A FORMALISM FOR THE DIMENSIONAL ANALYSIS OF TIME SERIES

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Abstract. We describe a theoretical formalism for the dimensional analysis of arbitrary stationary time series. We use this setting to study which properties are to be satisfied by a dimension concept in order to discern chaotic time series from white noise. In particular it follows that correlation dimensions can discriminate chaotic time series from white noise processes with L_∞–marginals, but not from arbitrary white noise. We also justify how the dimensional analysis can be put in practice using standard delay-embedding methods.

1. Introduction

The idea that some low-dimensional non-linear deterministic systems are able to emulate true stochastic dynamics is standard in economics and applied sciences. As a result, the modern analysis of time series has among its primary goals the distinction between such deterministic dynamics and genuine stochasticity from observed data. A main tool for this issue has been the use of some sort of dimension, in the belief that random processes are high-dimensional phenomena whereas interesting chaotic deterministic systems are low-dimensional.

The dimensional analysis of time series has been so far intimately linked to the *delay embedding method* [12, 14], which relies on the existence of an underlying smooth finite-dimensional dynamics where the data were recorded via a smooth observable (this is the *strange attractor hypothesis* (SAH)).

In this note we adopt a simple approach towards a general dimensional theory of arbitrary stationary time series, in particular, *without assuming* the SAH. The starting point is that, given a discrete time series $(u_i)_i$, the basic objects to look at are the finite dimensional probability distributions of the time series, here denoted by $\mu_{(m)}$, $m = 1, 2, \ldots$. Any sort of dimensional analysis on the series then aims to compute the numerical sequence of dimensions of these joint distributions. Since many different definitions of dimensions of a measure are available (see e.g. [13, 15]), a preliminary step consists of deciding what an *admissible* concept of dimension is (in order to do dimensional analysis). This is the main concern of section 2 below.

Our results apply to the correlation dimension, originally defined in [8], which is the most important dimension in the nonlinear analysis of economic time series (see e.g. [3, 4, 9, 6]). Correlation dimension is usually defined from a data set $\{x_i\}_i$

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as the scaling exponent of the (spatial) correlation statistic

$$C(r) = \lim_{n\to+\infty} \frac{1}{n^2}\mathrm{card}\{(i,j):\ \mathrm{dist}(x_i,x_j)<r,\ 1\le i,j\le n\}$$

with respect to r as r goes to zero. We use instead a theoretical approach to correlation dimension (see the definition in (3.1)) that was considered in [5] and in [13]. Results in [13] and [1] guarantee that, under rather general conditions, the correlation statistic above converges almost surely to the correlation integral on which the theoretical approach is built.

Say that a measure-dimension $\dim(\cdot)$ is *admissible* for dimensional analysis of stationary time series if it is monotonic (see properties **(3)** and **(3*)** below) and satisfies that

$$\begin{cases} \text{if } u_i \text{ is purely stochastic, then } \dim\mu_{(m)} = m, \text{ for } m=1,2,\dots, \\ \\ \text{if the SAH holds, then } \dim\mu_{(m)} = \dim\mu < +\infty, \text{ for all } m \text{ large enough.} \end{cases}$$

Here μ denotes the invariant measure of the hidden deterministic system; by *purely stochastic* we mean that the series is a realization of an independent stationary process composed of absolutely continuous random variables. (see Theorems 4.1 and 4.2 for precise statements). It is proved that Hausdorff and packing dimensions from fractal geometry [7, 10]are admissible. Also, correlation dimensions are admissible provided that the process has marginal densities which are essentially bounded. In section 3 we provide possible modifications of correlation dimensions to be admissible in the sense above.

A final concern is whether the information required to develop the proposed dimensional analysis can be recovered from the data series. Theorem 4.3 in section 4 justifies that the delay embedding method renders probability distributions that approximate the finite dimensional distributions of the process.

2. A framework to define dimensions of Borel measures

Let (X,d) be a metric space, $\mathcal{B}(X)$ denote the Borel σ-algebra in X, and $\mathcal{B}M(X)$ stand for the class of non-null finite Borel measures on X. A *dimension of measures* $\dim(\cdot)$ is a mapping from $\mathcal{B}M(X)$ to the non-negative reals that satisfies certain natural dimension–like properties. We consider the following basic list:

(1) (Boundedness) If $X=\mathbb{R}^m$, then $\dim\mu\le m$ for any $\mu\in\mathcal{B}M(\mathbb{R}^m)$.
(2) (Discrete measures) If $\mu\in\mathcal{B}M(X)$ is a discrete measure, then $\dim\mu=0$.
(3) (Monotonicity) If $\mu,\nu\in\mathcal{B}M(X)$ are such that μ is absolutely continuous with respect to ν, then $\dim\mu\ge\dim\nu$.
(4) (Lipschitz mappings) Let $g: X\mapsto Y$ be a Lipschitz mapping, $\mu\in\mathcal{B}M(X)$, and assume that the mapping dim is also defined in $\mathcal{B}M(Y)$, then $\dim(g_\sharp\mu)\le\dim\mu$, where $g_\sharp\mu\in\mathcal{B}M(Y)$ is the induced measure defined by $g_\sharp\mu(A)=\mu(g^{-1}(A))$, $A\in\mathcal{B}(Y)$.
(5) (Absolutely continuous measures) If $X=\mathbb{R}^m$, and $\mu\in\mathcal{B}M(\mathbb{R}^m)$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^m$; then $\dim\mu=m$.

Properties **(3)**, **(4)** and **(5)** are essential for the dimensional analysis of time series. In fact, in section 4 it is proved that any measure dimension satisfying those properties is admissible for the dimensional analysis of time series.

Further useful properties follow from the properties above, in particular property **(4)** implies

(6) (Bilipschitz invariance) If $g : X \mapsto Y$ is bi–Lipschitz (i.e. both g and g^{-1} are Lipschitz), then $\dim\mu = \dim \mu \circ g^{-1}$.

The list above plus some other natural properties and different implications among them were considered in [11].

It can be proved that important definitions of dimensions of measures from fractal geometry, namely Hausdorff and packing dimensions (see [10, 7] for their definitions and properties), satisfy properties **(1)** to **(5)**.

3. Correlation dimensions

The most widely used dimension in chaotic time series analysis has been the *correlation dimension*, denoted by $\beta(\cdot)$ here, introduced by Grassberger and Procaccia in [8]. The upper and lower correlation dimensions of $\mu \in \mathcal{B}M(X)$, as considered by Cutler [5], are defined by

$$\underline{\beta}(\mu) = \liminf_{r\to 0} \frac{\log \int \mu(B(x,r))d\mu}{\log r}; \qquad \overline{\beta}(\mu) = \limsup_{r\to 0} \frac{\log \int \mu(B(x,r))d\mu}{\log r}. \tag{3.1}$$

Correlation dimensions thus indicate the scaling behaviour of the expected masses of balls of radius r (usually called *correlation integrals* of μ) as r goes to zero. It turns out that correlation dimensions do not satisfy important properties of the list in section 1.

Theorem 3.1. *The correlation dimensions $\underline{\beta}$ and $\overline{\beta}$ do not satisfy properties* **(3)** *and* **(5)**

Proof. We outline here the proof given in [11], which consists on the construction of a Borel measure on the real line which is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$ (which obviously has correlation dimension one) but it has null correlation dimensions. Let $I \subset \mathbb{R}$ be the unit interval, $0 < a < 1$, and choose a sequence $\varepsilon_n > 0$ so that the intervals $I_n = [a^{n^2} - \varepsilon_n, a^{n^2} + \varepsilon_n]$ are pairwise disjoint. For $n \in \mathbb{N}$, define $\mu_n(A) = c_n\mathcal{L}^1(A \cap I_n)$ for $A \in \mathcal{B}M(I)$, so that $\mu_n(I) = a^n$, and thus $c_n = a^n/2\varepsilon_n$. It follows that $\underline{\beta}(\mu_n) = \overline{\beta}(\mu_n) = 1$ for all n. Let $\mu = \sum_{n\in\mathbb{N}} \mu_n \in \mathcal{B}M(I)$. For $r > 0$ small, we have

$$\int \mu(B(x,r))d\mu(x) \geq \frac{a^n}{1-a}\mu([0,r]) \geq \frac{a^n}{1-a}\sum_{i>n} a^i = \frac{a^{2n}}{(1-a)^2},$$

which in turn implies that $\underline{\beta}(\mu) \leq \overline{\beta}(\mu) = 0$. □

In order to proceed with a meaningful dimensional analysis using correlation dimensions properties **(3)** and **(5)** must be recovered somehow. This may be achieved by either weakening the requirements **(3)** and **(5)** or modifying the definitions of correlation dimensions. Both possibilities are explored in [11].

We first look at which properties standard correlation dimensions do satisfy. Weaker versions of **(3)** and **(5)** are naturally defined as follows.

(3*) If $\mu, \nu \in \mathcal{B}M(X)$ are such that μ has a density $h \in \mathrm{L}_\infty(\nu)$ with respect to ν, then $\dim\mu \geq \dim\nu$.

(5*) If $X = \mathbb{R}^m$ and $\mu \in \mathcal{B}M(X)$ has a density $h \in \mathrm{L}_\infty(\mathcal{L}^m)$, then $\dim\mu = m$.

Notice that **(3*)** is equivalent to the fact that there exists $C > 0$ such that $\mu(A) \leq C\nu(A)$ for $A \in \mathcal{B}(X)$, and thus requires a form of absolute continuity stronger than **(3)**. Theorem 3.2 below is the key result regarding the behaviour of correlation dimensions.

Theorem 3.2. [11] *The upper and lower correlation dimensions satisfy properties* **(1)**, **(2)**, **(3*)**, **(4)** *and* **(5*)**.

Regarding possible modifications of correlation dimensions, it turns out that the limit versions of correlation dimensions introduced by Y Pesin in [13] satisfy the full list **(1)**-**(5)**. Moreover, the following general result is proved in [11].

Theorem 3.3. *Let* dim *be a measure-dimension mapping satisfying properties* **(1)**, **(2)**, **(3*)**, **(4)**, *and* **(5*)**. *Then the modified dimension* $\dim_M$ *defined by*

$$\dim_M \mu = \lim_{\delta \to 0} \sup\{\dim \mu|_Z : Z \in \mathcal{B}(X),\ \mu(Z) \geq \mu(X) - \delta\} \tag{3.2}$$

for $\mu \in \mathcal{B}M(X)$, *satisfies properties* **(1)** *to* **(5)**.

4. Dimensional analysis of time series

This section concerns the role of dimension in the analysis of real–valued time series. We formulate the time series problem in terms of ergodic theory as follows. Let (X, f, μ) be a probabilistic dynamical system, that is, X is a metric space, μ is a probability measure in $\mathcal{B}M(X)$, and $f : X \mapsto X$ is a measurable μ-preserving mapping. We assume that the pair (f, μ) is ergodic. Let $h : X \mapsto \mathbb{R}$ be an observable of the system: h is a μ-measurable function such that $u_i = h(f^i(x_0)),\ \ i = 0, 1, 2, \dots,$ where $x_0 \in X$ is distributed according to μ. Any time series $(u_i)_i$ observed either in a smooth dynamical system or as a realization of a stochastic process is a particular case of the formulation above. Indeed, the *deterministic case* is obtained if

$$(\mathbf{D}) \begin{cases} X \text{ is a compact p} - \text{dimensional manifold,} \\ f \text{ is a } C^2 \text{ mapping,} \\ \text{and } h \text{ is } C^2. \end{cases}$$

The *stochastic case* arises if

$$(\mathbf{S}) \begin{cases} X = \mathbb{R}^\infty \text{ is the space of realizations of the process } U_i, \\ f \text{ is the shift mapping} \quad f((u_0, u_1, u_2, \dots)) = (u_1, u_2, \dots), \\ \text{and } h \text{ is given by} \quad h((u_0, u_1, u_2, \dots)) = u_0. \end{cases}$$

Notice that the f-invariance of the measure μ implies that the series u_i is strictly stationary. Recall that $\mu_{(m)}$ denotes the finite m-dimensional distribution of the series u_i.

If a measure dimension dim satisfies property **(4)** and thus property **(7)**, we may consider $\mathbb{R}^m$ endowed with the maximum norm, which is more convenient for computacional purposes.

Theorem 4.1 below addresses the case of the dimension of time series under the SAH.

Theorem 4.1. *Assume the hypotheses in* (D) *above, and let* $\dim(\cdot)$ *be a measure-dimension satisfying property* **(4)** *of section 2. For* $m \geq 2p + 1$, $\dim \mu_{(m)} = \dim \mu$ *generically.*

Proof. For $m \in \mathbb{N}$, let $J_m : X \mapsto \mathbb{R}^m$ be the 'delay mapping' defined by

$$J_m(x) = (h(x), h(f(x)), \ldots, h(f^{m-1}(x))).$$

The joint distribution $\mu_{(m)}$ of the series u_i satisfies $\mu_{(m)} = \mu \circ J_m^{-1}$. Takens theorem [14] implies that for $m \geq 2p+1$ the mapping J_{m_0} is generically an embedding onto $J_m(X)$. Assume that J_m is actuallly an embedding. Since X is compact, J_m is bi-Lipschitz, and property **(6)** of section 2 thus gives $\dim\mu_{(m)} = \dim\mu$. □

The theorem below deals with the dimensional analysis of white noise processes.

Theorem 4.2. *Let* $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty), \mu)$ *be the probability space of a real–valued stationary stochastic process* U_i*,* $i = 0, 1, 2, \ldots$*.*

i) *For any dimension mapping* dim *satisfying* **(4)** *we have* $\dim\mu_{(m)} \leq \dim\mu_{(m+1)}$ *for all* $m = 1, 2, \ldots$*.*

Assume that the process $\{U_i\}_i$ *is independent.*

ii) *If the random variables* U_i *have an* L_1*-density w.r.t.* $\mathcal{L}^1$ *and* dim *further satisfies* **(5)***, then* $\dim\mu_{(m)} = m$ *for all* $m = 1, 2, \ldots$*.*

iii) *If the* U_i*'s have an* L_∞*-density w.r.t.* $\mathcal{L}^1$ *and* dim *satisfies* **(5*)***, then* $\dim\mu_{(m)} = m$ *for* $m = 1, 2, \ldots$*.*

Proof. Let $g_m : \mathbb{R}^{m+1} \mapsto \mathbb{R}^m$ be the projection mapping

$$g_m((u_0, u_1, \ldots, u_m)) = (u_0, u_1, \ldots, u_{m-1}). \tag{4.1}$$

Since $\mu_{(m)} = \mu_{(m+1)} \circ g_m^{-1}$ and g_m is a contraction, claim **i)** follows from **(4)**.
Let μ_i denote the distribution of U_i. Since the variables U_i are independent, the finite dimensional distribution $\mu_{(m)}$ coincides with the cartesian product measure $\mu_0 \times \mu_1 \times \ldots \times \mu_{m-1}$ (it can be easily checked that they coincide over the class of m–dimensional rectangles and therefore over the class $\mathcal{B}(\mathbb{R}^m)$). Since every μ_i is absolutely continuous with respect to $\mathcal{L}^1$, the measure $\mu_0 \times \mu_1 \times \ldots \times \mu_{m-1}$ is absolutely continuous with respect the m–dimensional Lebesgue measure $\mathcal{L}^m$. It follows from **(5)** that $\dim\mu_{(m)} = \dim(\mu_0 \times \ldots \times \mu_{m-1}) = m$. This proves claim **ii)**. In the same way, claim **iii)** follows from **(5*)**. □

In view of Theorems 4.1 and 4.2 we considered as admissible a measure dimension that satisfies properties **(3)**, **(4)** and **(5)**. Notice that monotonicity is not required for both theorems to hold, it is a key property to compare the sizes of different measures. Notice that the correlation dimension satisfies the hypotheses of Theorems 4.1 and 4.2 (parts **i)** and **ii)**), and it is thus valid to discern deterministic time series from certain white noise processes: those with L_∞-marginals. Modified dimension mappings, according to definition (3.2), are also capable to discern determinism from arbitrary independent and identically distributed processes.

Since the analysis rests on the computation of $\dim\mu_{(m)}$, the reconstruction of the distributions $\mu_{(m)}$ becomes essential. This is, in a sense, a stochastic version of the problem solved by Takens theorem under the SAH. Theorem 4.3 below provides a solution for the reconstruction problem in the stochastic case, and also gives a meaningful interpretation of the delay embedding method when the SAH does not hold.

Theorem 4.3. (Measure-theoretic reconstruction theorem). *Assume the hypotheses in (S) above. For* $x_0 \in \mathbb{R}^\infty$ *and* $m \in \mathbb{N}$*, let* $\mu_{x_0,m,n}$ *denote the* n*-length sample measure defined by*

$$\mu_{x_0,m,n} = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{x_i^{(m)}},$$

where $x_i^{(m)} = (u_i, u_{i+1} \dots, u_{i+m-1})$ *for each* i *and* δ_x *stands for the Dirac measure at* x. *Then, for* μ*-a.e.* x_0, $\mu_{x_0,m,n} \to \mu_{(m)}$ *weakly for* $m = 1, 2, \dots$.

Proof. For $m \in \mathbb{N}$, let π_m be the projection mapping $\mathbb{R}^\infty \to \mathbb{R}^m$

$$\pi_m((u_0, u_1, \dots)) = (u_0, \dots, u_{m-1}),$$

and let $g : \mathbb{R}^m \to \mathbb{R}$ be $\mu_{(m)}$-integrable. For μ-a.e. $x_0 \in \mathbb{R}^\infty$ the Ergodic Theorem gives

$$\int g d\mu_{(m)} = \int g \circ \pi_m d\mu = \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(\pi_m(f^i x_0)) =$$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(x_i^{(m)}) = \lim_{n\to+\infty} \int g d\mu_{x_0,m,n}. \tag{4.2}$$

Let $\mathcal{Q}$ denote the class of subsets of $\mathbb{R}^m$ obtained as finite intersections of closed balls of $\mathbb{R}^m$ with rational radii and rational coordinates. Since the characteristic function of any $A \in \mathcal{Q}$ is $\mu_{(m)}$-integrable and $\mathcal{Q}$ is a countable set, we obtain from (4.2) that

$$\lim_{n\to+\infty} \mu_{x_0,m,n}(A) = \mu_{(m)}(A), \quad \text{for all } A \in \mathcal{Q}$$

for μ-a.e. x_0. Since every open set of $\mathbb{R}^m$ can be written as a finite or countable union of elements in $\mathcal{Q}$, [2, Theorem 2.2] implies that $\mu_{x_0,m,n}$ converges weakly to $\mu_{(m)}$ for μ-a.e. x_0. This holds for every m and the claim follows. □

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