Chapter 16: Simultaneous equations models

An obvious reason for the endogeneity of explanatory variables in a regression model is **simultaneity**: that is, one or more of the “explanatory” variables are jointly determined with the “dependent” variable. Models of this sort are known as **simultaneous equations models** (SEMs), and they are widely utilized in both applied microeconomics and macroeconomics. Each equation in a SEM should be a behavioral equation which describes how one or more economic agents will react to shocks or shifts in the exogenous explanatory variables, ceteris paribus. The simultaneously-determined variables often have an equilibrium interpretation, and we consider that these variables are only observed when the underlying model is in equilibrium. For instance, a demand curve relating the quantity demanded to the price of a good, as well as income, the prices of substitute commodities, etc. conceptually would express that quantity for a range of prices. But the only price-quantity pair that we observe is that resulting from market clearing, where the quantities supplied and demanded were matched, and an equilibrium price was struck. In the context of labor supply, we might relate aggregate hours to the average wage and additional explanatory factors:

$$h_i = \beta_0 + \beta_1 w_i + \beta_2 z_1 + u_i \tag{1}$$

where the unit of observation might be the county. This is a **structural equation**, or behavioral equation, relating labor supply to its causal factors: that is, it reflects the structure of the supply side of the labor market. This equation resembles many that we have considered earlier, and we might wonder why there would be any difficulty in estimating it. But if the data relate to an aggregate–such as the hours worked at the county level, in response to the average wage in the county–this equation poses problems that would not arise if, for instance, the unit of observation was the individual, derived from a survey. Although we can assume that the individual is a price- (or wage-) taker, we cannot assume that the average level of wages is exogenous to the labor market in Suffolk County. Rather, we must consider that it is determined within the market, affected by broader economic conditions. We might consider that the z variable expresses wage levels in other areas, which would cet.par. have an effect on the supply of labor in Suffolk County; higher wages in Middlesex County would lead to a reduction in labor supply in the Suffolk County labor market, cet. par.

To complete the model, we must add a specification of labor demand:

$$h_i = \gamma_0 + \gamma_1 w_i + \gamma_2 z_2 + \upsilon_i \tag{2}$$

where we model the quantity demanded of labor as a function of the average wage and additional factors that might shift the demand curve. Since the demand for labor is a derived demand, dependent on the cost of other factors of production, we might include some measure of factor cost (e.g. the cost of capital) as this equation's z variable. In this case, we would expect that a higher cost of capital would trigger substitution of labor for capital at every level of the wage, so that $\gamma_2 > 0$. Note that the supply equation represents the behavior of workers in the aggregate, while the demand equation represents the behavior of employers in the aggregate. In equilibrium, we would equate these two equations, and expect that at some level of equilibrium labor utilization and average wage that the labor market is equilibrated. These two equations then constitute a simultaneous equations model (SEM) of the labor market.

Neither of these equations may be consistently estimated via OLS, since the wage variable in each equation is correlated with the respective error term. How do we know this? Because these two equations can be solved and rewritten as two reduced form equations in the endogenous variables h_i and w_i. Each of those variables will depend on the exogenous variables in the entire system–z_1 and z_2–as well as the structural errors u_i and v_i. In general, any shock to either labor demand or supply will affect both the equilibrium quantity and price (wage). Even if we rewrote one of these equations to place the wage variable on the left hand side, this problem would persist: both endogenous variables in the system are jointly determined by the exogenous variables and structural shocks. Another implication of this structure is that we must have separate explanatory factors in the two equations. If $z_1 = z_2$, for instance, we would not be able to solve this system and uniquely identify its structural parameters. There must be factors that are unique to each structural equation that, for instance, shift the supply curve without shifting the demand curve.

The implication here is that even if we only care about one of these structural equations–for instance, we are tasked with modelling labor supply, and have no interest in working with the demand side of the market–we must be able to specify the other structural equations of the model. We need not estimate them, but we must be able to determine what measures they would contain. For instance, consider estimating the relationship between murder rate, number of police, and wealth for a number of cities. We might expect that both of those factors would reduce the murder rate, cet.par.: more police are available to apprehend murderers, and perhaps prevent murders, while we might expect that lower-income cities might have greater unrest and crime. But can we reasonably assume that the number of police (per capita) is exogenous to the murder rate? Probably not, in the sense that cities striving to reduce crime will spend more on police. Thus we might consider a second structural equation that expressed the number of police per capita as a function of a number of factors. We may have no interest in estimating this equation (which is behavioral, reflecting the behavior of city officials), but if we are to consistently estimate the former equation–the behavioral equation reflecting the behavior of murderers–we will have to specify the second equation as well, and collect data for its explanatory factors.

Simultaneity bias in OLS

What goes wrong if we use OLS to estimate a structural equation containing endogeneous explanatory variables? Consider the structural system:

$$
\begin{aligned}
y_1 &= \alpha_1 y_2 + \beta_1 z_1 + u_1 \qquad (3)\\
y_2 &= \alpha_2 y_1 + \beta_2 z_2 + u_2
\end{aligned}
$$

in which we are interested in estimating the first equation. Assume that the z variables are exogenous, in that each is uncorrelated with each of the error processes u. What is the correlation between y_2 and u_1? If we substitute the first equation into the second, we derive:

$$
y_2 = \alpha_2 \left(\alpha_1 y_2 + \beta_1 z_1 + u_1 \right) + \beta_2 z_2 + u_2 \qquad (4)
$$

$$
\left(1 - \alpha_2 \alpha_1 \right) y_2 = \alpha_2 \beta_1 z_1 + \beta_2 z_2 + \alpha_2 u_1 + u_2 \qquad (5)
$$

If we assume that $\alpha_2 \alpha_1 \neq 1$, we can derive the **reduced form** equation for y_2 as:

$$
y_2 = \pi_{21} z_1 + \pi_{22} z_2 + \upsilon_2 \qquad (6)
$$

where the reduced form error term $\upsilon_2 = \alpha_2 u_1 + u_2$. Thus y_2 depends on u_1, and estimation by OLS of the first equation in (3) will not yield consistent estimates. We can consistently estimate the reduced form equation (6) via OLS, and that in fact is an essential part of the strategy of the 2SLS estimator. But the parameters of the structural equation are nonlinear transformations of the reduced form parameters, so being able to estimate the reduced form parameters does not achieve the goal of providing us with point and interval estimates of the structural equation.

In this special case, we can evaluate the simultaneity bias that would result from improperly applying OLS to the original structural equation. The covariance of y_2 and u_1 is equal to the covariance of y_2 and $\upsilon_2 = \left[\alpha_2/\left(1-\alpha_2\alpha_1\right)E\left(u_1^2\right)\right] = \left[\alpha_2/\left(1-\alpha_2\alpha_1\right)\right]\sigma_1^2$. If we have some priors about the signs of the α parameters, we may sign the bias. Generally, it could be either positive or negative; that is, the OLS coefficient estimate could be larger or smaller than the correct estimate, but will not be equal to the population parameter in an expected sense unless the bracketed expression is zero. Note that this would happen if $\alpha_2 = 0$: that is, if y_2 was not simultaneously determined with y_1. But in that case, we do not have a simultaneous system; the model in that case is said to be a **recursive** system, which may be consistently estimated with OLS.

Identifying and estimating a structural equation

The tool that we will apply to consistently estimate structural equations such as (3) is one that we have seen before: two-stage least squares (2SLS). The application of 2SLS in a structural system is more straightforward than the general application of instrumental variables estimators, since the specification of the system makes clear what variables are available as instruments. Let us first consider a slightly different two-equation structural system:

$$\begin{aligned} q &= \alpha_1 p + \beta_1 z_1 + u_1 \\ q &= \alpha_2 p + u_2 \end{aligned} \tag{7}$$

We presume these equations describe the workings of a market, and that the equilibrium condition of market clearing has been imposed. Let q be per capita milk consumption at the county level, p be the average price of a gallon of milk in that county, and let z_1 be the price of cattle feed. The first structural equation is thus the supply equation, with $\alpha_1 > 0$ and $\beta_1 < 0$: that is, a higher cost of production will generally reduce the quantity supplied at the same price per gallon. The second equation is the demand equation, where we presume that $\alpha_2 < 0$, reflecting the slope of the demand curve in the $\{p, q\}$ plane. Given a random sample on $\{p, q, z_1\}$, what can we achieve? The demand equation is said to be identified–in fact, **exactly identified**–since one instrument is needed, and precisely one is available. z_1 is available because the demand for milk does not depend on the price of cattle feed, so we take advantage of an exclusion restriction that makes z_1 available to identify the demand curve. Intuitively, we can think of variations in z_1 shifting the supply curve up and down, tracing out the demand curve; in doing so, it makes it possible for us to estimate the structural parameters of the demand curve.

What about the supply curve? It, also, has a problem of simultaneity bias, but it turns out that the supply equation is **unidentified**. Given the model as we have laid it out, there is no variable available to serve as an instrument for p : that is, we need a variable that affects demand (and shifts the demand curve) but does not directly affect supply. In this case, no such variable is available, and we cannot apply the instrumental variables technique without an instrument. What if we went back to the drawing board, and realized that the price of orange juice should enter the demand equation–although it tastes terrible on corn flakes, orange juice might be a healthy substitute for quenching one's thirst? Then the supply curve would be identified–exactly identified–since we now would have a single instrument that served to shift demand but did not enter the supply relation. What if we also considered the price of beer as an additional demand factor? Then we would have two available instruments (presuming that each is appropriately correlated), and 2SLS would be used to "boil them down" into the single instrument needed. In that case, we would say that the supply curve would be **overidentified**.

The identification status of each structural equation thus hinges upon **exclusion restrictions**: our *a priori* statements that certain variables do not appear in certain structural equations. If they do not appear in a structural equation, they may be used as instruments to assist in identifying the parameters of that equation. For these variables to successfully identify the parameters, they must have nonzero population parameters in the equation in which they are included. Consider an example:

$$\begin{aligned} hours &= f_1\left(\log(wage), educ, age, kidslt6, nwifeinc\right) \\ \log(wage) &= f_2\left(hours, educ, xper, xper^2\right) \end{aligned} \quad (8)$$

The first equation is a labor supply relation, expressing the number of hours worked by a married woman as a function of her wage, education, age, the number of preschool children, and non-wage income (including spouses's earnings). The second equation is a labor demand equation, expressing the wage to be paid as a function of hours worked, the employee's education, and a polynomial in her work experience. The exclusion restructions indicate that the demand for labor does not depend on the worker's age (nor should it!), the presence of preschool kids, or other resources available to the worker. Likewise, we assume that the woman's willingness to participate in the market does not depend on her labor market experience. One instrument is needed to identify each equation; `age, kidslt6` and `nwifenc` are available to identify the supply equation, while `xper` and `xper`^{2} are available to identify the demand equation. This is the **order condition** for identfication, essentially counting instruments and variables to be instrumented; each equation is overidentified. But the order condition is only necessary; the sufficient condition is the **rank condition**, which essentially states that in the reduced-form equation:

$$\log(wage) = g\left(educ, age, kidslt6, nwifeinc, xper, xper^2\right) \tag{9}$$

at least one of the population coefficients on $\{xper, xper^2\}$ must be nonzero. But since we can consistently estimate this equation with OLS, we may generate sample estimates of those coefficients, and test the joint null that both coefficients are zero. If that null is rejected, then we satisfy the rank condition for the first equation, and we may proceed to estimate it via 2SLS. The equivalent condition for the demand equation is that at least one of the population coefficients $\{age, kidslt6, nwifeinc\}$ in the regression of $hours$ on the system's exogenous variables is nonzero. If any of those variables are significant in the equivalent reduced-form equation, it may be used as an instrument to estimate the demand equation via 2SLS.

The application of two-stage least squares (via Stata's `ivreg` command) involves identifying the endogenous explanatory variable(s), the exogenous variables that are included in each equation, and the instruments that are excluded from each equation. To satisfy the order condition, the list of (excluded) instruments must be at least as long as the list of endogenous explanatory variables. This logic carries over to structural equation systems with more than two endogenous variables / equations; a structural model may have any number of endogenous variables, each defined by an equation, and we can proceed to evaluate the identification status of each equation in turn, given the appropriate exclusion restrictions. Note that if an equation is **unidentified**, due to the lack of appropriate instruments, then no econometric technique may be used to estimate its parameters. In that case, we do not have knowledge that would allow us to "trace out" that equation's slope while we move along it.

Simultaneous equations models with time series

One of the most common applications of 2SLS in applied work is the estimation of structural time series models. For instance, consider a simple macro model:

$$C_t = \beta_0 + \beta_1 (Y_t - T_t) + \beta_2 r_t + u_{1t} \tag{10}$$

$$
\begin{aligned}
I_t &= \gamma_0 + \gamma_1 r_t + u_{2t} \\
Y_t &= C_t + I_t + G_t
\end{aligned}
$$

In this system, aggregate consumption each quarter is determined jointly with disposable income. Even if we assume that taxes are exogenous (and in fact they are responsive to income), the consumption function cannot be consistently estimated via OLS. If the interest rate is taken as exogenous (set, for instance, by monetary policy makers) then the investment equation may be consistently estimated via OLS. The third equation is an identity; it need not be estimated, and holds without error, but its presence makes explicit the simultaneous nature of the model. If r is exogenous, then we need one instrument to estimate the consumption function; government spending will suffice, and consumption will be exactly identified. If r is to be taken as endogenous, we would have to add at least one equation to the model to express how monetary policy reacts to economic conditions. We might also make the investment function more realistic by including dynamics–that investment depends on lagged income, for instance, Y_{t-1} (firms make investment spending plans based on the demand for their product). This would allow Y_{t-1}, a **predetermined variable**, to be used as an additional instrument in estimation of the consumption function. We may also use lags of exogenous variables–for instance, lagged taxes or government spending–as instruments in this context.

Although this only scratches the surface of a broad set of issues relating to the estimation of structural models with time series data, it should be clear that those models will generally require instrumental variables techniques such as 2SLS for the consistent estimation of their component relationships.