

Learning with Bounded Memory in Stochastic Models

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Abstract

Learning with bounded memory in stochastic frameworks is incomplete in the sense that the learning dynamics cannot converge to an REE. The properties of the dynamics arising from such rules are studied for models with steady states. If in standard linear models the REE is in a certain sense expectationally stable (E-stable), then the dynamics are asymptotically stationary and forecasts are unbiased. We also provide similar local results for a class of nonlinear models with small noise and their approximations.

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1 Introduction

There exists by now a sizeable literature that studies the dynamics of adaptive learning in macroeconomic and market equilibrium models. Two fundamental issues addressed in this literature are (i) can economic agents, who rely on on-line estimation rules and forecasting with data on relevant variables, learn to have rational expectations in the long run and (ii) what are the stable outcomes of such learning processes. The literature has been recently surveyed in (Evans and Honkapohja 1998) and (Marimon 1997).

A common starting point in this research is to postulate that economic agents behave like econometricians, i.e. they use standard econometric techniques to estimate the parameters of the stochastic process of the relevant variables and forecast the future values using these estimated parameter values. The assumed form of the stochastic process, the perceived law of motion (PLM), is taken to be correctly specified in the sense that with right parameter values it coincides with the rational expectations equilibria (REE) of interest.¹ In the most commonly studied frameworks learning is complete in the sense that the economy settles in an REE if the learning dynamics converges. For most circumstances the condition for the convergence of learning dynamics has turned out to be the so-called expectational stability (E-stability) condition. We will define E-stability precisely below.

The possibility of nonconvergence of learning dynamics has also been considered in the literature. It may be the case that the economy has no stable REE for particular values of the model parameters.² Another possibility is that learning dynamics is incomplete in the sense that it has no chance of converging to an REE for any parameter configuration, see e.g. Section 5 of (Evans and Honkapohja 1998) for a discussion and references. The incompleteness of learning may arise for different reasons. First, the PLM may be incorrectly specified. Second, the procedure for estimating the PLM may not yield exact convergence. Nevertheless, dynamics of incomplete learning may give a good approximation to actual economic data.³

Several papers in the literature have considered learning with a finite memory, and such rules have been shown to result in complete learning in various deterministic models. Given suitable values of structural parameters, the learning economy can indeed find an REE, see e.g. (Guesnerie and Woodford 1991), (Grandmont 1985), (Grandmont and Laroque 1986), (Balasko and Royer 1996) and (Grandmont 1998), though for other parameter values the REE may be unstable. In contrast, learning with a finite memory

¹Note that the dynamics are econometrically misspecified during learning, but the misspecification will disappear in the limit if the learning dynamics converges to an REE.

²This is discussed e.g. by (Grandmont and Laroque 1991), (Bullard 1994) and (Grandmont 1998).

³(Marcet and Nicolini 1998) argue that dynamics with certain type of incomplete learning provides a good description of the inflation processes in Latin America. (Sargent 1998) suggests that a similar form of incomplete learning may be an essential ingredient in the rise and decline of inflation in post-war America.

is known to lead to incomplete learning in the same models when random shocks are present. For example, if agents try to learn a steady state by computing a sample mean from a ζ nite data set of ζ xed length, the resulting dynamics cannot converge to a rational expectations solution for any parameter values when a random shock is present, see (Evans and Honkapohja 1995b).

This ζ nding invites a further study into the nature of incomplete learning with ζ nite memory when the economy is subject to random shocks. In this paper we show that, despite incompleteness, dynamics of learning can have several attractive properties in standard frameworks.⁴ Most importantly, E-stability has a key role for stationarity of the learning dynamics. Generally speaking, under E-stability the state of economy has a unique invariant distribution in the long run. Learning is then asymptotically unbiased in the sense that the mean of the ζ rst moment of the forecast is correct. There is also approximate convergence of the higher moments with the approximation improving as the support of the shock becomes small. Finally, we obtain some results on the influence of the memory length on the residual variance of the forecasts.

These properties seem relatively intuitive, but their precise statements require considerable care. In this paper we derive these results for standard frameworks, where agents try to learn a steady state. Several well-known models fall into the categories of models analyzed in this paper, and we start by discussing two examples.

Example 1. (The Muth market model) Consider a competitive market with a production lag. Demand is assumed to be a downward-sloping function of the market price, while supply depends on the expected price in consequence of a production lag. For simplicity, assume that suppliers are identical in their economic characteristics, including expectations and learning rules.

Postulate the demand function

$$q_t^d = C_1 - Bp_t$$

and the supply function

$$q_t^s = C_2 + DE_{t-1}^*p_t + v_t,$$

where $q_t^i, i = d, s$, denote quantities demanded and supplied, p_t is the market price, $E_{t-1}^*p_t$ denotes the (in general non-rational) price expectation of the suppliers, and v_t is an *iid* random shock with mean 0. B, C_1, C_2 and D are positive parameters.

Using equality of supply and demand, the reduced form of this model takes the form

$$p_t = \alpha + \beta E_{t-1}^*p_t + u_t, \tag{1}$$

where $u_t = -B^{-1}v_t$, $\alpha = B^{-1}(C_1 - C_2)$ and $\beta = -B^{-1}D$.⁵ A (stochastic) steady state equilibrium can be written in the form

$$\hat{p}_t = \frac{\alpha}{1 - \beta} + u_t.$$

⁴As is well-known, most dynamic economic models with expectations are self-referential. (Mitra 1999) considers the implications of bounded memory for some non-self-referential stochastic models.

⁵Some other models, e.g. a version of the Lucas' island model, also lead to the same reduced form.

To model learning it is postulated that agents think that the economy is in a steady state but do not know the value of the constant. In other words, they have a PLM of the form $p_t = A + u_t$ and they form an estimate of the value of A using past observations on prices. In this model the estimate is also the forecasted price $E_{t-1}^* p_t$. Computing the sample mean for a set of data is the standard statistical technique for estimating an unknown mean, so that a natural estimate of A at time t is given by

$$A_t = T^{-1} \sum_{i=1}^T p_{t-i}$$

if agents use past T prices in computing the sample mean. Substituting the estimate into (1) yields

$$p_t = \alpha + \frac{\beta}{T} \sum_{i=1}^T p_{t-i} + u_t \quad (2)$$

which is an $AR(T)$ process. The forecast for the equilibrium price is also $\frac{1}{T} \sum_{i=1}^T p_{t-i}$ which is a random variable with a nontrivial asymptotic variance if p_t follows (2). This shows that forecasts from finite-memory rules cannot converge to rational expectations equilibria. In this paper we are interested in the properties of the dynamics (2).

Example 2. Several common economic models lead to the reduced form

$$y_t = \alpha + \beta E_t^* y_{t+1} + v_t \quad (3)$$

in which the current value of the endogenous variable depends on its expected value for next period. Again v_t is an *iid* random shock. (Sometimes an exogenous non-*iid* variables is added to the reduced form. We omit it for simplicity.)

For example, the demand for money is assumed to be a linear function of expected inflation in the simple monetary inflation model. Assuming a constant nominal stock of money then yields (3) as the reduced form. Other examples leading to (3) are the model of a small open economy with purchasing power parity on prices and open interest rate parity, and the model of risk-neutral asset pricing in which the current asset price is the present value of expected price next period plus dividends.⁶

Model (3) has a stochastic steady state solution of the form $y_t = \hat{A} + v_t$, where $\hat{A} = \frac{\alpha}{1-\beta}$ and a natural learning rule estimating the constant, assumed unknown, is to use the sample mean from a set of past values of y_t , i.e. the estimate in period t is given by $A_t = T^{-1} \sum_{i=1}^T y_{t-i}$. (This assumes that current value of y_t is not used in the estimation. This avoids a simultaneity problem in the model.) Again the dynamics of learning can be described by an $AR(T)$ process.

These two examples have a convenient linearity property, and the learning dynamics can be analyzed by standard techniques from time series analysis. We will study the

⁶See (Evans and Honkapohja 1998), Section 3.3.1 for a more detailed discussion and references.

first and second moments of the learning dynamics described by the $AR(T)$ process in Section 2.

Nonlinear models with stochastic steady states also appear in the literature. In Section 3 we take up a general class of nonlinear models which was analyzed for complete learning by (Evans and Honkapohja 1995b). It turns out that, for models with small shocks, E-stability implies useful asymptotic properties for learning dynamics locally around a steady state when agents try to learn a (stochastic) steady state with a natural finite-memory rule. We also linearize the process and obtain an approximation which is an $ARMA(T, T)$ process.

Stationarity of this $ARMA$ approximation is briefly analyzed in Section 4. There we also consider a generalization of model (3) in Example 2 to incorporate observation errors. It is shown that E-stability yields stationarity of both processes.

Section 5 concludes.

2 Linear AR Models

2.1 Preliminaries

We start with the class of models mentioned in Examples 1 and 2 of the Introduction. Recall that these are of the following general form

$$x_t = \alpha + \beta E_t^* x_{t+1} + v_t \quad (4)$$

or

$$x_t = \alpha + \beta E_{t-1}^* x_t + v_t, \quad (5)$$

depending on the dating of the expectations and time period they concern. Here x_t is an endogenous variable, $E_t^* x_{t+1}$ is the subjective expectation of x_{t+1} held by agents at time t and v_t is a sequence of white noise shocks.

We focus on steady state solutions of models (4) and (5).⁷ These rational expectations solutions may be written as $x_t = \hat{A} + v_t$, where $\hat{A} = \frac{\alpha}{1-\beta}$. As discussed above, in modeling learning we postulate that agents think that they are in a steady state but do not know the value of the constant \hat{A} . In other words, they have a PLM of the form $x_t = A + u_t$ and they form an estimate of the value of A using past observations on x_t . In the class of models covered by Example 1 the estimate is also the forecast $E_{t-1}^* x_t$ (or $E_t^* x_{t+1}$ for the class covered by Example 2).

Before considering learning we define precisely the concept of E-stability which, as noted, will play a key role. With the PLM of the above form agents use the estimated value of the constant as their forecast. If A is the value of the forecast the temporary equilibrium or actual law of motion (ALM) of the economy is given by

$$x_t = \alpha + \beta A + u_t.$$

⁷As is well-known, (4) can have other solutions besides steady states.

This defines a mapping from the PLM to the ALM which takes the form $T(A) = \alpha + \beta A$. E-stability is defined by considering the ordinary differential equation

$$\frac{dA}{d\tau} = T(A) - A.$$

If this differential equation is locally asymptotically stable (l.a.s.) at the REE $\hat{A} = \frac{\alpha}{1-\beta}$, then the equilibrium is said to be weakly E-stable. The formal E-stability condition is $T'(\hat{A}) = \beta < 1$. This formulation of E-stability is closely connected to convergence of real-time learning schemes, see (Evans and Honkapohja 1998) for a detailed discussion.

This notion has been strengthened in several ways in the literature. For the results of this paper the concept of iterative E-stability turns out to be central. We say that the REE is iteratively E-stable if it is locally asymptotically stable in iterations of the T -map, i.e. if the difference equation

$$A_{n+1} = T(A_n)$$

is locally asymptotically stable at \hat{A} . The formal condition for iterative E-stability is $|T'(\hat{A})| = |\beta| < 1$ in this case.

The notion of iterative E-stability is related to concepts of rationalizability in game theory, and the connection between these concepts has been explored by (Guesnerie 1992) and (Evans and Guesnerie 1993) in the context of rational expectations. We also remark that another related concept is strong E-stability in which the E-stability is required to be robust to overparameterizations of the PLM of the agents. For the linear frameworks (4) and (5) of this section weak and strong E-stability happen to coincide, while for the nonlinear models in the next section the condition for iterative E-stability is identical to that of strong E-stability.⁸

2.2 Stationarity and Unbiasedness

After these preliminaries we begin to analyze learning dynamics with bounded memory for models (4) and (5). As noted above, computing the sample mean for a set of data is the standard way for estimating an unknown constant, so that an estimate of A at time t is given by

$$T^{-1} \sum_{i=1}^T x_{t-i} \tag{6}$$

if agents use past T prices in computing the sample mean. For some results we can in fact consider forecasting by a weighted sample mean, i.e.

$$\sum_{i=1}^T \mu_i x_{t-i}, \text{ where } \forall i : \mu_i \geq 0 \text{ and } \sum_{i=1}^T \mu_i = 1. \tag{7}$$

⁸Discussions of the E-stability concepts for different frameworks are given in (Evans and Honkapohja 1995a) and (Evans and Honkapohja 1998).

(6) is obviously a special case of (7).

Substituting the weighted mean into (4) or (5) yields

$$x_t = \alpha + \beta \sum_{i=1}^T \mu_i x_{t-i} + u_t \quad (8)$$

which is an $AR(T)$ process. The first question one needs to ask about such a process is whether it is stationary or not. This question is answered in the following proposition.

Proposition 1 (i) If the steady state is iteratively E-stable, i.e. $|\beta| < 1$, then x_t is (covariance) stationary for all $T \geq 1$.

(ii) If it is weakly E-unstable, i.e. $\beta > 1$, then the process is non-stationary.

Proof. Consider the following equation

$$1 - \beta \sum_{i=1}^T \mu_i z^i = 0 \quad (9)$$

We need the roots of (9) to be outside the unit circle for stationarity.

Suppose that $|\beta| < 1$. Then we have

$$1 < \left| \frac{1}{\beta} \right| = \left| \sum_{i=1}^T \mu_i z^i \right| \leq \sum_{i=1}^T \mu_i |z|^i, \quad (10)$$

where the final inequality follows from the triangle inequality. Suppose for a root z^\wedge we have $|z^\wedge| \leq 1$. Then for all i , $|z^\wedge|^i \leq 1$. It then follows from (10) that

$$1 < \sum_{i=1}^T \mu_i |z^\wedge|^i \leq \sum_{i=1}^T \mu_i = 1$$

which is a contradiction. This proves (i).

To prove (ii) consider the characteristic polynomial

$$p(\lambda) = \lambda^T - \beta \mu_1 \lambda^{T-1} - \beta \mu_2 \lambda^{T-2} - \dots - \beta \mu_{T-1} \lambda - \beta \mu_T \quad (11)$$

If $\beta > 1$ we have $p(1) < 0$, so that by continuity $p(\lambda)$ must have a root greater than one. This proves (ii). ■

Weak E-stability has also a further implication:

Corollary 2 If the steady state is weakly E-stable, i.e. $\beta < 1$, then in the case (6) of equal weights $\exists T^* : \forall T \geq T^*$ the process is stationary.

Proof. We consider only the case $\beta < -1$ due to Proposition 1. We basically replicate the proof of (Giona 1991). Consider the characteristic polynomial (11) which may be rewritten in this case as

$$p(\lambda) = \lambda^T + \frac{|\beta| (1 - \lambda^T)}{(1 - \lambda)T}$$

Define $q(\lambda) := (1-\lambda)p(\lambda)$. Observing that $\lambda = 1$ is not an eigenvalue (since $p(1) > 0$), the roots of $q(\lambda)$ are the same as that of $p(\lambda)$. The roots of $q(\lambda)$, on the other hand, are given by solving the equation

$$\lambda^T \left\{ -\lambda + \left(1 - \frac{|\beta|}{T}\right) \right\} + \frac{|\beta|}{T} = 0. \quad (12)$$

From (12) we have (on re-arranging)

$$|\lambda| = \left| 1 - \frac{|\beta|}{T} + \frac{|\beta|}{T\lambda^T} \right| \leq \left| 1 - \frac{|\beta|}{T} \right| + \frac{|\beta|}{T|\lambda^T|}$$

The proof now proceeds by contradiction. Assume that there exists some eigenvalue λ such that $|\lambda| > 1$. Then we can choose T^* such that $\forall T \geq T^*$ we have $\frac{|\beta|}{T} < 1$ and $|\lambda^T| > 2$. Consequently $\forall T \geq T^*$ it is true that

$$|\lambda| < 1 - \frac{|\beta|}{T} + \frac{|\beta|}{2T} = 1 - \frac{|\beta|}{2T} < 1$$

which contradicts $|\lambda| > 1$. Now let us assume that $\forall T \geq T^*$ there exists at least one eigenvalue on the unit circle, that is, $\lambda = e^{i\theta}$. In this case we have

$$1 = \left| \frac{|\beta|}{T} e^{-iT\theta} + \left(1 - \frac{|\beta|}{T}\right) \right|$$

This last equation implies $\theta = 0$, but we have already ruled out $\lambda = 1$ as an eigenvalue. Hence $\forall T \geq T^*$, all the eigenvalues are inside the unit circle. ■

The results demonstrate that iterative and weak E-stability are closely connected to stationarity properties for learning dynamics with natural ζ nite-memory rules for learning a stochastic steady state. With such rules exact convergence cannot obtain, but stationarity prevails if the underlying model has an iteratively E-stable rational expectations equilibrium and it may prevail even with just weak E-stability.

If the dynamics is stationary, it makes sense to consider further properties of learning with these ζ nite-memory rules. Inspecting (8) it is immediately seen that the unconditional mean of x_t converges to the rational steady state $\frac{\alpha}{1-\beta}$ under the postulated learning rule, and therefore the (unconditional) mean of the forecast also converges to the same value. Thus we have:

Proposition 3 If the dynamics (8) is stationary, then learning is asymptotically unbiased, i.e. the mean of the forecast $\sum_{i=1}^T \mu_i x_{t-i}$ converges to the steady state $\hat{A} = \frac{\alpha}{1-\beta}$.

2.3 Second Moment Properties

Here we are interested in determining the asymptotic variance of estimation errors when the process is stationary. We thus impose weak E-stability, i.e. $\beta < 1$ (and strengthen it if necessary). In order to compute this variance we ζ rst need to calculate the second

moments of the x_t process. This is a standard problem in time series econometrics, and one makes use of the Yule Walker equations, see e.g. Chapter 3 of (Hamilton 1994). The Yule Walker equations for this $AR(T)$ process yield a system of T simultaneous linear equations which can be solved for the i rst T auto-correlations of the process.

First define the i th auto-correlation as

$$\rho_i := \frac{Cov(x_{t-i}, x_t)}{Var(x_t)},$$

where $Cov(x_{t-i}, x_t)$ denotes the covariance between x_{t-i} and x_t and $Var(x_t)$ denotes the variance of x_t . To economize on notation we also define $a := \frac{\beta}{T}$. The Yule Walker equations in our case are

$$\begin{aligned} \rho_1 &= a + a(\rho_1 + \rho_2 + \dots + \rho_{T-1}) \\ \rho_2 &= a\rho_1 + a + a(\rho_1 + \rho_2 + \dots + \rho_{T-2}) \\ \rho_3 &= a(\rho_2 + \rho_1) + a + a(\rho_1 + \rho_2 + \dots + \rho_{T-3}) \\ &\dots\dots \\ \rho_{T-1} &= a(\rho_{T-2} + \rho_{T-3} + \dots + \rho_1) + a + a\rho_1 \\ \rho_T &= a(\rho_{T-1} + \rho_{T-2} + \dots + \rho_1) + a \end{aligned}$$

Proposition 4 If $|\beta| < 1$, then the above system of equations has a unique solution $\rho_i = \rho = \frac{\beta}{(1-\beta)T+\beta}$, for all i such that $1 \leq i \leq T$.

Proof. The T linear simultaneous equations need to be solved for the T unknowns $\rho_1, \rho_2, \dots, \rho_T$. However, on careful observation one sees that the following is true $\rho_1 = \rho_T$; $\rho_2 = \rho_{T-1}$; $\rho_3 = \rho_{T-2}$ or, in general, $\rho_j = \rho_{T-j+1}$.

This means that we can reduce the dimensionality of the equations to be solved for. As mentioned above, one can match the auto-correlations pairwise, so that we have to distinguish between two cases: when T is even and when T is odd. We i rst consider the case when T is even.

CASE 1: $T = 2M$; M is a positive integer greater than or equal to 2. In this case we can reduce the above set of T equations into M equations to solve for the M unknowns $\rho_1, \rho_2, \dots, \rho_M$. The resulting M equations are

$$\begin{aligned} \rho_1 &= a + a(\rho_1 + 2\rho_2 + \dots + 2\rho_M) \\ \rho_2 &= a\rho_1 + a + a(\rho_1 + \rho_2 + 2\rho_3 + \dots + 2\rho_M) \\ &\dots\dots \\ \rho_{M-1} &= a(\rho_{M-2} + \rho_{M-3} + \dots + \rho_1) + a + a(\rho_1 + \dots + \rho_{M-1} + 2\rho_M) \\ \rho_M &= a(\rho_{M-1} + \rho_{M-2} + \dots + \rho_1) + a + a(\rho_1 + \dots + \rho_{M-1} + \rho_M). \end{aligned}$$

There is an easy way to solve the above set of equations. First, subtract the second equation from the i rst to get $\rho_1 - \rho_2 = a(\rho_2 - \rho_1)$ or $(\rho_1 - \rho_2)(1 - a) = 0$. Since $|\beta| < 1$ implies that $|a| < 1$ for all $T \geq 1$, we get $\rho_1 = \rho_2$. Analogously, in general, subtracting

equation $j + 1$ from equation j (where $1 \leq j \leq M - 1$) one gets $(\rho_j - \rho_{j+1})(1 - a) = 0$, so that $\rho_j = \rho_{j+1}$.

This proves that all the auto-correlations are the same so that we can get the common value, say ρ , from a single equation. This yields $(1 - a - 2a(M - 1))\rho = a$ or

$$\rho = \frac{a}{1 - a - 2a(M - 1)} = \frac{a}{1 - a(T - 1)}.$$

CASE 2: $T = 2M + 1$; M is a positive integer greater than equal to 2. In this case we can reduce the above set of T equations into $M + 1$ equations to solve for the $M + 1$ unknowns $\rho_1, \rho_2, \dots, \rho_M, \rho_{M+1}$. The resulting $M + 1$ equations are

$$\begin{aligned} \rho_1 &= a + a(\rho_1 + 2\rho_2 + \dots + 2\rho_M + 2\rho_{M+1}) \\ \rho_2 &= a\rho_1 + a + a(\rho_1 + \rho_2 + 2\rho_3 + \dots + 2\rho_M + \rho_{M+1}) \\ &\dots\dots\dots \\ \rho_M &= a(\rho_{M-1} + \rho_{M-2} + \dots + \rho_1) + a + a(\rho_1 + \dots + \rho_M + \rho_{M+1}) \\ \rho_{M+1} &= a(\rho_M + \rho_{M-1} + \dots + \rho_1) + a + a(\rho_1 + \dots + \rho_{M-1} + \rho_M). \end{aligned}$$

Note that in this case we get an extra equation corresponding to the unmatched autocorrelation at lag $M + 1$. Here, analogously as for the ζ rst case, subtracting equation j from equation $j + 1$ for all $1 \leq j \leq M$ we get $\rho_j - \rho_{j+1} = -a\rho_j + a\rho_{j+1}$ which implies $(\rho_j - \rho_{j+1})(1 - a) = 0$, and since $a \neq 1$ we get $\rho_j = \rho_{j+1}$.

This proves that again we have $\rho_i = \rho$ for all $1 \leq i \leq M + 1$. Using this fact we can now easily get ρ from the ζ rst equation. This again gives us $\rho = \frac{a}{1 - a(T - 1)}$.

Note that this also shows that the solution is unique. So ζ nally we get the common value of ρ for all $T \geq 1$ as

$$\rho = \frac{a}{1 - a(T - 1)} = \frac{\frac{\beta}{T}}{1 - \frac{\beta}{T}(T - 1)} = \frac{\beta}{(1 - \beta)T + \beta}.$$

This proves the proposition for all $T \geq 4$. One can also check easily that the same is true for $T = 1, 2, 3$. ■

We are now in a position to get the asymptotic variance of x_t . First define γ_i to be the i th autocovariance, so that

$$\rho_i = \frac{\gamma_i}{\gamma_0},$$

where γ_0 is the asymptotic variance of x_t . From (Hamilton 1994), p. 59, we have $\gamma_0 = a \sum_{i=1}^T \gamma_i + \sigma^2 = a\gamma_0 \sum_{i=1}^T \rho_i + \sigma^2 = \gamma_0 aT\rho + \sigma^2 = \gamma_0 \left(\frac{\beta^2}{(1 - \beta)T + \beta} \right) + \sigma^2$. So ζ nally solving for γ_0 yields

$$\gamma_0 = \sigma^2 \left(1 - \frac{\beta^2}{(1 - \beta)T + \beta} \right)^{-1} = \sigma^2 \left(\frac{(1 - \beta)T + \beta}{(1 - \beta)(T + \beta)} \right).$$

Clearly γ_0 is decreasing in T and in σ^2 if the process is stationary.

We finally turn to the forecast error to see how it behaves with T . Denote the forecast error of the least squares estimate from the REE based on memory T as $Y_t(T)$. We have, by definition,

$$Y_t(T) = \sum_{i=1}^T \frac{1}{T} x_{t-i} - \hat{A}.$$

Proposition 5 The asymptotic variance of forecast error, $\text{Var}(Y_t(T))$, decreases monotonically with T and σ^2 . As $T \rightarrow \infty$, $\text{Var}(Y_t(T)) \rightarrow 0$.

Proof. $\text{Var}(Y_t(T)) = \text{Var}(\sum_{i=1}^T \frac{1}{T} x_{t-i})$. Thus

$$\begin{aligned} \text{Var}(Y_t(T)) &= \left(\frac{1}{T}\right)^2 \left[\sum_{i=1}^T \text{Var}(x_{t-i}) + \sum_{i=1}^T \sum_{j=1, i \neq j}^T \text{Cov}(x_{t-i}, x_{t-j}) \right] \\ &= \left(\frac{1}{T}\right)^2 \left[\sum_{i=1}^T \gamma_0 + 2 \sum_{i=1}^T \sum_{j=1, i < j}^T \text{Cov}(x_{t-i}, x_{t-j}) \right] \\ &= \left(\frac{1}{T}\right)^2 [T\gamma_0 + 2\gamma_0\rho\{(T-1) + (T-2) + \dots + 2 + 1\}] \\ &= \left(\frac{1}{T}\right)^2 [T\gamma_0 + 2\gamma_0\rho \frac{T(T-1)}{2}] = \frac{\gamma_0[1 + (T-1)\rho]}{T} = \frac{\sigma^2}{(1-\beta)(\beta+T)}. \end{aligned}$$

If $|\beta| < 1$, this is clearly decreasing monotonically in T and in σ^2 . Also note that $\lim_{T \rightarrow \infty} \text{Var}(Y_t(T)) = 0$. ■

2.4 Generalization to Higher Order Models

The preceding results can be easily generalized for the steady states of some higher order linear models. For example, consider the model

$$x_t = \alpha + \beta_0 E_{t-1}^* x_t + \beta_1 E_{t-1}^* x_{t+1} + v_t$$

first analyzed in (Evans 1985) for E-stability.

If the agents have a PLM of the stochastic steady-state form

$$x_t = a + v_t$$

the iterative E-stability condition is $|\beta_0 + \beta_1| < 1$.⁹ Assume now that agents make forecasts of the unknown constant a by computing the sample mean $a_t = \sum_{i=1}^T \frac{1}{T} x_{t-i}$

⁹See (Evans 1985). Evans' definition of E-stability is iterative E-stability in our terminology. (The distinction between E-stability and iterative E-stability was made only more recently.) Note that one could make a distinction between weak and strong iterative E-stability, but this is not needed in this paper.

from past data x_{t-1}, \dots, x_{t-T} and using the estimate as the forecast. The actual law of motion is given by

$$x_t = \alpha + (\beta_0 + \beta_1)a_t + v_t = \alpha + (\beta_0 + \beta_1) \sum_{i=1}^T \frac{1}{T} x_{t-i} + v_t,$$

which is an $AR(T)$ process. This process is a very minor modification to (8), and the above results apply for this framework.¹⁰

3 Nonlinear Models

3.1 Preliminaries

In this section we consider learning of a steady state for the class of nonlinear models

$$x_t = H(G(x_{t+1}, v_{t+1})^e, v_t), \quad (13)$$

discussed in (Evans and Honkapohja 1995b). Here H and G are given twice differentiable functions, x_t is the value of the (scalar) variable of interest at time t , and v_t is a sequence of independently and identically distributed random shocks with mean 0 and variance σ_v^2 . $G(x_{t+1}, v_{t+1})^e$ denotes the subjective expectations of $G(x_{t+1}, v_{t+1})$ formed in period t . We will introduce some further assumptions later.

A rational steady state is a function $x(v)$ such that

$$\forall v : x(v) = H(E_w G(x(w), w), v),$$

where the expectation E_w is taken with respect to a random variable which has the same distribution as the iid shocks v_t . For later purposes \hat{x} denotes the steady state of the corresponding nonstochastic model, i.e. $\hat{x} = H(G(\hat{x}, 0), 0)$. (Evans and Honkapohja 1995b) provide an existence theorem for this kind of steady state when the support of the shock v_t in (13) is sufficiently small.¹¹

Example 3. (The basic overlapping generations model with shocks.) In the basic overlapping generations (OG) model with production agents supply labor n_t and produce (perishable) output when young and consume c_{t+1} when old. The utility function of the representative agent of generation t is $U(c_{t+1}) - V(n_t)$. Holding money is the only means of saving, and there is a fixed quantity of money M_0 . Output is assumed to be equal to labor supply plus an additive productivity shock, so that output q_t is given by

$$q_t = n_t + \lambda_t,$$

¹⁰The new feature brought by the generalization is that this framework has other equilibria besides steady states.

¹¹To our knowledge, existence of the stochastic steady state equilibria has not been analyzed in full generality.

where λ_t is an *iid* positive productivity shock. The budget constraints are $p_{t+1}c_{t+1} = M_t$ and $p_t q_t = M_t$. The first-order condition plus the market clearing condition $q_{t+1} = c_{t+1}$ and $p_t/p_{t+1} = q_{t+1}/q_t$ yields

$$(n_t + \lambda_t)V'(n_t) = E_t^*((n_{t+1} + \lambda_{t+1})U'(n_{t+1} + \lambda_{t+1})).$$

Since $(n + \lambda)V'(n)$ is strictly increasing in n , and letting $v_t \equiv \lambda_t - E(\lambda_t)$, this equation can be solved for n_t . Letting $x_t \equiv n_t$ the model can be put in the standard form (13).

Returning to the general framework, suppose that agents are trying to learn the steady state. Agents have to forecast the quantity $G(x_{t+1}, v_{t+1})^e$ which is a constant $E_w G(x(w), w)$ in the steady state. The learning problem for the agents is to find this value. The data are given by the past observations $G(x_1, v_1), G(x_2, v_2), \dots, G(x_{t-1}, v_{t-1})$, and the agents are assumed to use the sample mean of these observations to forecast $G(x_{t+1}, v_{t+1})^e$.

We continue to focus on learning with a finite memory length T : At date t agents use T past observations to estimate $G(x_{t+1}, v_{t+1})^e$. The estimate and forecast at date t , δ_t , is given by

$$\delta_t = \sum_{i=1}^T \mu_i G(x_{t-i}, v_{t-i}),$$

where μ_i is a weight such that $\sum_{i=1}^T \mu_i = 1, \mu_i \geq 0$. The general results in this section hold for general weighting schemes, but in Section 4 attention will be focused on the most important case of the sample mean, i.e. the weights are equal $\mu_i = T^{-1}$.

Given the forecast δ_t , the actual law of motion of x_t from (13) is given by

$$x_t = H(\delta_t, v_t).$$

Substituting for δ_t in the above equation, we finally get the dynamical system

$$x_t = H\left(\sum_{i=1}^T \mu_i G(x_{t-i}, v_{t-i}), v_t\right) \quad (14)$$

(14) is the law of motion we are concerned with. This form can cover a wide variety of overlapping generations models with shocks to either preferences or technology. A special case arises if G is independent of its second argument.¹²

3.2 Markovian Formulation

We now start to analyze the process (14) for some general properties. First, observe that (14) can be written as a Markov process in the following manner. Define the state vector

$$X_{t-1} = (x_{t-1}, x_{t-2}, \dots, x_{t-T}, v_{t-1}, v_{t-2}, \dots, v_{t-T})'.$$

¹²For example, the dynamics in an overlapping generations model with a multiplicative shock to the disutility of labor and with agents trying to learn a stochastic steady state would follow a process in which the current value of the state variable depends only on the current shock and past states.

Then we can write

$$\begin{bmatrix} x_t \\ x_{t-1} \\ \dots \\ x_{t-T+1} \\ v_t \\ v_{t-1} \\ \dots \\ v_{t-T+1} \end{bmatrix} = \begin{bmatrix} \hat{H}(X_{t-1}, v_t) \\ X_{1,t-1} \\ \dots \\ X_{T-1,t-1} \\ v_t \\ X_{T+1,t-1} \\ \dots \\ X_{2T-1,t-1} \end{bmatrix}, \quad (15)$$

where

$$\hat{H}(X_{t-1}, v_t) \equiv H\left(\sum_{i=1}^T \mu_i G(x_{t-i}, v_{t-i}), v_t\right). \quad (16)$$

This can be written compactly as

$$X_t = F(X_{t-1}, v_t), \quad (17)$$

where the right hand side of (15) defines F .

Since X_{t-1} and v_t are independent, X_t is a Markov process with some state space $A \subset \mathbb{R}^{2T}$. The first question we study is whether there exists a unique invariant probability for (17) and whether any initial probability distribution converges to this invariant probability asymptotically. It should be emphasized that this analysis will be local since we will assume that the underlying noise is small (in a sense to be made precise shortly), since the existence of equilibria is known only for this case and since the linearization of (14) can be justified only in models with small noise.

3.3 Asymptotic Properties

It turns out that the existence of the unique invariant distribution with small noise and starting points in a neighborhood of the steady state can be established, if the steady state of the corresponding nonstochastic model is iteratively E-stable.¹³ To obtain this condition suppose that in the nonstochastic model agents have a PLM θ about the expectations $G(\hat{x}, 0)^e$, where \hat{x} is the unknown steady state. Then the T -map is given by

$$T(\theta) = G(H(\theta, 0), 0)$$

yielding the condition for iterative E-stability given in Condition 2 below.

We now proceed to the general analysis of (14) or (17). We make the following assumptions.

Condition 1 $v_t \in [-\varepsilon_v, \varepsilon_v]$ for all t .

Condition 2 (Iterative E-stability) $|D_1 H(G(\hat{x}, 0), 0) D_1 G(\hat{x}, 0)| < 1$.

¹³For this class of models iterative E-stability is in fact equivalent to strong E-stability, compare (Evans and Honkapohja 1995b).

We first show that with Conditions 1 and 2 and with ε_v small enough, the state space of the Markov process (17) may be assumed to be compact. This is the content of the following lemma.

Lemma 6 Assume Conditions 1 and 2 and ε_v small enough. Then the state space of (17), call it $N(\varepsilon_v) \subset A$, may be assumed to be compact.

Proof. We want to show that if $\|X_{t-1}\| \leq \varepsilon$ and $|v_t| \leq \varepsilon_v$, then we have $\|X_t\| \leq \varepsilon$, too. (The vector norm will be specified later.) Given the structure of (15) it is clear that it is basically the first component of the vector which is problematic (as the other components are merely definitions). The first component of X_t describes the process (14) for x_t . We will now prove that for suitable ε_x if $|x_{t-i}| \leq \varepsilon_x$ for all $i = 1, 2, \dots, T$ and $|v_{t-i}| \leq \varepsilon_v$ for all $i = 0, 1, 2, \dots, T$ then we have $|x_t| \leq \varepsilon_x$ too. By choosing some appropriate vector norm (like the max norm), this in turn shows that if $\|X_{t-1}\| \leq \varepsilon$ and $|v_t| \leq \varepsilon_v$, then we have $\|X_t\| \leq \varepsilon$, too.

Assume, without loss of generality, that the perfect foresight steady state is $\hat{x} = 0$. We linearize (14) around the vector $(0, 0, \dots, 0) \in \mathbb{R}^{2T+1}$ and get a second order residual term in the Taylor series expansion in the following manner:

$$x_t = \sum_{i=1}^T \alpha_i x_{t-i} + \beta_0 v_t + \sum_{i=1}^T \beta_i v_{t-i} + r(x_{t-1}, \dots, x_{t-T}, v_t, \dots, v_{t-T}), \quad (18)$$

where $\alpha_i = \mu_i D_1 H D_1 G$, $\beta_i = \mu_i D_1 H D_2 G$, for $i = 1, 2, \dots, T$ and $\beta_0 = D_2 H$; $D_i H$ being the partial derivative of H with respect to the i th argument, etc. These derivatives are evaluated at values corresponding to the nonstochastic steady state. Note also that, with the normalization $\hat{x} = 0$, the constant term in the Taylor series is 0.

Using the mean value theorem, the residual $r(\cdot)$ satisfies

$$\begin{aligned} |r(x_{t-1}, \dots, x_{t-T}, v_t, \dots, v_{t-T})| &\leq K(|x_{t-1}| + \dots + |x_{t-T}| + |v_t| + |v_{t-1}| + \dots \\ &+ |v_{t-T}|). \end{aligned}$$

Taking absolute values in (18) we get

$$\begin{aligned} |x_t| &\leq |\alpha_1| |x_{t-1}| + \dots + |\alpha_T| |x_{t-T}| + |\beta_0| |v_t| + |\beta_1| |v_{t-1}| + \dots + \\ &+ |\beta_T| |v_{t-T}| + |r(x_{t-1}, \dots, x_{t-T}, v_t, \dots, v_{t-T})| \\ &\leq \sum_{i=1}^T (K + |\alpha_i|) |x_{t-i}| + \sum_{i=1}^T (K + |\beta_i|) |v_{t-i}| + (K + |\beta_0|) |v_t| \\ &\leq (TK + |D_1 H D_1 G|) \varepsilon_x + (TK + |D_1 H D_2 G|) \varepsilon_v + (K + |D_2 H|) \varepsilon_v \end{aligned}$$

if $|x_{t-i}| \leq \varepsilon_x$ for all i .

Since $\frac{\partial r}{\partial x_{t-j}}$ and $\frac{\partial r}{\partial v_{t-j}}$ are zero when evaluated at the origin (which is the steady state here), the constant K can be made as close to zero as desired by restricting the analysis

to a small enough neighbourhood of the origin. We choose the neighborhood so that $1 - TK - |D_1HD_1G| > 0$. Then we choose ε_x such that

$$(TK + |D_1HD_1G|)\varepsilon_x + (TK + |D_1HD_2G|)\varepsilon_v + (K + |D_2H|)\varepsilon_v \leq \varepsilon_x$$

or in other words

$$\varepsilon_x \geq \frac{(TK + |D_1HD_2G| + K + |D_2H|)\varepsilon_v}{1 - TK - |D_1HD_1G|} \quad (19)$$

By condition 2 and the choice of K this inequality is well defined. This proves the lemma.¹⁴ ■

We now prove that under suitable assumptions there exists a unique invariant probability for (17) and that the n -step transition probability of this Markov process converges weakly to this invariant probability, as $n \rightarrow \infty$, for every point in the state space. We use the results of (Bhattacharyya and Lee 1988) to prove these assertions.

By Lemma 6 the state space $N(\varepsilon_v)$ of (17) may be assumed to be compact. Henceforth, we assume that ε_x in (19) is set so that equality holds. Thus $N(\varepsilon_v)$ is a compact metric space. We begin by proving the following lemma.

Lemma 7 There exists ε_v sufficiently small such that for all $v_t \in [-\varepsilon_v, \varepsilon_v] = N_v$ the process (17) is a strict contraction on $N(\varepsilon_v)$, i.e. $\|F(h, v) - F(0, v)\| < \|h\|$ for all $h \in N(\varepsilon_v), h \neq 0$.

Proof. We continue to assume, without any loss of generality, that the deterministic steady state is $(0, 0, \dots, 0)$. First, note that

$$F(h, v) - F(0, v) = DF(0, v)h + r(h, v)$$

Here DF denotes the Jacobian of F with respect to first (vector) argument. It then follows that

$$\begin{aligned} \|F(h, v) - F(0, v)\| &= \|DF(0, v)h + r(h, v)\| \\ &\leq \|DF(0, v)\| \|h\| + \|r(h, v)\| \end{aligned} \quad (20)$$

The $2T \times 2T$ Jacobian matrix $DF(0, 0)$ at the steady state takes the form

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where the $T \times T$ submatrices are given by

$$F_{11} = \begin{pmatrix} \mu_1\vartheta & \mu_2\vartheta & \mu_3\vartheta & \cdots & \mu_{T-1}\vartheta & \mu_T\vartheta \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

¹⁴We note here that in general the constant K will depend on T . In particular, the larger is T the smaller must K be. There are, however, special cases where a uniform result is obtainable when $\mu_i = o(T^{-1})$. This happens when the function $\hat{H}(X_{t-1}, v_t)$ is either independent of v_t or additive and linear in it.

$$F_{12} = \begin{pmatrix} \mu_1 \kappa & \cdots & \mu_T \kappa \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}, F_{21} = 0, F_{22} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Here we have introduced the notation $\vartheta = D_1 H D_1 G$, $\kappa = D_1 H D_2 G$.

Consider now the eigenvalues of $DF(0,0)$. From the partitioned form it is seen that the matrix $DF(0,0) - \lambda I_{2T \times 2T}$ is block triangular, so that its determinant is equal to the product of the determinants of $F_{11} - \lambda I_{T \times T}$ and $F_{22} - \lambda I_{T \times T}$. Thus the eigenvalues of $DF(0,0)$ consist of the eigenvalues of F_{11} and F_{22} . Zero is the only eigenvalue of F_{22} . The special form of F_{11} implies that its eigenvalues must satisfy the polynomial equation

$$p(\lambda) = \lambda^T - \vartheta \mu_1 \lambda^{T-1} - \dots - \vartheta \mu_{T-1} \lambda - \vartheta \mu_T = 0,$$

see e.g. (Hamilton 1994), Proposition 1.1. If $\lambda \neq 0$, dividing through by λ^T and setting $z = \lambda^{-1}$ we obtain equation (9) with $\beta = \vartheta$. The argument in Proposition 1 yields that the roots of this equation must have modulus less than one if $|\vartheta| < 1$.

Thus condition 2 implies that the spectral radius of the matrix $DF(0,0)$ is strictly less than one. Then there exists some matrix norm such that $\|DF(0,0)\| < 1$, see (Horn and Johnson 1985), Lemma 5.6.10. Now, by uniform continuity of the matrix norm in its elements, it follows that there exists ε_v sufficiently small such that for $\forall v_t \in [-\varepsilon_v, \varepsilon_v] = N_v : \|DF(0, v_t)\| < \delta < 1$. Moreover, $\|r(h, v)\| \leq \hat{\delta} \|h\|^2$ for some $\hat{\delta}$ since $r(h, v)$ is 2nd order in h .

It now follows from (20) that for all $\|h\|$ small enough we have

$$\|F(h, v) - F(0, v)\| \leq (\|DF(0, v)\| + \hat{\delta} \|h\|) \|h\| < \|h\|.$$

This proves that $F(., .)$ is a strict contraction on the state space $N(\varepsilon_v)$. ■

Proposition 8 There exists a unique invariant probability for the Markov process F on the state space $N(\varepsilon_v)$ and the n -step transition probability of F converges weakly to this invariant probability, as $n \rightarrow \infty$, for every point in $N(\varepsilon_v)$.

Proof. Lemmata 6 and 7 show that the conditions of Corollary 2.3 of (Bhattacharyya and Lee 1988) are satisfied on N . ■

This result demonstrates how, with small noise, iterative E-stability of the steady state of the nonstochastic model yields attractive limiting properties for the learning dynamics with small noise. Obviously, if $v_t \equiv 0$ for all t , iterative E-stability also guarantees local stability of the dynamical system (14) or (17).

3.4 Error Bounds

To obtain further information on the model (14), it appears necessary to revert to the linearization. This will be done in the next section. Here we briefly discuss error bounds for the residual term $r(x_{t-1}, \dots, x_{t-T}, v_t, \dots, v_{t-T})$ in the Taylor series expansion (18).

Rewrite (14) as

$$x_t = R(x_{t-1}, \dots, x_{t-T}, v_t, \dots, v_{t-T}) \quad (21)$$

Consider the linearization of (21). The residual $r(x_{t-1}, \dots, x_{t-T}, v_t, \dots, v_{t-T})$ in the Taylor series (18) consists of terms of the form $\frac{\partial^2 R}{\partial x_{t-i} \partial x_{t-j}}(\mathbf{X})(x_{t-i}x_{t-j})$, $\frac{\partial^2 R}{\partial v_{t-i} \partial v_{t-j}}(\mathbf{X})(v_{t-i}v_{t-j})$ and $\frac{\partial^2 R}{\partial x_{t-i} \partial v_{t-j}}(\mathbf{X})(x_{t-i}v_{t-j})$ at some point \mathbf{X} .¹⁵ Assuming that all the second order partial derivatives are bounded, it can be shown that the mean residual is bounded above by the expression $M\varepsilon_v^2$. For example,

$$\begin{aligned} & \left| E\left[\frac{\partial^2 R}{\partial x_{t-i} \partial v_{t-j}}(\mathbf{X})(x_{t-i}v_{t-j})\right] \right| \leq E \left| \frac{\partial^2 R}{\partial x_{t-i} \partial v_{t-j}}(\mathbf{X})(x_{t-i}v_{t-j}) \right| \\ & \leq M_1 E |x_{t-i}v_{t-j}| \leq M_1 \sqrt{E x_{t-i}^2 E v_{t-j}^2} = M_1 \varepsilon_x \varepsilon_v \leq M_2 \varepsilon_v^2, \end{aligned}$$

where M_i are constants. Likewise, the absolute value of the means of the other terms $\frac{\partial^2 R}{\partial v_{t-i} \partial v_{t-j}}(\mathbf{X})(v_{t-i}v_{t-j})$ and $\frac{\partial^2 R}{\partial x_{t-i} \partial x_{t-j}}(\mathbf{X})(x_{t-i}x_{t-j})$ are bounded by expressions of the same form.

Consider next the second order moments of x_t . Viewing the x_t process as a linear $ARMA(T, T)$ process plus a residual, these moments can be computed from the Yule Walker equations with error terms. If these residuals can be shown to be small, then the second order moments derived from the linearized process will be close to the true second order moments of x_t .

Recall that the Yule Walker equations are derived by multiplying x_t in (21) by x_{t-u} ($u = 1, 2, \dots, T$) and then taking expectations of both sides. Thus an individual representative term in the residual of these equations is of the form

$$\frac{\partial^2 R}{\partial v_{t-i} \partial x_{t-j}}(\mathbf{X})(v_{t-i}x_{t-j}x_{t-u})$$

with the absolute value of the mean bounded above by

$$\left| E\left[\frac{\partial^2 R}{\partial v_{t-i} \partial x_{t-j}}(\mathbf{X})(v_{t-i}x_{t-j}x_{t-u})\right] \right| \leq M_3 \varepsilon_v [E(x_{t-j}^4)E(x_{t-u}^4)]^{\frac{1}{4}} \leq M_4 \varepsilon_v^3$$

where we have applied Hylder's inequality twice. Here we have also used the fact that $x_t^2 \leq \varepsilon_x^2 \leq \tilde{M}\varepsilon_v^2$ for some \tilde{M} . Similar bounds can be found for the other terms involved in the residuals of the Yule Walker equations.

These considerations yield the following conclusion:

Remark 1 The mean absolute residuals in the Yule Walker equations are of third order in the support of the noise.

¹⁵Note that we have again assumed, w.l.o.g, that the nonstochastic steady state $\hat{x} = 0$.

4 Linear ARMA Models

In the preceding section it was seen that the linearization of the general nonlinear framework yielded an $ARMA(T, T)$ process as an approximation of the original model. In this section we study further the dynamics of the linearized process (but with the restriction that learning is based on estimation by the sample mean).

The significance of the $ARMA$ setup is not limited to this case. A closely related framework can arise in finite-memory learning in some linear frameworks when observation errors prevail. We present an example before proceeding to the main result.

Example 4. (A model with observation errors.) Consider model (3) as in example 2. Assume now that agents do not directly observe y_t , but only a variable $x_t = y_t + u_t$, where u_t represents the observation error, assumed to be *iid* with zero mean, constant variance σ_u^2 , and independent of the shock v_t . As before, v_t is assumed to be *iid* with zero mean and constant variance σ_v^2 .

Agents have the perceived law of motion

$$x_t = b + \text{noise}$$

and they estimate b by the sample mean from T past observations

$$b_t = T^{-1} \sum_{i=1}^T (y_{t-i} + u_{t-i}).$$

Substituting into (3) leads to a somewhat non-standard $ARMA(T, T)$ -type process

$$y_t = \alpha + \frac{\beta}{T} \sum_{i=1}^T (y_{t-i} + u_{t-i}) + v_t$$

describing the dynamics of learning with memory length T . Note that this is not a standard $ARMA(T, T)$ process in that the current shock v_t is permitted to have a different distribution from the lagged observation errors u_{t-i} . Such a process can nevertheless be tackled using standard time series techniques with minor modifications.

We now set up a framework that covers both this example and the linearization of the nonlinear model of Section 3. After centering both processes can be written as

$$z_t = a \sum_{i=1}^T z_{t-i} + cv_t + b \sum_{i=1}^T u_{t-i}, \quad (22)$$

where $z_t = y_t - Ey_t$, $a = \frac{\delta}{T}$, $b = \frac{\phi}{T}$. Formally in example 4 we have $\delta = \phi = \beta$, and in the linearized model (18) $v_t = u_t$ and $\delta = D_1HD_1G$, $\phi = D_1HD_2G$.

Proceeding with the general analysis, we are interested in finding the condition for stationarity of this process. Again iterative E-stability yields this property:

Proposition 9 If $|\delta| < 1$, then the process (22) is (covariance) stationary.

Proof. First, it is easily seen that the mean of z_t is zero. We then consider the two cases $v_t \neq u_t$ and $v_t = u_t$ separately.

Case 1: $v_t \neq u_t$. Using the lag operator and substituting in the values for a and b we can write (22) in the form

$$\left[1 - \frac{\delta}{T}(L + L^2 + \dots + L^T)\right]z_t = cv_t + \frac{\phi}{T}(L + L^2 + \dots + L^T)u_t.$$

Using the method of proof in Proposition 1 it is seen that the polynomial in the lag operator $1 - \frac{\delta}{T}(L + L^2 + \dots + L^T)$ in the left-hand side has all roots outside the unit circle if $|\delta| < 1$. Dividing both sides by this polynomial shows that z_t is the sum of two independent covariance-stationary processes

$$\frac{cv_t}{1 - \frac{\delta}{T}(L + L^2 + \dots + L^T)} \text{ and } \frac{\frac{\phi}{T}(L + L^2 + \dots + L^T)u_t}{1 - \frac{\delta}{T}(L + L^2 + \dots + L^T)}.$$

Therefore, it is itself covariance stationary.

Case 2: $v_t = u_t$. In this case, again using the method of proof in Proposition 1, it is easily seen that the process is stationary if $|\delta| < 1$. ■

The following further results are also evident.

Remark 2 If the condition for weak E-instability $\delta > 1$ holds, the process (22) is not stationary, and under weak E-stability $\delta < 1$ the process is stationary for T sufficiently large.

If the learning dynamics (22) is stationary, it is possible to proceed as in Section 2 and derive the asymptotic second moments of the process for z_t using the technique to derive Yule Walker equations. Given knowledge of these moments one can also obtain the variance of the forecast error in terms of the memory length T and the variance of the disturbances u_t and v_t . Unfortunately, it appears that unambiguous analytic results on properties of the variance of the forecast error, which would be comparable to Proposition 5, are not available. We leave the lengthy details and further analysis to another paper.

5 Concluding Remarks

Frameworks in which adaptive learning is incomplete are beginning to receive attention. This paper has provided basic analytical results for dynamics of adaptive learning when the learning rule has a finite memory and the presence of random shocks precludes exact convergence to the REE.

We focused on the case of learning a stochastic steady state. Our central result is that the E-stability principle, which plays a central role in situations of complete learning, as

discussed e.g. in (Evans and Honkapohja 1998), retains its importance in the analysis of incomplete learning, though it takes a new form. In our setup E-stability guarantees the stationarity of the dynamics of the learning economy and the unbiasedness of the forecasts.

Several open issues merit a further study. First, the nature of the linear approximation of the nonlinear framework in Section 3 is clearly worthy of a further analysis. Second, with incomplete learning it is possible to imagine criteria for choosing among different learning rules, so that they would be equilibria within some specified class of rules. (Evans and Honkapohja 1993) and (Brock and Hommes 1997) are examples of such view points. We expect to consider these issues in future work.

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