.

2.1 Primitives

The scheme has been simultaneously exploited by Mele (1998*a*,*b*) with minor changes. It is close to the framework of Cox, Ingersoll and Ross (1985*a*), with the exception that linear activities are absent here, and replaced by stocks. Such a modification is useful for obtaining stock-option restrictions as well as possible ramifications such as the ones presented in subsection 4.1.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T < \infty$, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ the P-augmentation of the natural filtration

$$\mathcal{F}_t^W = \sigma(W_s, s \le t)$$

generated by a Brownian motion in $\mathbb{R}^d$

$$W = \{W_t = (W_t^{(1)}, ..., W_t^{(d)})'\}_{t\in[0,T]},$$

(with $\mathcal{F} = \mathcal{F}_T$). We consider a diffusion state process

$$Y_t^{(h)} = Y_0^{(h)} + \int_0^t \hbar_h(u, Y_u)\mathrm{d}u + \int_0^t \sum_{j=1}^d \ell_{hj}(u, Y_u)\mathrm{d}W_u^{(j)} \quad (h = 1, ..., k), \tag{2.1}$$

where $\hbar_h(t, y)$ $(h = 1, ..., k)$ and $\ell_{hj}(t, y)$ $(h = 1, ..., k$ and $j = 1, ..., d)$ are progressively $\mathcal{F}_t$-measurable functions s.t. $[0, T] \times \mathbb{R}^k \mapsto \mathbb{R}$ and satisfy the usual regularity conditions ensuring a strong solution (Karatzas and Shreve (1991, def. 2.1, p.285)) to the preceding system.

Let $L^2_{0,T,d}(\Omega, \mathcal{F}, P)$ be the space of all the $\mathcal{F}_t$-adapted processes x in $\mathbb{R}^d$ which satisfy:

$$\int_0^T \mathrm{d}u \cdot \|x\|_u^2 < \infty \quad P\text{-a.s. with } P\left(\int_0^T \mathrm{d}u \cdot \|x\|_u^2 > 0\right) > 0,$$

and, for each $t \in [0, T]$,

$$\langle\sigma\rangle^\perp = \left\{x \in L^2_{0,T,d}(\Omega, \mathcal{F}, P) \ / \ \sigma_t \cdot x_t = 0 \text{ a.s.}\right\}.$$

Let

$$S_+ = \{S_t^+ = (S_t^{(0)}, ..., S_t^{(m)})'\}_{t\in[0,T]}$$

be the $\mathcal{F}_t$-adapted stochastic process representing the price of an accumulation factor $(S^{(0)})$ plus m primitive assets entitling to rights on the fruits, or dividends (the numéraire), of m trees $-$ as in the discrete-time model of Lucas (1978).

All assets are exchanged without frictions, and prices are *rational* formed, which means that $S_+ : ([0,T]\times\mathbb{R}^k) \mapsto \mathbb{R}^{m+1}_{++}$; we further suppose that S_+ belongs to $\mathcal{C}^{1,2}$. Let:

$$\overline{G}^{(i)} = \frac{S^{(i)}}{S^{(0)}} + \overline{z}^{(i)} \quad (i = 1, ..., m),$$

be the discounted gain process. Here, for each $i = 1, ..., m$, $\mathrm{d}\overline{z}^{(i)} = \frac{1}{S^{(0)}}\mathrm{d}z^{(i)}$, and:

$$z_t^{(i)} = \textstyle\int_0^t \zeta_s^{(i)}\mathrm{d}s,$$

with $\zeta^{(i)} = \{\zeta_t^{(i)}\}_{t\in[0,T]}$, an $\mathcal{F}_t$-adapted process, standing for the dividend process. An identically zero process is associated with the $0th$ asset − the accumulation factor − and, for each component, $\zeta_t^{(i)} = \zeta^{(i)}(t,y)$. The accumulator factor is taken to formally satisfy:

$$S_t^{(0)} = \exp(\textstyle\int_0^t r_s\mathrm{d}s), \quad t \in [0,T],$$

with $\{r_t\}_{t\in[0,T]}$ denoting the (default-) free interest rate process, an $\mathcal{F}_t$-adapted process satisfying:

$$E(\textstyle\int_0^T r_s\mathrm{d}s) < \infty.$$

We denote:

$$J = \begin{pmatrix} \ell_{11} & \cdots & \ell_{1d} \\ \vdots & \ddots & \vdots \\ \ell_{k1} & \cdots & \ell_{kd} \end{pmatrix}, \quad \frac{\partial S^{(i)}}{\partial Y} = \left(\frac{\partial S^{(i)}}{\partial y_1}, ..., \frac{\partial S^{(i)}}{\partial y_k}\right), \quad \frac{\partial S/S}{\partial Y} = \begin{pmatrix} \frac{\frac{\partial S^{(1)}}{\partial Y}}{S^{(1)}}, ..., \frac{\frac{\partial S^{(m)}}{\partial Y}}{S^{(m)}} \end{pmatrix}'.$$

By Itô's lemma,

$$\begin{aligned} S_t^{(i)} &= S_T^{(i)} \exp\Big(-\textstyle\int_t^T \mu_i(s,Y_s)\mathrm{d}s - \sum_{j=1}^d \int_t^T \sigma_{ij}(s,Y_s)\mathrm{d}W_s^{(j)} \\ &\qquad\qquad +\tfrac{1}{2}\textstyle\sum_{j=1}^d \int_t^T \sigma_{ij}^2(s,Y_s)\mathrm{d}s\Big) \\ &\quad + \textstyle\int_t^T \mathrm{d}s\cdot\zeta_s^{(i)} \exp\Big[-\int_t^s \mu_i(u,Y_u)\mathrm{d}u - \sum_{j=1}^d \int_t^s \sigma_{ij}(u,Y_u)\mathrm{d}W_u^{(j)} \\ &\qquad\qquad +\tfrac{1}{2}\textstyle\sum_{j=1}^d \int_t^s \sigma_{ij}^2(u,Y_u)\mathrm{d}u\Big]. \end{aligned} \tag{2.2}$$

Here, $\overline{\mu}^{(i)}(t,y)S^{(i)} = \frac{\partial S^{(i)}}{\partial t} + \sum_{h=1}^k \frac{\partial S^{(i)}}{\partial y_h}\hbar_h(t,y) + \frac{1}{2}\sum_{h=1}^k\sum_{j=1}^k \frac{\partial^2 S^{(i)}}{\partial y_h \partial y_j}\mathrm{cov}(y_h,y_j)$,[5] $\sigma_{ij}(t,y)S^{(i)} = \sum_{h=1}^k \frac{\partial S^{(i)}}{\partial y_h}\ell_{hj}(t,y)$ ($i = 1, ..., m$ and $j = 1, ..., d$) and $\mu_i(t,y) = \overline{\mu}_i(t,y) + \overline{\zeta}_i(t,y)$ ($i = 1, ..., m$); $\overline{\mu}$ is thus the average appreciation rate referring to the ith primitive asset, and $\overline{\zeta}_t^{(i)} = (\frac{\zeta}{S})_t^{(i)}$. Let σ denote the $m\times d$ matrix whose (i,j) entry is $\sigma_{ij}(t,y)$; this is:

$$\sigma = \frac{\partial S/S}{\partial Y}J,$$

[5] $\mathrm{cov}(y_h, y_j) = \mathrm{d}y_h\mathrm{d}y_j$, in the Itô's sense.

and is supposed to have a rank equal to $m \leq d$, $P \otimes \mathrm{d}t$-*a.s.*

Fruits can be continuously consumed between 0 and T; the consumption process is thus $c = \{c_t\}_{t \in [0,T]}$, a positive adapted process satisfying:

$$\int_0^T c_s \mathrm{d}s < \infty, \quad P\text{-}a.s..$$

Let

$$\pi = \{\pi_t = (\pi_t^{(1)}, ..., \pi_t^{(m)})'\}_{t \in [0,T]}, \quad \pi \in L^2_{0,T,m}(\Omega, \mathcal{F}, P),$$

be a *strategy*: here $\pi^{(i)}$ is the proportion of wealth invested in the ith primitive asset. The value V of a self-financing strategy satisfies:

$$\mathrm{d}V = (\pi'(\mu - 1_{m \times 1} r) + Vr - c)\mathrm{d}t + \pi' \sigma \mathrm{d}W. \tag{2.3}$$

We shall have occasion to say that markets are *complete* if and only if $m = d$ (see, e.g., Mele (1998*b*) for standard details justifying such a definition). Equilibrium is:

$$\pi = S_{m \times 1} \text{ and } c = \textstyle\sum_{i=1}^m \zeta^{(i)} \quad P \otimes \mathrm{d}t\text{-}a.s., \tag{2.4}$$

where $S_{m \times 1}$ contains the last m entries of S_+.

Let

$$\widetilde{\lambda} = \sigma'(\sigma\sigma')^{-1}(\mu - r).$$

Because σ has full rank, the preceding exists, and has the usual interpretation of a risk premium process. In fact, *all* processes belonging to the set:

$$\mathcal{P} = \left\{ \lambda \;/\; \lambda_t = \widetilde{\lambda}_t + \eta_t, \quad \eta \in \langle \sigma \rangle^{\perp} \right\}$$

are bounded and have the interpretation of risk premia processes. More precisely, by defining:

$$\widetilde{\xi}_T := \frac{\mathrm{d}\widetilde{Q}}{\mathrm{d}P} = \exp\left(- \int_0^T \widetilde{\lambda}_t' \mathrm{d}W_t - \tfrac{1}{2} \int_0^T \left\| \widetilde{\lambda} \right\|_t^2 \mathrm{d}t \right),$$

(the Radon-Nikodym derivative of $\widetilde{Q}$ with respect to P on $\mathcal{F}_T$) and the density process of any $Q \approx P$ on $(\Omega, \mathcal{F})$:

$$\xi_t = \widetilde{\xi}_t \cdot \exp\left(- \int_0^t \eta_u' \mathrm{d}W_u - \tfrac{1}{2} \int_0^t \|\eta\|_u^2 \mathrm{d}u \right) \quad (t \in [0, T]), \tag{2.5}$$

(a strictly positive martingale on P), one has that $Q \in \mathcal{Q}$ if and only iff it is of the form $Q(A) = E(1_A \xi_T)$, $\forall A \in \mathcal{F}_T$ (for the proof, adapt, for instance, lemma 3.4 p.429 in Shreve (1991) to the primitive security market model (2.2)). Here $\mathcal{Q}$ is the set of measures that are equivalent to P on $(\Omega, \mathcal{F})$ for $\overline{G}^{(i)}$ $(i = 1, ..., m)$. Further, it is well known that, under all conditions formulated until now, non-emptyness of $\mathcal{Q}$ implies absence of arbitrage opportunities (defined as, e.g., in def. 0.2.3 p.4 in Karatzas (1997)) and that the converse is also true (e.g., thm. 0.2.4 pp.4-7 in Karatzas (1997)). Finally, it is also well known that $\mathcal{Q}$ is a singleton if and only if markets are complete. In the stock option specification below, for instance, markets are incomplete due to stochastic volatility, and in the application of subsection 4.1, we shall consider the so-called Föllmer-Schweizer measure, or minimal martingale measure (Föllmer and Schweizer (1991)).

DEFINITION 2.1: The *minimal martingale measure* is defined to be: $\widetilde{Q}(A) = E(1_A \widetilde{\xi}_T)$, $\forall A \in \mathcal{F}_T$.

In the stochastic volatility setting, the economic interpretation of using $\widetilde{Q}$ is that the resulting model is one in which the risk associated to the fluctuations of stochastic volatility is not compensated, which is the hypothesis of Hull and White (1987). (See formula (2.11) below.) For the mathematical interpretation, the minimale martingale measure is the one which minimizes the Kullback-Leibler distance, or relative entropy, of the objective measure P with respect to any $Q \in \mathcal{Q}$.[6] This kind of results was shown to hold by Föllmer and Schweizer (1991), in the unidimensional case, but in a context more general than that of Brownian information.

2.2 Stock option restrictions

The first specialization of the model is immediate and produces a possible version of the well known stochastic volatility option pricing scheme.

We take $m = 1$, $k = d = 2$, and $\{S_t\}_{t \in [0,T]}$ as the (sole) stock price process, solution of the following stochastic differential equations (s.d.e.):

$$\begin{aligned} \mathrm{d}S_t &= S_t \cdot \left(\mu(t, S_t)\mathrm{d}t + \sigma_t \mathrm{d}W_t^{(1)}\right) \\ \mathrm{d}\nu_t &= \overline{\varphi}(t, \sigma_t)\mathrm{d}t + \overline{\psi}(t, \sigma_t)\mathrm{d}W_t^{(\sigma)} \end{aligned} \tag{2.6}$$

$$W_t^{(\sigma)} = \overline{\rho}(t, \sigma_t) W_t^{(1)} + \sqrt{1 - \overline{\rho}(t, \sigma_t)^2}\, W_t^{(2)}$$

for each $t \in [0, T]$. Here ν is a non-decreasing, continuously differentiable function of σ; $\mu(.)$, $\overline{\varphi}(.)$, $\overline{\psi}(.)$ are progressively $\mathcal{F}_t$-measurable functions respecting the conditions given for system (2.1); $\overline{\rho}(.)$ is also $\mathcal{F}_t$-measurable with modulus strictly less than one a.s.; ν is interpreted as the second state variable of the economy; finally, S is also interpreted as the first state variable of the economy.

Next, we define a *European contingent claim* as a non-degenerate square-integrable $\mathcal{F}_T$-measurable random variable $\widetilde{X} : \Omega \mapsto [0, \infty)$, and let H be the rational price function

[6] This is shown in Fornari and Mele (1998*b*, appendix A) for the general case in which information is diffusion information resumed by state variables.

of the claim:

$$H_t = H(t, S, \nu).$$

We restrict H to be in $\mathcal{C}^{1,2,2}$. By Itô's lemma, it thus satisfies:

$$\mathrm{d}H = \mathcal{D}^*[H]\mathrm{d}t + \frac{\partial H}{\partial S}\sigma S \mathrm{d}W^{(1)} + \frac{\partial H}{\partial \nu}\overline{\psi}\mathrm{d}W^{(\sigma)} \quad (t \in [0, T)),$$

with boundary condition:

$$H(T, x, \nu) = \widetilde{X}, \tag{2.7}$$

where $\mathcal{D}^*[.]$ is the Dynkin operator taken with respect to system (2.6).

We suppose that $\mathcal{Q}$ is also the set of the measures equivalent to P under which $\{(\frac{H}{S^{(0)}})_t\}_{t\in[0,T]}$ is an $\mathcal{F}_t$-$(Q \in \mathcal{Q})$-martingale as well.[7] By the Girsanov's theorem (Karatzas and Shreve (1991, thm 5.1, p.191)), for each $t \in [0, T)$,

$$\begin{array}{rcl} \mathrm{d}S &=& S \cdot \Big((\mu - \sigma\lambda^{(1)})\mathrm{d}t + \sigma_t \mathrm{d}(W^{(1)})^* \Big) \\ \mathrm{d}\nu &=& (\overline{\varphi} - \overline{\psi}\lambda^{(2)})\mathrm{d}t + \overline{\psi}\mathrm{d}(W^{(\sigma)})^* \\ \mathrm{d}H &=& \Big(\mathcal{D}^*[H] - \frac{\partial H}{\partial S}\sigma S\lambda^{(1)} - \frac{\partial H}{\partial \nu}\overline{\psi}\lambda^{(2)} \Big)\,\mathrm{d}t + \frac{\partial H}{\partial S}\sigma S \mathrm{d}(W^{(1)})^* + \frac{\partial H}{\partial \nu}\overline{\psi}\mathrm{d}(W^{(\sigma)})^* \end{array} \tag{2.8}$$

and the presumed $\mathcal{F}_t$-$(Q \in \mathcal{Q})$-martingale property of the discounted primitive asset price process,

$$\mu - \sigma\lambda^{(1)} = r \quad P \otimes \mathrm{d}t\text{-a.s.},$$

or,

$$r = \mu - \begin{pmatrix} \sigma & 0 \end{pmatrix} \begin{pmatrix} \lambda^{(1)} & \lambda^{(2)} \end{pmatrix}' = \mu - \overline{\sigma}\lambda \text{ (say)} \quad P \otimes \mathrm{d}t\text{-a.s.} \tag{2.9}$$

Here,

$$\lambda = \{\lambda_t = (\lambda^{(1)}, \lambda^{(2)})'_t\}_{t\in[0,T]},$$

is an $\mathcal{F}_t$-adapted process and satisfies

$$\textstyle\int_0^t \mathrm{d}u \cdot \|\lambda\|_u^2 < \infty \quad P\text{-a.s. } (t \in [0, T]),$$

and

$$W^* = \{W_t^* = ((W^{(1)})^*, (W^{(\sigma)})^*)'_t\}_{t\in[0,T]}$$

(with

$$(W^{(i)})_t^* = W_t^{(i)} + \textstyle\int_0^t \mathrm{d}u \cdot \lambda_u \quad (i = 1, \sigma) \ (t \in [0, T]))$$

is a standard two-dimensional $\mathcal{F}_t$-Brownian motion under the new probability measure $Q \in \mathcal{Q}$.

[7] Comments after assumption A.7 in appendix A of Fornari and Mele (1998*b*) justify such a position.

By the presumed $\mathcal{F}_t$-$(Q \in \mathcal{Q})$-martingale property of the discounted European claim price process,

$$H_t = E^Q \left(\frac{S_t^{(0)}}{S_T^{(0)}} \cdot \widetilde{X} \mid \mathcal{F}_t \right) \quad (Q \in \mathcal{Q}). \tag{2.10}$$

But since the solution of system (2.9) is indeterminate,

$$\text{card}(\mathcal{Q}) = \infty,$$

which shows that there exists an infinity of rational pricing functions, all induced by the risk premia belonging to the set $\mathcal{P} = \{\lambda \ / \ \lambda_t = \widetilde{\lambda}_t + \eta_t, \eta_t \in \langle \overline{\sigma} \rangle^{\perp}\}$. In other terms, relation (2.10) is the Feynman-Kac representation of the solution of the following partial differential equation:

$$\mathcal{D}[H] = rH,$$

with (2.7) as boundary condition (Karatzas and Shreve (1991, thm. 7.6, p.366)), which is not unique; here,

$$\mathcal{D}[.] \quad = \quad \mathcal{D}^*[.] - (\mu - r)S\tfrac{\partial}{\partial S}. - \overline{\psi}\lambda^{(2)}\tfrac{\partial}{\partial \nu}.,$$

the Dynkin operator applied to the first two eqs. of system (2.8), and non-uniqueness is due to the impossibility of recovering $\lambda^{(2)}$.

It is useful to compute the risk premium inducing the minimal martingale measure. This is:

$$\widetilde{\lambda} = \overline{\sigma}'(\overline{\sigma}\overline{\sigma}')^{-1}(\mu - r) = \begin{pmatrix} \frac{\mu - r}{\sigma} & 0 \end{pmatrix}. \tag{2.11}$$

While indeterminacy of the pricing function can be resolved by making reference to the existence of a representative agent (as in Wiggins (1987)),[8] we shall simply impose that risk-premia are as in (2.11) in subsection 4.1; see assumption 4.1 below. Such a position is standard in empirical studies.

We finally impose parametric restrictions. We take:

$$\begin{array}{rcl} \mu(t,s) & = & \underline{\mu} \\ \overline{\varphi}(t,\sigma) & = & \overline{\omega} - \varphi\sigma^{\delta} \\ \overline{\psi}(t,\sigma) & = & \psi\sigma^{\delta} \\ \overline{\rho}(t,\sigma) & = & \rho \end{array} \tag{2.12}$$

and:

$$\nu = \sigma^{\delta}, \tag{2.13}$$

where $\underline{\mu}, \overline{\omega}, \varphi, \psi, \rho$ and δ are real constants.

[8]See Mele (1998*b*) for a justification of the use of the representative agent in this context.

As it turns out, the approximation results in section 3 suggest that such restrictions caracterize diffusions to which ARCH models converge in distribution. Notice, also, that the Hull and White (1987) model corresponds to the special case in which $\overline{\omega} = 0$. This was first pointed out by Nelson and Foster (1994 p.20 eq. 4.21), though imprecisely. The reason for such a result is that the restriction $\overline{\omega} = 0$ implies that the variance process satisfies:

$$\mathrm{d} \ln \sigma^2 = A \mathrm{d}t + B \mathrm{d}W^{(\sigma)}$$

(where A, B are some real constants), *regardless* of the "volatility concept" δ. If $\overline{\omega} \neq 0$, however, the volatility concept matters.

2.3 Term structure restrictions

We now impose restrictions that identify a two-factor model for the term structure of interest rates. We let a representative agent behave as to maximize the expected flows of her instantaneous, logarithmic utility under the constraint of a generalized version of eq.(2.3); such a version also includes trading in zero-net supply assets, such as bonds. Please notice that we shall impose state-space restrictions that ensure that markets are complete, thus justifying the use of the representative agent object.

Under mild technical conditions (such as assumption A9 p.369 in Cox, Ingersoll, and Ross (1985*a*) (CIR*a*)), as well as the equilibrium conditions in (2.4), one has the following first order conditions (f.o.c.): $0 = u_c - \mathcal{J}_V$, and:

$$\left\{ \begin{array}{lll} 0 & = & (\mu - 1_m r)\mathcal{J}_V + (\sigma\sigma'\pi + \sigma\sigma_\ell'\pi_\ell)\mathcal{J}_{VV} + \sigma J'\mathcal{J}_{VY} \\ 0 & = & (\mu^\ell - 1_n r)\mathcal{J}_V + (\sigma_\ell\sigma_\ell'\pi_\ell + \sigma_\ell\sigma'\pi)\mathcal{J}_{VV} + \sigma_\ell' J\mathcal{J}_{VY} \end{array} \right. \tag{2.14}$$

where $(\mu^\ell, \sigma_\ell, \pi_\ell)$ – referring to n zero-net supply assets – are defined similarly to their counterparts (μ, σ, π) and $\mathcal{J} \equiv \mathcal{J}(t, V, Y)$ denotes indirect utility. We note that our conditions are qualitatively similar to the ones of CIR*a*. To obtain a theory of the term structure of interest rates, we can proceed along lines very similar to the ones presented in the Cox, Ingersoll, and Ross (1985*b*) (CIR*b*) paper. We use the first f.o.c. and derive the equilibrium interest rate:

$$r = \frac{\pi'\mu + \pi'\sigma\sigma'\pi\frac{\mathcal{J}_{VV}}{\mathcal{J}_V} + \pi'\sigma J'\frac{\mathcal{J}_{VY}}{\mathcal{J}_V}}{V}, \tag{2.15}$$

where we have used the equilibrium conditions $\pi_\ell = 0$ and $\pi' 1_m = V$. We are considering logarithmic preferences. It is well known that this implies $\frac{\mathcal{J}_V}{V\mathcal{J}_{VV}} = -1$ and $\mathcal{J}_{VY} = 0$ (no "hedging demand"). By plugging the resulting expression for the interest rate into the resulting first f.o.c., and using again the equilibrium condition $1_m'\pi = V$, we obtain the solution for $\frac{\pi}{V}$, which is:

$$\frac{\pi}{V} = (\sigma\sigma')^{-1}\mu + \frac{1 - 1_m'(\sigma\sigma')^{-1}\mu}{1_m'(\sigma\sigma')^{-1}1_m}(\sigma\sigma')^{-1}1_m, \tag{2.16}$$

which is qualitatively similar to the standard one presented by CIRb.

We now present our two-factor model. In a first step, we specialize the state space; in the second step, we impose the factor restrictions that are sufficient to get the primitives of the remainder.

- *1st step*: *construction of the primitive diffusion state model.* We posit that $k = d = 2$ and that the two state variables are solutions of the following system:

$$\mathrm{d}\begin{pmatrix} y_1 \\ y_2^\delta \end{pmatrix} = \begin{pmatrix} \kappa y_1 + \zeta \\ \widetilde{\omega} - \varphi y_2^\delta \end{pmatrix} \mathrm{d}t + \begin{pmatrix} \sqrt{y_1} y_2 & 0 \\ \psi y_2^\delta \rho & \psi y_2^\delta \sqrt{1-\rho^2} \end{pmatrix} \mathrm{d}\begin{pmatrix} W^{(1)} \\ W^{(2)} \end{pmatrix},$$

  where $\kappa, \zeta, \widetilde{\omega}, \varphi, \psi, \rho, \delta$ are constants. Such a setup differs from the Longstaff and Schwartz (1992) two factor model, and deserves some discussion. We wish to find out, as CIRb, an equilibrium expression in which the interest rate is a linear function of the first state variable. The justification is that such a representation yields a simple model in which the interest rate matches the kind of stochastic volatility presented in the following section; but a sufficient condition to obtain that model is modeling the second state variable as we did (see the following step).

- *2nd step*: *determining the factor restrictions.* Consistently with the motivation of the first step, we encompass the CIRb one-factor model, and suppose that the following factor restrictions hold in (2.2):

$$\begin{cases} \mu &= \widehat{\mu} y_1 \\ \sigma &= \widehat{\sigma}\sqrt{y_1} \end{cases} \tag{2.17}$$

  where $\widehat{\mu}$ and $\widehat{\sigma}$ are, respectively, a conformable vector and a conformable matrix of constants; see Appendix A. Substituting (2.17) into (2.15)-(2.16), we get the following expression for the equilibrium interest rate:

$$r = Ay_1, \quad A = \frac{1_m'(\widehat{\sigma}\widehat{\sigma}')^{-1}\widehat{\mu} - 1}{1_m'(\widehat{\sigma}\widehat{\sigma}')^{-1}1_m}.$$

  We suppose that $A > 0$. By differentiating,

$$\mathrm{d}r = (\kappa r + A\zeta)\mathrm{d}t + \sqrt{r}\nu \mathrm{d}W^{(1)};$$

  here,

$$\nu = \sqrt{A} y_2,$$

  and solves:

$$\mathrm{d}\nu^\delta = (\overline{\omega} - \varphi \nu^\delta)\mathrm{d}t + \psi \nu^\delta \mathrm{d}(\rho W^{(1)} + \sqrt{1-\rho^2} W^{(2)}),$$

  with $\overline{\omega} = \widetilde{\omega} A^{\frac{\delta}{2}}$.

In appendix A, we show that in this model there is an unique measure belonging to $\mathcal{Q}$: it has a density process of the form (2.5), with:

$$\lambda^{(i)} = \lambda_i \sqrt{r}, \tag{2.18}$$

where λ_i $(i = 1, 2)$ are two constants depending on $\widehat{\mu}$ and $\widehat{\sigma}$. In turn, this implies that the price of the bond has to satisfy a partial differential equation which is also reported in appendix A.

REMARK 2.1: In the setup of this section, the stock price and the short term interest rate can not display volatility dynamics that are both restricted to the last three relations in (2.12). If we accept the restrictions of this section, for instance, $\mathrm{d}S_i/S_i = \widehat{\mu}_i y_1 \mathrm{d}t + \widehat{\sigma}_i \sqrt{y_1} \mathrm{d}W$ $(i = 1, ..., m)$, where $\widehat{\mu}_i$ is scalar and $\widehat{\sigma}_i$ is in $\mathbb{R}^d$; hence, for $m = 1$, $\mathrm{d}S/S = \widehat{\mu} y_1 \mathrm{d}t + \widehat{\sigma} \sqrt{y_1} \mathrm{d}W$ (say), and if $y_1 = s$ (as in subsection 2.2), then $\mathrm{d}S/S = \widehat{\mu} S \mathrm{d}t + \widehat{\sigma} \sqrt{S} \mathrm{d}W$, which contradicts the restrictions (2.12) designed for the stock price.

A natural alternative consists in introducing one linear activity (as in CIRa, b) while allowing S to follow the same model of subsection 2.2. It is possible to show, however, that such a choice destroys the term structure model presented in this section.

Such difficulties are not inherent in the specific parametric restrictions in (2.12). In general, formulating tractable models for the short term interest rate implies very specific stock price restrictions.

3 Convergence foundations

We present here the convergence result. While initially motivated by, and then stated with reference to, the European stock option model given in subsection 2.2, the result is also useful in the term structure model to be implemented in subsection 4.2. See remark 4.1 below and appendix E.

Lamoureux and Lastrapes (1993) was the first paper in which the pricing of stock options with stochastic volatility was done with the help of the first approximation results of Nelson (1990) (see subsection 4.1 for further details). The approximation scheme was designed on the basis of the weak convergence[9] of the following discrete time GARCH(1,1)

[9] If (3.1) has to have a unique strong solution denoted as $\{\sigma_t^2\}_{t\geq 0}$, *weak convergence* of $\{{}_h\sigma_{hk}^2\}_{k=1,2,...}$ to $\{\sigma_t^2\}_{t\geq 0}$ means that the finite dimensional distributions of $\{{}_h\sigma_{hk}^2\}_{k=1,2,...}$ converge to those of $\{\sigma_t^2\}_{t\geq 0}$ as $h \downarrow 0$. See Stroock and Varadhan (1979). It turns out that the conditions demanded by Stroock and Varadhan (1979) are difficult to verify when studying the convergence of ARCH-type models (see Nelson (1990) and Fornari and Mele (1997a)). One then has to make reference to the conditions suggested by Nelson (1990).

process:

$$ {}_h\sigma^2_{h(k+1)} - {}_h\sigma^2_{hk} = \omega_h + (\alpha_h \cdot {}_h u^2_{hk} h^{-1} + \beta_h - 1)\, {}_h\sigma^2_{hk},\ \frac{{}_h u_{hk}}{\sqrt{h}} \sim N(0,1), $$

towards the unique strong solution of the stochastic differential equation:

$$ \mathrm{d}\sigma_t^2 = (\overline{\omega} - \varphi\sigma_t^2)\mathrm{d}t + \psi\sigma_t^2 \mathrm{d}W_t^{(2)} \quad (t \geq 0), \tag{3.1} $$

where h denotes sample frequency, ω_h, α_h and β_h are discrete time real parameters sequences (respecting conditions such as (3.7)-(3.9) below), and $\overline{\omega}, \varphi$ and ψ are real, non stochastic parameters; further, the process:

$$ \{{}_h\varepsilon_{hk}\}_{k=0,1,\ldots} = \{{}_h(u \cdot \sigma)_{hk}\}_{k=0,1,\ldots} $$

is interpreted as the error process of an observation model of the asset price.

Nelson (1990) also proved that the distribution of $\{{}_h\varepsilon_{hk}\}_{k=0,1,\ldots}$ is approximately a Student-t as $h \downarrow 0$ and $hk \to \infty$. Hence, non-normality can be accommodated even by a conditionally normal GARCH(1,1), a fact which was roughly known since Engle's (1982) seminal paper. However, empirical research has shown that fitted, standardized residuals are very often leptokurtic; this suggests to model $\{{}_h u_{hk}\}_{k=0,1,\ldots}$ as a process conditionally distributed according to a more flexible distribution (e.g., Bollerslev, Engle and Nelson (1994)). A second point is the apparent lack of tools by which one can choose among different volatility concepts, e.g. an ARCH based "variance" or "standard deviation" concept. These issues can easily be treated within an unifying framework, which is the A-PARCH model:

$$ \sigma_{t+1}^\delta = \omega + \alpha\left(|\varepsilon_t| - \gamma\varepsilon_t\right)^\delta + \beta\sigma_t^\delta, \tag{3.2} $$

in which one may further require that ε is g.e.d.$_{(\upsilon)}$ distributed in order to take account of possible conditional leptokurtosis:

$$ \varepsilon \sim \text{g.e.d.}_{(\upsilon)} = \frac{\upsilon \exp(-\frac{1}{2}\nabla_\upsilon^{-\upsilon}|\varepsilon|^\upsilon)}{2^{1+\upsilon^{-1}}\nabla_\upsilon\Gamma(\upsilon^{-1})}, \quad \nabla_\upsilon^2 \equiv \frac{\Gamma(\upsilon^{-1})}{2^{2/\upsilon}\Gamma(3\upsilon^{-1})}, \quad \upsilon > 0; $$

here, $\Gamma(.)$ is the Gamma function, $\omega > 0$, $\alpha, \beta \geq 0$, $-1 \leq \gamma \leq 1, \delta \in \mathbb{R}_{++}$, and γ allows for the leverage effect originally observed by Black (1976), and incorporated by Nelson (1991) in ARCH-type models.[10]

[10] As argued in Fornari and Mele (1997a), the *combination* of δ and υ should increase the flexibility of both the conditional and unconditional distributions of the error terms. In fact, while the conditionally normal GARCH gives rise to a stationary Student t, which is shaped by a single parameter, the conditionally g.e.d. A-PARCH implies a stationary *generalized* Student t for $\delta = \upsilon$, and a fairly general distribution for $\delta \neq \upsilon$, thus providing a potential better fit for the empirical distribution of asset price changes.

Consider the following approximating scheme:

$$
\begin{array}{lcl}
\ln_h S_{hk} & = & \ln_h S_{h(k-1)} + (\underline{\mu} - {}_h\sigma^2_{hk}/2)h + {}_h\varepsilon_{hk} \\
{}_h\varepsilon_{hk} & = & {}_hu_{hk} \cdot {}_h\sigma_{hk}, \quad \frac{{}_hu_{hk}}{\sqrt{h}} \sim \text{g.e.d.}_{(v)} \\
{}_h\sigma^{\delta}_{h(k+1)} - {}_h\sigma^{\delta}_{hk} & = & \omega_h + (\alpha_h \left| {}_hu_{hk} \right|^{\delta} (1 - \gamma_h s_k)^{\delta} h^{-\frac{\delta}{2}} + \beta_h - 1)_h\sigma^{\delta}_{hk}
\end{array}
\qquad (3.3)
$$

where $s_k = \text{sign}({}_hu_{hk})$,

$$
(\{\omega_h\}, \{\alpha_h\}, \{\beta_h\}) \in \mathbb{R}^3_+ \quad \text{and } \gamma_h \in [-1, +1] \quad (\forall h \in \mathbb{R}_+), \qquad (3.4)
$$

$\{{}_hS_{hk}\}_{k=0,1,\ldots}$ is the asset price process, and $\underline{\mu}$ is a bounded constant.

Fornari and Mele (1997*a*) give conditions under which this system converges weakly to model (2.6), with $\mu(.)$, $\overline{\varphi}(.)$, $\overline{\psi}(.)$ and ν as given by (2.12)-(2.13). Their convergence result does not clarify issues concerning the instantaneous correlation between an asset price process and its volatility. This is not disturbing whenever one is only interested in such issues as the stationary distribution of high frequency innovations. In fact, if standardized innovations are symmetrically distributed around zero (as is the case of the g.e.d.), distributional properties presented in Nelson (1990) or in Fornari and Mele (1997*a*,*b*) are robust in the presence of correlation.

However, if one wishes to use ARCH models to price assets when the presence of such a correlation is suspected, one has to extend the previous result to accommodate, endogenously, a correlation process. Such a level of analysis will eventually allow one to make the parametric link with the kind of models presented in the previous section. Further, it will lead to a simple and yet internally consistent way of estimating the correlation, thus avoiding procedures such as those in Lamoureux and Lastrapes (1993).[11]

We avoid here technicalities referring to the construction of the measure space in (3.3)-(3.4): these can be easily found in Nelson (1990), and are those exploited in Fornari and Mele (1997*a*). We only introduce notation for the filtration generated by $\{{}_hS_{h(i-1)}, {}_h\sigma^{\delta}_{hi}\}^k_{i=1}$, which is $\mathcal{F}_{hk}$, and which will be used in appendix B. Let the symbol $\Rightarrow$ denote weak convergence. Remaining notation is as in section 2.

[11] In the procedure followed by these authors, the estimator of the correlation was the regression coefficient of continuously compounded daily returns on the series of volatility obtained after the estimation of the *symmetric* GARCH(1,1) model.

Theorem 3.1: *Let*

$$m_{\delta,\upsilon} = \frac{2^{\frac{2\delta}{\upsilon}-1}\nabla_{\upsilon}^{2\delta}\Gamma(\frac{2\delta+1}{\upsilon})}{\Gamma(\upsilon^{-1})}, \quad n_{\delta,\upsilon} = \frac{2^{\frac{\delta}{\upsilon}-1}\nabla_{\upsilon}^{\delta}\Gamma(\frac{\delta+1}{\upsilon})}{\Gamma(\upsilon^{-1})},$$

and

$$\rho = \frac{\left((1-\gamma)^{\delta} - (1+\gamma)^{\delta}\right) \frac{2^{\frac{\delta-\upsilon+1}{\upsilon}}\nabla_{\upsilon}^{\delta+1}\Gamma(\frac{\delta+2}{\upsilon})}{\Gamma(\upsilon^{-1})}}{\sqrt{(m_{\delta,\upsilon} - n_{\delta,\upsilon}^2)\left((1-\gamma)^{2\delta} + (1+\gamma)^{2\delta}\right) - 2n_{\delta,\upsilon}^2(1-\gamma)^{\delta}(1+\gamma)^{\delta}}}, \tag{3.5}$$

and, for each h,

$$Z_h = (m_{\delta,\upsilon} - n_{\delta,\upsilon}^2)\left((1-\gamma_h)^{2\delta} + (1+\gamma_h)^{2\delta}\right) - 2n_{\delta,\upsilon}^2(1-\gamma_h)^{\delta}(1+\gamma_h)^{\delta}. \tag{3.6}$$

Next, suppose that the following conditions hold:

$$0 < \overline{\omega} = \lim_{h\downarrow 0} h^{-1}\omega_h < \infty, \tag{3.7}$$

$$-\varphi = \lim_{h\downarrow 0} h^{-1}(n_{\delta,\upsilon}((1-\gamma_h)^{\delta} + (1+\gamma_h)^{\delta})\alpha_h + \beta_h - 1) < \infty, \tag{3.8}$$

$$\psi^2 = \lim_{h\downarrow 0} h^{-1} \cdot Z_h \cdot \alpha_h^2 < \infty. \tag{3.9}$$

Consider, finally, the condition:

$$\gamma_h = \gamma, \quad \forall h. \tag{3.10}$$

Then,

$$\{{}_hS_{h(k-1)}, {}_h\sigma_{hk}^{\delta}\}_{k=0,1,\ldots} \Rightarrow \{S_t, \sigma_t^{\delta}\}_{t\geq 0} \quad \textit{as } h\downarrow 0,$$

where $\{S_t, \sigma_t^{\delta}\}_{t\geq 0}$ *are solutions of*:

$$\begin{aligned} \mathrm{d}\ln S_t &= (\underline{\mu} - \sigma_t^2/2)dt + \sigma_t \mathrm{d}W_t^{(1)}, \\ \mathrm{d}\sigma_t^{\delta} &= (\overline{\omega} - \varphi\sigma_t^{\delta})dt + \psi\sigma_t^{\delta}\cdot(\rho \mathrm{d}W_t^{(1)} + \sqrt{1-\rho^2}\mathrm{d}W_t^{(2)}). \end{aligned} \tag{3.11}$$

Proof: In Appendix B.

The preceding theorem shows that the variance of volatility is proportional to the square of volatility in eq.(3.11). Fornari and Mele (1997*b*) also provide examples of other models (the sign- and volatility- switching models introduced by Fornari and Mele (1996, 1997*b*)) converging weakly to diffusions in which the variance of volatility is a linear function of the square of volatility. Further, it can be shown that the modulus of ρ is strictly less than one in correspondence with reasonable values of $(\gamma, \delta, \upsilon)$ (see the numerical exercises in Fornari and Mele (1998*b*)). This last property implies that stochastic volatility models approximated by ARCH-type models generically induce incomplete markets in continuous time. (See Fornari and Mele (1998*b*) for a rigorous discussion.)

As noted, similar convergence results can be obtained in correspondence with diffusion processes designed to represent the instantaneous interest rate dynamics; see remark 4.1 below. Empirical evidence in Fornari and Mele (1995) combined with thm 3.1 then suggest that in this case instantaneous interest rate changes should be positively correlated with instantaneous volatility changes.

4 Implementation

4.1 Stock options

Since closed-form solutions for option prices with stochastic volatility are available in a limited number of cases, Monte Carlo schemes are often implemented in practice. Here we intend to illustrate how the previous convergence result can be used to implement such schemes.

We shall be concerned with a simple scheme in which European call options are to be evaluated through the model and restrictions in subsection 2.2. The first concern is the estimation of the parameters $\overline{\omega}, \varphi, \psi, \rho$ and δ. One easy procedure is to use conditions (3.5)-(3.10) (with the crude $h \equiv 1$) and directly get estimates. This amounts to using ARCH as direct approximators of stochastic volatility. Because the approximating scheme in thm.3.1 is essentially of Euler's type, such estimators are affected by a discretization bias. Nonetheless, such a bias can be corrected in a second step, by using model (3.3) as an auxiliary model in a simulation-based procedure. The argument is made more precise (in appendix E) with concern for the term structure application. In this subsection, we shall only have access to the first step of the procedure, which in turn is exactly the procedure that was followed by Lamoureux and Lastrapes (1993) with reference to the special case of the GARCH(1,1) model. Finally, to compare the results that can be obtained via the original Lamoureux-Lastrapes procedure with the ones that can be obtained here, we obtain estimates by using both the normal GARCH(1,1) model and the richer A-PARCH with g.e.d. errors.

The exercise consists in simulating the trajectory of the primitive asset price under one of the equivalent martingale measures $-$ i.e. the first two eqs. in (2.8) $-$ and, finally, in computing the price of the European call option as:

$$\widehat{H}_t = e^{-r(T-t)} \frac{1}{n} \sum_{i=1}^{n} (S_T^i - K)^+,$$

where S_T^i is the primitive price as of time T simulated at the i-th Monte Carlo round. One may now invoke a LLN and establish the convergence of $\widehat{H}_t$ to the Feynman-Kac representation of the solution (2.10). We further simplify the exercise by setting $r \equiv 0$.

As noted in subsections 2.1 and 2.2, we shall make the following:

Assumption 4.1: *In pricing European options, agents limit their attention to the minimal martingale measure.*

Our empirical results (to follow) suggest that the estimate of the intercept in the volatility equation is nil for all the models. With $\overline{\omega} = 0$,[12] eq. (3.11) is exactly discretized by:

$$\begin{array}{l} S_{t+1} = S_t \exp(-\frac{\sigma_t^2}{2} + \sigma_t \cdot \widetilde{u}_{t+1}) \quad (t = 1, ..., T) \\ \sigma_{t+1}^2 = \sigma_t^2 \exp(-\frac{2\varphi+\psi^2}{\delta} + \frac{2\psi}{\delta}\widetilde{n}_{t+1}) \quad (t = 1, ..., T) \end{array} \tag{4.1}$$

where we have used an exact approximation with unit steps; here $\{\widetilde{u}_t\}_{t=1}^T$ and $\{\widetilde{n}_t\}_{t=1}^T$ are jointly normal, with correlation ρ as that implied by relation (3.5), and $T = 20$ days, the maturity considered. In the case in which the GARCH(1,1) is used as approximator, we set $\delta = 2$ and $\rho = 0$ in system (4.1).

We make use of the Standard and Poor's 500 index of the New York Stock Exchange, observed daily between 1 January 1990 and 30 April 1996, a sample of 1600 observations. Before the models are estimated, autocorrelation in the series was removed through an autoregressive filter of the 5-th order. Let N be the sample size and $x_i = \ln \frac{\widehat{S}_i}{\widehat{S}_{i-1}}$, where $\widehat{S}$ is the filtered index as of time i ($i = 2, ..., N$). We consider the model:

$$x_i = k + \varepsilon_i, \ \varepsilon_i = (u \cdot \sigma)_i, \ u_i \approx \text{i.i. g.e.d.}_{(\upsilon)}, \quad (i = 2, ..., N) \tag{4.2}$$

with σ as in (3.2), k a real, non stochastic parameter, and (δ, υ) fixed at $(2, 2)$ in the case of the conditionally normal GARCH.

The log-likelihood of a single observation is:

$$\ell = \ln(\frac{\upsilon}{\nabla_\upsilon}) - (1 + \frac{1}{\upsilon}) \ln 2 - \ln \Gamma(\frac{1}{\upsilon}) - \frac{1}{2}\left(\frac{|\varepsilon|}{\nabla_\upsilon \sigma}\right)^\upsilon - \ln \sigma, \quad \nabla_\upsilon^2 \equiv \frac{\Gamma(\upsilon^{-1})}{2^{2/\upsilon}\Gamma(3\upsilon^{-1})},$$

which can be treated using the usual BHHH algorithm.

The estimation results are reported in Table 1. All parameters are significant at the standard level of confidence, except for the intercepts of (4.2) and the intercept of the two volatility equations. The estimates of the continuous time parameters, computed by means of form. (3.5)-(3.10), are reported in Table 2.

System (4.1) is started off at $S_1 = 781.79$, the April 30-th value, and $\sigma_1^2 = \{3.551 \cdot 10^{-5}; 3.912 \cdot 10^{-5}\}$ in the GARCH and A-PARCH case, respectively. With $T = 20$

[12] While such a restriction is dictated by our empirical results, the following equation:

$$\mathrm{d}\sigma_t^2 = (\frac{2\overline{\omega}}{\delta}\sigma_t^{2-\delta} - \frac{2\varphi + (1 - 2\delta^{-1})\psi^2}{\delta}\sigma_t^2)\mathrm{d}t + \frac{2\psi}{\delta}\sigma_t^2 \mathrm{d}\widetilde{W}_t^{(\sigma)},$$

should be discretized in the general case and used as an auxiliary model in an indirect-estimation procedure. Notice that in continuous-time, the "variance dynamics" are sensitive to the specific value of δ. See the discussion at the end of subsection 2.2.

days, the simulation was replicated 1000 times, which became 4000 by employing the antithetic variates technique. At the 20th step of the j-th simulation the price of the call is obtained as $(S_T^{(j)} - K)^+$. We then repeat the procedure a thousand times and calculate the average. We invert the sign of the random numbers and compute the average. We repeat the procedure 100 times and a single price of the option is obtained by regressing the 100 prices on a constant:

$$H_i = H + \epsilon_i \quad (i = 1, ..., 100).$$

Simulations are carried out using the two alternative parameters reported in Table 2. Table 3 reports the bias of the Hull and White model, under the null that the true generating process of the data is the A-PARCH. Such a bias of the alternatives is computed as follows. Let $\widehat{H}_G$ the estimate of H obtained using the normal GARCH parameters and $\widehat{H}_P$ the estimate of H obtained using the g.e.d.-A-PARCH parameters; the table then shows the following ratio:

$$\text{Bias} = \frac{\widehat{H}_G - \widehat{H}_P}{\widehat{H}_G} \cdot 100,$$

computed in correspondence of different strike prices, with moneyness ranging from 0.70 to 1.05. GARCH-based prices are higher than those based on the A-PARCH. The bias monotonously increases from in- to out-of-the-money options. From more than 110 percent for a moneyness equal to 1.05 it decreases to nearly 55 and 2.50 percent when $\frac{K}{S} = 1.0$ and 0.95 respectively. After this point, the bias drops to less than half a percentage point.

Ramifications

The preceding exercise was bounded to the computation of the option price under assumption 4.1. One might also be interested in computing hedging strategies. Unfortunately, the model we consider is incomplete and there is not a truly self-financing strategy. Following Föllmer and Schweizer (1991) and Hofmann, Platen, and Schweizer (1992), however, we can define "mean"-self-financing strategies, that is, strategies generating a hedging error which is a martingale under measures belonging to $\mathcal{Q}$; see Mele (1998*a*) for a recent theoretical treatment of such issues. It is possible to show that the strategy

$$\widetilde{\pi} = \frac{\partial H}{\partial S} S + \frac{\partial H}{\partial \nu} \cdot \overline{\psi} \cdot \overline{\rho} \cdot \sigma^{-1},$$

has the property in question (see Fornari and Mele (1998*b*) for details). The mathematical interpretation of $\widetilde{\pi}$ is that it makes the volatility of the resulting strategy value the best approximation (in projection terms) of the volatility of the European claim value. In fact, such a result generalizes a previous one obtained by Hofmann et al (1992) to the case of a non-zero $\overline{\rho}$. Using the ARCH parametrization (2.12)-(2.13), the preceding equality becomes

$$\widetilde{\pi} = \frac{\partial H}{\partial S} S + \frac{\partial H}{\partial \sigma^{\delta}} \psi \rho \sigma^{\delta - 1},$$

and can be numerically evaluated by usual methods, such as those exploited in the last section of Hofmann et al (1992).

4.2 The term structure of interest rates

The primitive in this subsection is the following stochastic differential equations system:

$$\begin{aligned} \mathrm{d}r &= (\iota - \theta r)\mathrm{d}t + \sqrt{r}\sigma \mathrm{d}W^{(1)} \\ \mathrm{d}\sigma^{\delta} &= (\overline{\omega} - \varphi \sigma^{\delta})\mathrm{d}t + \psi \sigma^{\delta} \mathrm{d}(\rho W^{(1)} + \sqrt{1-\rho^2} W^{(2)}) \end{aligned} \tag{4.3}$$

for each $t \in [0, T]$, where ι, θ are real-valued, non stochastic parameters. Remaining notation is as in subsection 2.3. In the notation of subsection 2.3, further, $\iota \equiv A\zeta$ and $\theta \equiv -\kappa$. We impose $\iota, \theta > 0$.

Remark 4.1: The approximation result in thm. 3.1 concerned the dynamics of the primitive asset price process. With nearly identically arguments, we might show the weak convergence of a suitably chosen discrete-time *interest rate* process towards the solution of system (4.3), with discrete-time volatility as in (3.3). See eqs.(4.5)-(4.6) below.

Let $B_T(r, \sigma^{\delta}, t)$ be the current rational price of a pure discount bond promising to pay one unit of numéraire at time T when the current instantaneous interest rate and its instantaneous volatility are r and σ^{δ}. System (4.3) is defined under the objective measure space. Arbitrage opportunities are ruled out by the existence of a measure equivalent to the objective measure which makes of $\{\frac{B_T(r,\sigma^{\delta},t)}{S_t^{(0)}}\}_{t \in [0,T]}$ a martingale. As argued in subsection 2.3 and constructively shown in appendix A, existence of the equivalent martingale measure in our model is ensured by existence of two processes exactly identified in (2.18); this implies the existence of a partial differential equation to be satisfied by the price of the bond. We are able to provide a solution based on iterated approximations. The result is the following:

Theorem 4.1: *Under the conditions given in appendix C, the sequence of functions obtained by iterated approximations*:

$$\widetilde{B}_T^{(n)}(r,\sigma^\delta) = \widetilde{\mathcal{B}}_T(r,\sigma^2) + \mathcal{K}[\widetilde{B}_T^{(n-1)}](r,\sigma^2)$$

converges to $\widetilde{B}_T(r,\sigma^\delta)$, *i.e.*:

$$\widetilde{B}_T(r,\sigma^\delta) = \widetilde{\mathcal{B}}_T(r,\sigma^2) + \lim_{N\to\infty} \sum_{n=1}^N \mathcal{K}^n[\widetilde{\mathcal{B}}_T](r,\sigma^2),$$

where $\mathcal{K}[.]$ *and* $\mathcal{K}^n[.]$ *are integral operators defined as in appendix C,* $\widetilde{B}_T(r,\sigma^\delta)$ *and* $\widetilde{\mathcal{B}}_T(r,\sigma^\delta)$ *are Laplace transforms of* $B_T(r,\sigma^\delta,t)$ *and*

$$\mathcal{B}_T(r,\sigma^2,t) = \exp\left[-D(\tau^t)r + F(\tau^t)\sigma^2 + U(\tau^t)\right]$$

with respect to t, *with* $D(.), F(.), U(.)$ *defined in appendix D and*:

$$\tau^t = T - t.$$

Up to a first order approximation,

$$B_T(r,\sigma^\delta,t) \simeq \mathcal{B}_T(r,\sigma^2,t) + \int_{\ {}^2_{++}} \int_t^T G(r,\sigma^2,t;r_+,\sigma_+^2,s)\mathcal{T}[\mathcal{B}_T](r_+,\sigma_+^2,s)\mathrm{d}s\mathrm{d}r_+\mathrm{d}\sigma_+^2, \tag{4.4}$$

where:

$$G(r,\sigma^2,t;r_+,\sigma_+^2,s) = \frac{1}{(2\pi)^2}\int\int e^{-ir_+\eta_1 - i\sigma_+^2\eta_2 - \widetilde{D}(\eta_1;\tau_s)r + \widetilde{F}(\eta_1,\eta_2;\tau_s)\sigma^2 + \widetilde{U}(\eta_1,\eta_2;\tau_s)}\mathrm{d}\eta_1\mathrm{d}\eta_2,$$

and the $\mathcal{T}[\mathcal{B}_T](.)$ *operator is given explicitely by*:

$$\begin{aligned}\mathcal{T}[\mathcal{B}_T](r,\sigma^2,t) = \ & \left[\phi\sigma^2(\lambda_1 - \lambda_3\sqrt{r}) + w(\sigma^{2-\delta}-1)\right]F(\tau^t)\mathcal{B}_T(r,\sigma^2,t) \\ & + \tfrac{1}{2}\left[\sigma^2(r-1)D^2(\tau^t) + \phi^2(\sigma^4-1)F^2(\tau^t)\right]\mathcal{B}_T(r,\sigma^2,t) \\ & - \left[\lambda_1(\sigma - r)\sigma + \rho\phi\sigma^3\sqrt{r}F(\tau^t)\right]D(\tau^t)\mathcal{B}_T(r,\sigma^2,t)\end{aligned}$$

with:

$$\begin{aligned}&\tau_s = s - t,\\ &w = \tfrac{2\overline{\omega}}{\delta},\\ &\phi = \tfrac{2\psi}{\delta},\\ &\lambda_3 = \lambda_1\rho + \lambda_2\sqrt{1-\rho^2},\end{aligned}$$

$i \equiv \sqrt{-1}$, *and* $\widetilde{D}(\eta_1;.), \widetilde{F}(\eta_1,\eta_2;.), \widetilde{U}(\eta_1,\eta_2;.)$ *are defined in appendix D. The first order correction is given in appendix D.*

Proof: In appendix D.

As shown in appendix D, G is the Arrow-Debreu state price associated with $\mathcal{B}_T$. It can be interpreted as the Green's function associated with a fundamental solution of the partial differential equation (D9) in appendix D. The model to which $\mathcal{B}_T$ corresponds is an affine model, and refers to an economy in which the instantaneous interest rate and its instantaneous volatility are jointly normal and independent. The idea of the preceding theorem is thus to start with a poor model, $\mathcal{B}_T$, which can nevertheless be exploited to get progressively more accurate approximations by its iteration under the action of $\mathcal{T}$ and G. For brevity, this iteration has been stopped at one in eq.(4.4).

The structural form of the risk premia process is crucial in determining the analytical solution. However, our choice has *not* been motivated by analytical convenience. It is just the result of the equilibrium model built up in subsection 2.3. In fact, following the guidelines of appendices C and D, one can price bonds in this framework under virtually any judicious specification of the risk premia.

The estimation of the model would take us beyond the aim of the paper, but we wish to give some details about a possible estimation procedure. Let $B_T(r, \sigma^\delta, t; a_+, \lambda)$ denote the bond price formula when the parameters are $a_+ = (\iota, \theta, \overline{\omega}, \varphi, \psi, \rho, \delta)'$ and $\lambda = (\lambda_1, \lambda_2)$. Recall that thm. 3.1 says that the model:

$$\begin{aligned} {}_h\sigma^\delta_{h(k+1)} - {}_h\sigma^\delta_{hk} &= \left[\omega_h - \left(1 - \beta_h - n_{\delta,\upsilon}((1-\gamma)^\delta + (1+\gamma)^\delta)\alpha_h\right)_h \sigma^\delta_{hk}\right] \\ &\quad + \alpha_h h^{-\frac{\delta}{2}} \cdot {}_h\sigma^\delta_{hk}\left[\left|{}_h u_{hk}\right|^\delta (1 - \gamma s_k)^\delta - E\left(\left|{}_h u_{hk}\right|^\delta (1-\gamma s_k)^\delta\right)\right] \end{aligned} \tag{4.5}$$

approximates the volatility process (3.11) when $\omega_h \to \overline{\omega} h$, $(1 - \beta_h - (n_{\delta,\upsilon}((1-\gamma)^\delta + (1+\gamma)^\delta)\alpha_h) \to \varphi h$ and $\alpha_h \to \sqrt{\frac{h}{(m_{\delta,\upsilon} - n^2_{\delta,\upsilon})\left((1-\gamma)^{2\delta} + (1+\gamma)^{2\delta}\right) - 2n^2_{\delta,\upsilon}(1-\gamma)^\delta(1+\gamma)^\delta}}\psi$. Here $E\left(\left|{}_h u_{hk}\right|^\delta (1-\gamma s_k)^\delta\right) = h^{\frac{\delta}{2}} n_{\delta,\upsilon}((1-\gamma)^\delta + (1+\gamma)^\delta)$ and $\frac{{}_h u_{hk}}{\sqrt{h}}$ is g.e.d.$_{(\upsilon)}$ for each $h > 0$. Similarly, by an extension of a convergence result in Fornari and Mele (1994 pp.308-309) (along the lines of thm. 3.1), the system composed by:

$$ {}_h r_{h(k+1)} - {}_h r_{hk} = \iota_h - \theta_h \cdot {}_h r_{hk} + {}_h\sigma_{h(k+1)}\sqrt{{}_h r_{hk}} \cdot {}_h u_{h(k+1)} \tag{4.6}$$

and (4.5) also converges weakly to (4.3) if $\iota_h \to \iota h$ and $\theta_h \to \theta h$.

This suggests the possibility to use an indirect estimator (Gouriéroux, Monfort, and Renault (1993)) of a_+ based on simulations of (4.5) and (4.6). Such a procedure encompasses that proposed by Broze, Scaillet, and Zakoïan (1995) by allowing the volatility of the interest rate to evolve in a stochastic and *autonomous* manner: in appendix E, we sketch the procedure that should be followed here. Finally, an estimator of λ can be the one that minimizes the squared differences between a given term structure and that predicted by the model. Alternatively, one can exactly fit an observed term structure with that predicted by the model by searching over λ, but this requires a monotonicity argument concerning both $\lambda_1 \mapsto B_T(r, \sigma^\delta, t; a_+, (., \lambda_2))$ and $\lambda_2 \mapsto B_T(r, \sigma^\delta, t; a_+, (\lambda_1, .))$.

Last, although the corrections in the bond pricing formula are in closed form, they must be computed by numerical integration algorithms; an alternative approach is to solve numerically the partial differential equation (D9) given in appendix D (Aït-Sahalia (1996), for instance, recently followed such an approach in a similar context). Our formula gives the choice between the two alternatives.

5 Conclusion

While conditionally heteroskedastic, the most usual approximations to continuous time stochastic volatility models are not exactly ARCH models (see, e.g., Melino and Turnbull (1990)); Ghysels, Harvey, and Renault (1996) provide an overview of the literature. As originally pointed by Nelson and Foster (1994), it also turns out that, typically, continuous time stochastic volatility models do not have the structural form required to be approximated by ARCH models. The occurrence of such events destroys the potential for ARCH to be useful as approximation or as auxiliary devices in continuous time models.

Concern of this article was to look for economies supporting equilibrium dynamics that are approximated by ARCH models. Within more or less constrained versions of such economies, we developed basic evaluation theory for stock options and the term structure of interest rates. Our theory has the attractive feature to be based on so general versions of ARCH models that it is expected to be consistent with past data analysis. While the versions of ARCH that have been used here are general, they are certainly *not* exhaustive. A generalization would be to expand the theory to include multivariate stochastic volatility models, jump-diffusion phenomena, and long-memory. This awaits future research.

Appendix A

CONSTRUCTION OF THE MARTINGALE MEASURE: By the second f.o.c. in (2.14), and (2.16),

$$\begin{aligned}\mu^{\ell}-1_n r &= \sigma_{\ell}\sigma'\left(\frac{\pi}{V}\right)\\ &= \sigma_{\ell}\left(\sigma'(\sigma\sigma')^{-1}\mu+\frac{1-1_m'(\sigma\sigma')^{-1}\mu}{1_m'(\sigma\sigma')^{-1}1_m}\sigma'(\sigma\sigma')^{-1}1_m\right).\end{aligned}\tag{A1}$$

This shows that the price system $\{\frac{B_t}{S_t^{(0)}}\}_{t\in[0,T]}$ is a $(\widehat{Q}\in\mathcal{Q})$-martingale, where $\widehat{Q}$ has density process:

$$\left.\frac{\mathrm{d}\widehat{Q}}{\mathrm{d}P}\right|_{\mathcal{F}_t}=\exp\left(-\int_0^t\widehat{\lambda}_u'\mathrm{d}W_u-\frac{1}{2}\int_0^t\left\|\widehat{\lambda}\right\|_u^2\mathrm{d}u\right),$$

and

$$\widehat{\lambda}=\sigma'(\sigma\sigma')^{-1}\mu+\frac{1-1_m'(\sigma\sigma')^{-1}\mu}{1_m'(\sigma\sigma')^{-1}1_m}\sigma'(\sigma\sigma')^{-1}1_m,$$

which has the interpretation of unit risk premium demanded to compensate the fluctuations of the d-dimensional Brownian motion; using the restrictions in (2.17), we get the solution:

$$\begin{aligned}\widehat{\lambda} &= \sqrt{y_1}\left(\widehat{\sigma}'(\widehat{\sigma}\widehat{\sigma}')^{-1}\widehat{\mu}+\frac{1-1_m'(\widehat{\sigma}\widehat{\sigma}')^{-1}\widehat{\mu}}{1_m'(\widehat{\sigma}\widehat{\sigma}')^{-1}1_m}\widehat{\sigma}'(\widehat{\sigma}\widehat{\sigma}')^{-1}1_m\right)\\ &\equiv \sqrt{y_1}\begin{pmatrix}\widetilde{\lambda}_1\\ \widetilde{\lambda}_2\end{pmatrix},\end{aligned}$$

where $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ are two constants. Note that under all of our assumptions, $\widehat{\lambda}$ satisfies the Novikov condition.

Finally, completeness is ensured by assuming that the number of stocks plus bonds equals two.

DERIVATION OF THE PARTIAL DIFFERENTIAL EQUATION: Denote the price of a (default) free-risk bond as B. By condition (A1),

$$\begin{aligned}&B_t+B_{y_1}(\kappa y_1+\zeta)+B_{y_2^\delta}(\widetilde{\omega}-\varphi y_2^\delta)\\ &+\frac{1}{2}\left(B_{y_1y_1}\mathrm{var}(y_1)+2B_{y_1y_2^\delta}\mathrm{cov}(y_1,y_2)+B_{y_2^\delta y_2^\delta}\mathrm{var}(y_2^\delta)\right)-rB\\ =\;&\frac{\partial B}{\partial Y}J\widehat{\lambda},\end{aligned}$$

with $\frac{\partial B}{\partial Y}\equiv\begin{pmatrix}B_{y_1} & B_{y_2^\delta}\end{pmatrix}$ and $J\equiv\begin{pmatrix}\sqrt{y_1}y_2 & 0\\ \psi y_2^\delta\rho & \psi y_2^\delta\sqrt{1-\rho^2}\end{pmatrix}$. In terms of (r,ν^δ), this is:

$$B_t+B_r(\iota-\theta r)+B_{\nu^\delta}(\overline{\omega}-\varphi\nu^\delta)$$

$$
\begin{aligned}
& +\frac{1}{2}\left(B_{rr}r\nu^2+2B_{r\nu^\delta}\psi\rho\sqrt{r}\nu^{\delta+1}+B_{\nu^\delta\nu^\delta}\psi^2\nu^{2\delta}\right)-rB \\
= \quad & B_r r\nu\lambda_1+B_{\nu^\delta}\left(\psi\rho\nu^\delta\sqrt{r}\lambda_1+\psi\sqrt{1-\rho^2}\nu^\delta\sqrt{r}\lambda_2\right) \qquad \text{(A2)}
\end{aligned}
$$

where $\iota \equiv A\zeta$, $\theta \equiv -\kappa$, $\lambda_i \equiv \frac{\widetilde{\lambda}_i}{\sqrt{A}}$ $(i=1,2)$.

In appendix C, we shall need to work with a pricing function of the form $B(r,\nu^2,t)$ for analytical purposes. We thus express (A2) equivalently as:

$$
\begin{aligned}
0 \quad = \quad & -rB-B_\tau+B_r(\iota-\theta r)+B_{\nu^2}(w\nu^{2-\delta}-\vartheta\nu^2)+\tfrac{1}{2}B_{rr}r\nu^2+\tfrac{1}{2}B_{\nu^2\nu^2}\phi^2\nu^4 \\
& +B_{r\nu^2}\sqrt{r}\nu^3\phi\rho-\left(\sqrt{r}\nu B_r+\phi\nu^2\rho B_{\nu^2}\right)\lambda^{(1)}-\phi\nu^2\sqrt{1-\rho^2}B_{\nu^2}\lambda^{(2)}, \qquad \text{(A3)}
\end{aligned}
$$

where $\lambda^{(i)}=\lambda_i\sqrt{r}$ and w,ϑ,ϕ are defined in thm.4.1 and appendix D.

Appendix B

Proof of Theorem 3.1: Conditions (3.7)-(3.9) are sufficient to establish the weak convergence of the stock price and volatility processes toward the solutions of the following stochastic differential equations:

$$
\mathrm{d}\ln S_t=(\underline{\mu}-\sigma_t^2/2)\mathrm{d}t+\sigma_t\mathrm{d}W_t^{(1)}
$$

$$
\mathrm{d}\sigma_t^\delta=(\overline{\omega}-\varphi\sigma_t^\delta)\mathrm{d}t+\psi\sigma_t^\delta\mathrm{d}W_t^{(\sigma)}
$$

where $\{W_t^{(j)}\}_{t\geq 0}$ $(j=1,\sigma)$ are $\mathcal{F}_t$-Brownian motions. This is constructively proven in Fornari and Mele (1997*a*, thm. 2.3 p. 209-211). It remains to show that $W_t^{(\sigma)}$ can be written as:

$$
W_t^{(\sigma)}=\rho W_t^{(1)}+\sqrt{1-\rho^2}W_t^{(2)} \quad (t\geq 0)
$$

with $\{W_t^{(2)}\}_{t\geq 0}$ another $\mathcal{F}_t$-Brownian motion. We are going to show that this is true thanks to the further restriction (3.10). It is sufficient to show that the limit:

$$
\lim_{h\downarrow 0} h^{-1}E\left[(\ln\ {}_hS_{hk}-\ \ln\ {}_hS_{h(k-1)})({}_h\sigma_{h(k+1)}^\delta-\ {}_h\sigma_{hk}^\delta)\mid \mathcal{F}_{hk}\right]
$$

is not ill-behaved. After that, an identification argument will do the work.

By (3.7)-(3.9), and the fact that ${}_hu_{hk}/\sqrt{h}$ is g.e.d.$_{(\upsilon)}$ for each h,

$$
\begin{aligned}
& \lim_{h\downarrow 0} h^{-1}E\left[(\ln\ {}_hS_{hk}-\ \ln\ {}_hS_{h(k-1)})({}_h\sigma_{h(k+1)}^\delta-\ {}_h\sigma_{hk}^\delta)\mid \mathcal{F}_{hk}\right] \\
= \quad & \lim_{h\downarrow 0} h^{-1}E\Big[((\underline{\mu}-\ {}_h\sigma_{hk}^2/2)h+\ {}_hu_{hk}\cdot {}_h\sigma_{hk}) \\
& \qquad \times(\omega_h+(\alpha_h\left|{}_hu_{hk}\right|^\delta(1-\gamma_h s_k)^\delta h^{-\frac{\delta}{2}}+\beta_h-1){}_h\sigma_{hk}^\delta)\mid \mathcal{F}_{hk}\Big] \\
= \quad & \lim_{h\downarrow 0} h^{-1}E\left[{}_hu_{hk}\left(\alpha_h\left|{}_hu_{hk}\right|^\delta(1-\gamma_h s_k)^\delta h^{-\frac{\delta}{2}}+\beta_h-1\right)\cdot\ {}_h\sigma_{hk}^{\delta+1}\mid \mathcal{F}_{hk}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{h\downarrow 0} h^{-1-\frac{\delta}{2}}\alpha_h \cdot E\left[{}_h u_{hk}\left|{}_h u_{hk}\right|^{\delta}(1-\gamma_h s_k)^{\delta}\cdot \ {}_h\sigma_{hk}^{\delta+1} \mid \mathcal{F}_{hk}\right] \\
&= \lim_{h\downarrow 0} \frac{\alpha_h}{\sqrt{h}}\left[\left((1-\gamma_h)^{\delta}-(1+\gamma_h)^{\delta}\right)\int_{+} x^{\delta+1}p(\mathrm{d}x)\right]\cdot \ {}_h\sigma_{hk}^{\delta+1},
\end{aligned}
$$

where $p(.)$ denotes the g.e.d.$_{(\upsilon)}$ density, or:

$$
\begin{aligned}
&\lim_{h\downarrow 0} h^{-1}E\left[(\ln \ {}_hS_{hk} - \ \ln \ {}_hS_{h(k-1)})({}_h\sigma_{h(k+1)}^{\delta} - \ {}_h\sigma_{hk}^{\delta}) \mid \mathcal{F}_{hk}\right] \\
= \ &\lim_{h\downarrow 0} \frac{\alpha_h}{\sqrt{h}}\left((1-\gamma_h)^{\delta}-(1+\gamma_h)^{\delta}\right)K\cdot \ {}_h\sigma_{hk}^{\delta+1};
\end{aligned}
\tag{B1}
$$

here,

$$
K = \frac{2^{\frac{\delta-\upsilon+1}{\upsilon}}\nabla_{\upsilon}^{\delta+1}\Gamma(\frac{\delta+2}{\upsilon})}{\Gamma(\upsilon^{-1})}.
$$

We claim that the r.h.s. of (B1) is bounded and bounded away from zero. To see this, notice that by condition (3.9),

$$
\psi^2 = \lim_{h\downarrow 0}\left(\frac{\alpha_h}{\sqrt{h}}\right)^2\cdot Z_h < \infty,
$$

and if $\gamma_h = \gamma$ for each h (condition (3.10)), then $Z_h = Z < \infty$ for each h, and:

$$
\lim_{h\downarrow 0}\left(\frac{\alpha_h}{\sqrt{h}}\right)^2 = \left(\frac{\psi}{\sqrt{Z}}\right)^2,
$$

which is bounded. By continuity, and (3.4),

$$
\lim_{h\downarrow 0}\frac{\alpha_h}{\sqrt{h}} = \frac{\psi}{\sqrt{Z}},
$$

which shows that (B1) is bounded and bounded away from zero:

$$
\lim_{h\downarrow 0} h^{-1}E\left[(\ln \ {}_hS_{hk} - \ \ln \ {}_hS_{h(k-1)})({}_h\sigma_{h(k+1)}^{\delta} - \ {}_h\sigma_{hk}^{\delta}) \mid \mathcal{F}_{hk}\right] = \frac{\psi\overline{K}}{\sqrt{Z}}\ \sigma^{\delta+1},
$$

where:

$$
\overline{K} = \left((1-\gamma)^{\delta}-(1+\gamma)^{\delta}\right)K.
$$

To identify ρ, we note that this has to solve the following equation: $\psi\rho = \frac{\psi}{\sqrt{Z}}\overline{K}$, from which we find, finally:

$$
\rho = \frac{\overline{K}}{\sqrt{Z}}.
$$

The proof is complete. *Q.E.D.*

Appendix C

Let $\mathbb{S}(Z)$ denote the Banach space of complex valued, bounded, continuous functions on a compact set $Z \subset \mathbb{R}^l$, endowed with the norm

$$\|m\| = \sup_{Z} |m| \, .$$

In appendix D, we shall be concerned with the solution of equations of the following form:

$$m(x) = n(x) + \textstyle\int_Z K(x,\xi) m(\xi) \mathrm{d}\xi, \tag{C1}$$

also known as Fredholm equations (e.g., Ruston (1986)); here $n \in \mathbb{S}(Z)$ and the kernel K is continuous on Z^2.

We aim at giving a condition ensuring existence and uniqueness of a solution $m \in \mathbb{S}(Z)$ to (C1).

Let $\mathcal{K}$ be the integral operator associated with K that makes a one-to-one correspondence between any function $v \in \mathbb{S}(Z)$ and the function

$$y = \mathcal{K}[v] \in \mathbb{S}(Z)$$

defined by

$$y(x) = \textstyle\int_Z K(x,\xi) v(\xi) \mathrm{d}\xi.$$

Note that the norm of $\mathcal{K}$ is

$$\|\mathcal{K}\| = \sup_{x \in Z} \textstyle\int_Z |K(x,\xi)| \, \mathrm{d}\xi,$$

that we suppose bounded from one:

$$\|\mathcal{K}\| < 1. \tag{C2}$$

In terms of $\mathcal{K}$, eq.(C1) is:

$$(I - \mathcal{K})[m] = n,$$

where $I[.]$ is the identity map.

Under condition (C2) the sequence

$$m_i = n + \mathcal{K}[m_{i-1}]$$

with $m_0 = 0$ (say), converges in $\mathbb{S}(Z)$ to the solution of (C1):

$$m = n + \textstyle\sum_{i=1}^{\infty} \mathcal{K}^i[n],$$

where $\mathcal{K}^i[.]$ is the integral operator associated with the ith iterate of the kernel K^i defined as $K^1 = K$ and as:

$$K^{j+1}(x,\xi) = \textstyle\int_{Z^j} K(x,\xi_1) K(\xi_1,\xi_2) \cdots K(\xi_j,\xi) \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_j$$

for $j \geq 1$.

Appendix D

Solution method: Pricing with Brownian information often translates into problems involving finding the scalar function $u \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0, T))$ which solves the following partial differential equation (p.d.e.):

$$\begin{cases} \mathcal{L}[u](x,t) = -q(x,t), \quad (x,t) \in \mathbb{R}^N \times [0,T) \\ u(x,T) = g(x), \quad x \in \mathbb{R}^N \end{cases} \tag{D1}$$

where

$$\mathcal{L}[u](x,t) \equiv \mathcal{D}[u](x,t) - r(x,t)u(x,t),$$

and

$$\mathcal{D}[u](x,t) \equiv u_t(x,t) + u_x(x,t)b(x,t) + \tfrac{1}{2}\mathrm{tr}[a(x,t)a'(x,t)u_{xx}(x,t)],$$

where, for an integer M, $a(x,t) : \mathbb{R}^N \times [0,T] \mapsto \mathbb{R}^{N\times M}$ (the space of the $N \times M$ real matrices), $b(x,t) : \mathbb{R}^N \times [0,T] \mapsto \mathbb{R}^N$, $(r,q)(x,t) : \mathbb{R}^N \times [0,T] \mapsto (\mathbb{R}_{++}, \mathbb{R})$, and subscripts denote partial derivatives. Usually, in this context, $u(.)$ represents the price of a European claim, b and a are drift and diffusion coefficients of a N-dimensional diffusion driven by M Brownian motions on the probability space $(\Omega, \mathcal{F}, Q)$, with $Q \in \mathcal{Q}$, where $\mathcal{Q}$ is as in the main text. The usual interpretation of $q(.)$ is that it represents the flow of dividends associated with the European claim. It is the objective of the first part of this appendix, however, to give (D1) and $q(.)$ a broader mathematical interpretation that is useful for the solution of our problem below. In what follows, we suppose standard mild regularity conditions to hold (essentially, uniform ellipticity on a; boundness on a, b; Hölder continuity on a, b, r, q; polynomial growth on g, q – see, e.g., Karatzas and Shreve (1991 p.366-369)), that are met in our problem below.

The starting point is the existence of a nonnegative function $G(t, x; \tau, \xi)$, the so-called *Green's Function*, defined for $0 \le t < \tau \le T, x \in \mathbb{R}^n$, such that the function:

$$\widehat{u}(x,t) = \int_{\ \ N} G(x,t;\xi,\tau)g(\xi)\mathrm{d}\xi, \quad 0 \le t < \tau, \quad x \in \mathbb{R}^N, \tag{D2}$$

is bounded, belongs to $\mathcal{C}^{2,1}$ and satisfies:

$$\begin{cases} \mathcal{L}[\widehat{u}](x,t) = 0, \quad (x,t) \in \mathbb{R}^N \times [0,\tau) \\ \lim_{t\uparrow\tau} \widehat{u}(x,t) = g(x), \quad x \in \mathbb{R}^N \end{cases} \tag{D3}$$

Further, by defining:

$$f(x,t) = G(x,t;\xi,\tau)$$

for fixed $(\xi, \tau) \in \mathbb{R}^N \times (0,T]$, one has that f satisfies (D3) in the backward variables (x,t), but with

$$\lim_{t\uparrow\tau} f(x,t) = \widehat{\delta}(x-\xi),$$

where $\widehat{\delta}(.)$ is the Dirac function (see, also, Arnold (1992 thm. 2.6.6, p.43)).

If $\mathcal{L}[\widehat{u}]=0$ is describing the no-arbitrage restriction of a securities market model, the economic interpretation of the Green's Function is that of the Arrow-Debreu state price of that model: it is the value as of time t in state x of a unit of numéraire at $\tau>t$ in state ξ. The preceding result then means that, if r does not depend on x, then the state price follows the same partial differential equation followed by the contingent claim. A proof of this proceeds along the following lines. Define $\widehat{u}$ by (D2); $\widehat{u}$ is arbitrage free if and only if there is a measure Q (say) equivalent to P on $(\Omega,\mathcal{F})$ under which:

$$\widehat{u}(x,t)=\int_{\mathbb{R}^N}\exp[-\textstyle\int_t^\tau r(s)\mathrm{d}s]Q(x,t;\xi,\tau)g(\xi)\mathrm{d}\xi,\quad 0\leq t\leq\tau,\quad x\in\mathbb{R}^N. \tag{D4}$$

By construction, Q is the solution of the following backward Kolmogorov equation:

$$0=Q_t(x,t;\xi,\tau)+Q_x(x,t;\xi,\tau)b(x,t)+\tfrac{1}{2}\mathrm{tr}[a(x,t)a'(x,t)Q_{xx}(x,t;\xi,\tau)]. \tag{D5}$$

Comparing (D2) with (D4), we see that it must be the case that:

$$G(x,t;\xi,\tau)=\exp[-\textstyle\int_t^\tau r(s)\mathrm{d}s]Q(x,t;\xi,\tau). \tag{D6}$$

Differentiating both sides of (D6) with respect to t, and using (D5), we obtain that $\mathcal{L}[f]=0$. This completes the proof.

Consider, next, eq.(D1) in its full generality. It is easily seen that if a function $G(x,t;\xi,\tau)$ satisfies, for fixed $(\tau,\xi)\in[0,T)\times\mathbb{R}^N$,

$$\begin{cases}\mathcal{L}[G](x,t;\xi,\tau)=0, & (x,t)\in\mathbb{R}^N\times[0,\tau)\\ G(x,\tau;\xi,\tau)=\widehat{\delta}(x-\xi), & x\in\mathbb{R}^N\end{cases}$$

in the backward variables (x,t), then the solution of (D1) can be written in the following form:

$$\begin{aligned}u(x,t)&=\int_{\mathbb{R}^N}\textstyle\int_t^\tau G(x,t;\xi_0,\tau_0)q(\xi_0,\tau_0)\mathrm{d}\tau_0\mathrm{d}\xi_0+\int_{\mathbb{R}^N}G(x,t;\xi,\tau)g(\xi)\mathrm{d}\xi\\&=\int_{\mathbb{R}^N}\textstyle\int_t^\tau G(x,t;\xi_0,\tau_0)q(\xi_0,\tau_0)\mathrm{d}\tau_0\mathrm{d}\xi_0+\widehat{u}(x,t).\end{aligned}$$

Next, let $\mathcal{T}[u]$ be an operator with the special property that:

$$\mathcal{T}[u](x,t)=q(x,t). \tag{D7}$$

If our p.d.e. can be re-written in the same form as (D1), with $q(.)$ as above, the computation of u becomes then tractable once we are given a *known* $\widehat{u}$:

$$\begin{aligned}u(x,t)&=\widehat{u}(x,t)+\int_{\mathbb{R}^N}\textstyle\int_t^\tau G(x,t;\xi_0,\tau_0)q(\xi_0,\tau_0)\mathrm{d}\tau_0\mathrm{d}\xi_0\\&=\widehat{u}(x,t)+\int_{\mathbb{R}^N}\textstyle\int_t^\tau G(x,t;\xi_0,\tau_0)\mathcal{T}[u](\xi_0,\tau_0)\mathrm{d}\tau_0\mathrm{d}\xi_0.\end{aligned}$$

Continuing:

$$u(x,t) = \widehat{u}(x,t) + \int_{\mathbb{R}^N} \int_t^T G(x,t;\xi_0,\tau_0)\mathcal{T}[\widehat{u}](\xi_0,\tau_0)\mathrm{d}\tau_0\mathrm{d}\xi_0 + \mathcal{R}_1(x,t),$$

where:

$$\mathcal{R}_1(x,t) = \int_{\mathbb{R}^N} \int_t^T G(x,t;\xi_0,\tau_0)\mathcal{T}\left[\int_{\mathbb{R}^N} \int_{\tau_0}^T G(\xi_0,\tau_0;\xi_1,\tau_1)\mathcal{T}[u](\xi_1,\tau_1)\mathrm{d}\tau_1\mathrm{d}\xi_1\right]\mathrm{d}\tau_0\mathrm{d}\xi_0.$$

The procedure can go on by applying the preceding functional iteration: iterating n times means that one has to consider an increasing sequence $\{\tau_i\}_{i=0}^n$ of integrals to intervene in the brackets of the above formula. In a second order correction, for instance,

$$u(x,t) = \widehat{u}(x,t) + \widehat{u}_1(x,t) + \widehat{u}_2(x,t) + \mathcal{R}_2(x,t)$$

where

$$\begin{array}{rcl} \widehat{u}_1(x,t) & = & \int_{\mathbb{R}^N} \int_t^T G(x,t;\xi_0,\tau_0)\mathcal{T}[\widehat{u}](\xi_0,\tau_0)\mathrm{d}\tau_0\mathrm{d}\xi_0 \\ \widehat{u}_2(x,t) & = & \int_{\mathbb{R}^N} \int_t^T G(x,t;\xi_0,\tau_0)\mathcal{T}\left[\int_{\mathbb{R}^N} \int_{\tau_0}^T G(\xi_0,\tau_0;\xi_1,\tau_1)\mathcal{T}[\widehat{u}](\xi_1,\tau_1)\mathrm{d}\tau_1\mathrm{d}\xi_1\right]\mathrm{d}\tau_0\mathrm{d}\xi_0 \end{array}$$

and

$$\begin{array}{rcl} \mathcal{R}_2(x,t) & = & \int_{\mathbb{R}^N} \int_t^T G(x,t;\xi_0,\tau_0)\mathcal{T}\left[\int_{\mathbb{R}^N} \int_{\tau_0}^T G(\xi_0,\tau_0;\xi_1,\tau_1)\right. \\ & & \left.\mathcal{T}\left[\int_{\mathbb{R}^N} \int_{\tau_1}^T G(\xi_1,\tau_1;\xi_2,\tau_2)\mathcal{T}[\widehat{u}](\xi_2,\tau_2)\mathrm{d}\tau_2\mathrm{d}\xi_2\right]\mathrm{d}\tau_1\mathrm{d}\xi_1\right]\mathrm{d}\tau_0\mathrm{d}\xi_0, \end{array}$$

and in formula (4.4) stated in thm. 4.1, we only considered a first order approximation:

$$u(x,t) \simeq \widehat{u}(x,t) + \widehat{u}_1(x,t).$$

The method thus consists in re-writing the original p.d.e. in a form that allows us to use $\widehat{u}$ to get progressively more accurate approximations of the true pricing function u. Convergence is ensured by condition (C2) in the problem in appendix C applied to the Laplace transform of $u(x,t)$ with respect to time t:

$$\widetilde{u}(x;a) = \int_0^\infty e^{-at}u(x,t)\mathrm{d}t. \tag{D8}$$

In practice, the choice of $\widehat{u}$ (which is in fact implied by the choice of *which* $\mathcal{T}[.]$ operator to use) should be a fair compromise between analytical convenience and informational richness of $\widehat{u}$. This is the principle followed during the course of the proof below.

Proof of Theorem 4.1: The starting point is the p.d.e. (A3). Also notice that eq.(A3) could alternatively be derived by noticing that, by Itô's lemma, the variance process satisfies:

$$\mathrm{d}\sigma^2 = \left(w\sigma^{2-\delta} - \vartheta\sigma^2\right)\mathrm{d}t + \phi\sigma^2\mathrm{d}W^{(\sigma)},$$

where w and ϕ have been defined in the theorem and:

$$\vartheta = \tfrac{2\varphi\delta-(2-\delta)\psi^2}{\delta^2}.$$

Because we are considering a rational price function, by Itô's lemma, the Girsanov's theorem and the Q-martingale property of $\frac{B}{S^{(0)}}$, we get exactly the p.d.e. (A3).

Plugging now λ into (A3) yields the following p.d.e.:

$$\begin{cases} (\mathcal{L}+\mathcal{T})[B_T](x,t) = 0, \quad (x,t) \in \mathbb{R}^2_{++} \times [0,T) \\ B_T(x,T) = 1, \quad x \in \mathbb{R}^2_{++} \end{cases} \tag{D9}$$

Here,

$$\begin{aligned} \mathcal{L}[B_T](r,\sigma^2,t) = \quad & -B_\tau - rB + (\iota - \theta r)B_r + (w - \vartheta\sigma^2)B_{\sigma^2} \\ & +\tfrac{1}{2}(\sigma^2 B_{rr} + \phi^2 B_{\sigma^2\sigma^2}) - \lambda_1\sigma^2 B_r - \lambda_2\phi\sigma^2 B_{\sigma^2}, \end{aligned}$$

$\tau = T - t$, and:

$$\begin{aligned} \mathcal{T}[B_T](r,\sigma^2,t) = \quad & [\phi\sigma^2(\lambda_2 - \lambda_3\sqrt{r}) + w(\sigma^{2-\delta} - 1)]B_{\sigma^2} + \tfrac{1}{2}\sigma^2(r-1)B_{rr} \\ & +\tfrac{1}{2}\phi^2(\sigma^4 - 1)B_{\sigma^2\sigma^2} + \lambda_1(\sigma - r)\sigma B_r + \rho\phi\sigma^3\sqrt{r}B_{r\sigma^2}, \end{aligned}$$

where λ_3 has been defined in the main text and $B_{xy} \equiv \frac{\partial^2}{\partial x \partial y} B_T(.)$.

Eq.(D9) can be recognized as a special case of the general scheme in (D1), with $\mathcal{T}[.]$ as in (D7). As suggested there, it is convenient to start with the following simpler problem. Solve for the following price:

$$\begin{cases} \mathcal{L}[\mathcal{B}_T](x,t) = 0, \quad (x,t) \in \mathbb{R}^2_{++} \times [0,T) \\ \mathcal{B}_T(x,T) = 1, \quad x \in \mathbb{R}^2_{++} \end{cases} \tag{D10}$$

and then compute the Green's function $G(x,t;\xi,T)$ associated with $\mathcal{B}_T$

$$\mathcal{B}_T(x,t) = \int_{\ \ 2} G(x,t;\xi,T)\mathrm{d}\xi.$$

This will eventually enable one to apply the functional iteration discussed in the first part of this appendix, obtaining, for instance, the first order approximation given in (4.4):

$$B_T(x,t) \simeq \mathcal{B}_T(x,t) + \int_{\ \ 2} \int_t^T G(x,t;\xi_0,\tau_0)\mathcal{T}[\mathcal{B}_T](\xi_0,\tau_0)\mathrm{d}\tau_0\mathrm{d}\xi_0,$$

where the first order correction is given by

$$\mathcal{R}_1(x,t) = \int_{\ \ 2} \int_t^T G(x,t;\xi_0,\tau_0)\mathcal{T}\left[\int_{\ \ 2} \int_{\tau_0}^T G(\xi_0,\tau_0;\xi_1,\tau_1)\mathcal{T}[\mathcal{B}_T](\xi_1,\tau_1)\mathrm{d}\tau_1\mathrm{d}\xi_1\right]\mathrm{d}\tau_0\mathrm{d}\xi_0,$$

or, more generally, obtaining progressively more accurate approximations, as in the first part of the theorem, with $\mathcal{K}$ as defined in appendix C and $\widetilde{B}_T$ and $\widetilde{\mathcal{B}}_T$ as in (D8).

The solution of eq.(D10) can be interpreted as a no-arbitrage price of a bond in the case in which the primitives satisfy the following 'special interest rate dynamics':

$$\begin{cases} \mathrm{d}r = (\iota - \theta r)\mathrm{d}t + \sigma \mathrm{d}W^{(A)} \\ \mathrm{d}\sigma^2 = (w - \vartheta\sigma^2)\mathrm{d}t + \phi \mathrm{d}W^{(B)} \end{cases}$$

where $W^{(i)}$ $(i = A, B)$ are two standard P-$\mathcal{F}_t$-Brownian motions, and the risk premia are $\lambda_1\sigma$ and $\lambda_2\sigma^2$. The solution of (D10) has been reported in the theorem, with $D(.), F(.)$ and $U(.)$ defined as:

$$D(\tau) = \frac{1-e^{-\theta\tau}}{\theta}$$

$$F(\tau) = \Upsilon_5 + \Upsilon_6 e^{-\Upsilon_1\tau} + \Upsilon_7 e^{-2\theta\tau} + \Upsilon_8 e^{-\theta\tau}$$

$$\begin{aligned} U(\tau) \quad = \quad & \Upsilon_9 + \Upsilon_{10}\tau + \Upsilon_{11}e^{-2\theta\tau} + \Upsilon_{12}e^{-2\Upsilon_1\tau} + \Upsilon_{13}e^{-4\theta\tau} + \Upsilon_{14}e^{-\Upsilon_1\tau} \\ & +\Upsilon_{15}e^{-\theta\tau} + \Upsilon_{16}e^{-(2\theta+\Upsilon_1)\tau} + \Upsilon_{17}e^{-(\theta+\Upsilon_1)\tau} + \Upsilon_{18}e^{-3\theta\tau} \end{aligned}$$

$$\Upsilon_1 = \vartheta + \phi\lambda_2$$

$$\Upsilon_2 = \theta^2\Upsilon_1$$

$$\Upsilon_3 = 1 + 2\theta\lambda_1$$

$$\Upsilon_4 = 1 + \theta\lambda_1$$

$$\Upsilon_5 = \frac{\Upsilon_3}{2\Upsilon_2}$$

$$\Upsilon_6 = \frac{\Upsilon_4}{\Upsilon_2-\theta^3} - \frac{1}{2\Upsilon_2-4\theta^3} - \Upsilon_5$$

$$\Upsilon_7 = \frac{1}{2\Upsilon_2-4\theta^3}$$

$$\Upsilon_8 = -\frac{\Upsilon_4}{\Upsilon_2-\theta^3}$$

$$\Upsilon_9 = \frac{\iota}{\theta^2} + w\left(\frac{\Upsilon_6}{\Upsilon_1} + \frac{\Upsilon_7}{2\theta} + \frac{\Upsilon_8}{\theta}\right) + \frac{1}{2}\phi^2\Xi$$

$$\Xi = \frac{\Upsilon_6^2}{2\Upsilon_1} + \frac{\Upsilon_7^2}{4\theta} + \frac{\Upsilon_8^2}{2\theta} + \frac{2\Upsilon_5\Upsilon_6}{\Upsilon_1} + \frac{\Upsilon_5\Upsilon_7}{\theta} + \frac{2\Upsilon_5\Upsilon_8}{\theta} + \frac{2\Upsilon_6\Upsilon_7}{2\theta+\Upsilon_1} + \frac{2\Upsilon_6\Upsilon_8}{\theta+\Upsilon_1} + \frac{2\Upsilon_7\Upsilon_8}{3\theta}$$

$$\Upsilon_{10} = \frac{1}{2}\phi^2\Upsilon_5^2 + w\Upsilon_5 - \frac{\iota}{\theta}$$

$$\Upsilon_{11} = -\frac{\phi^2\Upsilon_8^2}{4\theta} - \frac{\phi^2\Upsilon_5\Upsilon_7}{2\theta} - \frac{w\Upsilon_7}{2\theta}$$

$$\Upsilon_{12} = -\frac{\phi^2\Upsilon_6^2}{4\Upsilon_1}$$

$$\Upsilon_{13} = -\frac{\phi^2\Upsilon_7^2}{8\theta}$$

$$\Upsilon_{14} = -\frac{\phi^2\Upsilon_5\Upsilon_6}{\Upsilon_1} - \frac{w\Upsilon_6}{\Upsilon_1}$$

$$\Upsilon_{15} = -\frac{\phi^2\Upsilon_5\Upsilon_8}{\theta} - \frac{\iota}{\theta^2} - \frac{w\Upsilon_8}{\theta}$$

$$\Upsilon_{16} = -\frac{\phi^2\Upsilon_6\Upsilon_7}{2\theta+\Upsilon_1}$$

$$\Upsilon_{17} = -\frac{\phi^2 \Upsilon_6 \Upsilon_8}{\theta + \Upsilon_1}$$

$$\Upsilon_{18} = -\frac{\phi^2 \Upsilon_7 \Upsilon_8}{3\theta}$$

Next, we turn to the computation of the Green's function associated with $\mathcal{B}_T$; this is done by first computing its Fourier transform:

$$\widehat{G}(\eta_1, \eta_2; x_1, x_2, \tau) = \int\int e^{i\xi_1\eta_1 + i\xi_2\eta_2} G(x_1, x_2, t; \xi_1, \xi_2, T) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \quad (i \equiv \sqrt{-1}).$$

Because $G(x_1, x_2; x_1, x_2, \tau)$ satisfies the same partial differential equation satisfied by $\mathcal{B}_T(x_1, x_2, t)$, its Fourier transform will follow the same partial differential equation as well. To find its boundary behavior, we exploit the boundary behavior of $G(x_1, x_2, t; \xi_1, \xi_2, T)$, and find:

$$\begin{aligned}
\widehat{G}(\eta_1, \eta_2; x_1, x_2, 0) &= \int\int e^{i\xi_1\eta_1 + i\xi_2\eta_2} G(x_1, x_2, T; \xi_1, \xi_2, T) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&= \int\int \widehat{\delta}(x_1 - \xi_1)\widehat{\delta}(x_2 - \xi_2) e^{i\xi_1\eta_1 + i\xi_2\eta_2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \\
&\equiv \int\int \widehat{\delta}(x - \xi)\zeta(\xi)\mathrm{d}\xi \\
&= \zeta(x) \equiv e^{ix_1\eta_1 + ix_2\eta_2}. \qquad \text{(D11)}
\end{aligned}$$

We have thus to solve eq.(D10) (with $\widehat{G}$ replacing $\mathcal{B}_T$), but with (D11) serving as boundary condition. The solution is:

$$\widehat{G}(\eta_1, \eta_2; r, \sigma^2, \tau) = \exp\left(-\widetilde{D}(\eta_1; \tau)r + \widetilde{F}(\eta_1, \eta_2; \tau)\sigma^2 + \widetilde{U}(\eta_1, \eta_2; \tau)\right),$$

where:

$$\widetilde{D}(\eta_1; \tau) = \frac{1 - e^{-\theta\tau}}{\theta} - i\eta_1 e^{-\theta\tau}$$

$$\widetilde{F}(\eta_1, \eta_2; \tau) = \Upsilon_5 + \widetilde{\Upsilon}_6 e^{-\Upsilon_1\tau} + \widetilde{\Upsilon}_7 e^{-2\theta\tau} + \widetilde{\Upsilon}_8 e^{-\theta\tau}$$

$$\begin{aligned}
\widetilde{U}(\eta_1, \eta_2; \tau) &= \widetilde{\Upsilon}_9 + \Upsilon_{10}\tau + \widetilde{\Upsilon}_{11} e^{-2\theta\tau} + \widetilde{\Upsilon}_{12} e^{-2\Upsilon_1\tau} + \widetilde{\Upsilon}_{13} e^{-4\theta\tau} + \widetilde{\Upsilon}_{14} e^{-\Upsilon_1\tau} \\
&\quad + \widetilde{\Upsilon}_{15} e^{-\theta\tau} + \widetilde{\Upsilon}_{16} e^{-(2\theta + \Upsilon_1)\tau} + \widetilde{\Upsilon}_{17} e^{-(\theta + \Upsilon_1)\tau} + \widetilde{\Upsilon}_{18} e^{-3\theta\tau}
\end{aligned}$$

$$\widetilde{\Upsilon}_6 = \Upsilon_6 + i\eta_2 + \chi_1(i\eta_1) + \chi_2(i\eta_1)$$

$$\widetilde{\Upsilon}_7 = \Upsilon_7 - \chi_2(i\eta_1)$$

$$\widetilde{\Upsilon}_8 = \Upsilon_8 - \chi_1(i\eta_1)$$

$$\chi_1(i\eta_1) = \frac{\theta(\theta\lambda_1 - 1)}{\Upsilon_2 - \theta^3} i\eta_1$$

$$\chi_2(i\eta_1) = -\frac{\theta(\theta\eta_1^2 + 2i\eta_1)}{2(\Upsilon_2 - 2\theta^3)}$$

$$\widetilde{\Upsilon}_9 = \frac{\iota}{\theta^2} + w\left(\frac{\widetilde{\Upsilon}_6}{\Upsilon_1} + \frac{\widetilde{\Upsilon}_7}{2\theta} + \frac{\widetilde{\Upsilon}_8}{\theta}\right) + \frac{1}{2}\phi^2\widetilde{\Xi} + \frac{\iota}{\theta} \cdot i\eta_1$$

$$\widetilde{\Xi}=\frac{\widetilde{\Upsilon}_6^2}{2\Upsilon_1}+\frac{\widetilde{\Upsilon}_7^2}{4\theta}+\frac{\widetilde{\Upsilon}_8^2}{2\theta}+\frac{2\Upsilon_5\widetilde{\Upsilon}_6}{\Upsilon_1}+\frac{\Upsilon_5\widetilde{\Upsilon}_7}{\theta}+\frac{2\Upsilon_5\widetilde{\Upsilon}_8}{\theta}+\frac{2\widetilde{\Upsilon}_6\widetilde{\Upsilon}_7}{2\theta+\Upsilon_1}+\frac{2\widetilde{\Upsilon}_6\widetilde{\Upsilon}_8}{\theta+\Upsilon_1}+\frac{2\widetilde{\Upsilon}_7\widetilde{\Upsilon}_8}{3\theta}$$

$$\widetilde{\Upsilon}_{11}=-\frac{\phi^2\widetilde{\Upsilon}_8^2}{4\theta}-\frac{\phi^2\Upsilon_5\widetilde{\Upsilon}_7}{2\theta}-\frac{w\widetilde{\Upsilon}_7}{2\theta}$$

$$\widetilde{\Upsilon}_{12}=-\frac{\phi^2\widetilde{\Upsilon}_6^2}{4\Upsilon_1}$$

$$\widetilde{\Upsilon}_{13}=-\frac{\phi^2\widetilde{\Upsilon}_7^2}{8\theta}$$

$$\widetilde{\Upsilon}_{14}=-\frac{\phi^2\Upsilon_5\widetilde{\Upsilon}_6}{\Upsilon_1}-\frac{w\widetilde{\Upsilon}_6}{\Upsilon_1}$$

$$\widetilde{\Upsilon}_{15}=-\frac{\phi^2\Upsilon_5\widetilde{\Upsilon}_8}{\theta}-\frac{\iota}{\theta^2}-\frac{w\widetilde{\Upsilon}_8}{\theta}-\frac{\iota}{\theta}\cdot i\eta_1$$

$$\widetilde{\Upsilon}_{16}=-\frac{\phi^2\widetilde{\Upsilon}_6\widetilde{\Upsilon}_7}{2\theta+\Upsilon_1}$$

$$\widetilde{\Upsilon}_{17}=-\frac{\phi^2\widetilde{\Upsilon}_6\widetilde{\Upsilon}_8}{\theta+\Upsilon_1}$$

$$\widetilde{\Upsilon}_{18}=-\frac{\phi^2\widetilde{\Upsilon}_7\widetilde{\Upsilon}_8}{3\theta}$$

The Green's function can now be recovered by inverting its Fourier transform:

$$G(r_t,\sigma_t^2,t;r_T,\sigma_T^2,T)=\frac{1}{(2\pi)^2}\int\int e^{-ir_T\eta_1-i\sigma_T^2\eta_2}\widehat{G}(\eta_1,\eta_2;r_t,\sigma_t,\tau)\mathrm{d}\eta_1\mathrm{d}\eta_2,$$

which is the formula given in the theorem. *Q.E.D.*

Appendix E

Let $b_1^+=(\iota_1,\theta_1,\omega_1,\alpha_1,\beta_1,\gamma,\delta,\upsilon)'$ be the parameter vector in the following auxiliary model:

$$\begin{aligned}
r_t-r_{t-1} &= \iota_1-\theta_1 r_{t-1}+\sigma_t\sqrt{r_{t-1}}u_t\\
\sigma_{t+1}^\delta-\sigma_t^\delta &= \left[\omega_1-\left(1-\beta_1-n_{\delta,\upsilon}((1-\gamma)^\delta+(1+\gamma)^\delta)\alpha_1\right)\sigma_t^\delta\right]\\
&\quad+\alpha_1\sigma_t^\delta\left[|u_t|^\delta(1-\gamma s_t)^\delta-E\left(|u_t|^\delta(1-\gamma s_t)^\delta\right)\right]
\end{aligned}$$

and $\widehat{b}_1^+$ the ML estimator of b_1^+:

$$\widehat{b}_1^+=\arg\max_{b_1^+}\mathfrak{L}_N({}_1r;b_1^+),$$

where $\mathfrak{L}_N({}_1r;b_1^+)$ is the likelihood function implied by the model, ${}_1r$ is the observations set and N is the sample size. Define, also, b_1 as the vector containing the first five elements of b_1^+.

Consider simulating system (4.5)-(4.6) for small h. That is, rewrite (4.5)-(4.6) as:

$$\begin{cases}
{}_hr_{h(k+1)}-{}_hr_{hk} = (\iota-\theta\cdot{}_hr_{hk})h+{}_h\sigma_{h(k+1)}\sqrt{{}_hr_{hk}}\cdot{}_hu_{h(k+1)}\\
\\
{}_h\sigma_{h(k+1)}^\delta-{}_h\sigma_{hk}^\delta = (\overline{\omega}-\varphi\cdot{}_h\sigma_{hk}^\delta)h+\sqrt{\frac{h}{(m_{\delta,\upsilon}-n_{\delta,\upsilon}^2)\left((1-\gamma)^{2\delta}+(1+\gamma)^{2\delta}\right)-2n_{\delta,\upsilon}^2(1-\gamma)^\delta(1+\gamma)^\delta}}\psi\\
\qquad\times{}_h\sigma_{hk}^\delta\left[\left|\frac{{}_hu_{hk}}{\sqrt{h}}\right|^\delta(1-\gamma s_k)^\delta-E\left(\left|\frac{{}_hu_{hk}}{\sqrt{h}}\right|^\delta(1-\gamma s_k)^\delta\right)\right]
\end{cases}$$

set γ, δ, υ to their ML estimates $\widehat{\gamma}, \widehat{\delta}, \widehat{\upsilon}$, assign values to $a = (\iota, \theta, \overline{\omega}, \varphi, \psi)$, draw $\frac{{}_h u_{hk}}{\sqrt{h}}$ from the g.e.d. distribution, and eventually obtain ${}_{h,h}r^{(s)}(a) = \{{}_h r_{hk}^{(s)}(a)\}_{k=0}^{N/h}$, $s = 1, ..., S$ where S is the number of simulations. For each simulation, just retain the (N) numbers ${}_h r_{hk}^{(s)}(a)$ that correspond to integer indexes of time and then compute:

$$\widehat{b}_{1,s}^{(h)}(a) = \arg\max_{b_{1,s}^{(h)}} \mathfrak{L}_N({}_{1,h}r^{(s)}(a); b_{1,s}^{(h)}), \quad s = 1, ..., S,$$

where ${}_{1,h}r^{(s)}(.)$ denotes the set of the interest rate with integer indexes of time at simulation s and interval h, and $b_{1,s}^{(h)}$ is defined similarly to b_1.

The indirect estimator of a is:

$$\widehat{a} = \arg\min_a \left\| \widehat{b}_1 - \frac{1}{S} \sum_{s=1}^{S} \widehat{b}_{1,s}^{(h)}(a) \right\|^2,$$

the estimator of δ is $\widehat{\delta}$, and the estimator of ρ is obtained by plugging $\widehat{\gamma}, \widehat{\delta}, \widehat{\upsilon}$ into form. (3.5) of the main text. Asymptotics for $\widehat{a}$ can be obtained in a straight forward way; see Broze, Scaillet, and Zakoïan (1995).

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TABLES

Table 1:

Estimates of model (4.2)

	GARCH(1,1)	A-PARCH
k	0.003	0.003
ω	$1.724 \cdot 10^{-7}$	$1.092 \cdot 10^{-4}$
α	0.025	0.058
β	0.966	0.933
γ	$\equiv 0$	0.489
δ	$\equiv 2$	1.056
υ	$\equiv 2$	1.390

Table 2:

Continuous time parameters implied by the discrete time scheme

	GARCH(1,1)	A-PARCH
φ	0.009	0.022
ψ	0.035	0.051
$-\rho$	$\equiv 0$	0.606

Table 3:

Bias of the Hull and White model

Moneyness of the option ($\frac{K}{S}$)	Bias (%)
0.70	0.10
0.75	0.13
0.80	0.17
0.85	0.24
0.90	0.49
0.95	2.50
1.00	54.95
1.05	116.07